Chapter 3
When Einstein Meets Minkowski

Abstract Einstein's addition law of three-dimensional relativistically admissible velocities is the corner stone of Einstein's three-vector formalism of the special theory of relativity that he founded in 1905. Soon later, in 1908, special relativity was reformulated by Minkowski who introduced his elegant four-vector formalism. In this chapter, we present the harmonious interplay between Einstein's three-dimensional velocity addition and the Minkowskian four-vector formalism of Einstein's special theory of relativity, along with its relevant consequences to the study of hyperbolic geometry in Part II and hyperbolic triangle centers in Part III of the book.

3.1 Introduction

Einstein's addition law of three-dimensional relativistically admissible velocities, studied in Chap. 1, appears in [58, 60, 63, 64] as the corner stone of Einstein's three-vector formalism of the special theory of relativity that he founded in 1905 [12]. In 1908, special relativity was reformulated by Minkowski who introduced his elegant four-vector formalism [25, 59, 73]. The elegance and usefulness of the Minkowskian formalism posed an annoying problem: the concept of the relativistic mass, according to which mass is velocity dependent, seemed too wild, defying attempts to place it under the umbrella of the Minkowskian four-vector formalism of special relativity [2, 4, 42]. This intriguing puzzle challenges our mind in this chapter as well, in our mission to capture analogies that Euclidean and hyperbolic triangle centers share.

The study of Euclidean triangle centers can be approached by considering, in classical mechanics, the center of momentum of an isolated system of three non-interacting, uniformly moving massive particles with *Newtonian masses*, and with velocities in the Euclidean 3-space $\mathbb{R}^3$ of *Newtonian velocities*. A related example is found, for instance, in [24, p. 3], in which a classical mechanical interpretation of the Euclidean triangle centroid is presented.

In full analogy, the mission of this book is to approach the study of hyperbolic triangle centers by considering, in relativistic mechanics, the center of momentum

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of an isolated system of three noninteracting, uniformly moving massive particles with *relativistic masses*, and with relativistically admissible velocities in the ball $\mathbb{R}_c^3 \subset \mathbb{R}^3$ of *Einsteinian velocities*. But, in order to harness the relativistic mass for our mission, we must tame it by placing it under the umbrella of the Minkowskian four-vector formalism of special relativity.

Accordingly, the mission of this chapter is to show how the relativistic mass is tamed when Einstein, with his three-dimensional vector formalism, and Minkowski, with his four-dimensional vector formalism, meet.

3.2 Lorentz Transformation and Minkowski's Four-Velocity

Einstein addition underlies the Lorentz transformation group of special relativity. A Lorentz transformation is a linear transformation of spacetime coordinates that fixes the spacetime origin. A Lorentz boost, $L(\mathbf{v})$, is a Lorentz transformation without rotation, parametrized by a velocity parameter $\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}_c^3$.

Being linear, the Lorentz boost has a matrix representation $L_m(\mathbf{v})$, which turns out to be [40]

$$L_m(\mathbf{v}) = \begin{pmatrix} \gamma_{\mathbf{v}} & c^{-2}\gamma_{\mathbf{v}} v_1 & c^{-2}\gamma_{\mathbf{v}} v_2 & c^{-2}\gamma_{\mathbf{v}} v_3 \\ \gamma_{\mathbf{v}} v_1 & 1 + c^{-2}\frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}}+1} v_1^2 & c^{-2}\frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}}+1} v_1 v_2 & c^{-2}\frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}}+1} v_1 v_3 \\ \gamma_{\mathbf{v}} v_2 & c^{-2}\frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}}+1} v_1 v_2 & 1 + c^{-2}\frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}}+1} v_2^2 & c^{-2}\frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}}+1} v_2 v_3 \\ \gamma_{\mathbf{v}} v_3 & c^{-2}\frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}}+1} v_1 v_3 & c^{-2}\frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}}+1} v_2 v_3 & 1 + c^{-2}\frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}}+1} v_3^2 \end{pmatrix}. \tag{3.1}$$

Employing the matrix representation (3.1) of the Lorentz transformation boost, the Lorentz boost application to spacetime coordinates takes the form

$$L(\mathbf{v})\begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} = L_m(\mathbf{v})\begin{pmatrix} t \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} =: \begin{pmatrix} t' \\ x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} t' \\ \mathbf{x}' \end{pmatrix}, \tag{3.2}$$

where $\mathbf{v} = (v_1, v_2, v_3)^t \in \mathbb{R}_c^3$, $\mathbf{x} = (x_1, x_2, x_3)^t \in \mathbb{R}^3$, $\mathbf{x}' = (x_1', x_2', x_3')^t \in \mathbb{R}^3$, and $t, t' \in \mathbb{R}$, where exponent t denotes transposition.

In our approach to special relativity, analogies with classical results form the right tool. Hence, we emphasize that in the Newtonian limit of large vacuum speed of light c, $c \to \infty$, the Lorentz boost $L(\mathbf{v})$, (3.1)–(3.2), reduces to the Galilei boost $G(\mathbf{v})$, $\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$,

$$G(\mathbf{v})\begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} = \lim_{c\to\infty} L(\mathbf{v})\begin{pmatrix} t \\ \mathbf{x} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ v_1 & 1 & 0 & 0 \\ v_2 & 0 & 1 & 0 \\ v_3 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$= \begin{pmatrix} t \\ x_1 + v_1 t \\ x_2 + v_2 t \\ x_3 + v_3 t \end{pmatrix} = \begin{pmatrix} t \\ \mathbf{x} + \mathbf{v}t \end{pmatrix}, \tag{3.3}$$

where $\mathbf{x} = (x_1, x_2, x_3)^t \in \mathbb{R}^3$ and $t \in \mathbb{R}$.

As we see from (3.2)–(3.3), our spacetime coordinates are $(t, \mathbf{x})^t$ and, as a result, the Lorentz boost matrix representation $L_m(\mathbf{v})$ in (3.1) is *non-symmetric* for $c \neq 1$. In contrast, some authors present spacetime coordinates as $(ct, \mathbf{x})^t$, resulting in a *symmetric* Lorentz boost matrix representation found, for instance, in [27, (11.98), p. 541].

Since in our approach to special relativity analogies with classical results form the right tool, the representation of spacetime coordinates as $(t, \mathbf{x})^t$ is more advantageous than its representation as $(ct, \mathbf{x})^t$. Indeed, unlike the latter representation, the former representation of spacetime coordinates allows one to recover the Galilei boost from the Lorentz boost by taking the Newtonian limit of large speed of light c, as shown in the transition from (3.2) to (3.3).

As a result of adopting $(t, \mathbf{x})^t$ rather than $(ct, \mathbf{x})^t$ as our four-vector that represents four-position, our four-velocity is given by $(\gamma_{\mathbf{v}}, \gamma_{\mathbf{v}} \mathbf{v})$ rather than $(\gamma_{\mathbf{v}} c, \gamma_{\mathbf{v}} \mathbf{v})$, $\mathbf{v} \in \mathbb{R}^3_c$. Similarly, our four-momentum is given by

$$\begin{pmatrix} p_0 \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} \frac{E}{c^2} \\ \mathbf{p} \end{pmatrix} = m \begin{pmatrix} \gamma_{\mathbf{v}} \\ \gamma_{\mathbf{v}} \mathbf{v} \end{pmatrix} \tag{3.4}$$

rather than the standard four-momentum, which is given by $(p_0, \mathbf{p})^t = (E/c, \mathbf{p})^t = (m\gamma_{\mathbf{v}} c, m\gamma_{\mathbf{v}} \mathbf{v})^t$, as found in most relativity physics books. According to (3.4), the relativistically invariant mass (that is, rest mass) m of a particle is the ratio of the particle's four-momentum $(p_0, \mathbf{p})^t$ to its four-velocity $(\gamma_{\mathbf{v}}, \gamma_{\mathbf{v}} \mathbf{v})^t$.

For the sake of simplicity, and without loss of generality, some authors normalize the vacuum speed of light to $c = 1$ as, for instance, in [17]. We, however, prefer to leave c as a free positive parameter, enabling related modern results to be reduced to classical ones under the limit of large c, $c \to \infty$, as, for instance, in the transition from a Lorentz boost into a corresponding Galilei boost in (3.1)–(3.3).

