Chapter 4
Plasma Kinetic Theory: Vlasov–Maxwell and Related Equations

4.1 Mathematical Model

Plasma is an ionized gas of charged particles. Plasma is distinguished from usual gases in the sense that plasma particles give rise to essential electromagnetic fields. Hence, the usual Boltzmann kinetic approach that takes into account only paired collisions should be supplemented by influence of electromagnetic fields generated in plasma on the motion of plasma particles. The plasma inhomogeneity caused by the electromagnetic field (i.e. inhomogeneous distribution of charged plasma particles) results in generation of induced charges and currents. The latter in turn creates the electromagnetic field, that anew modifies the motion of plasma particles. Therefore the correct description of a system of plasma particles should meet the condition of self-consistency.

The analysis of an infinite system of equations of motion for all plasma particles is conventionally replaced by studying a distribution function of coordinates and impulses of all plasma particles. The key point here is that plasma is a gas, thus all plasma particles move independently.[1] Therefore one can use one-particle distribution function $f^\alpha(t, \boldsymbol{r}, \boldsymbol{p})$ that defines the probability to find a particle of α species with the impulse $\boldsymbol{p}$ at time t and point $\boldsymbol{r}$. The conservation of probability yields

$$\frac{df^\alpha}{dt} \equiv f_t^\alpha + \boldsymbol{r}_t f_{\boldsymbol{r}}^\alpha + \boldsymbol{p}_t f_{\boldsymbol{p}}^\alpha = 0.$$

[1]Usually it is assumed for gas particles that the energy of their interaction is small compared to their kinetic energy. Up to the order of magnitude the latter can be estimated as κT, where T is the temperature and κ is the Boltzmann constant. For charged plasma particles the energy of interaction is of the order of $e^2 N^{1/3}$, where $N^{-1/3}$ is the mean distance between particles, e is a charge and N is the number of particles in a unit volume. Hence the plasma demonstrates the gas property provided that

$$e^2 N^{1/3} \ll \kappa T.$$

This inequality holds for all real plasmas.

Y.N. Grigoriev et al., *Symmetries of Integro-Differential Equations,* Lecture Notes in Physics 806,
DOI 10.1007/978-90-481-3797-8_4,

Noting that $\boldsymbol{r}_t = \boldsymbol{v}$ is a particle velocity and $\boldsymbol{p}_t$ for charged particles is defined by the Lorentz force

$$e^{\alpha}\left\{\boldsymbol{E} + \frac{1}{c}[\boldsymbol{v} \times \boldsymbol{B}]\right\},$$

the equation for the distribution function for any plasma particle species takes the form:

$$f_t^{\alpha} + \boldsymbol{v} f_{\boldsymbol{r}}^{\alpha} + e^{\alpha}\left\{\boldsymbol{E} + \frac{1}{c}[\boldsymbol{v} \times \boldsymbol{B}]\right\} f_{\boldsymbol{p}}^{\alpha} = 0. \tag{4.1.1}$$

The charge and current densities are defined via distribution function

$$\begin{gathered} \rho = \sum_{\alpha} e_{\alpha} m_{\alpha}^3 \int \mathrm{d}\boldsymbol{v}\, f^{\alpha} \gamma^5, \\ \boldsymbol{j} = \sum_{\alpha} e_{\alpha} m_{\alpha}^3 \int \mathrm{d}\boldsymbol{v}\, f^{\alpha} \gamma^5 \boldsymbol{v}, \quad \gamma = \frac{1}{\sqrt{1 - \boldsymbol{v}^2/c^2}}, \end{gathered} \tag{4.1.2}$$

where summation is taken over all plasma particle species. These charge and current densities enter the field equations and define electric and magnetic fields in plasma in a self-consistent manner. Equation (4.1.1) in view of the field equations

$$\begin{gathered} \boldsymbol{B}_t + c\,\mathrm{rot}\,\boldsymbol{E} = 0; \quad \mathrm{div}\,\boldsymbol{E} = 4\pi\rho; \\ \boldsymbol{E}_t - c\,\mathrm{rot}\,\boldsymbol{B} + 4\pi\boldsymbol{j} = 0; \quad \mathrm{div}\,\boldsymbol{B} = 0, \end{gathered} \tag{4.1.3}$$

are known as *kinetic equations with a self-consistent field.* The efficiency of this equation for description of plasma properties was first demonstrated by Vlasov [1]. At present Vlasov's kinetic equation with a self-consistent field is a basic equation in the theory of a collisionless[2] plasma (e.g., hot plasma used in the plasma fusion experiments).

