Chapter 1
Einstein Gyrogroups

Abstract Einstein's addition law of three-dimensional relativistically admissible velocities is the corner stone of Einstein's three-vector formalism of the special theory of relativity that he founded in 1905. In this chapter, we present Einstein addition along with its role in nonassociative algebra and hyperbolic geometry. We make no demands upon readers of this book as to a prior acquaintance with either special relativity, nonassociative algebra or hyperbolic geometry.

1.1 Introduction

Einstein's addition law of three-dimensional relativistically admissible velocities appears in [58, 60, 63, 64] as the corner stone of Einstein's three-vector formalism of the special theory of relativity that he founded in 1905 [12, 34]. Einstein addition is presented in this chapter, along with the nonassociative algebraic structures that it encodes. These are the gyrocommutative gyrogroup structure and the gyrovector space structure that form a natural generalization of the common commutative group structure and vector space structure from associative algebra into nonassociative algebra.

It will turn out that the resulting Einstein gyrovector spaces, studied in Chap. 2, form the setting for the Cartesian–Beltrami–Klein ball model of hyperbolic geometry just as vector spaces form the setting for the standard Cartesian model of Euclidean geometry.

We make no demands upon readers of this book as to a prior acquaintance with either the hyperbolic geometry of Bolyai and Lobachevsky or the special theory of relativity of Einstein. Rather, we will present the modern and unknown in terms of analogies that they share with the classical and familiar. Accordingly, as a mathematical prerequisite for a fruitful reading of this book, familiarity with Euclidean geometry from the point of view of vectors and with basic elements of linear algebra and classical mechanics is assumed.

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1.2 Einstein Velocity Addition

Let c be any positive constant and let $(\mathbb{R}^n, +, \cdot)$ be the Euclidean n-space, $n = 1, 2, 3, \ldots$, equipped with the common vector addition, $+$, and inner product, $\cdot$. Furthermore, let

$$\mathbb{R}^n_c = \left\{\mathbf{v} \in \mathbb{R}^n : \|\mathbf{v}\| < c\right\} \tag{1.1}$$

be the c-ball of all relativistically admissible velocities of material particles. It is the open ball of radius c, centered at the origin of $\mathbb{R}^n$, consisting of all vectors $\mathbf{v}$ in $\mathbb{R}^n$ with magnitude $\|\mathbf{v}\|$ smaller than c.

Einstein velocity addition is a binary operation, $\oplus$, in the c-ball $\mathbb{R}^n_c$ of all relativistically admissible velocities, given by the equation [58], [49, (2.9.2)], [40, p. 55], [18],

$$\mathbf{u}\oplus\mathbf{v} = \frac{1}{1+\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}}\left\{\mathbf{u} + \frac{1}{\gamma_{\mathbf{u}}}\mathbf{v} + \frac{1}{c^2}\frac{\gamma_{\mathbf{u}}}{1+\gamma_{\mathbf{u}}}(\mathbf{u}\cdot\mathbf{v})\mathbf{u}\right\} \tag{1.2}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n_c$, where $\gamma_{\mathbf{u}}$ is the gamma factor given by the equation

$$\gamma_{\mathbf{v}} = \frac{1}{\sqrt{1-\frac{\|\mathbf{v}\|^2}{c^2}}} = \frac{1}{\sqrt{1-\frac{\mathbf{v}^2}{c^2}}} \tag{1.3}$$

Here $\mathbf{u}\cdot\mathbf{v}$ and $\|\mathbf{v}\|$ are the inner product and the norm in the ball, which the ball $\mathbb{R}^n_c$ inherits from its space $\mathbb{R}^n$, $\|\mathbf{v}\|^2 = \mathbf{v}\cdot\mathbf{v} = \mathbf{v}^2$. A nonempty set with a binary operation is called a *groupoid* so that, accordingly, the pair $(\mathbb{R}^n_c, \oplus)$ is an *Einstein groupoid*.

In the Newtonian limit of large c, $c \to \infty$, the ball $\mathbb{R}^n_c$ expands to the whole of its space $\mathbb{R}^n$, as we see from (1.1), and Einstein addition $\oplus$ in $\mathbb{R}^n_c$ reduces to the ordinary vector addition $+$ in $\mathbb{R}^n$, as we see from (1.2) and (1.3).

In physical applications, $\mathbb{R}^n = \mathbb{R}^3$ is the Euclidean 3-space, which is the space of all classical, Newtonian velocities, and $\mathbb{R}^n_c = \mathbb{R}^3_c \subset \mathbb{R}^3$ is the c-ball of $\mathbb{R}^3$ of all relativistically admissible, Einsteinian velocities. Furthermore, the constant c represents in physical applications the vacuum speed of light. Since we are interested in applications to geometry, we allow n to be any positive integer.

Einstein addition (1.2) of relativistically admissible velocities, with $n = 3$, was introduced by Einstein in his 1905 paper [12], [13, p. 141] that founded the special theory of relativity, where the magnitudes of the two sides of Einstein addition (1.2) are presented. One has to remember here that the Euclidean 3-vector algebra was not so widely known in 1905 and, consequently, was not used by Einstein. Einstein calculated in [12] the behavior of the velocity components parallel and orthogonal to the relative velocity between inertial systems, which is as close as one can get without vectors to the vectorial version (1.2) of Einstein addition.

We naturally use the abbreviation $\mathbf{u}\ominus\mathbf{v} = \mathbf{u}\oplus(-\mathbf{v})$ for Einstein subtraction, so that, for instance, $\mathbf{v}\ominus\mathbf{v} = \mathbf{0}$, $\ominus\mathbf{v} = \mathbf{0}\ominus\mathbf{v} = -\mathbf{v}$ and, in particular,

$$\ominus(\mathbf{u}\oplus\mathbf{v}) = \ominus\mathbf{u}\ominus\mathbf{v} \tag{1.4}$$

and

$$\ominus\mathbf{u}\oplus(\mathbf{u}\oplus\mathbf{v}) = \mathbf{v} \tag{1.5}$$

for all $\mathbf{u}, \mathbf{v}$ in the ball $\mathbb{R}^n_c$, in full analogy with vector addition and subtraction in $\mathbb{R}^n$. Identity (1.4) is known as the *automorphic inverse property*, and Identity (1.5) is known as the *left cancellation law* of Einstein addition [63]. We may note that Einstein addition does not obey the naive right counterpart of the left cancellation law (1.5) since, in general,

$$(\mathbf{u}\oplus\mathbf{v})\ominus\mathbf{v} \neq \mathbf{u} \tag{1.6}$$

However, this seemingly lack of a *right cancellation law* of Einstein addition is repaired in Sect. 1.9, p. 21.

Einstein addition and the gamma factor are related by the *gamma identity*,

$$\gamma_{\mathbf{u}\oplus\mathbf{v}} = \gamma_{\mathbf{u}}\gamma_{\mathbf{v}}\left(1 + \frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\right) \tag{1.7}$$

which can be equivalently written as

$$\gamma_{\ominus\mathbf{u}\oplus\mathbf{v}} = \gamma_{\mathbf{u}}\gamma_{\mathbf{v}}\left(1 - \frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\right) \tag{1.8}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n_c$. Here, (1.8) is obtained from (1.7) by replacing $\mathbf{u}$ by $\ominus\mathbf{u} = -\mathbf{u}$ in (1.7).

A frequently used identity that follows immediately from (1.3) is

$$\frac{\mathbf{v}^2}{c^2} = \frac{\|\mathbf{v}\|^2}{c^2} = \frac{\gamma_{\mathbf{v}}^2 - 1}{\gamma_{\mathbf{v}}^2} \tag{1.9}$$

and, similarly, a useful identity that follows immediately from (1.8) is

$$\frac{\mathbf{u}\cdot\mathbf{v}}{c^2} = 1 - \frac{\gamma_{\ominus\mathbf{u}\oplus\mathbf{v}}}{\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}} \tag{1.10}$$

It is the gamma identity (1.7) that signaled the emergence of hyperbolic geometry in special relativity when it was first studied by Sommerfeld [51] and Varičak [66, 67] in terms of *rapidities*, a term coined by Robb [47]. In fact, the gamma identity plays a role in hyperbolic geometry, analogous to the law of cosines in Euclidean geometry, as we will see in Sect. 6.3, p. 132. Historically, it formed the first link between special relativity and the hyperbolic geometry of Bolyai and Lobachevsky, recently leading to the novel trigonometry in hyperbolic geometry that became known as *gyrotrigonometry*, developed in [63, Chap. 12], [64, Chap. 4], [57, 62] and in Part II of this book.

Einstein addition is noncommutative. Indeed, while Einstein addition is commutative under the norm,

$$\|\mathbf{u}\oplus\mathbf{v}\| = \|\mathbf{v}\oplus\mathbf{u}\| \tag{1.11}$$

we have, in general,

$$\mathbf{u}\oplus\mathbf{v} \neq \mathbf{v}\oplus\mathbf{u} \tag{1.12}$$

for $\mathbf{u}, \mathbf{v} \in \mathbb{R}_c^n$. Moreover, Einstein addition is also nonassociative since, in general,

$$(\mathbf{u}\oplus\mathbf{v})\oplus\mathbf{w} \neq \mathbf{u}\oplus(\mathbf{v}\oplus\mathbf{w}) \tag{1.13}$$

for $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}_c^n$.

It seems that following the breakdown of commutativity and associativity in Einstein addition some mathematical regularity has been lost in the transition from Newton's velocity vector addition in $\mathbb{R}^n$ to Einstein's velocity addition (1.2) in $\mathbb{R}_c^n$. This is, however, not the case since Thomas gyration comes to the rescue, as we will see in Sect. 1.4. Owing to the presence of Thomas gyration, the Einstein groupoid $(\mathbb{R}_c^n, \oplus)$ has a grouplike structure [56] that we naturally call the *Einstein gyrogroup* [58]. The formal definition of the resulting abstract gyrogroup will be presented in Definition 1.5, p. 12.

1.3 Einstein Addition With Respect to Cartesian Coordinates

Like any physical law, Einstein velocity addition law (1.2) is coordinate independent. Indeed, it is presented in (1.2) in terms of vectors, noting that one of the great advantages of vectors is their ability to express results independent of any coordinate system.

However, in order to generate numerical and graphical demonstrations of physical laws, we need coordinates. Accordingly, we introduce Cartesian coordinates into the Euclidean n-space $\mathbb{R}^n$ and its ball $\mathbb{R}_c^n$, with respect to which we generate the graphs of this book. Introducing the Cartesian coordinate system Σ into $\mathbb{R}^n$ and $\mathbb{R}_c^n$, each point $P \in \mathbb{R}^n$ is given by an n-tuple

$$P = (x_1, x_2, \dots, x_n), \qquad x_1^2 + x_2^2 + \cdots + x_n^2 < \infty \tag{1.14}$$

of real numbers, which are the coordinates, or components, of P with respect to Σ. Similarly, each point $P \in \mathbb{R}_c^n$ is given by an n-tuple

$$P = (x_1, x_2, \dots, x_n), \qquad x_1^2 + x_2^2 + \cdots + x_n^2 < c^2 \tag{1.15}$$

of real numbers, which are the coordinates, or components of P with respect to Σ.

Equipped with a Cartesian coordinate system Σ and its standard vector addition given by component addition, along with its resulting scalar multiplication, $\mathbb{R}^n$ forms the standard Cartesian model of n-dimensional Euclidean geometry. In full analogy, equipped with a Cartesian coordinate system Σ and its Einstein addition, along with its resulting scalar multiplication (to be studied in Sect. 2.1), the ball $\mathbb{R}_c^n$ forms in this book the Cartesian–Beltrami–Klein ball model of n-dimensional hyperbolic geometry.

As an illustrative example, we present below the Einstein velocity addition law (1.2) in $\mathbb{R}^3_c$ with respect to a Cartesian coordinate system.

Let $\mathbb{R}^3_c$ be the c-ball of the Euclidean 3-space, equipped with a Cartesian coordinate system Σ. Accordingly, each point of the ball is represented by its coordinates $(x_1, x_2, x_3)^t$ (exponent t denotes transposition) with respect to Σ, satisfying the condition $x_1^2 + x_2^2 + x_3^2 < c^2$.

Furthermore, let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3_c$ be three points in $\mathbb{R}^3_c \subset \mathbb{R}^3$ given by their coordinates with respect to Σ,

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \qquad \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \qquad \mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \tag{1.16}$$

where

$$\mathbf{w} = \mathbf{u} \oplus \mathbf{v} \tag{1.17}$$

The dot product of $\mathbf{u}$ and $\mathbf{v}$ is given in Σ by the equation

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 \tag{1.18}$$

and the squared norm $\|\mathbf{v}\|^2 = \mathbf{v} \cdot \mathbf{v}$ of $\mathbf{v}$ is given by the equation

$$\|\mathbf{v}\|^2 = v_1^2 + v_2^2 + v_3^2 \tag{1.19}$$

Hence, it follows from the coordinate independent vector representation (1.2) of Einstein addition that the coordinate dependent Einstein addition (1.17) with respect to the Cartesian coordinate system Σ takes the form

$$\begin{aligned} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} &= \frac{1}{1 + \frac{u_1 v_1 + u_2 v_2 + u_3 v_3}{c^2}} \\ &\quad \times \left\{ \left[1 + \frac{1}{c^2} \frac{\gamma_{\mathbf{u}}}{1 + \gamma_{\mathbf{u}}} (u_1 v_1 + u_2 v_2 + u_3 v_3) \right] \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} + \frac{1}{\gamma_{\mathbf{u}}} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \right\} \end{aligned} \tag{1.20}$$

where

$$\gamma_{\mathbf{u}} = \frac{1}{\sqrt{1 - \frac{u_1^2 + u_2^2 + u_3^2}{c^2}}} \tag{1.21}$$

The three components of Einstein addition (1.17) are w_1, w_2 and w_3 in (1.20). For a two-dimensional illustration of Einstein addition (1.20) one may impose the condition $u_3 = v_3 = 0$, implying $w_3 = 0$.