The Lorentz boost (3.1)–(3.2) can be written vectorially in the form

$$L(\mathbf{u}) \begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} = \begin{pmatrix} \gamma_{\mathbf{u}} (t + \frac{1}{c^2} \mathbf{u} \cdot \mathbf{x}) \\ \gamma_{\mathbf{u}} \mathbf{u} t + \mathbf{x} + \frac{1}{c^2} \frac{\gamma_{\mathbf{u}}^2}{1+\gamma_{\mathbf{u}}} (\mathbf{u} \cdot \mathbf{x}) \mathbf{u} \end{pmatrix}. \tag{3.5}$$

Being written in a vector form, the Lorentz boost in (3.5) survives unimpaired in higher dimensions. Rewriting (3.5) in higher dimensional spaces, with $\mathbf{x} = \mathbf{v}t$, $\mathbf{u}, \mathbf{v} \in \mathbb{R}_c^n \subset \mathbb{R}^n$, we have

$$
\begin{aligned}
L(\mathbf{u}) \begin{pmatrix} t \\ \mathbf{v}t \end{pmatrix} &= \begin{pmatrix} \gamma_{\mathbf{u}} (t + \frac{1}{c^2}\mathbf{u}\cdot\mathbf{v}t) \\ \gamma_{\mathbf{u}} \mathbf{u}t + \mathbf{v}t + \frac{1}{c^2}\frac{\gamma_{\mathbf{u}}^2}{1+\gamma_{\mathbf{u}}}(\mathbf{u}\cdot\mathbf{v}t)\mathbf{u} \end{pmatrix} \\
&= \begin{pmatrix} \frac{\gamma_{\mathbf{u}\oplus\mathbf{v}}}{\gamma_{\mathbf{v}}} t \\ \frac{\gamma_{\mathbf{u}\oplus\mathbf{v}}}{\gamma_{\mathbf{v}}} (\mathbf{u}\oplus\mathbf{v})t \end{pmatrix} .
\end{aligned} \tag{3.6}
$$

Equation (3.6) reveals explicitly the way Einstein velocity addition underlies the Lorentz boost. The second equation in (3.6) follows from the first by (1.7), p. 5, and (1.2), p. 4.

The special case of $t = \gamma_{\mathbf{v}}$ in (3.6) proves useful, giving rise to the elegant identity

$$
L(\mathbf{u}) \begin{pmatrix} \gamma_{\mathbf{v}} \\ \gamma_{\mathbf{v}}\, \mathbf{v} \end{pmatrix} = \begin{pmatrix} \gamma_{\mathbf{u}\oplus\mathbf{v}} \\ \gamma_{\mathbf{u}\oplus\mathbf{v}} (\mathbf{u}\oplus\mathbf{v}) \end{pmatrix} \tag{3.7}
$$

of the Lorentz boost of four-velocities, $\mathbf{u}, \mathbf{v} \in \mathbb{R}_c^n$. Since in physical applications $n = 3$, in the context of n-dimensional special relativity we call $\mathbf{v}$ a three-vector and $(\gamma_{\mathbf{v}}, \gamma_{\mathbf{v}}\, \mathbf{v})^t$ a four-vector, etc.

The four-vector $m(\gamma_{\mathbf{v}}, \gamma_{\mathbf{v}}\, \mathbf{v})^t$ is the four-momentum of a particle with invariant mass (or, rest mass) m and velocity $\mathbf{v}$ relative to a given inertial rest frame $\Sigma_{\mathbf{0}}$. Let $\Sigma_{\ominus\mathbf{u}}$ be an inertial frame that moves with velocity $\ominus\mathbf{u} = -\mathbf{u}$ relative to the rest frame $\Sigma_{\mathbf{0}}$, $\mathbf{u}, \mathbf{v} \in \mathbb{R}_c^n$. Then, the particle with velocity $\mathbf{v}$ relative to $\Sigma_{\mathbf{0}}$ has velocity $\mathbf{u}\oplus\mathbf{v}$ relative to the frame $\Sigma_{\ominus\mathbf{u}}$. In full agreement and, owing to the linearity of the Lorentz boost, it follows from (3.7) that the four-momentum of the particle relative to the frame $\Sigma_{\ominus\mathbf{u}}$ is

$$
\begin{aligned}
L(\mathbf{u})m \begin{pmatrix} \gamma_{\mathbf{v}} \\ \gamma_{\mathbf{v}}\, \mathbf{v} \end{pmatrix} &= mL(\mathbf{u}) \begin{pmatrix} \gamma_{\mathbf{v}} \\ \gamma_{\mathbf{v}}\, \mathbf{v} \end{pmatrix} \\
&= m \begin{pmatrix} \gamma_{\mathbf{u}\oplus\mathbf{v}} \\ \gamma_{\mathbf{u}\oplus\mathbf{v}} (\mathbf{u}\oplus\mathbf{v}) \end{pmatrix} .
\end{aligned} \tag{3.8}
$$

It follows from the linearity of the Lorentz boost and from (3.7) that

$$
\begin{aligned}
L(\mathbf{w}) \sum_{k=1}^{N} m_k \begin{pmatrix} \gamma_{\mathbf{v}_k} \\ \gamma_{\mathbf{v}_k} \mathbf{v}_k \end{pmatrix} &= \sum_{k=1}^{N} m_k L(\mathbf{w}) \begin{pmatrix} \gamma_{\mathbf{v}_k} \\ \gamma_{\mathbf{v}_k} \mathbf{v}_k \end{pmatrix} \\
&= \sum_{k=1}^{N} m_k \begin{pmatrix} \gamma_{\mathbf{w}\oplus\mathbf{v}_k} \\ \gamma_{\mathbf{w}\oplus\mathbf{v}_k} (\mathbf{w}\oplus\mathbf{v}_k) \end{pmatrix}
\end{aligned}
$$

$$= \begin{pmatrix} \sum_{k=1}^{N} m_k \gamma_{\mathbf{w}\oplus\mathbf{v}_k} \\ \sum_{k=1}^{N} m_k \gamma_{\mathbf{w}\oplus\mathbf{v}_k} (\mathbf{w}\oplus\mathbf{v}_k) \end{pmatrix}. \tag{3.9}$$

The chain of equations (3.9) reveals the interplay of Einstein addition, $\oplus$, in $\mathbb{R}_c^n$ and vector addition, $+$, in $\mathbb{R}^n$ that appears implicitly in the sigma-notation for scalar and vector addition. This harmonious interplay between $\oplus$ and $+$, which is crucially important in our mission to determine hyperbolic triangle centers, reveals itself in (3.9) where Einstein's three-vector formalism of special relativity meets Minkowski's four-vector formalism of special relativity.

The (Minkowski) norm of a four-vector is Lorentz transformation invariant. The norm of the four-position $(t, \mathbf{x})^t$ is

$$\left\| \begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} \right\| = \sqrt{t^2 - \frac{\|\mathbf{x}\|^2}{c^2}} \tag{3.10}$$

and, accordingly, the norm of the four-velocity $(\gamma_{\mathbf{v}}, \gamma_{\mathbf{v}} \mathbf{v})^t$ is

$$\left\| \begin{pmatrix} \gamma_{\mathbf{v}} \\ \gamma_{\mathbf{v}} \mathbf{v} \end{pmatrix} \right\| = \gamma_{\mathbf{v}} \left\| \begin{pmatrix} 1 \\ \mathbf{v} \end{pmatrix} \right\| = \gamma_{\mathbf{v}} \sqrt{1 - \frac{\|\mathbf{v}\|^2}{c^2}} = 1. \tag{3.11}$$

3.3 Invariant Mass of a System of Particles

In obtaining the result in (3.8), we exploit the linearity of the Lorentz boost. We will now further exploit that linearity, demonstrated in (3.9), to obtain the relativistically invariant mass of a system of particles. Being observer's invariant, we refer the Newtonian, rest mass, m, to as the (relativistically) invariant mass, as opposed to the common relativistic mass, $m\gamma_{\mathbf{v}}$, which is observer's dependent.

Let

$$S = S(m_k, \mathbf{v}_k, \Sigma_{\mathbf{0}}, k = 1, \ldots, N) \tag{3.12}$$

be an isolated system of N noninteracting material particles the kth particle of which has invariant mass $m_k > 0$ and velocity $\mathbf{v}_k \in \mathbb{R}_c^n$ relative to an inertial frame $\Sigma_{\mathbf{0}}$, $k = 1, \ldots, N$.

Classically, the Newtonian mass m_{newton} of the system S equals the sum of the Newtonian masses of its constituent particles, that is,

$$m_{newton} = \sum_{k=1}^{N} m_k, \tag{3.13}$$

and it forms the total mass of the system. Relativistically, however, this need not be the case since dark matter may emerge, as we will see in Theorem 3.2 of Sect. 3.4.

Accordingly, we wish to determine the relativistically invariant mass m_0 of the system S, and the velocity $\mathbf{v}_0$ relative to $\Sigma_\mathbf{0}$ of a fictitious inertial frame, called the center of momentum frame, relative to which the three-momentum of S vanishes.