Meanwhile in describing the evolution of distribution functions frequently it is more convenient to use not standard Vlasov equations (4.1.1) with Euler velocity $\boldsymbol{v}$, but their hydrodynamic analogue [2–4] with Lagrangian velocity $\boldsymbol{w}$. At transition to Lagrangian notations instead of the Euler velocity $\boldsymbol{v}$ and the Euler momentum $\boldsymbol{p}$ for each particle species two vector functions are introduced, the velocity $\boldsymbol{V}^{\alpha}(t, \boldsymbol{r}, \boldsymbol{q})$ and the momentum $\boldsymbol{P}^{\alpha}(t, \boldsymbol{r}, \boldsymbol{q})$, depending upon the Lagrangian momentum $\boldsymbol{q}$ and Euler coordinates $\boldsymbol{r}$ and time t, and related to $\boldsymbol{v}$ and $\boldsymbol{p}$ by the formulas

$$\begin{gathered} \boldsymbol{v} = \boldsymbol{V}^{\alpha}(\boldsymbol{q}, \boldsymbol{r}, t), \quad \boldsymbol{p} = \boldsymbol{P}^{\alpha}(\boldsymbol{q}, \boldsymbol{r}, t), \\ \boldsymbol{V}^{\alpha} = c^2 \boldsymbol{P}^{\alpha}(m^2c^4 + c^2(\boldsymbol{P}^{\alpha})^2)^{-1/2}. \end{gathered} \tag{4.1.4}$$

[2]Equation (4.1.1) is approximate, as it neglects collisions of plasma particles. In view of particle collisions their motion becomes correlated. This effect leads to appearance of non-zero term in the right-hand side of (4.1.1), the so-called collision integral. However, the explicit form of the collision integral depends on particular conditions defined by the plasma properties in every concrete situation, and we will not discuss them here. In many particular problems collision effects can be neglected.

The change of variables (4.1.4) eliminates in the resulting equations for distribution functions for particle of each species the derivatives of these functions upon the Lagrangian momentum $\boldsymbol{q}$. Hence, Lagrangian formulation of the kinetic description of plasma is fulfilled via the equations of hydrodynamic type for the density $N^\alpha(t, \boldsymbol{r}, \boldsymbol{w})$ and the velocity $\boldsymbol{V}^\alpha(t, \boldsymbol{r}, \boldsymbol{w})$, which depend upon t, $\boldsymbol{r}$ and $\boldsymbol{w}$,

$$
\begin{gathered}
N_t^\alpha + \operatorname{div}(N^\alpha \boldsymbol{V}^\alpha) = 0, \\
\boldsymbol{V}_t^\alpha + (\boldsymbol{V}^\alpha \nabla)\boldsymbol{V}^\alpha \\
= \frac{e_\alpha}{m_\alpha}\sqrt{1 - \left(\frac{\boldsymbol{V}^\alpha}{c}\right)^2}\left\{\boldsymbol{E} + \frac{1}{c}[\boldsymbol{V}^\alpha \times \boldsymbol{B}] - \frac{1}{c^2}\boldsymbol{V}^\alpha(\boldsymbol{V}^\alpha \cdot \boldsymbol{E})\right\}.
\end{gathered}
\tag{4.1.5}
$$

Here the index α indicates the plasma particle species with the charge e_α and mass m_α and the charge and current densities, ρ and $\boldsymbol{j}$, are in turn determined by the motion of plasma particles:

$$
\begin{gathered}
\rho = \sum_\alpha e_\alpha m_\alpha^3 \int \mathrm{d}\boldsymbol{w} N^\alpha \Gamma^5, \\
\boldsymbol{j} = \sum_\alpha e_\alpha m_\alpha^3 \int \mathrm{d}\boldsymbol{w} N^\alpha \boldsymbol{V}^\alpha \Gamma^5, \quad \Gamma = \frac{1}{\sqrt{1 - (\boldsymbol{w}/c)^2}}.
\end{gathered}
\tag{4.1.6}
$$

It is typical, that (4.1.5) do not contain Lagrangian velocity $\boldsymbol{w}$ (or Lagrangian momentum $\boldsymbol{q}$) in explicit form. In order to find the dependence upon $\boldsymbol{q}$ one should solve these equations with the "initial" conditions $\boldsymbol{V}^\alpha = \boldsymbol{w}$, $N^\alpha = N_0^\alpha(t_0, \boldsymbol{r}, \boldsymbol{w})$ which hold for vanishing electric and magnetic fields $\boldsymbol{E} = \boldsymbol{B} = 0$ at some $t = t_0$. In particular, in homogeneous plasma "initial" conditions for the density N^α has the form $N_0^\alpha = n_{\alpha 0} f_0^\alpha(\boldsymbol{q})$, where the stationary and homogeneous function $f_0^\alpha(\boldsymbol{q})$ of Lagrangian momentum $\boldsymbol{q}$ coincides with the function $f_0^\alpha(\boldsymbol{p})$ of Euler momentum $\boldsymbol{p}$.