In the Newtonian–Euclidean limit, $c \to \infty$, the ball $\mathbb{R}_c^3$ expands to the Euclidean 3-space $\mathbb{R}^3$, and Einstein addition (1.20) reduces to the common vector addition in $\mathbb{R}^3$,

$$\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \tag{1.22}$$

1.4 Einstein Addition vs. Vector Addition

Vector addition, $+$, in $\mathbb{R}^n$ is both commutative and associative, satisfying

$$\begin{aligned} \mathbf{u}+\mathbf{v} &= \mathbf{v}+\mathbf{u}, && \text{(Commutative Law)} \\ \mathbf{u}+(\mathbf{v}+\mathbf{w}) &= (\mathbf{u}+\mathbf{v})+\mathbf{w} && \text{(Associative Law)} \end{aligned} \tag{1.23}$$

for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$. In contrast, Einstein addition, $\oplus$, in $\mathbb{R}_c^n$ is neither commutative nor associative.

In order to measure the extent to which Einstein addition deviates from associativity we introduce *gyrations*, which are maps that are *trivial* in the special cases when the application of $\oplus$ is associative. For any $\mathbf{u}, \mathbf{v} \in \mathbb{R}_c^n$, the gyration $\mathrm{gyr}[\mathbf{u}, \mathbf{v}]$ is a map of the Einstein groupoid $(\mathbb{R}_c^n, \oplus)$ onto itself. Gyrations $\mathrm{gyr}[\mathbf{u}, \mathbf{v}] \in \mathrm{Aut}(\mathbb{R}_c^3, \oplus)$, $\mathbf{u}, \mathbf{v} \in \mathbb{R}_c^3$, are defined in terms of Einstein addition by the equation

$$\mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{w} = \ominus(\mathbf{u}\oplus\mathbf{v})\oplus\bigl\{\mathbf{u}\oplus(\mathbf{v}\oplus\mathbf{w})\bigr\} \tag{1.24}$$

for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}_c^3$, and they turn out to be automorphisms of the Einstein groupoid $(\mathbb{R}_c^3, \oplus)$.

We recall that an automorphism of a groupoid $(S, \oplus)$ is a one-to-one map f of S onto itself that respects the binary operation, that is, $f(a\oplus b) = f(a)\oplus f(b)$ for all $a, b \in S$. The set of all automorphisms of a groupoid $(S, \oplus)$ forms a group, denoted $\mathrm{Aut}(S, \oplus)$. To emphasize that the gyrations of an Einstein gyrogroup $(\mathbb{R}_c^3, \oplus)$ are automorphisms of the gyrogroup, gyrations are also called *gyroautomorphisms*.

A gyration $\mathrm{gyr}[\mathbf{u}, \mathbf{v}]$, $\mathbf{u}, \mathbf{v} \in \mathbb{R}_c^3$, is *trivial* if $\mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{w} = \mathbf{w}$ for all $\mathbf{w} \in \mathbb{R}_c^3$. Thus, for instance, the gyrations $\mathrm{gyr}[\mathbf{0}, \mathbf{v}]$, $\mathrm{gyr}[\mathbf{v}, \mathbf{v}]$ and $\mathrm{gyr}[\mathbf{v}, \ominus\mathbf{v}]$ are trivial for all $\mathbf{v} \in \mathbb{R}_c^3$, as we see from (1.24).

Einstein gyrations, which possess their own rich structure, measure the extent to which Einstein addition deviates from commutativity and associativity as we see from the gyrocommutative and the gyroassociative laws of Einstein addition in the following identities [58, 60, 63]:

$$\begin{aligned}
\mathbf{u}\oplus\mathbf{v} &= \mathrm{gyr}[\mathbf{u},\mathbf{v}](\mathbf{v}\oplus\mathbf{u}), && \text{(Gyrocommutative Law)}\\
\mathbf{u}\oplus(\mathbf{v}\oplus\mathbf{w}) &= (\mathbf{u}\oplus\mathbf{v})\oplus \mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{w}, && \text{(Left Gyroassociative Law)}\\
(\mathbf{u}\oplus\mathbf{v})\oplus\mathbf{w} &= \mathbf{u}\oplus\big(\mathbf{v}\oplus \mathrm{gyr}[\mathbf{v},\mathbf{u}]\mathbf{w}\big), && \text{(Right Gyroassociative Law)}\\
\mathrm{gyr}[\mathbf{u}\oplus\mathbf{v},\mathbf{v}] &= \mathrm{gyr}[\mathbf{u},\mathbf{v}], && \text{(Gyration Left Loop Property)}\\
\mathrm{gyr}[\mathbf{u},\mathbf{v}\oplus\mathbf{u}] &= \mathrm{gyr}[\mathbf{u},\mathbf{v}], && \text{(Gyration Right Loop Property)}\\
\mathrm{gyr}[\ominus\mathbf{u},\ominus\mathbf{v}] &= \mathrm{gyr}[\mathbf{u},\mathbf{v}], && \text{(Gyration Even Property)}\\
\big(\mathrm{gyr}[\mathbf{u},\mathbf{v}]\big)^{-1} &= \mathrm{gyr}[\mathbf{v},\mathbf{u}], && \text{(Gyration Inversion Law)}
\end{aligned} \tag{1.25}$$

for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n_c$.

Einstein addition is thus regulated by gyrations to which it gives rise owing to its nonassociativity, so that Einstein addition and its gyrations are inextricably linked. The resulting gyrocommutative gyrogroup structure of Einstein addition was discovered in 1988 [55]. Interestingly, (Thomas) gyrations are the mathematical abstraction of the relativistic effect known as *Thomas precession* [63, Sect. 10.3].

The loop properties in (1.25) present important gyration identities. These two gyration identities are, however, just the tip of a giant iceberg. Many other useful gyration identities are studied in [58, 60, 63] and will be studied in the sequel.

1.5 Gyrations

Owing to its nonassociativity, Einstein addition gives rise in (1.24) to gyrations

$$\mathrm{gyr}[\mathbf{u},\mathbf{v}] : \mathbb{R}^n_c \to \mathbb{R}^n_c \tag{1.26}$$

for any $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n_c$ in an Einstein groupoid $(\mathbb{R}^n_c, \oplus)$. Gyrations, in turn, regulate Einstein addition, endowing it with the rich structure of a gyrocommutative gyrogroup, as we will see in Sect. 1.6, and a gyrovector space, as we will see in Sect. 2.1. Clearly, gyrations measure the extent to which Einstein addition is nonassociative, where associativity corresponds to trivial gyrations.

An explicit presentation of the gyrations of Einstein groupoids $(\mathbb{R}^n_c, \oplus)$ is, therefore, desirable. Indeed, the gyration equation (1.24) can be manipulated into the equation

$$\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{w} = \mathbf{w} + \frac{A\mathbf{u} + B\mathbf{v}}{D} \tag{1.27}$$

where

$$\begin{aligned}
A &= -\frac{1}{c^2}\frac{\gamma_{\mathbf{u}}^2}{(\gamma_{\mathbf{u}}+1)}(\gamma_{\mathbf{v}}-1)(\mathbf{u}\cdot\mathbf{w}) + \frac{1}{c^2}\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}(\mathbf{v}\cdot\mathbf{w}) \\
&\quad + \frac{2}{c^4}\frac{\gamma_{\mathbf{u}}^2\gamma_{\mathbf{v}}^2}{(\gamma_{\mathbf{u}}+1)(\gamma_{\mathbf{v}}+1)}(\mathbf{u}\cdot\mathbf{v})(\mathbf{v}\cdot\mathbf{w}) \\
B &= -\frac{1}{c^2}\frac{\gamma_{\mathbf{v}}}{\gamma_{\mathbf{v}}+1}\big\{\gamma_{\mathbf{u}}(\gamma_{\mathbf{v}}+1)(\mathbf{u}\cdot\mathbf{w}) + (\gamma_{\mathbf{u}}-1)\gamma_{\mathbf{v}}(\mathbf{v}\cdot\mathbf{w})\big\} \\
D &= \gamma_{\mathbf{u}}\gamma_{\mathbf{v}}\left(1+\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\right) + 1 = \gamma_{\mathbf{u}\oplus\mathbf{v}} + 1 > 1
\end{aligned} \tag{1.28}$$

for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}_c^n$. Allowing $\mathbf{w} \in \mathbb{R}^n \supset \mathbb{R}_c^n$ in (1.27)–(1.28), gyrations $\mathrm{gyr}[\mathbf{u}, \mathbf{v}]$ are expendable from maps of $\mathbb{R}_c^n$ to linear maps of $\mathbb{R}^n$ for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}_c^n$.

In each of the three special cases when (i) $\mathbf{u} = \mathbf{0}$, or (ii) $\mathbf{v} = \mathbf{0}$, or (iii) $\mathbf{u}$ and $\mathbf{v}$ are parallel in $\mathbb{R}^n$, $\mathbf{u}\|\mathbf{v}$, we have $A\mathbf{u} + B\mathbf{v} = \mathbf{0}$ so that $\mathrm{gyr}[\mathbf{u}, \mathbf{v}]$ is trivial. Thus, we have

$$\begin{aligned}
\mathrm{gyr}[\mathbf{0}, \mathbf{v}]\mathbf{w} &= \mathbf{w} \\
\mathrm{gyr}[\mathbf{u}, \mathbf{0}]\mathbf{w} &= \mathbf{w} \\
\mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{w} &= \mathbf{w}, \qquad \mathbf{u}\|\mathbf{v}
\end{aligned} \tag{1.29}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}_c^n$, and all $\mathbf{w} \in \mathbb{R}^n$.

It follows from (1.27) that

$$\mathrm{gyr}[\mathbf{v}, \mathbf{u}]\big(\mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{w}\big) = \mathbf{w} \tag{1.30}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}_c^n$, $\mathbf{w} \in \mathbb{R}^n$, so that gyrations are invertible linear maps of $\mathbb{R}^n$, the inverse, $\mathrm{gyr}^{-1}[\mathbf{u}, \mathbf{v}]$, of $\mathrm{gyr}[\mathbf{u}, \mathbf{v}]$ being $\mathrm{gyr}[\mathbf{v}, \mathbf{u}]$. We thus have the gyration inversion property

$$\mathrm{gyr}^{-1}[\mathbf{u}, \mathbf{v}] = \mathrm{gyr}[\mathbf{v}, \mathbf{u}] \tag{1.31}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}_c^n$.

Gyrations keep the inner product of elements of the ball $\mathbb{R}_c^n$ invariant, that is,

$$\mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{a}\cdot \mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{b} = \mathbf{a}\cdot\mathbf{b} \tag{1.32}$$

for all $\mathbf{a}, \mathbf{b}, \mathbf{u}, \mathbf{v} \in \mathbb{R}_c^n$. Hence, $\mathrm{gyr}[\mathbf{u}, \mathbf{v}]$ is an *isometry* of $\mathbb{R}_c^n$, keeping the norm of elements of the ball $\mathbb{R}_c^n$ invariant,

$$\|\mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{w}\| = \|\mathbf{w}\| \tag{1.33}$$

Accordingly, $\mathrm{gyr}[\mathbf{u}, \mathbf{v}]$ represents a rotation of the ball $\mathbb{R}_c^n$ about its origin for any $\mathbf{u}, \mathbf{v} \in \mathbb{R}_c^n$.

The invertible self-map $\mathrm{gyr}[\mathbf{u}, \mathbf{v}]$ of $\mathbb{R}_c^n$ respects Einstein addition in $\mathbb{R}_c^n$,

$$\mathrm{gyr}[\mathbf{u}, \mathbf{v}](\mathbf{a}\oplus\mathbf{b}) = \mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{a}\oplus \mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{b} \tag{1.34}$$

for all $\mathbf{a}, \mathbf{b}, \mathbf{u}, \mathbf{v} \in \mathbb{R}^n_c$, so that $\mathrm{gyr}[\mathbf{u}, \mathbf{v}]$ is an automorphism of the Einstein groupoid $(\mathbb{R}^n_c, \oplus)$.

1.6 From Einstein Velocity Addition to Gyrogroups

Taking the key features of the Einstein groupoid $(\mathbb{R}^n_c, \oplus)$ as axioms, and guided by analogies with groups, we are led to the formal gyrogroup definition in which gyrogroups turn out to form a most natural generalization of groups. Definitions related to groups and gyrogroups thus follow.

Definition 1.1 (Groups) A groupoid $(G, +)$ is a group if its binary operation satisfies the following axioms. In G there is at least one element, 0, called a left identity, satisfying

(G1)

$$0 + a = a$$

for all $a \in G$. There is an element $0 \in G$ satisfying Axiom (G1) such that for each $a \in G$ there is an element $-a \in G$, called a left inverse of a, satisfying

(G2)

$$-a + a = 0$$

Moreover, the binary operation obeys the associative law

(G3)

$$(a + b) + c = a + (b + c)$$

for all $a, b, c \in G$.

Groups are classified into commutative and noncommutative groups.

Definition 1.2 (Commutative Groups) A group $(G, +)$ is commutative if its binary operation obeys the commutative law

(G6)

$$a + b = b + a$$

for all $a, b \in G$.

Definition 1.3 (Subgroups) A subset H of a subgroup $(G, +)$ is a subgroup of G if it is nonempty, and H is closed under group compositions and inverses in G, that is, $x, y \in H$ implies $x + y \in H$ and $-x \in H$.

Theorem 1.4 (The Subgroup Criterion) *A subset H of a group G is a subgroup if and only if* (i) *H is nonempty, and* (ii) *$x, y \in H$ implies $x - y \in H$.*

For a proof of the Subgroup Criterion see any book on group theory.

Definition 1.5 (Gyrogroups) A groupoid $(G, \oplus)$ is a gyrogroup if its binary operation satisfies the following axioms. In G there is at least one element, 0, called a left identity, satisfying

(G1)

$$0\oplus a = a$$

for all $a \in G$. There is an element $0 \in G$ satisfying Axiom (G1) such that for each $a \in G$ there is an element $\ominus a \in G$, called a left inverse of a, satisfying

(G2)

$$\ominus a\oplus a = 0$$

Moreover, for any $a, b, c \in G$ there exists a unique element $\mathrm{gyr}[a, b]c \in G$ such that the binary operation obeys the left gyroassociative law

(G3)

$$a\oplus(b\oplus c) = (a\oplus b)\oplus \mathrm{gyr}[a, b]c$$

The map $\mathrm{gyr}[a, b] : G \to G$ given by $c \mapsto \mathrm{gyr}[a, b]c$ is an automorphism of the groupoid $(G, \oplus)$, that is,

(G4)

$$\mathrm{gyr}[a, b] \in \mathrm{Aut}(G, \oplus)$$

and the automorphism $\mathrm{gyr}[a, b]$ of G is called the gyroautomorphism, or the gyration, of G generated by $a, b \in G$. The operator $\mathrm{gyr} : G \times G \to \mathrm{Aut}(G, \oplus)$ is called the gyrator of G. Finally, the gyroautomorphism $\mathrm{gyr}[a, b]$ generated by any $a, b \in G$ possesses the left loop property

(G5)

$$\mathrm{gyr}[a, b] = \mathrm{gyr}[a\oplus b, b]$$

The gyrogroup axioms (G1)–(G5) in Definition 1.5 are classified into three classes:

1. The first pair of axioms, (G1) and (G2), is a reminiscent of the group axioms.
2. The last pair of axioms, (G4) and (G5), presents the gyrator axioms.
3. The middle axiom, (G3), is a hybrid axiom linking the two pairs of axioms in (1) and (2).