Assuming that the four-momentum is additive, the sum of the four-momenta of the N particles of the system S gives the four-momentum $(m_0\gamma_{\mathbf{v}_0}, m_0\gamma_{\mathbf{v}_0}\mathbf{v}_0)^t$ of S. Accordingly,

$$\sum_{k=1}^{N} m_k \begin{pmatrix} \gamma_{\mathbf{v}_k} \\ \gamma_{\mathbf{v}_k}\mathbf{v}_k \end{pmatrix} = m_0 \begin{pmatrix} \gamma_{\mathbf{v}_0} \\ \gamma_{\mathbf{v}_0}\mathbf{v}_0 \end{pmatrix}, \tag{3.14}$$

where

(i) The invariant masses $m_k > 0$ and the velocities $\mathbf{v}_k \in \mathbb{R}^n_c$, $k = 1, \ldots, N$, relative to $\Sigma_\mathbf{0}$ of the constituent particles of S are given, while
(ii) The invariant mass m_0 of S and the velocity $\mathbf{v}_0$ of the center of momentum frame of S relative to $\Sigma_\mathbf{0}$ are to be determined uniquely by the *Resultant Relativistically Invariant Mass Theorem*, which is Theorem 3.2 in Sect. 3.4

If $m_0 > 0$ and $\mathbf{v}_0 \in \mathbb{R}^n_c$ that satisfy (3.14) exist then, as anticipated, the three-momentum of the system S relative to its center of momentum frame vanishes since, by (3.8) and (3.14), the four-momentum of S relative to its center of momentum frame is given by

$$\begin{aligned} L(\ominus\mathbf{v}_0)\sum_{k=1}^{N} m_k \begin{pmatrix} \gamma_{\mathbf{v}_k} \\ \gamma_{\mathbf{v}_k}\mathbf{v}_k \end{pmatrix} &= L(\ominus\mathbf{v}_0) m_0 \begin{pmatrix} \gamma_{\mathbf{v}_0} \\ \gamma_{\mathbf{v}_0}\mathbf{v}_0 \end{pmatrix} \\ &= m_0 \begin{pmatrix} \gamma_{\ominus\mathbf{v}_0\oplus\mathbf{v}_0} \\ \gamma_{\ominus\mathbf{v}_0\oplus\mathbf{v}_0}(\ominus\mathbf{v}_0\oplus\mathbf{v}_0) \end{pmatrix} = m_0 \begin{pmatrix} 1 \\ \mathbf{0} \end{pmatrix}, \end{aligned} \tag{3.15}$$

noting that $\gamma_{\ominus\mathbf{v}_0\oplus\mathbf{v}_0} = \gamma_\mathbf{0} = 1$.

3.4 The Resultant Relativistically Invariant Mass Theorem

Lemma 3.1 below presents an identity that we need for the proof of the *Resultant Relativistically Invariant Mass Theorem* 3.2.

Lemma 3.1 *Let N be any positive integer, and let $m_k \in \mathbb{R}$ and $\mathbf{v}_k \in \mathbb{R}^n_c$, $k = 1, \ldots, N$, be N real numbers and N points of an Einstein gyrogroup $\mathbb{R}^n_c = (\mathbb{R}^n_c, \oplus)$. Then*

$$\left(\sum_{k=1}^{N} m_k \gamma_{\mathbf{v}_k} \frac{\mathbf{v}_k}{c}\right)^2 = \left(\sum_{k=1}^{N} m_k \gamma_{\mathbf{v}_k}\right)^2 - \left\{\left(\sum_{k=1}^{N} m_k\right)^2 + 2\sum_{\substack{j,k=1\\ j<k}}^{h} m_j m_k (\gamma_{\ominus\mathbf{v}_j\oplus\mathbf{v}_k} - 1)\right\}. \tag{3.16}$$

Proof The proof is given by the following chain of equations, which are numbered for subsequent explanation:

$$
\begin{aligned}
&\left(\sum_{k=1}^{N} m_k \gamma_{\mathbf{v}_k} \frac{\mathbf{v}_k}{c}\right)^2 \\
&\overbrace{=\!=}^{(1)} \sum_{k=1}^{N} m_k^2 \gamma_{\mathbf{v}_k}^2 \frac{\mathbf{v}_k^2}{c^2} + 2 \sum_{\substack{j,k=1\\ j<k}}^{h} m_j m_k \gamma_{\mathbf{v}_j} \gamma_{\mathbf{v}_k} \frac{\mathbf{v}_j \cdot \mathbf{v}_k}{c^2} \\
&\overbrace{=\!=}^{(2)} \sum_{k=1}^{N} m_k^2 (\gamma_{\mathbf{v}_k}^2 - 1) + 2 \sum_{\substack{j,k=1\\ j<k}}^{h} m_j m_k (\gamma_{\mathbf{v}_j} \gamma_{\mathbf{v}_k} - \gamma_{\ominus \mathbf{v}_j \oplus \mathbf{v}_k}) \\
&\overbrace{=\!=}^{(3)} \sum_{k=1}^{N} m_k^2 \gamma_{\mathbf{v}_k}^2 - \sum_{k=1}^{N} m_k^2 + 2 \sum_{\substack{j,k=1\\ j<k}}^{h} m_j m_k \gamma_{\mathbf{v}_j} \gamma_{\mathbf{v}_k} \\
&\qquad - 2 \sum_{\substack{j,k=1\\ j<k}}^{h} m_j m_k \gamma_{\ominus \mathbf{v}_j \oplus \mathbf{v}_k} \\
&\overbrace{=\!=}^{(4)} \left(\sum_{k=1}^{N} m_k \gamma_{\mathbf{v}_k}\right)^2 - \left\{\sum_{k=1}^{N} m_k^2 + 2 \sum_{\substack{j,k=1\\ j<k}}^{h} m_j m_k \gamma_{\ominus \mathbf{v}_j \oplus \mathbf{v}_k}\right\} \\
&\overbrace{=\!=}^{(5)} \left(\sum_{k=1}^{N} m_k \gamma_{\mathbf{v}_k}\right)^2 - \left\{\left(\sum_{k=1}^{N} m_k\right)^2 + 2 \sum_{\substack{j,k=1\\ j<k}}^{h} m_j m_k (\gamma_{\ominus \mathbf{v}_j \oplus \mathbf{v}_k} - 1)\right\}.
\end{aligned}
\tag{3.17}
$$

The assumption $\mathbf{v}_k \in \mathbb{R}_c^n$ implies, by (1.3), p. 4, that all gamma factors in (3.16)–(3.17) are real and greater than 1. Derivation of the numbered equalities in (3.17) follows:

1. This equation is obtained by an expansion of the square of a sum of vectors in $\mathbb{R}^n$.
2. Follows from Item 1 by (1.9)–(1.10), p. 5.
3. Follows from Item 2 by an obvious expansion.
4. Follows from Item 3 by an expansion of the square of a sum of real numbers.
5. Follows from Item 4 by an expansion of another square of a sum of real numbers. □

Einstein velocity addition law (1.2), p. 4, admits the following theorem:

Theorem 3.2 (Resultant Relativistically Invariant Mass Theorem) *Let* $(\mathbb{R}_c^n, \oplus)$ *be an Einstein gyrogroup, and let* $m_k \in \mathbb{R}$ *and* $\mathbf{v}_k \in \mathbb{R}_c^n$, $k = 1, 2, \ldots, N$, *be* N *real* *numbers and N elements of* $\mathbb{R}_c^n$ *satisfying*

$$\sum_{k=1}^{N} m_k \gamma_{\mathbf{v}_k} \neq 0. \tag{3.18}$$

Furthermore, let

$$\sum_{k=1}^{N} m_k \begin{pmatrix} \gamma_{\mathbf{v}_k} \\ \gamma_{\mathbf{v}_k} \mathbf{v}_k \end{pmatrix} = m_0 \begin{pmatrix} \gamma_{\mathbf{v}_0} \\ \gamma_{\mathbf{v}_0} \mathbf{v}_0 \end{pmatrix} \tag{3.19}$$

be an $(n+1)$*-vector equation for the two unknowns* $m_0 \in \mathbb{R}$ *and* $\mathbf{v}_0 \in \mathbb{R}^n$.

Then (3.19) *possesses a unique solution* $(m_0, \mathbf{v}_0)$, $m_0 \neq 0$, $\mathbf{v}_0 \in \mathbb{R}_c^n$, *satisfying the following three identities for all* $\mathbf{w} \in \mathbb{R}_c^n$ (*including, in particular, the interesting special case of* $\mathbf{w} = \mathbf{0}$):

$$\mathbf{w} \oplus \mathbf{v}_0 = \frac{\sum_{k=1}^{N} m_k \gamma_{\mathbf{w} \oplus \mathbf{v}_k} (\mathbf{w} \oplus \mathbf{v}_k)}{\sum_{k=1}^{N} m_k \gamma_{\mathbf{w} \oplus \mathbf{v}_k}}, \tag{3.20}$$

$$\gamma_{\mathbf{w} \oplus \mathbf{v}_0} = \frac{\sum_{k=1}^{N} m_k \gamma_{\mathbf{w} \oplus \mathbf{v}_k}}{m_0}, \tag{3.21}$$

$$\gamma_{\mathbf{w} \oplus \mathbf{v}_0} (\mathbf{w} \oplus \mathbf{v}_0) = \frac{\sum_{k=1}^{N} m_k \gamma_{\mathbf{w} \oplus \mathbf{v}_k} (\mathbf{w} \oplus \mathbf{v}_0)}{m_0}, \tag{3.22}$$

where

$$m_0 = \sqrt{\left(\sum_{k=1}^{N} m_k\right)^2 + 2 \sum_{\substack{j,k=1 \\ j<k}}^{N} m_j m_k (\gamma_{\ominus(\mathbf{w} \oplus \mathbf{v}_j) \oplus (\mathbf{w} \oplus \mathbf{v}_k)} - 1)}. \tag{3.23}$$

Proof Following (3.18), we assume, without loss of generality, that

$$\sum_{k=1}^{N} m_k \gamma_{\mathbf{v}_k} > 0 \tag{3.24}$$

(otherwise, we replace each m_k by $-m_k$, resulting in the replacement of m_0 by $-m_0$). Let us consider the following four equations, (3.25)–(3.28), which are specialized from (3.20)–(3.23) by taking $\mathbf{w} = \mathbf{0}$:

$$\mathbf{v}_0 = \frac{\sum_{k=1}^{N} m_k \gamma_{\mathbf{v}_k} \mathbf{v}_k}{\sum_{k=1}^{N} m_k \gamma_{\mathbf{v}_k}}, \tag{3.25}$$

$$\gamma_{\mathbf{v}_0} = \frac{\sum_{k=1}^{N} m_k \gamma_{\mathbf{v}_k}}{m_0}, \tag{3.26}$$

$$\gamma_{\mathbf{v}_0}\mathbf{v}_0 = \frac{\sum_{k=1}^{N} m_k \gamma_{\mathbf{v}_k} \mathbf{v}_0}{m_0}, \tag{3.27}$$

where

$$m_0 = \sqrt{\left(\sum_{k=1}^{N} m_k\right)^2 + 2\sum_{\substack{j,k=1\\ j<k}}^{N} m_j m_k (\gamma_{\ominus \mathbf{v}_j \oplus \mathbf{v}_k} - 1)}. \tag{3.28}$$

The proof of this theorem consists of two parts. In the first part of the proof, we show that if (3.19) for the unknowns $\mathbf{v}_0 \in \mathbb{R}^n$ and $m_0 \in \mathbb{R}$ possesses a solution, then the solution must be given uniquely by $\mathbf{v}_0$ of (3.25) and m_0 of (3.28), with $\mathbf{v}_0 \in \mathbb{R}^n_c$ and $m_0 > 0$, satisfying (3.26)–(3.27).