Given the density $N^\alpha(t, \boldsymbol{r}, \boldsymbol{w})$ and the velocity $\boldsymbol{V}^\alpha(t, \boldsymbol{r}, \boldsymbol{w})$ that depend upon Lagrangian momentum the particles distribution function in Euler representation is restored with the help of the following relations (the index of particles species in these formulas is omitted)

$$
\begin{gathered}
N(t, \boldsymbol{r}, \boldsymbol{q}) = f\big(\boldsymbol{p} = \boldsymbol{P}(\boldsymbol{q}, \boldsymbol{r}, t), \boldsymbol{r}, t\big) \det\left(\frac{\partial P_i}{\partial q_j}\right), \quad \boldsymbol{v} = c^2 \boldsymbol{p}(m^2c^4 + c^2\boldsymbol{p}^2)^{-1/2}, \\
\boldsymbol{w} = c^2 \boldsymbol{q}(m^2c^4 + c^2\boldsymbol{q}^2)^{-1/2}, \quad \boldsymbol{V} = c^2 \boldsymbol{P}(m^2c^4 + c^2\boldsymbol{P}^2)^{-1/2}.
\end{gathered}
\tag{4.1.7}
$$

The system of equations (4.1.4)–(4.1.7), (4.1.3) presents the Lagrangian formulation [2–4] of Vlasov–Maxwell equations, in which (4.1.6) appear as non-local material relations.

The search for particular solutions of the joint system of Vlasov–Maxwell equations (4.1.1)–(4.1.3) or its Lagrangian formulation (4.1.5)–(4.1.7) is very important both in theoretical treatment and practical applications. The group analysis of the system of Vlasov–Maxwell equations, which forms the essence of this chapter, offers a nice opportunity in constructing these solutions. The main obstacle in finding symmetry group for systems (4.1.1)–(4.1.3) and (4.1.5)–(4.1.7) with the help of a standard Lie algorithm is the non-locality of material relations (4.1.2) and (4.1.6). The first successful attempt in this field [5] deals with calculating the continuous point Lie group for the system (4.1.1)–(4.1.3) in the one-dimensional non-relativistic approximation of homogeneous electron plasma using the methods of moments. On the contrary we will follow a general algorithm [6–8] based on the direct method of calculation of symmetries, described in Chaps. 2 and 4.

This chapter is structured as follows. Section 4.2 introduces an approach for calculating symmetries of integro-differential equations used in this chapter. In Sect. 4.3.1 we describe in details the application of the general algorithm to the most simple one-dimensional non-relativistic model of one-component charged electron plasma that arises from (4.1.1)–(4.1.3) while treating only one particle species (electrons) and in one-dimensional plane geometry. We also neglect relativistic effects here. We consider this model since it is physically simple and informative from the group standpoint. The model has the same characteristic features as the complete three-dimensional system of kinetic equations for collisionless relativistic electron–ion plasma. The only difference is in the a smaller amount of calculations necessary for constructing and solving the group determining equations. For this reason, the next models are analyzed in less detail.

In the next sections we present the result of group analysis for the successively complicated systems that take into account other plasma species (Sects. 4.3.3, 4.3.4), relativistic effects (Sects. 4.3.2, 4.3.4), the presence of stationary or moving ion background (Sects. 4.3.5, 4.3.6). We also consider the so-called quasi-neutral approximation for plasma particles dynamics (Sect. 4.3.7). Symmetry of plasma kinetic equations in three dimensional geometry is analyzed for electron gas in Sect. 4.4.1 and for electron–ion plasma in Sect. 4.4.2. We also discuss the symmetry of plasma kinetic equations in Lagrangian variables (Sect. 4.5).

The special section is devoted to symmetry of Benney equations (Sect. 4.6). Here we apply both our algorithm and method of moments to demonstrate the incompleteness of the algorithm to describe all the admitted symmetries.

Section 4.7 is devoted mainly to particular problems that illustrate the efficiency of the symmetry approach to integro-differential equations to find solutions to various particular problems of interest. Here we especially draw attention to symmetries known in mathematical physics as *renormgroup symmetries*. Section 4.7 demonstrates the method of their construction as well as examples of applications.

4.2 Definition and Infinitesimal Test

To extend the classical Lie algorithm to integro-differential equations it appears necessary to resolve several problems. First, one should define the local one-parameter transformation group G for the nonlocal (integro-differential) equations and formulate the invariance criteria that lead to determining equations, which appear also nonlocal. Secondly, and a procedure of solving nonlocal determining equations should be described.

