Chapter 10
Epilogue

Abstract The success of the use of Einstein addition along with Einstein velocity dependent relativistic mass in the determination of various hyperbolic triangle centers is demonstrated in this book. But, looking at the relativistic velocity addition law and its underlying hyperbolic geometry through the lens of cosmological stellar aberration leads to a startling conclusion: relativistic velocities add in the cosmos according to the gyroparallelogram addition law of hyperbolic geometry, which is commutative. This Epilogue of the book may thus serve as the Prologue for the future of Einstein's special relativity theory as a theory regulated by the hyperbolic geometry of Bolyai and Lobachevsky.

10.1 Introduction

The hero of this book is Albert Einstein, whose addition law forms the powerful and elegant tool that enables hyperbolic triangle centers and relations between them to be determined. He introduced his addition law (1.2), p. 4, in his 1905 paper that founded the special theory of relativity [12]. We note that the Euclidean 3-vector algebra was not so widely known in 1905 and, consequently, was not used by Einstein. Indeed, Einstein calculated in [12] the behavior of the velocity components parallel and orthogonal to the relative velocity between inertial systems, which is as close as one can get without vectors to the vectorial version (1.2) that we use in this book. Soon later, in 1908–1910, Vladimir Varičak demonstrated in [66, 67] that Einstein velocity addition law has interpretation in hyperbolic geometry.

Being neither commutative nor associative, Einstein addition law seemed to be structureless until its gyro-structure was discovered by the author in 1988 [55]. Einstein's failure to recognize and advance the extraordinarily rich nonassociative algebraic structure that his addition law encodes contributed to the eclipse of his velocity addition law, creating a void that could be filled only with Minkowskian relativity.

Minkowskian relativity is the reformulation of Einsteinian relativity based on the Lorentz transformation of four-vectors rather than Einstein addition of three-vectors. The term "Minkowskian relativity" was coined by Lewis Pyenson in [43,

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Fundamental Theories of Physics 166,
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p. 146]. The historical struggle between Einsteinian relativity and Minkowskian relativity is skillfully described by the renowned historian of relativity physics Scott Walter in [70] where, for the first time, the term "Minkowskian relativity" appears in a title. According to Walter [72], the reason we have space–time formalism today is that Minkowski's friend, Sommerfeld, took it upon himself to rewrite Minkowski's formalism, and to make it look like ordinary vector analysis. The basic distinction between Einsteinian special relativity and Minkowskian special relativity is that in the former Einstein velocity addition law and its three-vectors form the primitive concept from which the Lorentz transformation group is derived, while in the latter it is the Lorentz transformation group and its four-vectors that form the primitive concept.

The gyrovector space approach to special relativity in this book and in its forerunners [58, 60, 63, 64] tilts the balance away from Minkowskian special relativity toward Einsteinian special relativity. The success of Einstein addition law as a powerful tool in the search for hyperbolic triangle centers is Einstein's triumph. Surprisingly, the success of Einstein addition law in hyperbolic geometry also indicates that this extraordinarily useful law is incomplete, as explained in the sequel.

Ignoring forces, do uniform relativistically admissible velocities in the Universe add according to Einstein velocity addition law? Owing to analogies that (i) the Einstein addition law of Einsteinian, relativistically admissible, velocities and (ii) the vector addition law of Newtonian, classical velocities share, it is commonly accepted that uniform velocities in the Universe add according to Einstein addition law. This belief is partially supported by Fizeau's 1851 experiment [38] that validates experimentally the physical significance of Einstein velocity addition law for parallel velocities.

However, in addition to the remarkable analogies that Einstein velocity addition law of Einsteinian velocities and the common vector addition law of Newtonian velocities share, there is a remarkable disanalogy:

1. The common vector addition law gives rise to the well-known triangle and parallelogram addition law, and the resulting triangle and parallelogram addition law coincide with the common vector addition law.
2. In full analogy, Einstein velocity addition law gives rise to a *gyrotriangle* and a *gyroparallelogram addition law* as well, as we have seen in Sect. 5.3, p. 120, and as one can see, in more detail, in [63]. In contrast, however, the resulting gyrotriangle and gyroparallelogram addition law do not coincide with Einstein addition. Thus, for instance, Einstein gyroparallelogram addition law, (5.11), p. 124, is commutative while, in contrast, Einstein addition law, (1.2), p. 4, is noncommutative.