In the second part of the proof, we show that $\mathbf{v}_0$ of (3.25) and m_0 of (3.28), indeed, form a solution of (3.19) for the unknowns $\mathbf{v}_0 \in \mathbb{R}^n_c$ and $m_0 > 0$, and that the solution satisfies (3.20)–(3.23).

Part I In this part of the proof, we assume that there exist $m_0 \in \mathbb{R}$ and $\mathbf{v}_0 \in \mathbb{R}^n$ that satisfy (3.19). Then, the norms of the two sides of (3.19) are equal while, by (3.11), the norm of the right-hand side of (3.19) is m_0. Hence, the norm of the left-hand side of (3.19) equals m_0 as well, obtaining the following chain of equations, which are numbered for subsequent explanation:

$$\begin{aligned}
m_0^2 &\overbrace{=\!=}^{(1)} \left\| \sum_{k=1}^{N} m_k \begin{pmatrix} \gamma_{\mathbf{v}_k} \\ \gamma_{\mathbf{v}_k}\mathbf{v}_k \end{pmatrix} \right\|^2 \\
&\overbrace{=\!=}^{(2)} \left\| \begin{pmatrix} \sum_{k=1}^{N} m_k \gamma_{\mathbf{v}_k} \\ \sum_{k=1}^{N} m_k \gamma_{\mathbf{v}_k}\mathbf{v}_k \end{pmatrix} \right\|^2 \\
&\overbrace{=\!=}^{(3)} \left(\sum_{k=1}^{N} m_k \gamma_{\mathbf{v}_k}\right)^2 - \left(\sum_{k=1}^{N} m_k \gamma_{\mathbf{v}_k} \frac{\mathbf{v}_k}{c}\right)^2 \\
&\overbrace{=\!=}^{(4)} \left(\sum_{k=1}^{N} m_k\right)^2 + 2\sum_{\substack{j,k=1\\ j<k}}^{h} m_j m_k \gamma_{\ominus \mathbf{v}_j \oplus \mathbf{v}_k} - 1.
\end{aligned} \tag{3.29}$$

Derivation of the numbered equalities in (3.29) follows:

1. This equation follows from the result that the norm of the left-hand side of (3.19) equals the norm of the right-hand side of (3.19), the latter being m_0 by (3.11).
2. Follows from Item 1 by the common "four-vector" addition of $(n+1)$-vectors (where $n = 3$ in physical applications).
3. Follows from Item 2 by (3.10).
4. Follows from Item 3 by Identity (3.16) of Lemma 3.1.

It follows from the upper entry of (3.19), along with Assumption (3.24) that

$$m_0 > 0. \tag{3.30}$$

We thus obtained in (3.29) the desired equation, (3.28), for m_0.

Hence, if $m_0 \in \mathbb{R}$ and $\mathbf{v}_0 \in \mathbb{R}^n$ that satisfy (3.19) exist, m_0 is positive and must be given by (3.28). Clearly, $\mathbf{v}_0 \in \mathbb{R}^n_c$ since $\gamma_{\mathbf{v}_0}$ is real, as we see from (3.19).

By assumption, $\mathbf{v}_0$ satisfies (3.19). Equation (3.19) is equivalent to two equations, formed by the upper entry and by the lower entry of (3.19). Dividing the lower entry of (3.19) by its upper entry, noting that $m_0 \neq 0$ by (3.30), we obtain (3.25).

Similarly, dividing the upper entry of (3.19) by $m_0 > 0$ we obtain (3.26), and dividing the lower entry of (3.19) by $m_0 > 0$ we obtain (3.27).

Hence, if $m_0 \in \mathbb{R}$ and $\mathbf{v}_0 \in \mathbb{R}^n$ that satisfy (3.19) exist, then $m_0 > 0$, $\mathbf{v}_0 \in \mathbb{R}^n_c$, and they must be given by (3.28) and (3.25), and satisfy (3.26)–(3.27).

Part II In Part I, we have shown that if (3.19) possesses a solution for the unknowns $\mathbf{v}_0 \in \mathbb{R}^n$ and $m_0 \in \mathbb{R}$, then $\mathbf{v}_0 \in \mathbb{R}^n_c$ is given uniquely by (3.25) and $m_0 > 0$ is given uniquely by (3.28), satisfying (3.26)–(3.27). We will now show that, indeed, $\mathbf{v}_0 \in \mathbb{R}^n_c$, given by (3.25), and $m_0 > 0$, given by (3.28), form a solution of (3.19), and that the solution satisfies (3.20)–(3.23). Accordingly, in this second part of the proof we assume that $\mathbf{v}_0 \in \mathbb{R}^n_c$ and $m_0 > 0$ are given by (3.25) and (3.28), and we will prove that the pair $(m_0, \mathbf{v}_0)$ forms a solution of (3.19).

It follows from Identity (3.16) of Lemma 3.1, along with m_0 of (3.28) that

$$\left(\sum_{k=1}^{N} m_k \gamma_{\mathbf{v}_k} \frac{\mathbf{v}_k}{c}\right)^2 = \left(\sum_{k=1}^{N} m_k \gamma_{\mathbf{v}_k}\right)^2 - m_0^2. \tag{3.31}$$

Hence, by (3.25) and (3.31), we have the following chain of equations, which are numbered for subsequent explanation:

$$\begin{aligned}
\frac{\mathbf{v}_0^2}{c^2} &\overbrace{=\!=}^{(1)} \frac{(\sum_{k=1}^{N} m_k \gamma_{\mathbf{v}_k} \frac{\mathbf{v}_k}{c})^2}{(\sum_{k=1}^{N} m_k \gamma_{\mathbf{v}_k})^2} \\
&\overbrace{=\!=}^{(2)} \frac{(\sum_{k=1}^{N} m_k \gamma_{\mathbf{v}_k})^2 - m_0^2}{(\sum_{k=1}^{N} m_k \gamma_{\mathbf{v}_k})^2} \\
&=\!= 1 - \frac{m_0^2}{(\sum_{k=1}^{N} m_k \gamma_{\mathbf{v}_k})^2}.
\end{aligned} \tag{3.32}$$

Derivation of the numbered equalities in (3.32) follows:

1. This equation is given by Assumption (3.25).
2. Follows from Item 1 by (3.31).

It follows from (3.32) that

$$\gamma_{\mathbf{v}_0} = \frac{1}{\sqrt{1-\frac{\mathbf{v}_0^2}{c^2}}} = \frac{\sum_{k=1}^{N} m_k \gamma_{\mathbf{v}_k}}{m_0}, \tag{3.33}$$

thus verifying (3.26).

Following (3.26) and (3.25), we have

$$\gamma_{\mathbf{v}_0} \mathbf{v}_0 = \frac{\sum_{k=1}^{N} m_k \gamma_{\mathbf{v}_k} \mathbf{v}_k}{m_0}, \tag{3.34}$$

thus verifying (3.27).

Finally, (3.33) implies that m_0 and $\mathbf{v}_0$ satisfy the upper entry of (3.19) and, similarly, (3.34) implies that m_0 and $\mathbf{v}_0$ satisfy the lower entry of (3.19). Hence, the pair consisting of m_0 and $\mathbf{v}_0$ forms a solution of (3.19). We have thus shown that $\mathbf{v}_0 \in \mathbb{R}_c^n$ and $m_0 > 0$ given by (3.25) and (3.28) form a solution of (3.19), and that this solution satisfies (3.26)–(3.28).

To complete the proof it remains to show that the pair $(m_0, \mathbf{v}_0)$ satisfies (3.20)–(3.23) as well.

Let us first show that m_0 given by (3.28) is given by (3.23) as well. Indeed, following (1.3), p. 4, (1.33), p. 10, and (1.65), p. 18, we have

$$\gamma_{\ominus(\mathbf{w}\oplus\mathbf{v}_j)\oplus(\mathbf{w}\oplus\mathbf{v}_k)} = \gamma_{\ominus\mathbf{v}_j\oplus\mathbf{v}_k}, \tag{3.35}$$

implying that the right-hand sides of (3.23) and (3.28) are equal, so that m_0 is independent of $\mathbf{w} \in \mathbb{R}_c^n$, as desired. As such, m_0 is given by each of (3.23) and (3.28).