4.2.1 Definition of Symmetry Group

Let an integro-differential equation under consideration be expressed as a zero equality for some functional (here we indicate only one argument for a function f), defined for $x_1 \le x \le x_2$,

$$F[f(x)] = 0, \tag{4.2.8}$$

and let G be a local one-parameter group that transforms f to $\tilde{f}(x, a)$,

$$\tilde{f}(x, a) = f + a\varkappa + o(a), \quad \tilde{x} = x. \tag{4.2.9}$$

Here we use the canonical group representation hence independent variables x do not vary. Then the local group G of point transformations (4.2.9) is called a symmetry group of integro-differential equations (4.2.8) iff for any a the function F does not vary [9] (see also Chap. 2),

$$F[\tilde{f}(x, a)] = 0. \tag{4.2.10}$$

Differentiating (4.2.10) with respect to group parameter a and assuming $a \to 0$ gives the invariance criterion in the infinitesimal form akin to (1.1.31) in Chap. 1. In view of the canonical form of transformations (4.2.9) the functional F depends upon a via $\tilde{f}$. Therefore to find the infinitesimal invariance criterion we should calculate the derivative $\mathrm{d}F/\mathrm{d}a$.

4.2.2 Variational Derivative for Functionals

Let $f(x, a)$ be a differentiable function with respect to a, $f(x, a)$ and $\partial f(x, a)/\partial a$ continuous functions for $a \ge 0$, $x_1 \le x \le x_2$. The derivative $\mathrm{d}F/\mathrm{d}a$ [10]

$$\frac{\mathrm{d}}{\mathrm{d}a} F[f(x, a)] = \delta F\left[f(x, a);\, f'_a(x, a)\right], \tag{4.2.11}$$

is given by variation of the functional δF, defined as a linear in δf part of a difference

$$\delta F = F[f + \delta f] - F[f].$$

Let $F[f(x, a)]$ be a differentiable functional (recall that the functional F, defined on the interval $[x_1, x_2]$, is called a differentiable functional [10] if it has the first derivative in each point of this interval). Then the last formula is rewritten in the following form

$$\delta F = \int_{x_1}^{x_2} F'[f(x); q]\delta f(q)\,\mathrm{d}q. \tag{4.2.12}$$

Here the derivative $F'[f(x); y] = \delta F/\delta f(y)$ of the differentiable functional F with respect to a function f in the point y is defined via the principal (linear) part of an increment of the functional as a limit (if it exists) (see [10]):

$$\frac{\delta F[f]}{\delta f(y)} = \lim_{\varepsilon\to 0} \frac{\{F[f+\delta f_\varepsilon] - F[f]\}}{\int_\Delta \mathrm{d}y \delta f_\varepsilon(y)}; \quad y \in [x_1, x_2]. \tag{4.2.13}$$

In (4.2.13) the infinitesimal variation $\delta f_\varepsilon(y) \geq 0$ is a continuously differentiable function given on fixed interval $\Delta = [x_1, x_2]$ which differs from zero only in ε-vicinity of a point y, and the norm $\|\delta f_\varepsilon\|_{C^1} \to 0$ at $\varepsilon \to 0$.

Example 4.2.1 Let $b(y)$ be a continuous function and $F[f]$ a linear functional

$$F[f] = \int_{x_1}^{x_2} b(y) f(y)\,\mathrm{d}y.$$

By δf_ε denote a variation that differs from zero only in ε-vicinity of a point q. Then using the mean value theorem

$$F[f+\delta f_\varepsilon] - F[f] = \int_{x_1}^{x_2} b(y)\delta f_\varepsilon\,\mathrm{d}y = b(q)\int_{x_1}^{x_2} \delta f_\varepsilon\,\mathrm{d}y,$$

we get the variation derivative

$$\frac{\delta F[f]}{\delta f(q)} = \lim_{\varepsilon\to 0} b(q) \frac{\int_\Delta \delta f_\varepsilon(y)\,\mathrm{d}y}{\int_\Delta \delta f_\varepsilon(y)\,\mathrm{d}y} = b(q). \tag{4.2.14}$$

Choosing $b(y) = 1/(\sqrt{2\pi}\sigma)\exp(-(y-y_0)^2/2\sigma^2)$ we obtain

$$\frac{\delta F[f]}{\delta f(q)} = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(y-y_0)^2}{2\sigma^2}\right). \tag{4.2.15}$$

In the limit $\sigma \to 0$ we have $b(y) \to \delta(y-y_0)$, $F[f] \to f(y_0)$ and hence

$$\frac{\delta f(y_0)}{\delta f(q)} = \delta(y_0 - q). \tag{4.2.16}$$

4.2.3 *Infinitesimal Criterion*

According to Sect. 4.2.1 to write the infinitesimal criterion for the symmetry group for nonlocal equations one should differentiate (4.2.10) with respect to group parameter a and assume $a \to 0$, i.e. calculate the limit of the derivative $\mathrm{d}F/\mathrm{d}a$ for vanishing a. Combining (4.2.11), (4.2.12) and assuming $a \to 0$ in view of (4.2.9) we get

$$\left.\frac{\mathrm{d}F[\tilde f]}{\mathrm{d}a}\right|_{a=0} = \int \varkappa(y) \frac{\delta F[f(x)]}{\delta f(y)}\,\mathrm{d}y \equiv YF, \tag{4.2.17}$$

where we have introduced the generator Y defined by its action on function F as follows:

$$Y(F) = \int \varkappa(y) \frac{\delta F}{\delta f(y)} \, \mathrm{d}y.$$

We will write this operator formally in the form

$$Y = \int \varkappa(y) \frac{\delta}{\delta f(y)} \, \mathrm{d}y. \tag{4.2.18}$$