As in group theory, we use the notation $a\ominus b = a\oplus(\ominus b)$ in gyrogroup theory as well.

In full analogy with groups, gyrogroups are classified into gyrocommutative and non-gyrocommutative gyrogroups.

Definition 1.6 (Gyrocommutative Gyrogroups) A gyrogroup $(G, \oplus)$ is gyrocommutative if its binary operation obeys the gyrocommutative law

(G6)

$$a \oplus b = \text{gyr}[a, b](b \oplus a)$$

for all $a, b \in G$.

In order to capture analogies with groups, we introduce into the abstract gyrogroup $(G, \oplus)$ a second binary operation, $\boxplus$, called the gyrogroup *cooperation*, or coaddition.

Definition 1.7 (The Gyrogroup Cooperation (Coaddition)) Let $(G, \oplus)$ be a gyrogroup. The gyrogroup cooperation (or, coaddition), $\boxplus$, is a second binary operation in G related to the gyrogroup operation (or, addition), $\oplus$, by the equation

$$a \boxplus b = a \oplus \text{gyr}[a, \ominus b]b \tag{1.35}$$

for all $a, b \in G$.

Naturally, we use the notation $a \boxminus b = a \boxplus (\ominus b)$ where $\ominus b = -b$, so that

$$a \boxminus b = a \ominus \text{gyr}[a, b]b \tag{1.36}$$

The gyrogroup cooperation is commutative if and only if the gyrogroup operation is gyrocommutative, as we will see in Theorem 1.33, p. 35.

Hence, in particular, Einstein coaddition $\boxplus = \boxplus_E$ is commutative since Einstein addition $\oplus = \oplus_E$ is gyrocommutative. Indeed, Einstein coaddition $\boxplus = \boxplus_E$ is commutative, given explicitly by the equation [63, pp. 92–93]

$$\mathbf{u} \boxplus \mathbf{v} = \frac{\gamma_\mathbf{u} + \gamma_\mathbf{v}}{\gamma_\mathbf{u}^2 + \gamma_\mathbf{v}^2 + \gamma_\mathbf{u}\gamma_\mathbf{v}(1 + \frac{\mathbf{u}\cdot\mathbf{v}}{c^2}) - 1}(\gamma_\mathbf{u}\mathbf{u} + \gamma_\mathbf{v}\mathbf{v}) = 2\otimes\frac{\gamma_\mathbf{u}\mathbf{u} + \gamma_\mathbf{v}\mathbf{v}}{\gamma_\mathbf{u} + \gamma_\mathbf{v}} \tag{1.37}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n_c$ where, by scalar multiplication definition, $2\otimes\mathbf{v} = \mathbf{v}\oplus\mathbf{v}$.

While it is clear how to define a right identity and a right inverse in a gyrogroup, the existence of such elements is not presumed. Indeed, the existence of a unique identity and a unique inverse, both left and right, is a consequence of the gyrogroup axioms, as the following theorem shows, along with other immediate results.

Theorem 1.8 (First Gyrogroup Properties) *Let $(G, \oplus)$ be a gyrogroup. For any elements $a, b, c, x \in G$ we have*:

1. *If $a\oplus b = a\oplus c$, then $b = c$ (general left cancellation law; see item (9) below).*
2. $\text{gyr}[0, a] = I$ *for any left identity* 0 *in* G.
3. $\text{gyr}[x, a] = I$ *for any left inverse x of a in G.*
4. $\text{gyr}[a, a] = I$.
5. *There is a left identity which is a right identity.*
6. *There is only one left identity.*
7. *Every left inverse is a right inverse.*
8. *There is only one left inverse, $\ominus a$, of a, and $\ominus(\ominus a) = a$.*

9. *The Left Cancellation Law*:

$$\ominus a \oplus (a \oplus b) = b. \tag{1.38}$$

10. *The Gyrator Identity*:

$$\mathrm{gyr}[a,b]x = \ominus(a \oplus b) \oplus \{a \oplus (b \oplus x)\}. \tag{1.39}$$

11. $\mathrm{gyr}[a,b]0 = 0$.
12. $\mathrm{gyr}[a,b](\ominus x) = \ominus\,\mathrm{gyr}[a,b]x$.
13. $\mathrm{gyr}[a,0] = I$.

Proof

1. Let x be a left inverse of a corresponding to a left identity, 0, in G. We have $x \oplus (a \oplus b) = x \oplus (a \oplus c)$, implying $(x \oplus a) \oplus \mathrm{gyr}[x,a]b = (x \oplus a) \oplus \mathrm{gyr}[x,a]c$ by left gyroassociativity. Since 0 is a left identity, $\mathrm{gyr}[x,a]b = \mathrm{gyr}[x,a]c$. Since automorphisms are bijective, $b = c$.
2. By left gyroassociativity we have for any left identity 0 of G, $a \oplus x = 0 \oplus (a \oplus x) = (0 \oplus a) \oplus \mathrm{gyr}[0,a]x = a \oplus \mathrm{gyr}[0,a]x$. Hence, by Item 1 above we have $x = \mathrm{gyr}[0,a]x$ for all $x \in G$ so that $\mathrm{gyr}[0,a] = I$.
3. By the left loop property and by Item 2 above we have $\mathrm{gyr}[x,a] = \mathrm{gyr}[x \oplus a, a] = \mathrm{gyr}[0,a] = I$.
4. Follows from an application of the left loop property and Item 2 above.
5. Let x be a left inverse of a corresponding to a left identity, 0, of G. Then by left gyroassociativity and Item 3 above, $x \oplus (a \oplus 0) = (x \oplus a) \oplus \mathrm{gyr}[x,a]0 = 0 \oplus 0 = 0 = x \oplus a$. Hence, by (1), $a \oplus 0 = a$ for all $a \in G$ so that 0 is a right identity.
6. Suppose 0 and 0^* are two left identities, one of which, say 0, is also a right identity. Then $0 = 0^* \oplus 0 = 0^*$.
7. Let x be a left inverse of a. Then $x \oplus (a \oplus x) = (x \oplus a) \oplus \mathrm{gyr}[x,a]x = 0 \oplus x = x = x \oplus 0$, by left gyroassociativity, (G2) of Definition 1.5 and Items 3, 5, 6 above. By Item 1, we have $a \oplus x = 0$ so that x is a right inverse of a.
8. Suppose x and y are left inverses of a. By Item 7 above, they are also right inverses, so $a \oplus x = 0 = a \oplus y$. By Item 1, $x = y$. Let $\ominus a$ be the resulting unique inverse of a. Then $\ominus a \oplus a = 0$ so that the inverse $\ominus(\ominus a)$ of $\ominus a$ is a.
9. By left gyroassociativity and by Item 3, we have

$$\ominus a \oplus (a \oplus b) = (\ominus a \oplus a) \oplus \mathrm{gyr}[\ominus a, a]b = b \tag{1.40}$$

10. By an application of the left cancellation law in Item 9 to the left gyroassociative law (G3) in Definition 1.5, we obtain the result in Item 10.
11. We obtain Item 11 from Item 10 with $x = 0$.
12. Since $\mathrm{gyr}[a,b]$ is an automorphism of $(G, \oplus)$, we have from Item 11

$$\mathrm{gyr}[a,b](\ominus x) \oplus \mathrm{gyr}[a,b]x = \mathrm{gyr}[a,b](\ominus x \oplus x) = \mathrm{gyr}[a,b]0 = 0 \tag{1.41}$$

and hence the result.
13. We obtain Item 13 from Item 10 with $b = 0$, and a left cancellation, Item 9. □

1.7 Elements of Gyrogroup Theory

Einstein gyrogroups $(G, \oplus)$ possess the *gyroautomorphic inverse property*, according to which $\ominus(a \oplus b) = \ominus a \ominus b$ for all $a, b \in G$. In general, however, $\ominus(a \oplus b) \neq \ominus a \ominus b$ in other gyrogroups. Hence, the following theorem is interesting.

Theorem 1.9 (Gyrosum Inversion Law) *For any two elements a, b of a gyrogroup $(G, \oplus)$ we have the gyrosum inversion law*

$$\ominus(a \oplus b) = \mathrm{gyr}[a, b](\ominus b \ominus a) \tag{1.42}$$

Proof By the gyrator identity in Theorem 1.8(10) and a left cancellation, Theorem 1.8(9), we have

$$\begin{aligned} \mathrm{gyr}[a, b](\ominus b \ominus a) &= \ominus(a \oplus b) \oplus \big(a \oplus \big(b \oplus (\ominus b \ominus a)\big)\big) \\ &= \ominus(a \oplus b) \oplus (a \ominus a) \\ &= \ominus(a \oplus b) \end{aligned} \tag{1.43}$$

□

Theorem 1.10 *For any two elements, a and b, of a gyrogroup $(G, \oplus)$, we have*

$$\begin{aligned} \mathrm{gyr}[a, b]b &= \ominus\big\{\ominus(a \oplus b) \oplus a\big\} \\ \mathrm{gyr}[a, \ominus b]b &= \ominus(a \ominus b) \oplus a \end{aligned} \tag{1.44}$$

Proof The first identity in (1.44) follows from Theorem 1.8(10) with $x = \ominus b$, and Theorem 1.8(12), and the second part of Theorem 1.8(8). The second identity in (1.44) follows from the first one by replacing b by $\ominus b$. □

A nested gyroautomorphism is a gyration generated by points that depend on another gyration. Thus, for instance, some gyrations in (1.45)–(1.47) below are nested.

Theorem 1.11 *Any three elements a, b, c of a gyrogroup $(G, \oplus)$ satisfy the nested gyroautomorphism identities*

$$\mathrm{gyr}[a, b \oplus c]\,\mathrm{gyr}[b, c] = \mathrm{gyr}\big[a \oplus b, \mathrm{gyr}[a, b]c\big]\,\mathrm{gyr}[a, b] \tag{1.45}$$

$$\mathrm{gyr}\big[a \oplus b, \ominus\,\mathrm{gyr}[a, b]b\big]\,\mathrm{gyr}[a, b] = I \tag{1.46}$$

$$\mathrm{gyr}\big[a, \ominus\,\mathrm{gyr}[a, b]b\big]\,\mathrm{gyr}[a, b] = I \tag{1.47}$$

and the gyroautomorphism product identities

$$\mathrm{gyr}[\ominus a, a \oplus b]\,\mathrm{gyr}[a, b] = I \tag{1.48}$$

$$\mathrm{gyr}[b, a \oplus b]\,\mathrm{gyr}[a, b] = I \tag{1.49}$$

Proof By two successive applications of the left gyroassociative law in two different ways, we obtain the following two chains of equations for all $a, b, c, x \in G$,

$$\begin{aligned} a\oplus\big(b\oplus(c\oplus x)\big) &= a\oplus\big((b\oplus c)\oplus \mathrm{gyr}[b,c]x\big) \\ &= \big(a\oplus(b\oplus c)\big)\oplus \mathrm{gyr}[a,b\oplus c]\,\mathrm{gyr}[b,c]x \end{aligned} \tag{1.50}$$

and

$$\begin{aligned} a\oplus\big(b\oplus(c\oplus x)\big) &= (a\oplus b)\oplus \mathrm{gyr}[a,b](c\oplus x) \\ &= (a\oplus b)\oplus\big(\mathrm{gyr}[a,b]c\oplus \mathrm{gyr}[a,b]x\big) \\ &= \big((a\oplus b)\oplus \mathrm{gyr}[a,b]c\big)\oplus \mathrm{gyr}\big[a\oplus b, \mathrm{gyr}[a,b]c\big]\,\mathrm{gyr}[a,b]x \\ &= \big(a\oplus(b\oplus c)\big)\oplus \mathrm{gyr}\big[a\oplus b, \mathrm{gyr}[a,b]c\big]\,\mathrm{gyr}[a,b]x \end{aligned} \tag{1.51}$$

By comparing the extreme right-hand sides of these two chains of equations, and by employing the left cancellation law, Theorem 1.8(1), we obtain the identity

$$\mathrm{gyr}[a,b\oplus c]\,\mathrm{gyr}[b,c]x = \mathrm{gyr}\big[a\oplus b, \mathrm{gyr}[a,b]c\big]\,\mathrm{gyr}[a,b]x \tag{1.52}$$

for all $x \in G$, thus verifying (1.45).

In the special case when $c = \ominus b$, (1.45) reduces to (1.46), noting that the left-hand side of (1.45) becomes trivial owing to Items (2) and (3) of Theorem 1.8.

Identity (1.47) results from the following chain of equations, which are numbered for subsequent derivation:

$$\begin{aligned} I &\overset{(1)}{\Longleftarrow\!\!=} \mathrm{gyr}\big[a\oplus b, \ominus \mathrm{gyr}[a,b]b\big]\,\mathrm{gyr}[a,b] \\ &\overset{(2)}{=\!=} \mathrm{gyr}\big[(a\oplus b)\ominus \mathrm{gyr}[a,b]b, \ominus \mathrm{gyr}[a,b]b\big]\,\mathrm{gyr}[a,b] \\ &\overset{(3)}{=\!=} \mathrm{gyr}\big[a\oplus(b\ominus b), \ominus \mathrm{gyr}[a,b]b\big]\,\mathrm{gyr}[a,b] \\ &\overset{(4)}{=\!=} \mathrm{gyr}\big[a, \ominus \mathrm{gyr}[a,b]b\big]\,\mathrm{gyr}[a,b] \end{aligned} \tag{1.53}$$

Derivation of the numbered equalities in (1.53) follows:

1. Follows from (1.46).
2. Follows from Item 1 by the left loop property.
3. Follows from Item 2 by the left gyroassociative law. Indeed, an application of the left gyroassociative law to the first entry of the left gyration in (3) gives the first entry of the left gyration in (2), that is, $a\oplus(b\ominus b) = (a\oplus b)\ominus \mathrm{gyr}[a,b]b$.
4. Follows from Item 3 immediately, since $b\ominus b = 0$.