We have thus shown that the unique solution of (3.19) is formed by $\mathbf{v}_0 \in \mathbb{R}_c^n$ and $m_0 > 0$ that are given by (3.25) and (3.28), and that the solution satisfies (3.26)–(3.27). It, therefore, remains to show that the solution satisfies (3.20)–(3.22) as well.

Applying the Lorentz boost $L(\mathbf{w})$, $\mathbf{w} \in \mathbb{R}_c^n$, to each side of (3.19), we have

$$L(\mathbf{w}) \sum_{k=1}^{N} m_k \begin{pmatrix} \gamma_{\mathbf{v}_k} \\ \gamma_{\mathbf{v}_k} \mathbf{v}_k \end{pmatrix} = L(\mathbf{w}) m_0 \begin{pmatrix} \gamma_{\mathbf{v}_0} \\ \gamma_{\mathbf{v}_0} \mathbf{v}_0 \end{pmatrix}. \tag{3.36}$$

Following the linearity of the Lorentz boost, illustrated in (3.8) and (3.9), (3.36) can be written as

$$\sum_{k=1}^{N} m_k \begin{pmatrix} \gamma_{\mathbf{w}\oplus\mathbf{v}_k} \\ \gamma_{\mathbf{w}\oplus\mathbf{v}_k}(\mathbf{w}\oplus\mathbf{v}_k) \end{pmatrix} = m_0 \begin{pmatrix} \gamma_{\mathbf{w}\oplus\mathbf{v}_0} \\ \gamma_{\mathbf{w}\oplus\mathbf{v}_0}(\mathbf{w}\oplus\mathbf{v}_0\ ) \end{pmatrix}. \tag{3.37}$$

Equation (3.37) is identical with (3.19) in which $\mathbf{v}_k \in \mathbb{R}_c^n$ is replaced by $\mathbf{w}\oplus\mathbf{v}_k \in \mathbb{R}_c^n$, $k = 0, 1, \ldots, N$.

But, the unique solution of (3.19) is the pair $(m_0 > 0, \mathbf{v}_0 \in \mathbb{R}_c^n)$ that satisfies (3.25)–(3.28). Hence, the unique solution of (3.37) is the pair $(m_0 > 0, \mathbf{w}\oplus\mathbf{v}_0 \in \mathbb{R}_c^n)$

that satisfies (3.20)–(3.23). Hence, the unique solution ($m_0 > 0, \mathbf{v}_0 \in \mathbb{R}_c^n$) of (3.19) satisfies not only (3.25)–(3.28) but, more generally, (3.20)–(3.23), and the proof is complete. □

In physical applications to particle systems, the real numbers m_k in Theorem 3.2 represent particle masses. As such, they are positive so that Assumption (3.24) is satisfied. However, anticipating applications of Theorem 3.2 to barycentric coordinates in hyperbolic geometry, in Chap. 4, we need the validity of Theorem 3.2 for real numbers m_k that need not be positive as well.

We have thus established in Theorem 3.2 the following four results concerning an isolated system S, (3.12),

$$S = S(m_k, \mathbf{v}_k, \Sigma_{\mathbf{0}}, k = 1, \dots, N) \tag{3.38}$$

of N noninteracting material particles the kth particle of which has invariant mass $m_k > 0$ and velocity $\mathbf{v}_k \in \mathbb{R}_c^n$ relative to an inertial frame $\Sigma_{\mathbf{0}}$, $k = 1, \dots, N$:

1. The relativistically invariant (or, rest) mass m_0 of the system S is given by

$$m_0 = \sqrt{\left(\sum_{k=1}^{N} m_k\right)^2 + 2\sum_{\substack{j,k=1\\ j<k}}^{N} m_j m_k (\gamma_{\ominus\mathbf{v}_j \oplus \mathbf{v}_k} - 1)} \tag{3.39}$$

   according to (3.23) with $\mathbf{w} = \mathbf{0}$.
2. The relativistic mass of the system S is

$$m_0 \gamma_{\mathbf{v}_0} \tag{3.40}$$

   relative to the rest frame $\Sigma_{\mathbf{0}}$, where $\mathbf{v}_0$ is the velocity of the center of momentum frame of S relative to $\Sigma_{\mathbf{0}}$, given by

$$\mathbf{v}_0 = \frac{\sum_{k=1}^{N} m_k \gamma_{\mathbf{v}_k} \mathbf{v}_k}{\sum_{k=1}^{N} m_k \gamma_{\mathbf{v}_k}} \tag{3.41}$$

   according to (3.20) with $\mathbf{w} = \mathbf{0}$.
3. Like energy and momentum, the relativistic mass is additive, that is, in particular for the system S relative to the rest frame $\Sigma_{\mathbf{0}}$, by (3.21) with $\mathbf{w} = \mathbf{0}$,

$$m_0 \gamma_{\mathbf{v}_0} = \sum_{k=1}^{N} m_k \gamma_{\mathbf{v}_k} . \tag{3.42}$$

4. The relativistic mass $m_0 \gamma_{\mathbf{v}_0}$ of a system meshes extraordinarily well with the Minkowskian four-vector formalism of special relativity. In particular, for the system S relative to the rest frame $\Sigma_{\mathbf{0}}$, we have, by (3.19),

$$\sum_{k=1}^{N} \begin{pmatrix} m_k \gamma_{\mathbf{v}_k} \\ m_k \gamma_{\mathbf{v}_k} \mathbf{v}_k \end{pmatrix} = \begin{pmatrix} m_0 \gamma_{\mathbf{v}_0} \\ m_0 \gamma_{\mathbf{v}_0} \mathbf{v}_0 \end{pmatrix}, \tag{3.43}$$

where m_0 and $\mathbf{v}_0$ are given uniquely by (3.39) and (3.41).

Thus, the relativistically invariant mass m_0 of a particle system S in (3.39) gives rise to its associated relativistic mass $m_0\gamma_{\mathbf{v}_0}$ relative to the rest frame $\Sigma_{\mathbf{0}}$. The latter, in turn, brings in (3.43) the concept of the relativistic mass into conformity with the Minkowskian four-vector formalism of special relativity. Moreover, we will see in Sects. 3.7 and 3.8 that the relativistically invariant mass m_0 of a particle system S provides a natural interpretation of observations in astrophysics and in particle physics.

To appreciate the power and elegance of Theorem 3.2 in relativistic mechanics in terms of novel analogies that it shares with familiar results in classical mechanics, we present below the classical counterpart, Theorem 3.3, of Theorem 3.2. The latter is obtained from the former by approaching the Newtonian limit when c tends to infinity. The resulting Theorem 3.3 is immediate, and its importance in classical mechanics is well-known.

Theorem 3.3 (Resultant Newtonian Invariant Mass Theorem) *Let* $(\mathbb{R}^n, +)$ *be a Euclidean* n*-space, and let* $m_k \in \mathbb{R}$ *and* $\mathbf{v}_k \in \mathbb{R}^n$, $k = 1, 2, \dots, N$, *be* N *real numbers and* N *elements of* $\mathbb{R}^n$ *satisfying*

$$\sum_{k=1}^{N} m_k \neq 0. \tag{3.44}$$

Furthermore, let

$$\sum_{k=1}^{N} m_k \begin{pmatrix} 1 \\ \mathbf{v}_k \end{pmatrix} = m_0 \begin{pmatrix} 1 \\ \mathbf{v}_0 \end{pmatrix} \tag{3.45}$$

be an $(n+1)$*-vector equation for the two unknowns* $m_0 \in \mathbb{R}$ *and* $\mathbf{v}_0 \in \mathbb{R}^n$.

Then (3.45) *possesses a unique solution* $(m_0, \mathbf{v}_0)$, $m_0 \neq 0$, *satisfying the following equations for all* $\mathbf{w} \in \mathbb{R}^n$ (*including, in particular, the interesting special case of* $\mathbf{w} = \mathbf{0}$):

$$\mathbf{w} + \mathbf{v}_0 = \frac{\sum_{k=1}^{N} m_k(\mathbf{w} + \mathbf{v}_k)}{\sum_{k=1}^{N} m_k} \tag{3.46}$$

and

$$m_0 = \sum_{k=1}^{N} m_k. \tag{3.47}$$

Proof While the proof of Theorem 3.3 is trivial, our point is to present a proof that emphasizes how Theorem 3.3 is derived from Theorem 3.2. Indeed, in the limit as $c \to \infty$, the results of Theorem 3.2 tend to corresponding results of Theorem 3.3, noting that in this limit gamma factors tend to 1. Accordingly, Theorem 3.3 is a special case of Theorem 3.2 corresponding to $c = \infty$. $\square$

In physical applications to particle systems, the real numbers m_k in Theorem 3.3 represent particle masses and, hence, they are positive. However, anticipating applications of Theorem 3.3 to barycentric coordinates in Euclidean geometry, in Chap. 4, we need the validity of Theorem 3.3 for real numbers m_k that need not be positive as well.