Hence, the invariance criterion for F with respect to the admitted group can be expressed in an infinitesimal form using the canonical group operator Y,

$$YF\big|_{F=0} = 0, \tag{4.2.19}$$

which generalizes the action of a standard canonical group operator (see formula (1.5.7) in Chap. 1) not only on differential functions but on *functionals* as well using variational differentiation in the definition of Y [7]. One can verify by direct calculation that the action of Y on any differential function and its derivatives, e.g., f and $f_x, \ldots$ produces the usual result: $Yf = \varkappa$, $Yf_x = D_x(\varkappa)$ and so on. Hence, if F describe usual differential equations then formulas (4.2.19) lead to standard local determining equations, while for F having the form of integro-differential equations formulas (4.2.19) can be treated as *nonlocal* determining equations as they depend both on local and nonlocal variables. As we treat local and nonlocal variables in determining equations as independent it is possible to separate these equations into local and nonlocal. The procedure of solving local determining equations is fulfilled in a standard way using Lie algorithm based on splitting the system of over-determined equations with respect to local variables and their derivatives. As a result we get expressions for coordinates of group generator that define the so-called group of *intermediate* symmetry [7]. In the similar manner the solution of nonlocal determining equations is fulfilled using the information borrowed from an intermediate symmetry and by "variational" splitting of nonlocal determining equations using the procedure of variational differentiation. Therefore, the algorithm of finding symmetries of nonlocal equations appears as an algorithmic procedure that consists of a sequence of several steps: (a) defining the set of local group variables, (b) constructing determining equations on basis of the infinitesimal criterion of invariance, that employs the generalization of the definition of the canonical operator, (c) separating determining equations into local and nonlocal, (d) solving local determining equations using a standard Lie algorithm, (e) solving nonlocal determining equations using the procedure of variational differentiation.

4.2.4 Prolongation on Nonlocal Variables

To complete we describe the procedure of prolongation of a symmetry group on nonlocal variables, say in the form of the integral relation

$$J(u) = \int \mathscr{F}(u(z)) \, \mathrm{d}z. \tag{4.2.20}$$

To fulfill this procedure one should first rewrite the operator, say Y, in a canonical form and then formally prolong this operator on the nonlocal variable J

$$Y + \varkappa^J \partial_J \equiv \varkappa \partial_u + \varkappa^J \partial_J. \tag{4.2.21}$$

The integral relation between $\varkappa$ and $\varkappa^J$ is obtained by applying the generator (4.2.21) to the *definition* of J, i.e. to (4.2.20). Substituting the explicit expression for the coordinate $\varkappa$ of the known operator Y and calculating integrals obtained gives the desired coordinate $\varkappa^J$,

$$\begin{aligned} \varkappa^J &= \int \frac{\delta J(u)}{\delta u(z)} \varkappa(z)\, \mathrm{d}z \\ &\equiv \int \frac{\delta \mathscr{F}(u(z'))}{\delta u(z)} \varkappa(z)\, \mathrm{d}z\, \mathrm{d}z' = \int \mathscr{F}_u \varkappa(z)\, \mathrm{d}z. \end{aligned} \tag{4.2.22}$$

4.3 Symmetry of Plasma Kinetic Equations in One-Dimensional Approximation

This section discuss the symmetry of Vlasov–Maxwell equations (4.1.1)–(4.1.3) for plane (one-dimensional) geometry. We start with the case of a one component non-relativistic electron plasma (electron gas) and proceed with a set of different models, including multi-component plasma, relativistic plasma, plasma with neutralizing moving and stationary ion background.

4.3.1 Non-relativistic Electron Gas

Consider the system of Vlasov–Maxwell equations (4.1.1)–(4.1.3) for charged electron gas. In case of non-relativistic motion of electrons in the self-consistent electric field E the one-dimensional Vlasov kinetic equation for the distribution function f is written as follows:

$$f_t + v f_x + \frac{e}{m} E f_v = 0. \tag{4.3.1}$$

Here the potential field E obeys the Poisson equation and the corresponding Maxwell equation

$$E_x = 4\pi\rho, \quad E_t + 4\pi j = 0, \tag{4.3.2}$$

and charge density ρ and current density j are expressed as the integrals

$$\rho = em \int \mathrm{d}v f, \quad j = em \int \mathrm{d}v f v \tag{4.3.3}$$

over electron velocities. Momentarily, we will assume that the charge e and mass m of the electron (parameters of the system) are invariants. The dependent variables E, j, and ρ are functions of two arguments, time t and coordinate x,