To verify (1.48), we consider the special case of (1.45) when $b = \ominus a$, obtaining

$$\mathrm{gyr}[a, \ominus a \oplus c]\,\mathrm{gyr}[\ominus a, c] = \mathrm{gyr}\big[0, \mathrm{gyr}[a, \ominus a]c\big]\,\mathrm{gyr}[a, \ominus a] = I \tag{1.54}$$

where the second identity in (1.54) follows from Items (2) and (3) of Theorem 1.8. Replacing a by $\ominus a$ and c by b in (1.54), we obtain (1.48).

Finally, (1.49) is derived from (1.48) by an application of the left loop property to the first gyroautomorphism in (1.48) followed by a left cancellation, Theorem 1.8(9). Accordingly,

$$\begin{aligned} I &= \mathrm{gyr}[\ominus a, a \oplus b]\,\mathrm{gyr}[a, b] \\ &= \mathrm{gyr}\big[\ominus a \oplus (a \oplus b), a \oplus b\big]\,\mathrm{gyr}[a, b] \\ &= \mathrm{gyr}[b, a \oplus b]\,\mathrm{gyr}[a, b] \end{aligned} \tag{1.55}$$

□

The nested gyroautomorphism identity (1.47) in Theorem 1.11 allows the equation that defines the coaddition ⊞ to be dualized with its corresponding equation in which the roles of the binary operations ⊞ and ⊕ are interchanged, as shown in the following theorem:

Theorem 1.12 *Let* $(G, \oplus)$ *be a gyrogroup with cooperation* ⊞ *given in Definition* 1.7, *p.* 13, *by the equation*

$$a \boxplus b = a \oplus \mathrm{gyr}[a, \ominus b]b \tag{1.56}$$

Then

$$a \oplus b = a \boxplus \mathrm{gyr}[a, b]b \tag{1.57}$$

Proof Let a and b be any two elements of G. By (1.56) and (1.47), we have

$$\begin{aligned} a \boxplus \mathrm{gyr}[a, b]b &= a \oplus \mathrm{gyr}\big[a, \ominus \mathrm{gyr}[a, b]b\big]\,\mathrm{gyr}[a, b]b \\ &= a \oplus b \end{aligned} \tag{1.58}$$

thus verifying (1.57). □

We naturally use the notation

$$a \boxminus b = a \boxplus (\ominus b) \tag{1.59}$$

in a gyrogroup $(G, \oplus)$, so that, by (1.59), (1.56) and Theorem 1.8(12),

$$\begin{aligned} a \boxminus b &= a \boxplus (\ominus b) \\ &= a \oplus \mathrm{gyr}[a, b](\ominus b) \\ &= a \ominus \mathrm{gyr}[a, b]b \end{aligned} \tag{1.60}$$

and hence

$$a \boxminus a = a \ominus a = 0 \tag{1.61}$$

as it should be. Identity (1.61), in turn, implies the equality between the inverses of $a \in G$ with respect to $\oplus$ and $\boxplus$,

$$\boxminus a = \ominus a \tag{1.62}$$

for all $a \in G$.

Theorem 1.13 *Let* $(G, \oplus)$ *be a gyrogroup. Then*

$$(\ominus a \oplus b) \oplus \mathrm{gyr}[\ominus a, b](\ominus b \oplus c) = \ominus a \oplus c \tag{1.63}$$

for all $a, b, c \in G$.

Proof By the left gyroassociative law and the left cancellation law, and using the notation $d = \ominus b \oplus c$, we have

$$\begin{aligned}(\ominus a \oplus b) \oplus \mathrm{gyr}[\ominus a, b](\ominus b \oplus c) &= (\ominus a \oplus b) \oplus \mathrm{gyr}[\ominus a, b]d \\ &= \ominus a \oplus (b \oplus d) \\ &= \ominus a \oplus \big(b \oplus (\ominus b \oplus c)\big) \\ &= \ominus a \oplus c \end{aligned} \tag{1.64}$$

□

Theorem 1.14 (The Gyrotranslation Theorem, I) *Let* $(G, \oplus)$ *be a gyrogroup. Then*

$$\ominus(\ominus a \oplus b) \oplus (\ominus a \oplus c) = \mathrm{gyr}[\ominus a, b](\ominus b \oplus c) \tag{1.65}$$

for all $a, b, c \in G$.

Proof Identity (1.65) is a rearrangement of Identity (1.63) obtained by a left cancellation. □

The importance of Identity (1.65) lies in the analogy it shares with its group counterpart, $-(-a + b) + (-a + c) = -b + c$ in any group $(\mathit{Group}, +)$.

The identity of Theorem 1.13 can readily be generalized to any number of terms, for instance,

$$(\ominus a \oplus b) \oplus \mathrm{gyr}[\ominus a, b]\big\{(\ominus b \oplus c) \oplus \mathrm{gyr}[\ominus b, c](\ominus c \oplus d)\big\} = \ominus a \oplus d \tag{1.66}$$

which generalizes the obvious group identity $(-a + b) + (-b + c) + (-c + d) = -a + d$ in any group $(\mathit{Group}, +)$.

1.8 The Two Basic Equations of Gyrogroups

The two basic equations of gyrogroup theory are

$$a \oplus x = b \tag{1.67}$$

and

$$x \oplus a = b \tag{1.68}$$

for $a, b, x \in G$, each for the unknown x in a gyrogroup $(G, \oplus)$.

Let x be a solution of the first basic equation, (1.67). Then we have by (1.67) and the left cancellation law, Theorem 1.8(9),

$$\ominus a \oplus b = \ominus a \oplus (a \oplus x) = x \tag{1.69}$$

Hence, if a solution x of (1.67) exists then it must be given by $x = \ominus a \oplus b$, as we see from (1.69).

Conversely, $x = \ominus a \oplus b$ is, indeed, a solution of (1.67) as wee see by substituting $x = \ominus a \oplus b$ into (1.67) and applying the left cancellation law in Theorem 1.8(9). Hence, the gyrogroup equation (1.67) possesses the unique solution $x = \ominus a \oplus b$.

The solution of the second basic gyrogroup equation, (1.68), is quiet different from that of the first, (1.67), owing to the noncommutativity of the gyrogroup operation. Let x be a solution of (1.68). Then we have the following chain of equations, which are numbered for subsequent derivation:

$$\begin{aligned} x &\overbrace{=\!=\!=}^{(1)} x \oplus 0 \\ &\overbrace{=\!=\!=}^{(2)} x \oplus (a \ominus a) \\ &\overbrace{=\!=\!=}^{(3)} (x \oplus a) \oplus \mathrm{gyr}[x, a](\ominus a) \\ &\overbrace{=\!=\!=}^{(4)} (x \oplus a) \ominus \mathrm{gyr}[x, a]a \\ &\overbrace{=\!=\!=}^{(5)} (x \oplus a) \ominus \mathrm{gyr}[x \oplus a, a]a \\ &\overbrace{=\!=\!=}^{(6)} b \ominus \mathrm{gyr}[b, a]a \overbrace{=\!=\!=}^{(7)} b \boxminus a \end{aligned} \tag{1.70}$$

Derivation of the numbered equalities in (1.70) follows:

1. Follows from the existence of a unique identity element, 0, in the gyrogroup $(G, \oplus)$ by Theorem 1.8.
2. Follows from the existence of a unique inverse element $\ominus a$ of a in the gyrogroup $(G, \oplus)$ by Theorem 1.8.
3. Follows from Item 2 by the left gyroassociative law in Axiom (G3) of gyrogroups in Definition 1.5, p. 12.

4. Follows from Item 3 by Theorem 1.8(12).
5. Follows from Item 4 by the left loop property (G5) of gyrogroups in Definition 1.5.
6. Follows from Item 5 by the assumption that x is a solution of (1.68).
7. Follows from Item 6 by (1.60).

Hence, if a solution x of (1.68) exists then it must be given by $x = b \boxminus a$, as we see from (1.70).

Conversely, $x = b \boxminus a$ is, indeed, a solution of (1.68), as we see from the following chain of equations:

$$\begin{aligned}
x \oplus a &\overbrace{=\!=\!=}^{(1)} (b \boxminus a) \oplus a \\
&\overbrace{=\!=\!=}^{(2)} \big(b \ominus \mathrm{gyr}[b, a]a\big) \oplus a \\
&\overbrace{=\!=\!=}^{(3)} \big(b \ominus \mathrm{gyr}[b, a]a\big) \oplus \mathrm{gyr}\big[b, \ominus \mathrm{gyr}[b, a]\big]\, \mathrm{gyr}[b, a]a \\
&\overbrace{=\!=\!=}^{(4)} b \oplus \big(\ominus \mathrm{gyr}[b, a]a \oplus \mathrm{gyr}[b, a]a\big) \\
&\overbrace{=\!=\!=}^{(5)} b \oplus 0 \\
&\overbrace{=\!=\!=}^{(6)} b
\end{aligned} \tag{1.71}$$

Derivation of the numbered equalities in (1.71) follows:

1. Follows from the assumption that $x = b \boxminus a$.
2. Follows from Item 1 by (1.60).
3. Follows from Item 2 by Identity (1.47) of Theorem 1.11, according to which the gyration product applied to a in (3) is trivial.
4. Follows from Item 3 by the left gyroassociative law. Indeed, an application of the left gyroassociative law to (4) results in (3).
5. Follows from Item 4 since $\ominus \mathrm{gyr}[b, a]a$ is the unique inverse of $\mathrm{gyr}[b, a]a$.
6. Follows from Item 5 since 0 is the unique identity element of the gyrogroup $(G, \oplus)$.

Formalizing the results of this section, we have the following theorem:

Theorem 1.15 (The Two Basic Gyrogroup Equations) *Let $(G, \oplus)$ be a gyrogroup, and let $a, b \in G$. The unique solution of the equation*

$$a \oplus x = b \tag{1.72}$$

in G for the unknown x is

$$x = \ominus a \oplus b \tag{1.73}$$

and the unique solution of the equation

$$x \oplus a = b \tag{1.74}$$

in G for the unknown x is

$$x = b \boxminus a \tag{1.75}$$

Let $(G, \oplus)$ be a gyrogroup, and let $a \in G$. The maps λ_a and ρ_a of G, given by

$$\begin{aligned} \lambda_a\colon \quad G \to G, \qquad \lambda_a\colon \quad g \mapsto a \oplus g \\ \rho_a\colon \quad G \to G, \qquad \rho_a\colon \quad g \mapsto g \oplus a \end{aligned} \tag{1.76}$$

are called, respectively, a *left gyrotranslation* of G by a and a *right gyrotranslation* of G by a. Theorem 1.15 asserts that each of these transformations of G is bijective, that is, it maps G onto itself in a one-to-one manner.

1.9 The Basic Gyrogroup Cancellation Laws

The basic cancellation laws of gyrogroup theory are obtained in this section from the basic equations of gyrogroups solved in Sect. 1.8. Substituting the solution (1.73) into its equation (1.72), we obtain the left cancellation law

$$a \oplus (\ominus a \oplus b) = b \tag{1.77}$$

for all $a, b \in G$, already verified in Theorem 1.8(9).

Similarly, substituting the solution (1.75) into its equation (1.74), we obtain the first right cancellation law

$$(b \boxminus a) \oplus a = b \tag{1.78}$$

for all $a, b \in G$. The latter can be dualized, obtaining the second right cancellation law

$$(b \ominus a) \boxplus a = b \tag{1.79}$$

for all $a, b \in G$. Indeed, (1.79) results from the following chain of equations

$$\begin{aligned} b &= b \oplus 0 \\ &= b \oplus (\ominus a \oplus a) \\ &= (b \ominus a) \oplus \operatorname{gyr}[b, \ominus a]a \\ &= (b \ominus a) \oplus \operatorname{gyr}[b \ominus a, \ominus a]a \\ &= (b \ominus a) \boxplus a \end{aligned} \tag{1.80}$$

where we employ the left gyroassociative law, the left loop property, and the definition of the gyrogroup cooperation. Identities (1.77)–(1.79) form the three basic cancellation laws of gyrogroup theory.

1.10 Commuting Automorphisms with Gyroautomorphisms

In this section, we will find that automorphisms of a gyrogroup commute with its gyroautomorphisms in a special, interesting way.