Identity (3.46) of Theorem 3.3 is immediate. Yet, it is geometrically important. The geometric importance of the validity of (3.46) for all $\mathbf{w} \in \mathbb{R}^n$ lies on its implication that the velocity $\mathbf{v}_0$ of the center of momentum frame of a particle system relative to a given inertial rest frame in classical mechanics is independent of the choice of the origin of the classical velocity space $\mathbb{R}^n$ with its underlying standard model of Euclidean geometry.

Unlike Identity (3.46) of Theorem 3.3, which is immediate, its counterpart in Theorem 3.2, Identity (3.20), is not immediate. Yet, in full analogy with Theorem 3.3, the validity of Identity (3.20) in Theorem 3.2 for all $\mathbf{w} \in \mathbb{R}_c^n$ is geometrically important. This geometric importance of Identity (3.20) lies on its implication that the velocity $\mathbf{v}_0$ of the center of momentum frame of a particle system relative to a given inertial rest frame in relativistic mechanics is independent of the choice of the origin of the relativistic velocity space $\mathbb{R}_c^n$ with its underlying Cartesian–Beltrami–Klein ball model of hyperbolic geometry.

3.5 Mass and Velocity of Particle Systems

In this section, we emphasize the analogies that the classical mass and velocity of a particle system share with their relativistic counterparts. Let

$$\begin{aligned} S_{newton} &= S_{newton}\big(m_k, \mathbf{v}_k \in \mathbb{R}^n, \Sigma_\mathbf{0}, k = 1, \ldots, N\big), \\ S_{einstein} &= S_{einstein}\big(m_k, \mathbf{v}_k \in \mathbb{R}_c^n, \Sigma_\mathbf{0}, k = 1, \ldots, N\big) \end{aligned} \tag{3.48}$$

be a Newtonian particle system and its corresponding Einsteinian particle system. Each of these is an isolated system of N noninteracting material particles the kth particle of which has invariant mass $m_k > 0$ and velocity $\mathbf{v}_k$ relative to a rest frame $\Sigma_\mathbf{0}$. These velocities are Newtonian, $\mathbf{v}_k \in \mathbb{R}^n$, for the Newtonian system S_{newton} and Einsteinian, $\mathbf{v}_k \in \mathbb{R}_c^n$, for the Einsteinian system $S_{einstein}$.

The mass m_0 in Identity (3.45) of Theorem 3.3 is the Newtonian mass of the Newtonian particle system S_{newton}. It is given by, (3.47),

$$m_0 = \sum_{k=1}^{N} m_k. \tag{3.49}$$

The velocity $\mathbf{v}_0 \in \mathbb{R}^n$ in Identity (3.45) is the Newtonian velocity of the center of momentum frame of the system S_{newton} relative to $\Sigma_\mathbf{0}$. It is given by (3.46) with $\mathbf{w} = \mathbf{0}$,

$$\mathbf{v}_0 = \frac{\sum_{k=1}^{N} m_k \mathbf{v}_k}{\sum_{k=1}^{N} m_k} \tag{3.50}$$

satisfying (3.46), i.e.,

$$\mathbf{w} + \mathbf{v}_0 = \frac{\sum_{k=1}^{N} m_k(\mathbf{w} + \mathbf{v}_k)}{\sum_{k=1}^{N} m_k} \tag{3.51}$$

for all $\mathbf{w} \in \mathbb{R}^n$.

In full analogy, the mass m_0 in Identity (3.19), p. 66, of Theorem 3.2 is the Einsteinian rest mass of the Einsteinian particle system $S_{einstein}$. It is given by (3.28) as

$$m_0 = \sqrt{\left(\sum_{k=1}^{N} m_k\right)^2 + 2\sum_{\substack{j,k=1\\ j<k}}^{N} m_j m_k(\gamma_{\ominus\mathbf{v}_j\oplus\mathbf{v}_k} - 1)} \tag{3.52}$$

satisfying (3.23), i.e.,

$$m_0 = \sqrt{\left(\sum_{k=1}^{N} m_k\right)^2 + 2\sum_{\substack{j,k=1\\ j<k}}^{N} m_j m_k(\gamma_{\ominus(\mathbf{w}\oplus\mathbf{v}_j)\oplus(\mathbf{w}\oplus\mathbf{v}_k)} - 1)} \tag{3.53}$$

for all $\mathbf{w} \in \mathbb{R}_c^n$.

The velocity $\mathbf{v}_0 \in \mathbb{R}_c^n$ in Identity (3.19) is the Einsteinian velocity of the center of momentum frame of the system $S_{einstein}$ relative to $\Sigma_{\mathbf{0}}$. It is given by (3.25) as

$$\mathbf{v}_0 = \frac{\sum_{k=1}^{N} m_k \gamma_{\mathbf{v}_k} \mathbf{v}_k}{\sum_{k=1}^{N} m_k \gamma_{\mathbf{v}_k}} \tag{3.54}$$

satisfying (3.20), i.e.,

$$\mathbf{w}\oplus\mathbf{v}_0 = \frac{\sum_{k=1}^{N} m_k \gamma_{\mathbf{w}\oplus\mathbf{v}_k}(\mathbf{w}\oplus\mathbf{v}_k)}{\sum_{k=1}^{N} m_k \gamma_{\mathbf{w}\oplus\mathbf{v}_k}} \tag{3.55}$$

for all $\mathbf{w} \in \mathbb{R}_c^n$.

Accordingly, the relativistic mass of the system $S_{einstein}$ is $m_0\gamma_{\mathbf{v}_0}$.

In the Newtonian limit of large c, $c \to \infty$, gamma factors tend to 1. Hence, in that limit the relativistically invariant rest mass m_0 in (3.52)–(3.53) tends to its Newtonian counterpart m_0 in (3.49), and the relativistic center of momentum velocity $\mathbf{v}_0 \in \mathbb{R}_c^n$ in (3.54)–(3.55) tends to its corresponding classical center of momentum velocity $\mathbf{v}_0 \in \mathbb{R}^n$ in (3.50)–(3.51).

3.6 The Relativistic Mass is Additive

Suppose that the system S, (3.38), is made up of M subsystems each itself a system of particles. Let $m_{0,p}$ be the relativistically invariant mass and $\mathbf{v}_{0,p}$ the center of momentum frame velocity of the pth subsystem, $p = 1, \dots, M$, so that the relativistic mass of the pth subsystem is $m_{0,p}\gamma_{\mathbf{v}_{0,p}}$.

Then, the relativistic mass $m_0\gamma_{\mathbf{v}_0}$ of the system S, given by (3.39)–(3.41), is additive, that is, it possesses the relativistic mass additivity property

$$m_0\gamma_{\mathbf{v}_0} = \sum_{p=1}^{M} m_{0,p}\gamma_{\mathbf{v}_{0,p}}. \tag{3.56}$$

For simplicity, we prove the relativistic mass additivity property (3.56) for the case of $M = 2$ subsystems, the proof for any $M > 2$ being similar.

Let us, therefore, view the system S of N particles, (3.38)–(3.41), with $N \ge 3$, as a system of the two subsystems S_1 and S_2,

$$\begin{aligned} S_1 &= S_1(m_k, \mathbf{v}_k, \Sigma_{\mathbf{0}}, k = 1, \dots, N_1), \\ S_2 &= S_2(m_k, \mathbf{v}_k, \Sigma_{\mathbf{0}}, k = N_1 + 1, \dots, N) \end{aligned} \tag{3.57}$$

for any fixed N_1, $1 < N_1 < N$.

Then, the relativistically invariant masses $m_{0,1}$ and $m_{0,2}$ of the subsystems S_1 and S_2 and their center of momentum frame velocities, $\mathbf{v}_{0,1}$ and $\mathbf{v}_{0,2}$ relative to $\Sigma_{\mathbf{0}}$, respectively, are

$$\begin{aligned} m_{0,1} &= \sqrt{\left(\sum_{k=1}^{N_1} m_k\right)^2 + 2\sum_{\substack{j,k=1\\ j<k}}^{N_1} m_j m_k(\gamma_{\ominus\mathbf{v}_j\oplus\mathbf{v}_k} - 1)}, \\ m_{0,2} &= \sqrt{\left(\sum_{k=N_1+1}^{N} m_k\right)^2 + 2\sum_{\substack{j,k=N_1+1\\ j<k}}^{N} m_j m_k(\gamma_{\ominus\mathbf{v}_j\oplus\mathbf{v}_k} - 1)}, \end{aligned} \tag{3.58}$$

and

$$\begin{aligned} \mathbf{v}_{0,1} &= \frac{\sum_{k=1}^{N_1} m_k\gamma_{\mathbf{v}_k}\mathbf{v}_k}{\sum_{k=1}^{N_1} m_k\gamma_{\mathbf{v}_k}}, \\ \mathbf{v}_{0,2} &= \frac{\sum_{k=N_1+1}^{N} m_k\gamma_{\mathbf{v}_k}\mathbf{v}_k}{\sum_{k=N_1+1}^{N} m_k\gamma_{\mathbf{v}_k}}, \end{aligned} \tag{3.59}$$

possessing the relativistic mass additivity property

$$m_{0,1}\gamma_{\mathbf{v}_{0,1}} + m_{0,2}\gamma_{\mathbf{v}_{0,2}} = m_0\gamma_{\mathbf{v}_0}. \tag{3.60}$$

The proof of the additivity property (3.60) follows from (3.42) immediately. Indeed, by applying the identity in (3.42) to each of the particle systems S_1, S_2 and S, we have the chain of equations

$$
\begin{aligned}
m_{0,1}\gamma_{\mathbf{v}_{0,1}} + m_{0,2}\gamma_{\mathbf{v}_{0,2}} &= \sum_{k=1}^{N_1} m_k\gamma_{\mathbf{v}_k} + \sum_{k=N_1+1}^{N} m_k\gamma_{\mathbf{v}_k} \\
&= \sum_{k=1}^{N} m_k\gamma_{\mathbf{v}_k} \\
&= m_0\gamma_{\mathbf{v}_0} \,. \qquad (3.61)
\end{aligned}
$$

The relativistically invariant mass m_0 of a system of particles S, (3.38)–(3.41), leads to its associated relativistic mass $m_0\gamma_{\mathbf{v}_0}$, (3.42). We thus see that, in special relativity, while relativistically invariant mass m_0 is not additive [36], relativistic mass, $m_0\gamma_{\mathbf{v}_0}$ is additive.