$$E = E(t,x), \quad j = j(t,x), \quad \rho = \rho(t,x), \tag{4.3.4}$$

and the distribution function

$$f = f(t, x, v) \tag{4.3.5}$$

has three arguments, t, x and electron velocity v. It follows from (4.3.4) that electric field intensity E, current density j, and charge density ρ are independent of electron velocity v. Hence we have three additional differential constraints

$$E_v = 0, \quad j_v = 0, \quad \rho_v = 0, \tag{4.3.6}$$

which should be used in group analysis of the system (4.3.1)–(4.3.3) as well as compatibility condition for the field equations (4.3.2), known as continuity equation,

$$\rho_t - j_x = 0. \tag{4.3.7}$$

The coordinates ξ and η of the Lie point symmetry group generator

$$X = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \xi^3 \frac{\partial}{\partial v} + \eta^1 \frac{\partial}{\partial f} + \eta^2 \frac{\partial}{\partial E} + \eta^3 \frac{\partial}{\partial j} + \eta^4 \frac{\partial}{\partial \rho}, \tag{4.3.8}$$

are considered as functions of the seven variables

$$t,\ x,\ v,\ f,\ E,\ J,\ \rho. \tag{4.3.9}$$

These coordinates are solutions to the *determining equations*, which, in turn, appear as necessary and sufficient conditions for the invariance of system (4.3.1)–(4.3.3), (4.3.6) with respect to the group with generator (4.3.8). Local (differential) determining equations can be stated and solved directly in terms of the generator (4.3.8). In this section we however use the canonical form

$$Y = \varkappa^1 \frac{\partial}{\partial f} + \varkappa^2 \frac{\partial}{\partial E} + \varkappa^3 \frac{\partial}{\partial j} + \varkappa^4 \frac{\partial}{\partial \rho}, \tag{4.3.10}$$

for the generator (4.3.3), which offers substantial advantages in the group analysis of the complete system of Vlasov–Maxwell equations because of non-locality of the system.

4.3.1.1 Non-relativistic Electron Gas: The Solution to the Local Determining Equations

The invariance conditions for Vlasov kinetic equation (4.3.1), the field equations (4.3.2), and (4.3.6) with respect to the group with canonical generator (4.3.10) are given by the six local determining equations

$$\begin{gathered} D_t(\varkappa^1) + v D_x(\varkappa^1) + \frac{e}{m} E D_v(\varkappa^1) + \frac{e}{m} \varkappa^2 f_v = 0, \\ D_x(\varkappa^2) = 4\pi \varkappa^4, \quad D_t(\varkappa^2) = -4\pi \varkappa^3, \\ D_v(\varkappa^2) = 0, \quad D_v(\varkappa^3) = 0, \quad D_v(\varkappa^4) = 0, \end{gathered} \tag{4.3.11}$$

which should be solved with taking into account the fact that the group variables (4.3.9) and the corresponding derivatives are related by the manifold (4.3.1)–(4.3.3), (4.3.6) and (4.3.7). Here we use the standard notations (see, e.g. Chap. 1) for the operator of total differentiation D_i with respect to the group variable indicated by the subscript. For example, the operator D_v of total differentiation with respect to v is given by

$$D_v \equiv \frac{\partial}{\partial v} + f_v \frac{\partial}{\partial f} + f_{vt} \frac{\partial}{\partial f_t} + f_{vx} \frac{\partial}{\partial f_x} + f_{vv} \frac{\partial}{\partial f_v} + \cdots . \tag{4.3.12}$$

The solution of the system of local determining equations (4.3.11) is given by the following formulas for the coordinates $\varkappa$ of the generator (4.3.10):

$$\begin{aligned}
\varkappa^1 &= \eta^1 - f_t \xi^1 - f_x \left[x\left(A_4 + \frac{1}{2}\xi_t^1 \right) + \beta \right] - f_v \left[v\left(A_4 - \frac{1}{2}\xi_t^1 \right) + \frac{1}{2} x \xi_{tt}^1 + \beta_t \right], \\
\varkappa^2 &= E\left[A_4 - \frac{3}{2}\xi_t^1 \right] + \frac{m}{e}\left[\frac{1}{2} x \xi_{ttt}^1 + \beta_{tt} \right] - E_t \xi^1 - E_x \left[x\left(A_4 + \frac{1}{2}\xi_t^1 \right) + \beta \right], \\
\varkappa^3 &= j\left[A_4 - \frac{5}{2}\xi_t^1 \right] + \rho\left[\frac{1}{2} x \xi_{tt}^1 + \beta_t \right] + \frac{3}{8\pi} E \xi_{tt}^1 \\
&\quad - \frac{m}{4\pi e}\left[\frac{1}{2} x \xi_{tttt}^1 + \beta_{ttt} \right] - j_t \xi^1 - j_x \left[x\left(A_4 + \frac{1}{2}\xi_t^1 \right) + \beta \right], \\
\varkappa^4 &= -2\rho \xi_t^1 + \frac{m}{8\pi e} \xi_{ttt}^1 - \rho_t \xi^1 - \rho_x \left[x\left(A_4 + \frac{1}{2}\xi_t^1 \right) + \beta \right].
\end{aligned} \tag{4.3.13}$$