Theorem 1.16 *For any two elements* a, b *of a gyrogroup* $(G, \oplus)$ *and any automorphism* A *of* $(G, \oplus)$, $A \in \mathrm{Aut}(G, \oplus)$,

$$A\,\mathrm{gyr}[a, b] = \mathrm{gyr}[Aa, Ab]A \tag{1.81}$$

Proof For any three elements $a, b, x \in (G, \oplus)$ and any automorphism $A \in \mathrm{Aut}(G, \oplus)$, we have, by the left gyroassociative law,

$$\begin{aligned}(Aa\oplus Ab)\oplus A\,\mathrm{gyr}[a, b]x &= A\bigl((a\oplus b)\oplus \mathrm{gyr}[a, b]x\bigr)\\ &= A\bigl(a\oplus(b\oplus x)\bigr)\\ &= Aa\oplus(Ab\oplus Ax)\\ &= (Aa\oplus Ab)\oplus \mathrm{gyr}[Aa, Ab]Ax\end{aligned} \tag{1.82}$$

Hence, by a left cancellation, Theorem 1.8(1),

$$A\,\mathrm{gyr}[a, b]x = \mathrm{gyr}[Aa, Ab]Ax$$

for all $x \in G$, implying (1.81). □

Theorem 1.17 *Let* a, b *be any two elements of a gyrogroup* $(G, \oplus)$ *and let* $A \in \mathrm{Aut}(G)$ *be an automorphism of* G. *Then*

$$\mathrm{gyr}[a, b] = \mathrm{gyr}[Aa, Ab] \tag{1.83}$$

if and only if the automorphisms A *and* $\mathrm{gyr}[a, b]$ *are commutative.*

Proof If $\mathrm{gyr}[Aa, Ab] = \mathrm{gyr}[a, b]$ then by Theorem 1.16 the automorphisms $\mathrm{gyr}[a, b]$ and A commute, $A\,\mathrm{gyr}[a, b] = \mathrm{gyr}[Aa, Ab]A = \mathrm{gyr}[a, b]A$. Conversely, if $\mathrm{gyr}[a, b]$ and A commute then by Theorem 1.16 $\mathrm{gyr}[Aa, Ab] = A\,\mathrm{gyr}[a, b]A^{-1} = \mathrm{gyr}[a, b]AA^{-1} = \mathrm{gyr}[a, b]$. □

As a simple, but useful, consequence of Theorem 1.17 we note the elegant identity

$$\mathrm{gyr}\bigl[\mathrm{gyr}[a, b]a, \mathrm{gyr}[a, b]b\bigr] = \mathrm{gyr}[a, b] \tag{1.84}$$

1.11 The Gyrosemidirect Product

Definition 1.18 (Gyroautomorphism Groups, Gyrosemidirect Product) Let $G = (G, \oplus)$ be a gyrogroup, and let $\mathrm{Aut}(G) = \mathrm{Aut}(G, \oplus)$ be the automorphism group of G. A gyroautomorphism group, $\mathrm{Aut}_0(G)$, of G is any subgroup of $\mathrm{Aut}(G)$ containing all the gyroautomorphisms $\mathrm{gyr}[a, b]$ of G, $a, b \in G$. The *gyrosemidirect product group*

$$G \times \mathrm{Aut}_0(G) \tag{1.85}$$

of a gyrogroup G and any gyroautomorphism group, $\mathrm{Aut}_0(G)$ of G, is a group of pairs (x, X), where $x \in G$ and $X \in \mathrm{Aut}_0(G)$, with operation given by the *gyrosemidirect product*

$$(x, X)(y, Y) = \big(x \oplus Xy, \mathrm{gyr}[x, Xy]XY\big) \tag{1.86}$$

It is anticipated in Definition 1.18 that the gyrosemidirect product set (1.85) of a gyrogroup and any one of its gyroautomorphism groups is a set that forms a group with group operation given by the gyrosemidirect product (1.86). The following theorem shows that this is indeed the case.

Theorem 1.19 *Let* $(G, \oplus)$ *be a gyrogroup, and let* $\mathrm{Aut}_0(G, \oplus)$ *be a gyroautomorphism group of* G. *Then the gyrosemidirect product* $G \times \mathrm{Aut}_0(G)$ *is a group, with group operation given by the gyrosemidirect product* (1.86).

Proof We will show that the set $G \times \mathrm{Aut}_0(G)$ with its binary operation (1.86) satisfies the group axioms.

(i) Existence of a left identity: A left identity element of $G \times \mathrm{Aut}_0(G)$ is the pair $(0, I)$, where $0 \in G$ is the identity element of G, and $I \in \mathrm{Aut}_0(G)$ is the identity automorphism of G. Indeed,

$$(0, I)(a, A) = \big(0 \oplus Ia, \mathrm{gyr}[0, Ia]IA\big) = (a, A) \tag{1.87}$$

noting that the gyration in (1.87) is trivial by Theorem 1.8(2).

(ii) Existence of a left inverse: Let $A^{-1} \in \mathrm{Aut}_0(G)$ be the inverse automorphism of $A \in \mathrm{Aut}_0(G)$. Then, by the gyrosemidirect product (1.86) we have

$$\big(\ominus A^{-1}a, A^{-1}\big)(a, A) = \big(\ominus A^{-1}a \oplus A^{-1}a, \mathrm{gyr}\big[\ominus A^{-1}a, A^{-1}a\big]A^{-1}A\big) = (0, I) \tag{1.88}$$

Hence, a left inverse of $(a, A) \in G \times \mathrm{Aut}_0(G)$ is the pair $(\ominus A^{-1}a, A^{-1})$,

$$(a, A)^{-1} = \big(\ominus A^{-1}a, A^{-1}\big) \tag{1.89}$$

(iii) Validity of the associative law: We have to show that the successive products in (1.90) and in (1.91) below are equal.

On the one hand, we have

$$\begin{aligned}
&(a_1, A_1)\big((a_2, A_2)(a_3, A_3)\big) \\
&\quad = (a_1, A_1)\big(a_2 \oplus A_2 a_3, \operatorname{gyr}[a_2, A_2 a_3] A_2 A_3\big) \\
&\quad = \big(a_1 \oplus A_1(a_2 \oplus A_2 a_3), \operatorname{gyr}\big[a_1, A_1(a_2 \oplus A_2 a_3)\big] A_1 \operatorname{gyr}[a_2, A_2 a_3] A_2 A_3\big) \\
&\quad = \big(a_1 \oplus (A_1 a_2 \oplus A_1 A_2 a_3), \\
&\qquad \operatorname{gyr}[a_1, A_1 a_2 \oplus A_1 A_2 a_3] \operatorname{gyr}[A_1 a_2, A_1 A_2 a_3] A_1 A_2 A_3\big)
\end{aligned} \tag{1.90}$$

where we employ the gyrosemidirect product (1.86) and the commuting law (1.81).

On the other hand, we have

$$\begin{aligned}
&\big((a_1, A_1)(a_2, A_2)\big)(a_3, A_3) \\
&\quad = \big(a_1 \oplus A_1 a_2, \operatorname{gyr}[a_1, A_1 a_2] A_1 A_2\big)(a_3, A_3) \\
&\quad = \big((a_1 \oplus A_1 a_2) \oplus \operatorname{gyr}[a_1, A_1 a_2] A_1 A_2 a_3, \\
&\qquad \operatorname{gyr}\big[a_1 \oplus A_1 a_2, \operatorname{gyr}[a_1, A_1 a_2] A_1 A_2 a_3\big] \operatorname{gyr}[a_1, A_1 a_2] A_1 A_2 A_3\big)
\end{aligned} \tag{1.91}$$

where we employ (1.86).

In order to show that the gyrosemidirect products in (1.90) and (1.91) are equal, using the notation

$$\begin{aligned}
a_1 &= a \\
A_1 a_2 &= b \\
A_1 A_2 A_3 &= c
\end{aligned} \tag{1.92}$$

we have to establish the identity

$$\begin{aligned}
&\big(a \oplus (b \oplus c), \operatorname{gyr}[a, b \oplus c] \operatorname{gyr}[b, c] A_1 A_2 A_3\big) \\
&\quad = \big((a \oplus b) \oplus \operatorname{gyr}[a, b]c, \operatorname{gyr}\big[a \oplus b, \operatorname{gyr}[a, b]c\big] \operatorname{gyr}[a, b] A_1 A_2 A_3\big)
\end{aligned} \tag{1.93}$$

This identity between two pairs is equivalent to the two identities between their corresponding entries,

$$\begin{aligned}
a \oplus (b \oplus c) &= (a \oplus b) \oplus \operatorname{gyr}[a, b]c \\
\operatorname{gyr}[a, b \oplus c] \operatorname{gyr}[b, c] &= \operatorname{gyr}\big[a \oplus b, \operatorname{gyr}[a, b]c\big] \operatorname{gyr}[a, b]
\end{aligned} \tag{1.94}$$

The first identity is valid, being the left gyroassociative law, and the second identity is valid by (1.45), p. 15. □

Instructively, a second proof of Theorem 1.19 is given below.

Proof A one-to-one map of a set Q_1 onto a set Q_2 is said to be bijective and, accordingly, the map is called a bijection. The set of all bijections of a set Q onto itself forms a group under bijection composition. Let S be the group of all bijections of the set G onto itself under bijection composition. Let each element

$$(a, A) \in S_0 := G \times \mathrm{Aut}_0(G) \tag{1.95}$$

act bijectively on the gyrogroup $(G, \oplus)$ according to the equation

$$(a, A)g = a \oplus Ag \tag{1.96}$$

the unique inverse of (a, A) in $S_0 = G \times \mathrm{Aut}_0(G)$ being, by (1.89),

$$(a, A)^{-1} = \left(\ominus A^{-1}a, A^{-1}\right) \tag{1.97}$$

Being a set of special bijections of G onto itself, given by (1.96), S_0 is a subset of the group S, $S_0 \subset S$. Employing the subgroup criterion in Theorem 1.4, p. 11, we will show that, under bijection composition, S_0 is a subgroup of the group S.

Two successive bijections $(a, A), (b, B) \in S_0$ of G are equivalent to a single bijection $(c, C) \in S_0$ according to the following chain of equations. Employing successively the bijection (1.96) along with the left gyroassociative law we have

$$\begin{aligned}
(a, A)(b, B)g &= (a, A)(b \oplus Bg) \\
&= a \oplus A(b \oplus Bg) \\
&= a \oplus (Ab \oplus ABg) \\
&= (a \oplus Ab) \oplus \mathrm{gyr}[a, Ab]ABg \\
&= \left(a \oplus Ab, \mathrm{gyr}[a, Ab]AB\right)g \\
&=: (c, C)g
\end{aligned} \tag{1.98}$$

for all $g \in G$, $(a, A), (b, B) \in S_0$.

It follows from (1.98) that bijection composition in S_0 is given by the gyrosemidirect product, (1.86),

$$(a, A)(b, B) = \left(a \oplus Ab, \mathrm{gyr}[a, Ab]AB\right) \tag{1.99}$$

Finally, for any $(a, A), (b, B) \in S_0$ we have by (1.97) and (1.99),

$$\begin{aligned}
(a, A)(b, B)^{-1} &= (a, A)\left(\ominus B^{-1}b, B^{-1}\right) \\
&= \left(a \ominus AB^{-1}b, \mathrm{gyr}\left[a, \ominus AB^{-1}b\right]AB^{-1}\right) \\
&\in S_0
\end{aligned} \tag{1.100}$$

Hence, by the subgroup criterion in Theorem 1.4, p. 11, the subset S_0 of the group S of all bijections of G onto itself is a subgroup under bijection composition. But, bijection composition in S_0 is given by the gyrosemidirect product (1.99). Hence, as desired, the set $S_0 = G \times \mathrm{Aut}_0(G)$ with composition given by the gyrosemidirect product (1.99) forms a group. □

The gyrosemidirect product group enables problems in gyrogroups to be converted to the group setting thus gaining access to the powerful group theoretic techniques. Illustrative examples for the use of gyrosemidirect product groups are provided by the proof of the following two Theorems 1.20 and 1.21.

Theorem 1.20 *Let* $(G, \oplus)$ *be a gyrogroup, let* $a, b \in G$ *be any two elements of* G*, and let* $Y \in \mathrm{Aut}(G)$ *be any automorphism of* $(G, \oplus)$*. Then, the unique solution of the automorphism equation*

$$Y = \ominus\,\mathrm{gyr}[b, Xa]X \tag{1.101}$$

for the unknown automorphism $X \in \mathrm{Aut}(G)$ *is*

$$X = \ominus\,\mathrm{gyr}[b, Ya]Y \tag{1.102}$$

Proof Let X be a solution of (1.101), and let $x \in G$ be given by the equation

$$x = b \boxminus Xa \tag{1.103}$$

so that, by a right cancellation, (1.78), $b = x \oplus Xa$.

Then we have the following gyrosemidirect product

$$\begin{aligned}(x, X)(a, I) &= \big(x \oplus Xa, \mathrm{gyr}[x, Xa]X\big)\\ &= \big(x \oplus Xa, \mathrm{gyr}[x \oplus Xa, Xa]X\big)\\ &= \big(b, \mathrm{gyr}[b, Xa]X\big)\\ &= (b, \ominus Y)\end{aligned} \tag{1.104}$$

so that

$$\begin{aligned}(x, X) &= (b, \ominus Y)(a, I)^{-1}\\ &= (b, \ominus Y)(\ominus a, I)\\ &= \big(b \oplus Ya, \ominus\,\mathrm{gyr}[b, Ya]Y\big)\end{aligned} \tag{1.105}$$

Comparing the second entries of the extreme sides of (1.105) we have

$$X = \ominus\,\mathrm{gyr}[b, Ya]Y \tag{1.106}$$

Hence, if a solution X of (1.101) exists, then it must be given by (1.102).

Conversely, the automorphism X in (1.106) is, indeed, a solution of (1.101) as we see by substituting X from (1.106) into the right-hand side of (1.101), and employing the nested gyration identity (1.47), p. 15. Indeed,

$$\ominus\,\mathrm{gyr}[b, Xa]X = \mathrm{gyr}\big[b, \ominus\,\mathrm{gyr}[b, Ya]Ya\big]\,\mathrm{gyr}[b, Ya]Y = Y \tag{1.107}$$

as desired. □

1.12 Basic Gyration Properties

The most important basic gyration properties that we establish in this section are the *gyration even property*

$$\mathrm{gyr}[\ominus a, \ominus b] = \mathrm{gyr}[a, b] \tag{1.108}$$

and the *gyroautomorphism inversion law*

$$\mathrm{gyr}^{-1}[a, b] = \mathrm{gyr}[b, a] \tag{1.109}$$

for any two elements a and b of a gyrogroup $(G, \oplus)$, where $\mathrm{gyr}^{-1}[a, b] = (\mathrm{gyr}[a, b])^{-1}$ is the inverse of the gyration $\mathrm{gyr}[a, b]$.

Theorem 1.21 (Gyrosum Inversion, Gyroautomorphism Inversion) *For any two elements a, b of a gyrogroup $(G, \oplus)$ we have the gyrosum inversion law*

$$\ominus(a\oplus b) = \mathrm{gyr}[a, b](\ominus b\ominus a) \tag{1.110}$$

and the gyroautomorphism inversion law

$$\mathrm{gyr}^{-1}[a, b] = \mathrm{gyr}[\ominus b, \ominus a] \tag{1.111}$$

Proof Let $\mathrm{Aut}_0(G)$ be any gyroautomorphism group of $(G, \oplus)$, and let $G \times \mathrm{Aut}_0(G)$ be the gyrosemidirect product of the gyrogroup G and the group $\mathrm{Aut}_0(G)$ according to Definition 1.18. Being a group, the product of two elements of the gyrosemidirect product group $G \times \mathrm{Aut}_0(G)$ has a unique inverse. This inverse can be calculated in two different ways.