3.7 The Relativistically Invariant Mass of a System in Astrophysics

The resultant relativistically invariant mass m_0, (3.39),

$$
m_0 \;=\; \sqrt{\left(\sum_{k=1}^{N} m_k\right)^2 + 2\sum_{\substack{j,k=1\\ j<k}}^{N} m_j m_k(\gamma_{\ominus\mathbf{v}_j\oplus\mathbf{v}_k} - 1)} \qquad (3.62)
$$

of a particle system $S = S(m_k, \mathbf{v}_k, \Sigma_{\mathbf{0}}, N)$ comprises two distinct kinds of relativistically invariant mass that represent the Newtonian contribution and the relativistic contribution. These two distinct kinds of mass are:

1. The Newtonian mass m_{newton},

$$
m_{newton} := \sum_{k=1}^{N} m_k \qquad (3.63)
$$

   which is the sum of the invariant, rest masses of the particles that constitute the system S, as in (3.13).
2. The dark mass m_{dark},

$$
m_{dark} := \sqrt{2\sum_{\substack{j,k=1\\ j<k}}^{N} m_j m_k(\gamma_{\ominus\mathbf{v}_j\oplus\mathbf{v}_k} - 1)} . \qquad (3.64)
$$

The dark mass of a particle system S, given by (3.64), depends on the *velocity dispersion* of S, that is, on the spread of internal velocities $\mathbf{v}_{jk} = \ominus\mathbf{v}_j\oplus\mathbf{v}_k$, $1 \le j <$

$k \le N$, of the constituent particles of S relative to each other. In other words, the dark mass in (3.64) measures the extent to which the system S deviates away from rigidity. Gravitationally, dark mass behaves just like ordinary mass, as postulated in cosmology [6, p. 37]. However, it is undetectable by all means other than gravity since it is fictitious, or virtual, in the sense that it is generated solely by relative motion between constituent objects of the system.

We thus see from (3.64) that the dark mass of a system may be viewed as a measure of mass that results solely from the velocity dispersion of the system. In astrophysics, the velocity dispersion of stars or galaxies in a cluster is estimated by measuring the radial velocities of selected constituents. Once the velocity distribution is known, the cluster's mass is calculated by using the *virial theorem* [5].

If one attributes the dark mass m_{dark} of a galaxy to the mass of a central dark object, a black hole, a correlation should result between the mass of a black hole and the velocity dispersion of its host galaxy. Indeed, Ferrarese and Merritt reported in 2000 that "The masses of supermassive black holes correlate almost perfectly with the velocity dispersion of their host bulges" [16]; and Gebhardt et al. remarked that the resulting relation is of interest "because it implies that central black hole mass is constrained and closely related to properties of the host galaxy's bulge" [19]. A recent improved version of the black hole mass (M)–velocity dispersion (σ) relation (called the M–σ relation) and black hole mass–luminosity (L) relation (called the M–L relation) was reported by Gültekin et al. [23].

Dark matter was introduced into cosmology as an ad hoc postulate, hypothesized to provide observed missing gravitational force [7]. In contrast, dark mass emerges here as a consequence of the covariance of Einstein's special theory of relativity, and it stems from relative motion between constituent objects of a system. All relative velocities between the constituent particles of a *rigid* system vanish, so that if the system S is rigid, then $\ominus\mathbf{v}_j \oplus \mathbf{v}_k = \mathbf{0}$, $j, k = 1, \dots, N$. This, in turn, implies by (3.64) that the dark mass of a rigid system vanishes.

The mass m_{newton} and the dark mass m_{dark} of a system S are relativistically invariant, and are composed according to the Pythagorean formula

$$m_0 = \sqrt{m_{newton}^2 + m_{dark}^2}, \tag{3.65}$$

giving rise to the invariant resultant rest mass m_0 of the system S in (3.62)–(3.64).

3.8 The Relativistically Invariant Mass of a System in Particle Physics

Following the four-momentum in (3.4), p. 61, and the four-vector norm (3.10)–(3.11) we have

$$\left\| \begin{pmatrix} p_0 \\ \mathbf{p} \end{pmatrix} \right\| = \sqrt{\frac{E^2}{c^4} - \frac{\|\mathbf{p}\|^2}{c^2}} = m, \tag{3.66}$$

where, by (3.4),

$$\begin{aligned} p_0 &= m\gamma_{\mathbf{v}}, \\ E &= m\gamma_{\mathbf{v}} c^2, \\ \mathbf{p} &= m\gamma_{\mathbf{v}} \mathbf{v}. \end{aligned} \tag{3.67}$$

Assuming that both energy, E, and three-momentum, $\mathbf{p}$, are additive is equivalent to assuming that the four-momentum is additive. The latter assumption, in turn, led us to identity (3.14), p. 64, that we now write as

$$\begin{pmatrix} \frac{E}{c^2} \\ \mathbf{p} \end{pmatrix} = \sum_{k=1}^{N} \begin{pmatrix} \frac{E_k}{c^2} \\ \mathbf{p}_k \end{pmatrix} = \sum_{k=1}^{N} m_k \begin{pmatrix} \gamma_{\mathbf{v}_k} \\ \gamma_{\mathbf{v}_k} \mathbf{v}_k \end{pmatrix} = m_0 \begin{pmatrix} \gamma_{\mathbf{v}_0} \\ \gamma_{\mathbf{v}_0} \mathbf{v}_0 \end{pmatrix}. \tag{3.68}$$

Here, in (3.68),

$$\begin{aligned} E_k &= m_k \gamma_{\mathbf{v}_k} c^2, \\ \mathbf{p}_k &= m_k \gamma_{\mathbf{v}_k} \mathbf{v}_k \end{aligned} \tag{3.69}$$

are the energy and momentum of the kth particle of the system S, $k = 1, \ldots, N$, and accordingly,

$$\begin{aligned} E &= \sum_{k=1}^{N} E_k, \\ \mathbf{p} &= \sum_{k=1}^{N} \mathbf{p}_k \end{aligned} \tag{3.70}$$

are the energy and momentum of the system S.

Furthermore, as in (3.14), $\mathbf{v}_0$ is the velocity of the center of momentum frame of S relative to the rest frame $\Sigma_{\mathbf{0}}$, and m_0 is the resultant invariant mass of S.

Noting (3.11), p. 63, the norms of the two extreme sides of (3.68) give the equation

$$m_0 = \sqrt{\frac{E^2}{c^4} - \frac{\|\mathbf{p}\|^2}{c^2}} \tag{3.71}$$

where E and $\mathbf{p}$ are given by (3.70). Identity (3.71) demonstrates, by the relativistic four-vector formalism, that the resultant mass m_0 of a particle system S in (3.39) is relativistically invariant, being the norm of a four-vector.

Identity (3.71), written equivalently as

$$E^2 = m_0^2 c^4 + \|\mathbf{p}\|^2 c^2 \tag{3.72}$$

is known in particle physics as the *energy–momentum relation*. For a particle in its inertial rest frame, where $\mathbf{p} = \mathbf{0}$, Relation (3.72) reduces to Einstein's famous formula

$$E = m_0 c^2. \tag{3.73}$$

The energy–momentum relation (3.71)–(3.72) is used in particle physics to calculate the relativistically invariant mass m_0 of a system of particles in terms of the total energy E and momentum $\mathbf{p}$ of the system.