This remarkable disanalogy raises a natural question: Ignoring forces, do uniform velocities in the Universe add (i) according to Einstein velocity addition law, or (ii) according to the gyrotriangle and the gyroparallelogram addition law to which Einstein addition law gives rise?

The answer to the question lies in the *stellar aberration* effect. Stellar aberration results from the velocity of the Earth in its annual orbit about the Sun, discovered by the English astronomer James Bradley in the 1720s [53].

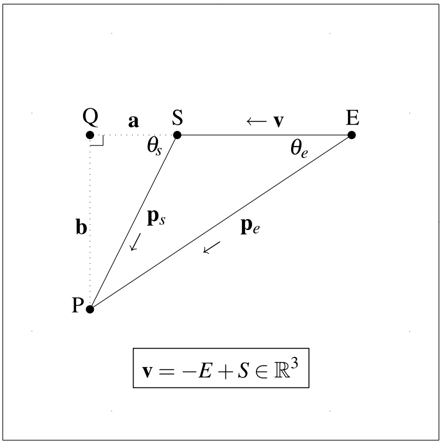

Fig. 10.1 Classical Stellar Aberration and Particle Aberration: Classical interpretation in terms of the triangle law or, equivalently, the parallelogram law, of addition of Newtonian velocities in the standard model of 3-dimensional Euclidean geometry $(\mathbb{R}^3, +)$. Two dimensions are shown for graphical clarity. Here $+$ is the common vector addition in $\mathbb{R}^3$. A particle P moves with Newtonian velocity $\mathbf{p}_e$ ($\mathbf{p}_s$) relative to the Earth E (the Sun S), making an angle θ_e (θ_s) with the Newtonian velocity $\mathbf{v}$ of the Sun S relative to the Earth E. In order to calculate the Newtonian, classical, particle aberration $\theta_s - \theta_e$, the Euclidean triangle ESP is augmented into the Euclidean right-angled triangle EQP, allowing elementary trigonometry to be employed. Points are given by their orthogonal Cartesian coordinates (x, y, z), $x^2 + y^2 + z^2 < \infty$. The coordinates are not shown. The Euclidity of $(\mathbb{R}^3, +)$ is determined by the Euclidean metric in which the distance between two points $A, B \in \mathbb{R}^3$ is $\|-A + B\|$. Classical stellar aberration is a special classical particle aberration when the particle is a photon emitted from a star

10.2 Stellar Aberration

Stellar aberration in the Universe, described in Figs. 10.1–10.2, is our laboratory, and we are the experimenters asking whether relativistic velocities add

(i) According to Einstein velocity addition law (1.2), p. 4, or
(ii) According to Einstein gyroparallelogram addition law (5.11), p. 124

Fortunately, the cosmological phenomenon of *stellar aberration* comes to the rescue. Owing to the validity of well-known relativistic stellar aberration formulas, we will find here that

1. Einsteinian, relativistically admissible velocities that need not be parallel do not add according to Einstein velocity addition law. Rather, they are gyrovectors that add according to the gyroparallelogram addition law (5.11), Fig. 5.4, p. 123, which is commutative, just as
2. Newtonian, classical velocities are vectors that add according to the common parallelogram addition law

To set the stage for presenting the relativistic stellar aberration, we begin with the presentation of the *classical particle aberration*, Fig. 10.1, which will be extended to a presentation of the *relativistic particle aberration*, Fig. 10.2, by gyro-analogies.