On the one hand, the inverse of the left-hand side of the gyrosemidirect product

$$(a, I)(b, I) = \big(a\oplus b, \mathrm{gyr}[a, b]\big) \tag{1.112}$$

in $G \times \mathrm{Aut}_0(G)$ is

$$\begin{aligned}(b, I)^{-1}(a, I)^{-1} &= (\ominus b, I)(\ominus a, I)\\ &= \big(\ominus b\ominus a, \mathrm{gyr}[\ominus b, \ominus a]\big)\end{aligned} \tag{1.113}$$

On the other hand, the inverse of the right-hand side of the product (1.112) is, by (1.97),

$$\big(\ominus \operatorname{gyr}^{-1}[a,b](a\oplus b), \operatorname{gyr}^{-1}[a,b]\big) \tag{1.114}$$

for all $a, b \in G$. Comparing corresponding entries in (1.113) and (1.114), we have

$$\ominus b\ominus a = \ominus \operatorname{gyr}^{-1}[a,b](a\oplus b) \tag{1.115}$$

and

$$\operatorname{gyr}[\ominus b, \ominus a] = \operatorname{gyr}^{-1}[a,b] \tag{1.116}$$

Eliminating $\operatorname{gyr}^{-1}[a,b]$ between (1.115) and (1.116), we have

$$\ominus b\ominus a = \ominus \operatorname{gyr}[\ominus b, \ominus a](a\oplus b) \tag{1.117}$$

Replacing (a,b) by $(\ominus b, \ominus a)$, (1.117) becomes

$$a\oplus b = \ominus \operatorname{gyr}[a,b](\ominus b\ominus a) \tag{1.118}$$

Identities (1.118) and (1.116) complete the proof. □

Instructively, the gyrosum inversion law (1.110) is verified here as a by-product along with the gyroautomorphism inversion law (1.111) in Theorem 1.21 in terms of the gyrosemidirect product group. A direct proof of (1.110) is, however, simpler as we see in Theorem 1.9, p. 15.

Theorem 1.22 *Let $(G, \oplus)$ be a gyrogroup. Then for all $a, b \in G$*

$$\operatorname{gyr}^{-1}[a,b] = \operatorname{gyr}\big[a, \ominus \operatorname{gyr}[a,b]b\big] \tag{1.119}$$

$$\operatorname{gyr}^{-1}[a,b] = \operatorname{gyr}[\ominus a, a\oplus b] \tag{1.120}$$

$$\operatorname{gyr}^{-1}[a,b] = \operatorname{gyr}[b, a\oplus b] \tag{1.121}$$

$$\operatorname{gyr}[a,b] = \operatorname{gyr}[b, \ominus b\ominus a] \tag{1.122}$$

$$\operatorname{gyr}[a,b] = \operatorname{gyr}[\ominus a, \ominus b\ominus a] \tag{1.123}$$

$$\operatorname{gyr}[a,b] = \operatorname{gyr}\big[\ominus(a\oplus b), a\big] \tag{1.124}$$

Proof Identity (1.119) follows from (1.47).

Identity (1.120) follows from (1.48).

Identity (1.121) results from an application to (1.120) of the left loop property followed by a left cancellation.

Identity (1.122) follows from the gyroautomorphism inversion law (1.111) and from (1.120),

$$\operatorname{gyr}[a,b] = \operatorname{gyr}^{-1}[\ominus b, \ominus a] = \operatorname{gyr}[b, \ominus b\ominus a] \tag{1.125}$$

Identity (1.123) follows from an application, to the right-hand side of (1.122), of the left loop property followed by a left cancellation.

Identity (1.124) follows by inverting (1.120) by means of the gyroautomorphism inversion law (1.111). □

Theorem 1.23 (The Gyration Inversion Law; The Gyration Even Property) *The gyroautomorphisms of any gyrogroup* $(G, \oplus)$ *obey the gyration inversion law*

$$\mathrm{gyr}^{-1}[a, b] = \mathrm{gyr}[b, a] \tag{1.126}$$

and possess the gyration even property

$$\mathrm{gyr}[\ominus a, \ominus b] = \mathrm{gyr}[a, b] \tag{1.127}$$

satisfying the four mutually equivalent nested gyroautomorphism identities

$$\begin{aligned} \mathrm{gyr}\big[b, \ominus\, \mathrm{gyr}[b, a]a\big] &= \mathrm{gyr}[a, b] \\ \mathrm{gyr}\big[b, \mathrm{gyr}[b, \ominus a]a\big] &= \mathrm{gyr}[a, \ominus b] \\ \mathrm{gyr}\big[\ominus\, \mathrm{gyr}[a, b]b, a\big] &= \mathrm{gyr}[a, b] \\ \mathrm{gyr}\big[\mathrm{gyr}[a, \ominus b]b, a\big] &= \mathrm{gyr}[a, \ominus b] \end{aligned} \tag{1.128}$$

for all $a, b \in G$.

Proof By the left loop property and (1.121), we have

$$\begin{aligned} \mathrm{gyr}^{-1}[a \oplus b, b] &= \mathrm{gyr}^{-1}[a, b] \\ &= \mathrm{gyr}[b, a \oplus b] \end{aligned} \tag{1.129}$$

for all $a, b \in G$. Let us substitute $a = c \boxminus b$ into (1.129), so that by a right cancellation $a \oplus b = c$, obtaining the identity

$$\mathrm{gyr}^{-1}[c, b] = \mathrm{gyr}[b, c] \tag{1.130}$$

for all $c, b \in G$. Renaming c in (1.130) as a, we obtain (1.126), as desired.

Identity (1.127) results from (1.111) and (1.126),

$$\begin{aligned} \mathrm{gyr}[\ominus a, \ominus b] &= \mathrm{gyr}^{-1}[b, a] \\ &= \mathrm{gyr}[a, b] \end{aligned} \tag{1.131}$$

Finally, the first identity in (1.128) follows from (1.119) and (1.126).

By means of the gyroautomorphism inversion law (1.126), the third identity in (1.128) is equivalent to the first one.

The second (fourth) identity in (1.128) follows from the first (third) by replacing a by $\ominus a$ (or, alternatively, by replacing b by $\ominus b$), noting that gyrations are even by (1.127). □

The left gyroassociative law and the left loop property of gyrogroups admit right counterparts, as we see from the following theorem.

Theorem 1.24 *For any three elements a, b, and c of a gyrogroup $(G, \oplus)$ we have*

(i) $(a\oplus b)\oplus c = a\oplus(b\oplus \mathrm{gyr}[b, a]c)$ (*Right Gyroassociative Law*).
(ii) $\mathrm{gyr}[a, b] = \mathrm{gyr}[a, b\oplus a]$ (*Right Loop Property*).

Proof The right gyroassociative law follows from the left gyroassociative law and the gyration inversion law (1.126) of gyroautomorphisms,

$$\begin{aligned} a\oplus\bigl(b\oplus \mathrm{gyr}[b, a]c\bigr) &= (a\oplus b)\oplus \mathrm{gyr}[a, b]\,\mathrm{gyr}[b, a]c \\ &= (a\oplus b)\oplus c \end{aligned} \tag{1.132}$$

The right loop property results from (1.121) and the gyration inversion law (1.126),

$$\begin{aligned} \mathrm{gyr}[b, a\oplus b] &= \mathrm{gyr}^{-1}[a, b] \\ &= \mathrm{gyr}[b, a] \end{aligned} \tag{1.133}$$

□

The right cancellation law allows the loop property to be dualized in the following theorem.

Theorem 1.25 (The Coloop Property—Left and Right) *Let $(G, \oplus)$ be a gyrogroup. Then*

$$\begin{aligned} \mathrm{gyr}[a, b] &= \mathrm{gyr}[a \boxminus b, b] \quad (\textit{Left Coloop Property}) \\ \mathrm{gyr}[a, b] &= \mathrm{gyr}[a, b \boxminus a] \quad (\textit{Right Coloop Property}) \end{aligned}$$

for all $a, b \in G$.

Proof The proof follows from an application of the left and the right loop property followed by a right cancellation,

$$\begin{aligned} \mathrm{gyr}[a \boxminus b, b] &= \mathrm{gyr}\bigl[(a \boxminus b)\oplus b, b\bigr] = \mathrm{gyr}[a, b] \\ \mathrm{gyr}[a, b \boxminus a] &= \mathrm{gyr}\bigl[a, (b \boxminus a)\oplus a\bigr] = \mathrm{gyr}[a, b] \end{aligned} \tag{1.134}$$

□

A right and a left loop give rise to the identities in the following theorem.

Theorem 1.26 *Let $(G, \oplus)$ be a gyrogroup. Then*

$$\begin{aligned} \mathrm{gyr}[a\oplus b, \ominus a] &= \mathrm{gyr}[a, b] \\ \mathrm{gyr}[\ominus a, a\oplus b] &= \mathrm{gyr}[b, a] \end{aligned} \tag{1.135}$$

for all $a, b \in G$.

Proof By a right loop, a left cancellation and a left loop we have

$$\begin{aligned}\mathrm{gyr}[a\oplus b, \ominus a] &= \mathrm{gyr}\big[a\oplus b, \ominus a\oplus(a\oplus b)\big]\\ &= \mathrm{gyr}[a\oplus b, b]\\ &= \mathrm{gyr}[a, b]\end{aligned} \tag{1.136}$$

thus verifying the first identity in (1.135). The second identity in (1.135) follows from the first one by gyroautomorphism inversion, (1.126). □

In general, $\ominus(a\oplus b) \neq \ominus a\ominus b$ in a gyrogroup $(G, \oplus)$. In fact, we have $\ominus(a\oplus b) = \ominus a\ominus b$ for all $a, b \in G$ if and only if the gyrogroup $(G, \oplus)$ is gyrocommutative, as we see from Theorem 1.32, p. 34, of Chap. 2 on gyrocommutative gyrogroups. In this sense, the gyrogroup cooperation $\boxplus$ conducts itself more properly than its associated gyrogroup operation, as we see from the following theorem.

Theorem 1.27 (The Cogyroautomorphic Inverse Property) *Any gyrogroup* $(G, \oplus)$ *possesses the cogyroautomorphic inverse property*,

$$\ominus(a \boxplus b) = (\ominus b) \boxplus (\ominus a) \tag{1.137}$$

for any $a, b \in G$.

Proof We verify (1.137) in the following chain of equations, which are numbered for subsequent derivation:

$$\begin{aligned}a \boxplus b &\overbrace{=\!=\!=}^{(1)} a\oplus \mathrm{gyr}[a, \ominus b]b\\ &\overbrace{=\!=\!=}^{(2)} \ominus \mathrm{gyr}\big[a, \mathrm{gyr}[a, \ominus b]b\big]\big\{\ominus \mathrm{gyr}[a, \ominus b]b\ominus a\big\}\\ &\overbrace{=\!=\!=}^{(3)} \mathrm{gyr}\big[a, \mathrm{gyr}[a, \ominus b]b\big]\big\{\ominus\big(\ominus \mathrm{gyr}[a, \ominus b]b\ominus a\big)\big\}\\ &\overbrace{=\!=\!=}^{(4)} \mathrm{gyr}\big[a, \ominus \mathrm{gyr}[a, \ominus b](\ominus b)\big]\big\{\ominus\big(\ominus \mathrm{gyr}[a, \ominus b]b\ominus a\big)\big\}\\ &\overbrace{=\!=\!=}^{(5)} \mathrm{gyr}^{-1}[a, \ominus b]\big\{\ominus\big(\ominus \mathrm{gyr}[a, \ominus b]b\ominus a\big)\big\}\\ &\overbrace{=\!=\!=}^{(6)} \ominus\big(\ominus b\ominus \mathrm{gyr}^{-1}[a, \ominus b]a\big)\\ &\overbrace{=\!=\!=}^{(7)} \ominus\big\{\ominus b\ominus \mathrm{gyr}[b, \ominus a]a\big\}\\ &\overbrace{=\!=\!=}^{(8)} \ominus\big\{(\ominus b) \boxplus (\ominus a)\big\}\end{aligned} \tag{1.138}$$

Inverting both extreme sides of (1.138) we obtain the desired identity (1.137).

Derivation of the numbered equalities in (1.138) follows:

1. Follows from Definition 1.7, p. 13, of the gyrogroup cooperation $\boxplus$.
2. Follows from Item 1 by the gyrosum inversion, (1.110).
3. Follows from Item 2 by Theorem 1.8(12) applied to the term $\{\ldots\}$ in (2).
4. Follows from Item 3 by Theorem 1.8(12) applied to b, that is, $\mathrm{gyr}[a, \ominus b]b = \ominus\,\mathrm{gyr}[a, \ominus b](\ominus b)$.
5. Follows from Item 4 by Identity (1.119) of Theorem 1.22.
6. Follows from Item 5 by distributing the gyroautomorphism $\mathrm{gyr}^{-1}[a, \ominus b]$ over each of the two terms in $\{\ldots\}$.
7. Follows from Item 6 by the gyroautomorphism inversion law (1.111).
8. Follows from Item 7 by Definition 1.7, p. 13, of the gyrogroup cooperation $\boxplus$.

□

Theorem 1.28 *Let* $(G, \oplus)$ *be a gyrogroup. Then*

$$a\oplus\big\{(\ominus a\oplus b)\oplus a\big\} = b \boxplus a \tag{1.139}$$

for all $a, b \in G$.

Proof The proof rests on the following chain of equations, which are numbered for subsequent explanation:

$$\begin{aligned} a\oplus\big\{(\ominus a\oplus b)\oplus a\big\} &\overset{(1)}{=} \big\{a\oplus(\ominus a\oplus b)\big\}\oplus \mathrm{gyr}[a, \ominus a\oplus b]a \\ &\overset{(2)}{=} b\oplus \mathrm{gyr}[b, \ominus a\oplus b]a \\ &\overset{(3)}{=} b\oplus \mathrm{gyr}[b, \ominus a]a \\ &\overset{(4)}{=} b \boxplus a \end{aligned} \tag{1.140}$$

The derivation of the equalities in (1.140) follows.

1. Follows from the left gyroassociative law.
2. Follows from Item 1 by a left cancellation, and by a left loop followed by a left cancellation.
3. Follows from Item 2 by a right loop, that is, an application of the right loop property to (3) gives (2).
4. Follows from Item 3 by Definition 1.7, p. 13, of the gyrogroup cooperation $\boxplus$.