As an illustrative example, let us consider two particles with rest (or, Newtonian) masses m_1 and m_2, and velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ relative to an inertial rest frame $\Sigma_\mathbf{0}$, respectively. If these particles were to collide and stick, the rest mass m_0 and the velocity $\mathbf{v}_0$ relative to $\Sigma_\mathbf{0}$ of the resulting composite particle would satisfy the four-momentum conservation law (3.14), p. 64, that is,

$$m_0 \begin{pmatrix} \gamma_{\mathbf{v}_0} \\ \gamma_{\mathbf{v}_0} \mathbf{v}_0 \end{pmatrix} = m_1 \begin{pmatrix} \gamma_{\mathbf{v}_1} \\ \gamma_{\mathbf{v}_1} \mathbf{v}_1 \end{pmatrix} + m_2 \begin{pmatrix} \gamma_{\mathbf{v}_2} \\ \gamma_{\mathbf{v}_2} \mathbf{v}_2 \end{pmatrix}. \tag{3.74}$$

Hence, by (3.39) and (3.65),

$$\begin{aligned} m_0 &= \sqrt{(m_1 + m_2)^2 + 2m_1 m_2 (\gamma_{\ominus \mathbf{v}_1 \oplus \mathbf{v}_2} - 1)} \\ &= \sqrt{m_{newton}^2 + m_{dark}^2}, \end{aligned} \tag{3.75}$$

where

$$\begin{aligned} m_{newton} &= m_1 + m_2, \\ m_{dark} &= 2m_1 m_2 (\gamma_{\ominus \mathbf{v}_1 \oplus \mathbf{v}_2} - 1) > 0, \end{aligned} \tag{3.76}$$

and, by (3.41),

$$\mathbf{v}_0 = \frac{m_1 \gamma_{\mathbf{v}_1} \mathbf{v}_1 + m_2 \gamma_{\mathbf{v}_2} \mathbf{v}_2}{m_1 \gamma_{\mathbf{v}_1} + m_2 \gamma_{\mathbf{v}_2}}. \tag{3.77}$$

Hence, the relativistic mass of the composite particle is $m_0 \gamma_{\mathbf{v}_0}$, where m_0 is given by (3.75), and $\mathbf{v}_0$ is given by (3.77).

It is clear from (3.75)–(3.76) that the Newtonian mass, m_{newton}, is conserved during the collision. It is only the total invariant mass, m_0, which is increased following the collision owing to the emergence of the dark mass m_{dark}.

Examples of particles that collide and stick, as described in (3.74)–(3.77), are observed in experimental searches for new particles in high-energy particle colliders.

We thus see that owing to the introduction of the relativistically invariant mass m_0 in (3.62), along with its Newtonian and dark mass components in (3.63)–(3.65), the concept of the relativistic mass fits well under the umbrella of the four-vector formalism of special relativity. Moreover, we see that the resulting dark mass emerges naturally not only in the interpretation of observations in astrophysics, demonstrated qualitatively in Sect. 3.7, but also in the interpretation of observations in particle physics, demonstrated qualitatively in this section. Naturally, we will find in

Sect. 4.2 of Chap. 4 that the relativistically invariant mass m_0 in (3.62) is what we need for the introduction of barycentric coordinates into hyperbolic geometry. The latter, in turn, is what we need for the determination of hyperbolic triangle centers.

3.9 Remarkable Analogies

In this section, we emphasize the analogies in Theorems 3.2, p. 65, and 3.3, p. 71, that the classical mass and center of momentum velocity of a particle system in (3.78a)–(3.78d) below share with their relativistic counterparts in (3.79a)–(3.79d) below.

Seeking a way to place the relativistic mass $m_0\gamma_{\mathbf{v}_0}$ of a particle system S under the umbrella of the Minkowskian four-vector formalism of special relativity, we have uncovered the novel, relativistically invariant, or rest, mass m_0 of a particle system, presented in (3.79d) below. Furthermore, following the discovery of m_0 in (3.62), we have uncovered remarkable analogies that Newtonian and Einsteinian mechanics share.

To see the analogies clearly, let us consider the following well known classical results, (3.78a)–(3.78d) below, which are involved in the determination of the Newtonian resultant mass m_0 and the classical center of momentum velocity of a Newtonian system of particles, and to which we will subsequently present our Einsteinian analogs that have been discovered in Theorem 3.2. Let

$$S = S(m_k, \mathbf{v}_k, \Sigma_{\mathbf{0}}, k = 1, \ldots, N), \quad \mathbf{v}_k \in \mathbb{R}^n \tag{3.78a}$$

be an isolated Newtonian system of N noninteracting material particles the kth particle of which has mass m_k and Newtonian uniform velocity $\mathbf{v}_k$ relative to an inertial frame $\Sigma_{\mathbf{0}}$, $k = 1, \ldots, N$. Furthermore, let m_0 be the resultant mass of S, considered as the mass of a virtual particle located at the center of momentum of S, and let $\mathbf{v}_0$ be the Newtonian velocity relative to $\Sigma_{\mathbf{0}}$ of the Newtonian center of momentum frame of S. Then we have the following well-known identities:

$$1 = \frac{1}{m_0}\sum_{k=1}^{N} m_k \tag{3.78b}$$

and

$$\begin{aligned} \mathbf{v}_0 &= \frac{1}{m_0}\sum_{k=1}^{N} m_k\mathbf{v}_k, \\ \mathbf{w} + \mathbf{v}_0 &= \frac{1}{m_0}\sum_{k=1}^{N} m_k(\mathbf{w} + \mathbf{v}_k), \end{aligned} \tag{3.78c}$$

where the binary operation $+$ is the common vector addition in $\mathbb{R}^n$, and where

$$m_0 = \sum_{k=1}^{N} m_k \tag{3.78d}$$

for $\mathbf{v}, \mathbf{w}_k \in \mathbb{R}^3$, $m_k > 0$, $k = 0, 1, \ldots, N$.

In full analogy with (3.78a), let

$$S = S(m_k, \mathbf{v}_k, \Sigma_{\mathbf{0}}, k = 1, \ldots, N), \quad \mathbf{v}_k \in \mathbb{R}^n_c \tag{3.79a}$$

be an isolated Einsteinian system of N noninteracting material particles the kth particle of which has invariant mass m_k and Einsteinian uniform velocity $\mathbf{v}_k$ relative to an inertial frame $\Sigma_{\mathbf{0}}$, $k = 1, \ldots, N$. Furthermore, let m_0 be the resultant mass of S, considered as the mass of a virtual particle located at the center of mass of S (calculated in (3.29)), and let $\mathbf{v}_0$ be the Einsteinian velocity relative to $\Sigma_{\mathbf{0}}$ of the Einsteinian center of momentum of the Einsteinian system S. Then, as shown in Theorem 3.2, the relativistic analogs of the Newtonian expressions in (3.78b)–(3.78d) are, respectively, the following Einsteinian expressions in (3.79b)–(3.79d),

$$\begin{aligned} \gamma_{\mathbf{v}_0} &= \frac{1}{m_0} \sum_{k=1}^{N} m_k \gamma_{\mathbf{v}_k}, \\ \gamma_{\mathbf{u}\oplus\mathbf{v}_0} &= \frac{1}{m_0} \sum_{k=1}^{N} m_k \gamma_{\mathbf{u}\oplus\mathbf{v}_k}, \end{aligned} \tag{3.79b}$$

and

$$\begin{aligned} \gamma_{\mathbf{v}_0} \mathbf{v}_0 &= \frac{1}{m_0} \sum_{k=1}^{N} m_k \gamma_{\mathbf{v}_k} \mathbf{v}_k, \\ \gamma_{\mathbf{w}\oplus\mathbf{v}_0} (\mathbf{w}\oplus\mathbf{v}_0) &= \frac{1}{m_0} \sum_{k=1}^{N} m_k \gamma_{\mathbf{w}\oplus\mathbf{v}_k} (\mathbf{w}\oplus\mathbf{v}_k), \end{aligned} \tag{3.79c}$$

where the binary operation $\oplus$ is the Einstein velocity addition in $\mathbb{R}^n_c$, given by (1.2), p. 4, and where

$$m_0 = \sqrt{\left(\sum_{k=1}^{N} m_k\right)^2 + 2 \sum_{\substack{j,k=1 \\ j<k}}^{N} m_j m_k (\gamma_{\ominus\mathbf{v}_j\oplus\mathbf{v}_k} - 1)} \tag{3.79d}$$

for $\mathbf{w}, \mathbf{v}_k \in \mathbb{R}^3_c$, $m_k > 0$, $k = 0, 1, \ldots, N$. Here m_0 is the relativistic invariant mass of the Einsteinian system S, supposed concentrated at the relativistic center of mass of S, and $\mathbf{v}_0$ is the Einsteinian velocity relative to $\Sigma_{\mathbf{0}}$ of the Einsteinian center of momentum frame of the Einsteinian system S.

To conform with the Minkowskian four-vector formalism of special relativity, both m_0 and $\mathbf{v}_0$ are determined in Theorem 3.2 as the unique solution of the Minkowskian four-vector equation (3.19).

We finally wrote (3.62) as (3.65), i.e.,

$$m_0 = \sqrt{m_{newton}^2 + m_{dark}^2}, \tag{3.80}$$

viewing the relativistically invariant, or rest, mass m_0 of the system S as a Pythagorean composition of the Newtonian rest mass, m_{newton} and the dark mass, m_{dark} of S. The mass m_{dark} is *dark* in the sense that it is the mass of virtual matter that does not collide and does not emit radiation. Following observations in cosmology, one may postulate that our dark mass reveals its presence only gravitationally. We have shown qualitatively that (3.80) explains observations in both astrophysics and particle physics.

We should remark that the presence of our dark mass is predicted by theoretic special relativistic techniques. Hence, it need not account for the whole mass of dark matter observed by astrophysicists in the cosmos because there could be contributions from general relativistic considerations and, perhaps, other unknown sources.

3.10 Problems

Problem 3.1 Matrix Representation of the Lorentz Boost:
Show that the Lorentz boost $L(\mathbf{u})$, given vectorially by (3.5), p. 61, is a linear map that possesses the matrix representation (3.1), p. 60.