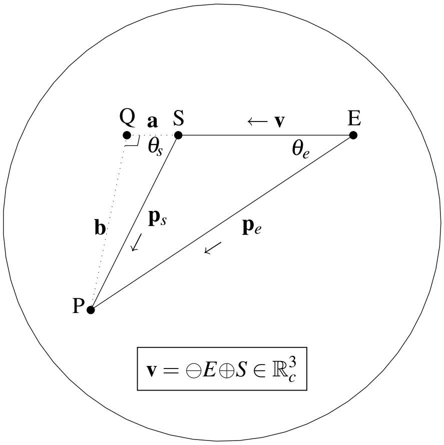

Fig. 10.2 Relativistic Stellar Aberration and Particle Aberration: Relativistic interpretation in terms of the gyrotriangle law or, equivalently, the gyroparallelogram law, of addition of Einsteinian velocities in the Beltrami–Klein ball model of 3-dimensional hyperbolic geometry $(\mathbb{R}_c^3, \oplus)$. Here $\oplus$ is Einstein addition, (1.2), p. 4, in the c-ball $\mathbb{R}_c^3 \subset \mathbb{R}^3$. A particle P moves with Einsteinian velocity $\mathbf{p}_e$ ($\mathbf{p}_s$) relative to the Earth E (the Sun S), making an angle θ_e (θ_s) with the Einsteinian velocity $\mathbf{v}$ of the Sun S relative to the Earth E. In order to calculate the Einsteinian, relativistic, particle aberration $\theta_s - \theta_e$, the gyrotriangle ESP is augmented into the right-gyroangled gyrotriangle EQP, allowing elementary gyrotrigonometry to be employed. Points are given by their orthogonal Cartesian coordinates (x, y, z), $x^2 + y^2 + z^2 < c^2$. The coordinates are not shown. The hyperbolicity of $(\mathbb{R}_c^3, \oplus)$ is determined by the hyperbolic gyrometric in which the distance between two points $A, B \in \mathbb{R}_c^3$ is $\|\ominus A \oplus B\|$. Relativistic stellar aberration is a special relativistic particle aberration when the particle is a photon emitted from a star

Figures 10.1 and 10.2 present, respectively, the Newtonian velocity space $\mathbb{R}^3$ of classical velocities, and the Einsteinian velocity space $\mathbb{R}_c^3$ of relativistically admissible velocities, along with several of their points, where only two dimensions are shown for clarity. The origin, O, of each of these two velocity spaces, not shown in Figs. 10.1–10.2, is arbitrarily selected, representing an arbitrarily selected inertial rest frame $\Sigma_{\mathbf{0}}$.

Points of a velocity space represent uniform velocities relative to the rest frame $\Sigma_{\mathbf{0}}$. In particular, the points E, S and P in Figs. 10.1–10.2 represent, respectively, the velocity of the Earth, the Sun, and a Particle (emitted, for instance, from a star) relative to the rest frame $\Sigma_{\mathbf{0}}$.

Accordingly, the Newtonian velocity vector $\mathbf{v}$ of the Sun relative to the Earth is

$$\mathbf{v} = -E + S \tag{10.1a}$$

and the Einsteinian velocity gyrovector $\mathbf{v}$ of the Sun relative to the Earth is

$$\mathbf{v} = \ominus E \oplus S \tag{10.1b}$$

as shown, respectively, in Figs. 10.1–10.2.

Similarly, the particle P moves uniformly relative to the Earth and relative to the Sun with respective Newtonian velocities, Fig. 10.1,

$$\begin{aligned}\mathbf{P}_e &= -E + P,\\ \mathbf{P}_s &= -S + P.\end{aligned} \tag{10.2a}$$

In full analogy with (10.2a), the Einsteinian velocities of the particle P relative to the Earth and relative to the Sun are, Fig. 10.2,

$$\begin{aligned}\mathbf{P}_e &= \ominus E \oplus P\\ \mathbf{P}_s &= \ominus S \oplus P\end{aligned} \tag{10.2b}$$

The Newtonian velocities (10.2a) of the particle make angles θ_e and θ_s, respectively, with the Newtonian velocity $\mathbf{v}$ in (10.1a), as shown in Fig. 10.1.

Similarly, the Einsteinian velocities (10.2b) of the particle make gyroangles θ_e and θ_s, respectively, with the Einsteinian velocity $\mathbf{v}$ in (10.1b), as shown in Fig. 10.2.

Following Fig. 10.1, *classical particle aberration* is the angular change $\theta_s - \theta_e$ in the apparent direction of a moving particle caused by the motion with Newtonian relative velocity $\mathbf{v}$, (10.1a), between E and S. A relationship between the angles θ_s and θ_e is called a *classical particle aberration formula*.

Similarly, following Fig. 10.2, *relativistic particle aberration* is the gyroangular change $\theta_s - \theta_e$ in the apparent direction of a moving particle caused by the motion with Einsteinian relative velocity $\mathbf{v}$, (10.1b), between E and S. A relationship between the gyroangles θ_s and θ_e is called a *relativistic particle aberration formula*.