□

1.13 An Advanced Gyrogroup Equation

As an example, we present in Theorem 1.29 below an advanced gyrogroup equation and its unique solution. The equation is advanced in the sense that its unknown appears in the equation both directly, and indirectly in the argument of a gyration.

Theorem 1.29 *Let*

$$c = \mathrm{gyr}[b, \ominus x]x \tag{1.141}$$

be an equation for the unknown x *in a gyrogroup* $(G, \oplus)$. *The unique solution of* (1.141) *is*

$$x = \ominus\big(\ominus b \ominus (c \boxminus b)\big) \tag{1.142}$$

Proof If a solution x to the gyrogroup equation (1.141) exists then, by (1.141) and by the second identity in (1.44), p. 15,

$$c = \mathrm{gyr}[b, \ominus x]x = \ominus(b \ominus x) \oplus b \tag{1.143}$$

Applying a right cancellation to (1.143), we obtain

$$\ominus(b \ominus x) = c \boxminus b \tag{1.144}$$

or, equivalently, by gyro-sign inversion,

$$b \ominus x = \ominus(c \boxminus b) \tag{1.145}$$

so that, by a left cancellation

$$\ominus x = \ominus b \ominus (c \boxminus b) \tag{1.146}$$

implying, by gyro-sign inversion,

$$x = \ominus\big(\ominus b \ominus (c \boxminus b)\big) \tag{1.147}$$

Hence, if a solution x to (1.141) exists, it must be given by (1.147).

Conversely, x given by (1.147) is a solution. Indeed, the substitution of x of (1.147) into the right-hand side of (1.143) results in c, as we see in the following chain of equations, which are numbered for subsequent derivation:

$$\begin{aligned}
\mathrm{gyr}[b, \ominus x]x &\overbrace{=\!=\!=}^{(1)} \ominus \mathrm{gyr}\big[b, \ominus b \ominus (c \boxminus b)\big]\big\{\ominus b \ominus (c \boxminus b)\big\} \\
&\overbrace{=\!=\!=}^{(2)} \ominus\big\{b \oplus \big(\ominus b \ominus (c \boxminus b)\big)\big\} \oplus b \\
&\overbrace{=\!=\!=}^{(3)} \ominus \mathrm{gyr}\big[\ominus\big\{b \oplus \big(\ominus b \ominus (c \boxminus b)\big)\big\}, b\big]\big(\ominus b \oplus \big\{b \oplus \big(\ominus b \ominus (c \boxminus b)\big)\big\}\big)
\end{aligned}$$

$$\overbrace{=\!=}^{(4)} \ominus \mathrm{gyr}\big[b\oplus\big(\ominus b\ominus(c\boxminus b)\big), \ominus b\big]\big\{\ominus b\ominus(c\boxminus b)\big\}$$
$$\overbrace{=\!=}^{(5)} \ominus \mathrm{gyr}\big[b, \ominus b\ominus(c\boxminus b)\big]\big\{\ominus b\ominus(c\boxminus b)\big\}$$
$$\overbrace{=\!=}^{(6)} \ominus \mathrm{gyr}\big[\ominus(c\boxminus b), \ominus b\big]\big\{\ominus b\ominus(c\boxminus b)\big\}$$
$$\overbrace{=\!=}^{(7)} \ominus \mathrm{gyr}[c\boxminus b, b]\big\{\ominus b\ominus(c\boxminus b)\big\}$$
$$\overbrace{=\!=}^{(8)} (c\boxminus b)\oplus b$$
$$\overbrace{=\!=}^{(9)} c \tag{1.148}$$

Derivation of the numbered equalities in (1.148) follows:

1. Follows from the substitution of x from (1.147), and from Theorem 1.8(12).
2. Follows from Item 1 by the first identity in (1.44), p. 15.
3. Follows from Item 2 by the gyrosum inversion law (1.42), p. 15.
4. Follows from Item 3 by the gyration even property and by a left cancellation.
5. Follows from Item 4 by the first identity in (1.135).
6. Follows from Item 5 by the second identity in (1.135).
7. Follows from Item 6 by the gyration even property.
8. Follows from Item 7 by the gyrosum inversion law (1.42), p. 15.
9. Follows from Item 8 by a right cancellation. □

Corollary 1.30 *Let* $(G, \oplus)$ *be a gyrogroup. The map*

$$a \mapsto c = \mathrm{gyr}[b, \ominus a]a \tag{1.149}$$

of G into itself is bijective so that when a runs over all the elements of G its image, c, runs over all the elements of G as well, for any given element $b \in G$.

Proof It follows immediately from Theorem 1.29 that, for any given $b \in G$, the map (1.149) is bijective and, hence, the result of the corollary. □

1.14 Gyrocommutative Gyrogroups

Definition 1.31 (Gyroautomorphic Inverse Property) A gyrogroup $(G, \oplus)$ possesses the *gyroautomorphic inverse property* if for all $a, b \in G$,

$$\ominus(a\oplus b) = \ominus a\ominus b \tag{1.150}$$

Theorem 1.32 (The Gyroautomorphic Inverse Property) *A gyrogroup is gyrocommutative if and only if it possesses the gyroautomorphic inverse property.*

Proof Let $(G, \oplus)$ be a gyrogroup possessing the gyroautomorphic inverse property. Then the gyrosum inversion law (1.42), p. 15, specializes, by means of Theorem 1.8(12), p. 13, to the gyrocommutative law $(G6)$ in Definition 1.6, p. 12,

$$\begin{aligned} a\oplus b &= \ominus \operatorname{gyr}[a, b](\ominus b \ominus a) \\ &= \operatorname{gyr}[a, b]\big\{\ominus(\ominus b \ominus a)\big\} \\ &= \operatorname{gyr}[a, b](b\oplus a) \end{aligned} \tag{1.151}$$

for all $a, b \in G$.

Conversely, if the gyrocommutative law is valid then by Theorem 1.8(12) and the gyrosum inversion law, (1.42), p. 15, we have

$$\operatorname{gyr}[a, b]\big\{\ominus(\ominus b\ominus a)\big\} = \ominus \operatorname{gyr}[a, b](\ominus b\ominus a) = a\oplus b = \operatorname{gyr}[a, b](b\oplus a) \tag{1.152}$$

so that, by eliminating the gyroautomorphism $\operatorname{gyr}[a, b]$ on both extreme sides of (1.152) and inverting the gyro-sign, we recover the gyroautomorphic inverse property,

$$\ominus(b\oplus a) = \ominus b\ominus a \tag{1.153}$$

for all $a, b \in G$. □

Theorem 1.33 *The gyrogroup cooperation $\boxplus$ of a gyrogroup $(G, \oplus)$ is commutative if and only if the gyrogroup $(G, \oplus)$ is gyrocommutative.*

Proof For any $a, b \in G$ we have, by Equality (7) of the chain of equations (1.138) in the proof of Theorem 1.27, p. 31,

$$a \boxplus b = \ominus\big(\ominus b\ominus \operatorname{gyr}[b, \ominus a]a\big) \tag{1.154}$$

But by definition,

$$b \boxplus a = b\oplus \operatorname{gyr}[b, \ominus a]a \tag{1.155}$$

Hence

$$a \boxplus b = b \boxplus a \tag{1.156}$$

for all $a, b \in G$ if and only if

$$\ominus(\ominus b\ominus c) = b\oplus c \tag{1.157}$$

for all $a, b \in G$, where

$$c = \operatorname{gyr}[b, \ominus a]a \tag{1.158}$$

as we see from (1.154) and (1.155). But the self-map of G that takes a to c in (1.158),

$$a \mapsto \operatorname{gyr}[b, \ominus a]a = c \tag{1.159}$$

for any given $b \in G$ is bijective, by Corollary 1.30, p. 34. Hence, the commutative relation (1.156) for $\boxplus$ holds for all $a, b \in G$ if and only if (1.157) holds for all $b, c \in G$. The latter, in turn, is the gyroautomorphic inverse property that, by Theorem 1.32, is equivalent to the gyrocommutativity of the gyrogroup $(G, \oplus)$. Hence, (1.156) holds for all $a, b \in G$ if and only if the gyrogroup $(G, \oplus)$ is gyrocommutative. □

Theorem 1.34 *Let $(G, \oplus)$ be a gyrocommutative gyrogroup. Then*

$$\mathrm{gyr}[a, b]\,\mathrm{gyr}[b \oplus a, c] = \mathrm{gyr}[a, b \oplus c]\,\mathrm{gyr}[b, c] \tag{1.160}$$

for all $a, b, c \in G$.

Proof Using the notation $g_{a,b} = \mathrm{gyr}[a, b]$ whenever convenient, we have by Theorem 1.16, p. 22, by the gyrocommutative law, and by Identity (1.45), p. 15,

$$\begin{aligned}\mathrm{gyr}[a, b]\,\mathrm{gyr}[b \oplus a, c] &= \mathrm{gyr}\big[g_{a,b}(b \oplus a), g_{a,b}c\big]\,\mathrm{gyr}[a, b]\\ &= \mathrm{gyr}\big[a \oplus b, \mathrm{gyr}[a, b]c\big]\,\mathrm{gyr}[a, b]\\ &= \mathrm{gyr}[a, b \oplus c]\,\mathrm{gyr}[b, c]\end{aligned} \tag{1.161}$$

□

Theorem 1.35 *Let $a, b, c \in G$ be any three elements of a gyrocommutative gyrogroup $(G, \oplus)$, and let $d \in G$ be determined by the "gyroparallelogram condition"*

$$d = (b \boxplus c) \ominus a \tag{1.162}$$

Then, the elements a, b, c and d satisfy the telescopic gyration identity

$$\mathrm{gyr}[a, \ominus b]\,\mathrm{gyr}[b, \ominus c]\,\mathrm{gyr}[c, \ominus d] = \mathrm{gyr}[a, \ominus d] \tag{1.163}$$

for all $a, b, c \in G$.

Proof By Identity (1.160), along with an application of the right and the left loop property, we have

$$\mathrm{gyr}\big[a', b' \oplus a'\big]\,\mathrm{gyr}\big[b' \oplus a', c'\big] = \mathrm{gyr}\big[a', b' \oplus c'\big]\,\mathrm{gyr}\big[b' \oplus c', c'\big] \tag{1.164}$$

Let

$$\begin{aligned} a &= \ominus c' \\ c &= \ominus a' \\ b &= b' \oplus a' \end{aligned} \tag{1.165}$$

so that, by the third equation in (1.165) and by (1.162), we have

$$\begin{aligned} b'\oplus c' &= \bigl(b \boxminus a'\bigr)\oplus c' \\ &= (b \boxplus c)\ominus a \\ &= d \end{aligned} \tag{1.166}$$

Then (1.164), expressed in terms of a, b, c, d in (1.165)–(1.166), takes the form

$$\mathrm{gyr}[\ominus c, b]\,\mathrm{gyr}[b, \ominus a] = \mathrm{gyr}[\ominus c, d]\,\mathrm{gyr}[d, \ominus a] \tag{1.167}$$

Inverting both sides of (1.167) by means of the gyration inversion law (1.126), p. 29, and the gyration even property (1.127), p. 29, we obtain the identity

$$\mathrm{gyr}[a, \ominus b]\,\mathrm{gyr}[b, \ominus c] = \mathrm{gyr}[a, \ominus d]\,\mathrm{gyr}[d, \ominus c] \tag{1.168}$$

from which the telescopic gyration identity (1.163) follows immediately, by a gyration inversion and the gyration even property. □

Theorem 1.36 *The gyroparallelogram condition* (1.162),

$$d = (b \boxplus c)\ominus a \tag{1.169}$$

in a gyrocommutative gyrogroup $(G, \oplus)$ *is equivalent to the identity*

$$\ominus c\oplus d = \mathrm{gyr}[c, \ominus b](b\ominus a) \tag{1.170}$$

Proof In a gyrocommutative gyrogroup $(G, \oplus)$, the gyroparallelogram condition (1.162) implies the following chain of equations, which are numbered for subsequent derivation.

$$\begin{aligned} d &\overbrace{=\!=\!=}^{(1)} (b \boxplus c)\ominus a \\ &\overbrace{=\!=\!=}^{(2)} (c \boxplus b)\ominus \mathrm{gyr}[b, \ominus c]\,\mathrm{gyr}[c, \ominus b]a \\ &\overbrace{=\!=\!=}^{(3)} (c \boxplus b)\ominus \mathrm{gyr}\bigl[c, \mathrm{gyr}[c, \ominus b]b\bigr]\,\mathrm{gyr}[c, \ominus b]a \\ &\overbrace{=\!=\!=}^{(4)} \bigl(c\oplus \mathrm{gyr}[c, \ominus b]b\bigr)\ominus \mathrm{gyr}\bigl[c, \mathrm{gyr}[c, \ominus b]b\bigr]\,\mathrm{gyr}[c, \ominus b]a \\ &\overbrace{=\!=\!=}^{(5)} c\oplus\bigl(\mathrm{gyr}[c, \ominus b]b\ominus \mathrm{gyr}[c, \ominus b]a\bigr) \\ &\overbrace{=\!=\!=}^{(6)} c\oplus \mathrm{gyr}[c, \ominus b](b\ominus a) \end{aligned} \tag{1.171}$$

for all $a, b, c \in G$. Derivation of the numbered equalities in (1.171) follows:

1. This is the gyroparallelogram condition (1.162).
2. Follows from Item 1 (i) since the gyrogroup cooperation $\boxplus$ is commutative, by Theorem 1.33, p. 35, and (ii) since, by gyration inversion along with the gyration even property in (1.126)–(1.127), p. 29, the gyration product applied to a in (2) is trivial.
3. Follows from Item 2 by the second nested gyration identity in (1.128), p. 29.
4. Follows from Item 3 by Definition 1.7, p. 13, of the gyrogroup cooperation $\boxplus$.
5. Follows from Item 4 by the left gyroassociative law. Indeed, an application of the left gyroassociative law to (5) results in (4).
6. Follows from Item 5 since gyroautomorphisms respect their gyrogroup operation.

Finally, (1.170) follows from (1.171) by a left cancellation, moving c from the extreme right-hand side of (1.171) to its extreme left-hand side. □

Theorem 1.37 *Let $(G, \oplus)$ be a gyrocommutative gyrogroup. Then*

$$\text{gyr}[a, b]\{b\oplus(a\oplus c)\} = (a\oplus b)\oplus c \tag{1.172}$$

for all $a, b, c \in G$.