The coordinates (4.3.13) depend upon three arbitrary functions

$$\xi(t), \quad \beta(t), \quad \eta^1(f) \tag{4.3.14}$$

and A_4 is an arbitrary constant. The group symmetry with the generator (4.3.10) and coordinates (4.3.13) admitted by the system of equations (4.3.1), (4.3.2), (4.3.6) will be referred to as the *intermediate* group symmetry of the complete system of the self-consistent field equations (4.3.1)–(4.3.3). The symmetry is generated only by the differential equations in the integro-differential Vlasov–Maxwell system and does not take into account integral terms in the material equations (4.3.3), which determine charge and current densities of electrons. The intermediate group symmetry (4.3.10), (4.3.13) plays an auxiliary role in obtaining the final equations for the coordinates ξ and η of the generator (4.3.8) of the desired Lie group. The charge density ρ and the current density j have a concrete physical meaning. By introducing them as independent group variables in the set (4.3.9) along with t, x, v, f, and E, we divide the group analysis of the local and the nonlocal part of the Vlasov–Maxwell system into two stages. The intermediate symmetry (4.3.10), (4.3.13) completes the local group analysis of the system. In what follows we shall see that the nonlocal determining equations appearing as invariance conditions for the material equations (4.3.3) with respect to the sought Lie group eliminate the arbitrary dependence of ξ, β and η^1 on t and f.

4.3.1.2 Non-relativistic Electron Gas: Nonlocal Determining Equations and Their Solutions

Since the material equations (4.3.3) are nonlocal (they involve integration of the distribution function f and of the product vf over the electron velocity v), the differentiation with respect to f in the first term of the canonical generator (4.3.10) should be generalized so as to act not only on functions of f but also on linear functionals (4.3.3) of f. Hence, we represent this term as the integral of variational derivative with respect to f with weight $\varkappa^1$ over electron velocity v:

$$\varkappa^1 \frac{\partial}{\partial f} \equiv \int \mathrm{d}v \varkappa^1(v) \frac{\delta}{\delta f(v)}. \tag{4.3.15}$$

For brevity, we indicate only the integration variable v as an argument of f and of $\varkappa^1$. Our shorthand notation implies that the coordinate $\varkappa^1(v)$ of the canonical generator in (4.3.15) stands for the following extended expression in (4.3.13), depending on integration variable v:

$$\varkappa^1(v) \equiv \eta^1(f(t,x,v)) - \xi^1 f_t(t,x,v) - \left[x\left(A_4 + \frac{1}{2}\xi_t^1\right) + \beta\right] f_x(t,x,v) \\ - \left[v\left(A_4 - \frac{1}{2}\xi_t^1\right) + \frac{1}{2}x\xi_{tt}^1 + \beta_t\right] f_v(t,x,v). \tag{4.3.16}$$

When applied to functions of f, the operator of the differentiation with respect to f in (4.3.15) gives the usual result, i.e. it coincides with the ordinary differentiation with respect to f. When applied to linear functionals of f, i.e., to the charge and current densities (4.3.3), the derivative in (4.3.15) permits us to introduce the variational derivative on the right-hand side in (4.3.15) under the integral over v together with the coordinate $\varkappa^1$ of the canonical generator (4.3.10).

Substituting (4.3.15) in (4.3.10) and using the a well-known identity

$$\frac{\delta f(v)}{\delta f(v')} = \delta\left(v - v'\right), \tag{4.3.17}$$

where δ is the Dirac delta-function we obtain the invariance conditions for the integral material equations (4.3.3) with respect to the Lie group with canonical generator (4.3.10), the nonlocal determining equations [7, 8]

$$\varkappa^4 - em \int \mathrm{d}v \varkappa^1 = 0, \quad \varkappa^3 - em \int \mathrm{d}v v \varkappa^1 = 0. \tag{4.3.18}$$

The integration in (4.3.18) is over all values of v, just as in (4.3.3) and (4.3.15). Let us consider the first of the two determining equations in (4.3.18) in more detail. Substituting the coordinates $\varkappa^1$ and $\varkappa^4$ from (4.3.13) into the determining equations in question and taking into account (4.3.3) for charge density ρ, we reduce the determining equations to the simple form

$$em \int \mathrm{d}v \left[\eta^1(f(v)) + f(v)\mathscr{K}(t)\right] - \frac{m}{8\pi e}\xi_{ttt}^1(t) = 0. \tag{4.3.19}$$