In order to uncover classical particle aberration formulas, we draw the altitude PQ from vertex P to side ES (extended if necessary) obtaining the right-angled triangle EQP in Fig. 10.1. The latter, in turn, enables the parallelogram law and the triangle law of Newtonian velocity addition, and the triangle equality and trigonometry, to be applied, obtaining the following two, mutually equivalent, classical particle aberration formulas:

$$\begin{aligned}\cot\theta_e &= \cot\theta_s + \frac{v}{p_s \sin\theta_s},\\ \cot\theta_s &= \cot\theta_e - \frac{v}{p_e \sin\theta_e}.\end{aligned} \tag{10.3a}$$

The resulting classical particle aberration formulas (10.3a) are in full agreement with formulas available in the literature; see, for instance, [54, (134), p. 147].

The details of obtaining the classical particle aberration formulas (10.3a), illustrated in Fig. 10.1, are presented in [63, Chap. 13] and, hence, will not be presented here.

In full analogy, in order to uncover relativistic particle aberration formulas, we draw the altitude PQ from vertex P to side ES (extended if necessary) obtaining the right gyrotriangle EQP in Fig. 10.2. The latter, in turn, enables the gyrotriangle law and the gyroparallelogram law of Einsteinian velocity addition, and the gyrotriangle equality and gyrotrigonometry, to be applied, obtaining the following two, mutually equivalent, relativistic particle aberration formulas:

$$\begin{aligned}\cot\theta_e &= \gamma_v\left(\cot\theta_s + \frac{v}{p_s\sin\theta_s}\right),\\ \cot\theta_s &= \gamma_v\left(\cot\theta_e - \frac{v}{p_e\sin\theta_e}\right).\end{aligned} \tag{10.3b}$$

The resulting relativistic particle aberration formulas (10.3b) are in full agreement with formulas available in the literature; see, for instance, [45, p. 53], [46, p. 86] and [33, pp. 12–14].

The details of obtaining the relativistic particle aberration formulas (10.3b), illustrated in Fig. 10.2, are presented in [63, Chap. 13] and, hence, will not be presented here.

In Euclidean geometry, the triangle law and the parallelogram law of vector addition are equivalent. In full analogy, their gyro-counterparts are equivalent as well, as explained in [64, Sect. 4.3] in detail.

The equivalence of the two equations in (10.3a) implies $p_s\sin\theta_s = p_e\sin\theta_e$, thus recovering the law of sines

$$\frac{p_s}{\sin\theta_e} = \frac{p_e}{\sin\theta_s} \tag{10.4a}$$

for the Euclidean triangle *ESP* in Fig. 10.1, noting that $\sin\theta_s = \sin(\pi-\theta_s)$.

In full analogy, the equivalence of the two equations in (10.3b) implies the relativistic law of gyrosines (6.44), p. 140,

$$\frac{\gamma_{p_s}p_s}{\sin\theta_e} = \frac{\gamma_{p_e}p_e}{\sin\theta_s} \tag{10.4b}$$

for the gyrotriangle *ESP* in Fig. 10.2, noting that $\sin\theta_s = \sin(\pi-\theta_s)$.

Here, we have described the way to recover the well-known classical particle aberration formulas (10.3a) by employing trigonometry, the triangle equality, the triangle addition law, and the parallelogram addition law of Newtonian velocities.

In full analogy, we have also described here the way to recover the well-known relativistic particle aberration formulas (10.3b) by employing gyrotrigonometry, the gyrotriangle equality, the gyrotriangle addition law, and the gyroparallelogram addition law (5.11), p. 124, of Einsteinian velocities.

In contrast, the well-known relativistic particle aberration formulas (10.3b) are obtained in the literature by employing the Lorentz transformation group of special relativity.

What is remarkable here is that the relativistic particle aberration formulas (10.3b), which are commonly obtained in the literature by Lorentz transformation considerations, are recovered here by gyrotrigonometry and the gyroparallelogram addition law of Einsteinian velocities, in full analogy with the recovery of their classical counterparts. This remarkable way of recovering the particle aberration formulas (10.3a), (10.3b) demonstrates that since special relativity is governed by the Lorentz transformation group, Einsteinian velocities in special relativity add according to the gyroparallelogram addition law, just as Newtonian velocities add according to the parallelogram addition law.