Proof By the left gyroassociative law and by the gyrocommutative law we have the chain of equations

$$\begin{aligned} b\oplus(a\oplus c) &= (b\oplus a)\oplus \text{gyr}[b, a]c \\ &= \text{gyr}[b, a](a\oplus b)\oplus \text{gyr}[b, a]c \\ &= \text{gyr}[b, a]\{(a\oplus b)\oplus c\} \end{aligned} \tag{1.173}$$

from which (1.172) is derived by the gyroautomorphism inversion law (1.111), p. 27. □

The special case of Theorem 1.37 corresponding to $c = \ominus a$ gives rise to a new cancellation law in gyrocommutative gyrogroups, called the *left-right cancellation law*.

Theorem 1.38 (The Left-Right Cancellation Law) *Let $(G, \oplus)$ be a gyrocommutative gyrogroup. Then*

$$(a\oplus b)\ominus a = \text{gyr}[a, b]b \tag{1.174}$$

for all $a, b, c \in G$.

Proof Identity (1.174) follows from (1.44), p. 15, and the gyroautomorphic inverse property (1.150), p. 34. Alternatively, Identity (1.174) is equivalent to the special case of (1.173) when $c = \ominus a$. □

The left-right cancellation law (1.174) is not a complete cancellation since the echo of the "canceled" a remains in the argument of the involved gyroautomorphism.

Theorem 1.39 *Let* $(G, \oplus)$ *be a gyrocommutative gyrogroup. Then*

$$a \oplus \bigl\{(\ominus a \oplus b) \oplus a\bigr\} = a \boxplus b \tag{1.175}$$

for all $a, b \in G$.

Proof By Theorem 1.28, p. 32, and Theorem 1.33,

$$a \oplus \bigl\{(\ominus a \oplus b) \oplus a\bigr\} = b \boxplus a = a \boxplus b \tag{1.176}$$

□

Theorem 1.40 *Let* $(G, \oplus)$ *be a gyrocommutative gyrogroup. Then*

$$a \boxplus (a \oplus b) = a \oplus (b \oplus a) \tag{1.177}$$

for all $a, b \in G$.

Proof By a left cancellation and Theorem 1.39, we have

$$a \oplus (b \oplus a) = a \oplus \bigl(\bigl\{\ominus a \oplus (a \oplus b)\bigr\} \oplus a\bigr) = a \boxplus (a \oplus b) \tag{1.178}$$

□

Theorem 1.41 (The Gyrotranslation Theorem, II) *Let* $(G, \oplus)$ *be a gyrocommutative gyrogroup. For all* $a, b, c \in G$,

$$\begin{aligned} \ominus(a \oplus b) \oplus (a \oplus c) &= \mathrm{gyr}[a, b](\ominus b \oplus c) \\ (a \oplus b) \ominus (a \oplus c) &= \mathrm{gyr}[a, b](b \ominus c) \end{aligned} \tag{1.179}$$

Proof The first identity in (1.179) follows from the Gyrotranslation Theorem 1.14, p. 18, with a replaced by $\ominus a$. Hence, it is valid in nongyrocommutative gyrogroups as well. The second identity in (1.179) follows from the first by the gyroautomorphic inverse property of gyrocommutative gyrogroups, Theorem 1.32, p. 34. Hence, it is valid in gyrocommutative gyrogroups. □

The following theorem gives an elegant gyration identity in which the product of three telescopic gyrations is equivalent to a single gyration.

Theorem 1.42 *Let* $a, b, c \in G$ *be any three elements of a gyrocommutative gyrogroup* $(G, \oplus)$. *Then*,

$$\mathrm{gyr}[\ominus a \oplus b, a \ominus c] = \mathrm{gyr}[a, \ominus b]\,\mathrm{gyr}[b, \ominus c]\,\mathrm{gyr}[c, \ominus a] \tag{1.180}$$

Proof By Theorem 1.16, p. 22, and by the gyrocommutative law, we have

$$\begin{aligned}\mathrm{gyr}[a,b]\,\mathrm{gyr}[b\oplus a,c] &= \mathrm{gyr}\big[\mathrm{gyr}[a,b](b\oplus a),\mathrm{gyr}[a,b]c\big]\,\mathrm{gyr}[a,b]\\ &= \mathrm{gyr}\big[a\oplus b,\mathrm{gyr}[a,b]c\big]\,\mathrm{gyr}[a,b]\end{aligned} \tag{1.181}$$

Hence, Identity (1.45) in Theorem 1.11, p. 15, can be written as

$$\mathrm{gyr}[a,b\oplus c]\,\mathrm{gyr}[b,c] = \mathrm{gyr}[a,b]\,\mathrm{gyr}[b\oplus a,c] \tag{1.182}$$

By gyroautomorphism inversion, the latter can be written as

$$\mathrm{gyr}[a,b\oplus c] = \mathrm{gyr}[a,b]\,\mathrm{gyr}[b\oplus a,c]\,\mathrm{gyr}[c,b] \tag{1.183}$$

Using the notation $b\oplus a = d$, which implies $a = \ominus b\oplus d$, Identity (1.183) becomes, by means of Theorem 1.26, p. 30,

$$\begin{aligned}\mathrm{gyr}[\ominus b\oplus d,b\oplus c] &= \mathrm{gyr}[\ominus b\oplus d,b]\,\mathrm{gyr}[d,c]\,\mathrm{gyr}[c,b]\\ &= \mathrm{gyr}[\ominus b,d]\,\mathrm{gyr}[d,c]\,\mathrm{gyr}[c,b]\end{aligned} \tag{1.184}$$

Renaming the elements $b,c,d\in G$, $(b,c,d)\to(\ominus a,c,\ominus b)$, (1.184) becomes

$$\mathrm{gyr}[a\ominus b,\ominus a\oplus c] = \mathrm{gyr}[a,\ominus b]\,\mathrm{gyr}[\ominus b,c]\,\mathrm{gyr}[c,\ominus a] \tag{1.185}$$

By means of the gyroautomorphic inverse property, Theorem 1.32, p. 34, and the gyration even property (1.127) in Theorem 1.23, p. 29, Identity (1.185) can be written, finally, in the desired form (1.180). □

The special case of Theorem 1.42 when $c = \ominus b$ is interesting, giving rise to the following theorem.

Theorem 1.43 *Let* $(G,\oplus)$ *be a gyrocommutative gyrogroup. Then*

$$\mathrm{gyr}[a,\ominus b] = \mathrm{gyr}[\ominus a\oplus b,a\oplus b]\,\mathrm{gyr}[a,b] \tag{1.186}$$

Proof Owing to the gyration inversion law in Theorem 1.23, p. 29, Identity (1.180) can be written as

$$\mathrm{gyr}[\ominus a\oplus b,a\ominus c]\,\mathrm{gyr}[\ominus a,c] = \mathrm{gyr}[a,\ominus b]\,\mathrm{gyr}[b,\ominus c] \tag{1.187}$$

from which the result (1.186) of the theorem follows in the spacial case when $c = \ominus b$ by applying the gyration even property. □

Identity (1.186) is interesting since it relates the gyration $\mathrm{gyr}[a,\ominus b]$ to the gyration $\mathrm{gyr}[a,b]$. Furthermore, it gives rise, by gyration inversion and the gyration even property, to the following elegant gyration identity,

$$\mathrm{gyr}[\ominus a,b]\,\mathrm{gyr}[b,a] = \mathrm{gyr}[\ominus a\oplus b,a\oplus b] \tag{1.188}$$

in which the product of two gyrations is equivalent to a single gyration.

As an application of Theorem 1.42, and for later reference, we present the following Theorem 1.44. This theorem, the proof of which involves a long chain of gyrocommutative gyrogroup identities, will prove crucially important for the introduction of hyperbolic vectors, called gyrovectors, into hyperbolic geometry in Chap. 3.

Theorem 1.44 *Let* $(G, \oplus)$ *be a gyrocommutative gyrogroup. Then*

$$(x \oplus a) \boxplus (x \oplus b) = x \oplus \{(a \boxplus b) \oplus x\} \tag{1.189}$$

for all $a, b, x \in G$.

Before presenting the proof of Theorem 1.44, it would be instructive to present an immediate corollary of the theorem. Renaming the elements $x, a, b \in G$ of the identity, (1.189), of Theorem 1.44 as $\ominus a, b, c$, respectively, we obtain the *parallelogram addition law* as a corollary of Theorem 1.44:

Corollary 1.45 (The Gyroparallelogram Addition Law) *Let* a, b, c *be any three elements of a gyrocommutative gyrogroup* $(G, \oplus)$, *and let* $d \in G$ *be given by the gyroparallelogram condition*, (1.162),

$$d = (b \boxplus c) \ominus a \tag{1.190}$$

Then, the points $a, b, d, c \in G$ *satisfy the gyroparallelogram addition law*

$$(\ominus a \oplus b) \boxplus (\ominus a \oplus c) = (\ominus a \oplus d) \tag{1.191}$$

The term *gyroparallelogram addition law* in Corollary 1.45 will be justified in Sect. 5.4, p. 123, in terms of analogies it shares with the common parallelogram addition law of Euclidean geometry.

Proof The proof of Theorem 1.44 is given by the following chain of equations, which are numbered for subsequent derivation.

$$\begin{aligned}
(x \oplus a) \boxplus (x \oplus b) &\overbrace{=\!=}^{(1)} (x \oplus a) \oplus \mathrm{gyr}\big[x \oplus a, \ominus(x \oplus b)\big](x \oplus b) \\
&\overbrace{=\!=}^{(2)} (x \oplus a) \oplus \mathrm{gyr}[x \oplus a, \ominus x \ominus b](x \oplus b) \\
&\overbrace{=\!=}^{(3)} x \oplus \big\{a \oplus \mathrm{gyr}[a, x]\, \mathrm{gyr}[x \oplus a, \ominus x \ominus b](x \oplus b)\big\} \\
&\overbrace{=\!=}^{(4)} x \oplus \big\{a \oplus \mathrm{gyr}[a, x]\, \mathrm{gyr}[\ominus x, \ominus a]\, \mathrm{gyr}[a, \ominus b]\, \mathrm{gyr}[b, x](x \oplus b)\big\} \\
&\overbrace{=\!=}^{(5)} x \oplus \big\{a \oplus \mathrm{gyr}[a, x]\, \mathrm{gyr}[x, a]\, \mathrm{gyr}[a, \ominus b](b \oplus x)\big\}
\end{aligned}$$

$$
\begin{aligned}
&\overbrace{=\!=}^{(6)} x\oplus\{a\oplus \text{gyr}[a,\ominus b](b\oplus x)\}\\
&\overbrace{=\!=}^{(7)} x\oplus\{a\oplus(\text{gyr}[a,\ominus b]b\oplus \text{gyr}[a,\ominus b]x)\}\\
&\overbrace{=\!=}^{(8)} x\oplus\{(a\oplus \text{gyr}[a,\ominus b]b)\oplus \text{gyr}\big[a,\text{gyr}[a,\ominus b]b\big]\,\text{gyr}[a,\ominus b]x\}\\
&\overbrace{=\!=}^{(9)} x\oplus\{(a\oplus \text{gyr}[a,\ominus b]b)\oplus \text{gyr}[b,\ominus a]\,\text{gyr}[a,\ominus b]x\}\\
&\overbrace{=\!=}^{(10)} x\oplus\{(a\oplus \text{gyr}[a,\ominus b]b)\oplus x\}\\
&\overbrace{=\!=}^{(11)} x\oplus\{(a\boxplus b)\oplus x\}
\end{aligned}
\tag{1.192}
$$

Derivation of the numbered equalities in (1.192) follows:

1. Follows from Definition 1.7, p. 13, of the gyrogroup cooperation $\boxplus$.
2. Follows from Item 1 by the gyroautomorphic inverse property, Theorem 1.32, p. 34.
3. Follows from Item 2 by the right gyroassociative law.
4. Follows from Item 3 by Identity (1.180) of Theorem 1.42, thus providing an elegant example for an application of that theorem.
5. Follows from Item 4 by the gyration even property, and by the gyrocommutative law.
6. Follows from item 5 by the gyration inversion law (1.126), p. 29.
7. Follows from Item 6 by expanding the gyration application term by term.
8. Follows from Item 7 by the left gyroassociative law.
9. Follows from item 8 by the second identity in (1.128), p. 29.
10. Follows from Item 9 by the gyration even property and the gyration inversion law in Theorem 1.23, p. 29, implying $\text{gyr}[b,\ominus a]\,\text{gyr}[a,\ominus b] = I$.
11. Follows from Item 10 by Definition 1.7, p. 13, of the gyrogroup cooperation $\boxplus$. □

1.15 Problems

Problem 1.1 The Automorphic Inverse Property:
Verify the automorphic inverse property (1.4), p. 4, of Einstein addition.

Problem 1.2 The Left Cancellation Law:
Verify the left cancellation law of Einstein addition (1.5), p. 5. You may use a computer software for symbolic manipulation, like Mathematica or Maple.

Problem 1.3 The Gamma Identity:
Verify the gamma identity (1.7), p. 5. You may use a computer software for symbolic manipulation, like Mathematica or Maple.

Problem 1.4 The Algebra of Einstein Addition:
Verify the algebraic laws of Einstein addition in (1.25), p. 9. You may use a computer software for symbolic manipulation, like Mathematica or Maple.

Remark Note that in the proof of gyration identities, gyrations must be applied to a generic element. Thus, for instance, in order to verify the gyration left loop property $\mathrm{gyr}[\mathbf{u}\oplus\mathbf{v}, \mathbf{v}] = \mathrm{gyr}[\mathbf{u}, \mathbf{v}]$ for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}_c^n$ one has to show that $\mathrm{gyr}[\mathbf{u}\oplus\mathbf{v}, \mathbf{v}]\mathbf{w} = \mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{w}$ for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}_c^n$.

Problem 1.5 Gyrations in Explicit Form:
Verify identity (1.27), p. 9. You may use a computer software for symbolic manipulation, like Mathematica or Maple.

Problem 1.6 Einstein Coaddition:
Prove that the gyrogroup coaddition definition (1.35), p. 13, specializes to Einstein coaddition (1.37).