Hence, any experiment that supports the validity of the relativistic particle aberration formulas (10.3b), amounts to an experiment that supports the validity of the gyroparallelogram addition law (5.11), p. 124, of Einsteinian velocities.

In the special case when the particle P in Fig. 10.2 is a photon emitted from a star, the Einsteinian speed of the photon relative to both E and S is $p_e = p_s = c$, and the relativistic particle aberration formulas (10.3b) reduce to the corresponding *stellar aberration formulas*:

$$\begin{aligned} \cot\theta_e &= \gamma_v \frac{\cos\theta_s + v/c}{\sin\theta_s}, \\ \cot\theta_s &= \gamma_v \frac{\cos\theta_e - v/c}{\sin\theta_e}. \end{aligned} \tag{10.5}$$

The discovery of stellar aberration, which results from the velocity of the Earth in its annual orbit about the Sun, by the English astronomer James Bradley in the 1720s, is described, for instance, in [53].

A high precision test of the validity of the stellar aberration formulas (10.5) in special relativity has recently been obtained as a byproduct of the "GP-B" gyroscope experiment. Indeed, the validity of the stellar aberration formulas (10.5) is central for the success of the "GP-B" gyroscope experiment developed by NASA and Stanford University [15] to test two unverified predictions of Einstein's general theory of relativity [14, 20].

The GP-B space gyroscopes encountered two kinds of stellar aberration. Orbital aberration with 97.5-minute period of ±5.1856 arc-seconds that results from the motion of the gyroscopes around the Earth, and annual aberration with one year period of about ±20.4958 arc-seconds that results from the motion of the Earth (and the gyroscopes) around the Sun. These aberrations, calculated by methods of special relativity, were used to calibrate the gyroscopes and their accompanying instruments.

If the "GP-B" gyroscope experiment proves successful, it could be considered as an experimental evidence of the validity of the stellar aberration formulas (10.5) and, hence, the validity of the relativistic particle aberration formulas (10.3b) as well. The experimental significance of (10.3b), in turn, could be considered as an experimental evidence of the validity of the gyroparallelogram addition law of Einsteinian velocities. Indeed, the preliminary analysis of data has confirmed the theoretical prediction of the "GP-B" gyroscope experiment [21], so that the experiment seems to prove successful.

The success of "GP-B", thus, establishes experimentally the validity of the gyroparallelogram law (5.11), p. 124, as the commutative addition law of uniform, relativistically admissible velocities. The geometric significance of our special relativistic approach to hyperbolic geometry is, thus, associated with physical-experimental significance as well.

10.3 On the Future of Special Relativity and Hyperbolic Geometry

It is hoped that following the demonstration that Einstein addition forms an extraordinarily powerful, elegant tool for studying hyperbolic geometry, readers of this book will come along and join us in the hunt for more hyperbolic triangle centers and relations between them by means of Einstein addition.

Relativity today is no longer only a matter of pure science aimed at understanding the fundamental laws of Nature and the structure of the Universe. Rather, it has reached the status of applied technology in everyday life, as Claus Lammerzahl points out in detail in [32]. There are many technologies whose good operation requires one to take into account relativistic effects. The best known of these technologies is, perhaps, the Global Positioning System (GPS).

Yet, as Z.K. Silagadze notes in [50],

> "The teaching of special relativity, however, still follows its presentation as it unfolded historically, trying to convince the audience of this teaching that Newtonian physics is natural but incorrect and special relativity is its paradoxical but correct amendment."
>
> Z.K. Silagadze, 2008

Interest in Euclidean triangle centers has long history, indicating that, following this book, hyperbolic triangle centers will prove quite popular and challenging as well [3, 11, 52, 68]. While creating momentum for the exploration of hyperbolic triangle centers, this book contributes to modernizing and popularizing the teaching of Einstein's special relativity theory along with its underlying hyperbolic geometry of Bolyai and Lobachevsky. In the resulting modernized special relativity, Einstein addition is a primitive concept from which the Minkowskian formalism is derived, and the Einstein relativistic mass is a concept that interplays harmoniously with the underlying hyperbolic geometry of the Einsteinian three-vector formalism as well as with the Minkowskian four-vector formalism of special relativity.