The coefficient $\mathscr{K}$ in the product $\mathscr{K} f$ in the integrand on the left-hand side in (4.3.19) is independent of v and f; specifically, we have

$$\mathscr{K}(t) = A_4 + \frac{3}{2}\xi_t^1. \tag{4.3.20}$$

The derivation of the determining equations (4.3.19) involves integrating by parts with respect to v, which removes the derivative f_v from the integrand in the nonlocal term in the original determining equations. The resultant antiderivative f is assumed to vanish at the ends of the infinite integration interval, that is,

$$f \to 0, \quad v \to \pm\infty. \tag{4.3.21}$$

The determining equations (4.3.19) is a linear nonhomogeneous integral equation for η^1, which can easily be solved. According to the general ideas of Lie technique, (4.3.19), as well as any determining equation, is an identity with respect to the group variable f. Therefore, it remains valid after differentiating with respect to f. Since the determining equations (4.3.19) is an integral equation, we should use variational differentiation with respect to f rather than ordinary differentiation. Taking into account that nonhomogeneous term proportional to ξ^1_{ttt} in (4.3.19) is independent of f, we obtain:

$$\frac{\delta}{\delta f(v')} \int \mathrm{d}v \big[\eta^1\big(f(v)\big) + f(v)\mathscr{K}(t)\big] = 0. \tag{4.3.22}$$

The nonlocal equation (4.3.19), which is an identity with respect to f, should be combined with its differential corollary (4.3.22) in the sense that a solution to (4.3.22) is also a solution to (4.3.19). Introducing the variational derivative in (4.3.22) under the integral over v,

$$\int \mathrm{d}v \big\{\eta^1_f + \mathscr{K}\big\} \frac{\delta f(v)}{\delta f(v')} = 0. \tag{4.3.23}$$

and evaluating the integral over v with the aid of the delta-function (4.3.17) that appears in the integrand, as a consequence of (4.3.19), we obtain a first-order ordinary differential equation for the dependence of the coordinate η^1 of the determining equations (4.3.8) on f:

$$\eta^1_f + \mathscr{K} = 0. \tag{4.3.24}$$

Its solution depends on one arbitrary constant

$$\eta^1 = -\mathscr{K} f + A. \tag{4.3.25}$$

Since the coordinate η^1 is independent of t, we immediately obtain the condition

$$\mathscr{K}_t = 0 \tag{4.3.26}$$

imposed on the coefficient $\mathscr{K}$ of the determining equations (4.3.19). It follows from (4.3.20) and (4.3.26) that

$$\xi^1_{tt} = 0. \tag{4.3.27}$$

As was mentioned above, it is necessary to consider (4.3.25) for η^1, appearing as a direct consequence of variational differentiation (4.3.9) of the determining equations (4.3.19) with respect to the distribution function f, together with (4.3.19):

$$em \int_{-\infty}^{+\infty} \mathrm{d}v A - \frac{m}{8\pi e} \xi^1_{ttt}(t) = 0. \tag{4.3.28}$$

In view of (4.3.27), the second term on the left-hand side in (4.3.28) is zero. Therefore, (4.3.28) is reduced to

$$Aem \int_{-\infty}^{+\infty} \mathrm{d}v = 0, \tag{4.3.29}$$

whence follows that the integration constant A in (4.3.25) is zero, that is, we have

$$\eta^1 = -\mathscr{K} f. \tag{4.3.30}$$

The integration of (4.3.30) yields

$$\xi^1(t) = A_1 + 2A_3 t. \tag{4.3.31}$$

We insert the explicit formula (4.3.31) for the dependence of ξ^1 on t into expression (4.3.20) for the coefficient $\mathscr{K}$ and obtain

$$\mathscr{K} = 3A_3 + A_4, \tag{4.3.32}$$

whence follows the definite expression for the coordinate

$$\eta^1(f) = -(3A_3 + A_4) f. \tag{4.3.33}$$

Equations (4.3.31) and (4.3.33) are the basic result of solving the first nonlocal determining equations in (4.3.18) and define explicit dependence of ξ and η on t and f in the intermediate group symmetry (4.3.13). The second nonlocal determining equations in system (4.3.18) pertains to the invariance of electron current density with respect to the admitted Lie group. By substituting the extended expressions for the coordinates $\varkappa^3$ and $\varkappa^1$ from (4.3.13) into this determining equations, we easily reduce it to the following linear nonhomogeneous integral equation for η^1, which is similar to (4.3.19):

$$em \int \mathrm{d}v v \left(\eta^1 + f\mathscr{K}\right) + \frac{3}{8\pi} E\xi^1_{tt} - \frac{mx}{8\pi e} \xi^1_{tttt} - \frac{m}{4\pi e} \beta_{ttt} = 0. \tag{4.3.34}$$

The passage from (4.3.18) to (4.3.34) involves integration by parts with respect to v. Here we take into account conditions (4.3.21), which state that the electron distribution function f decays rapidly for large velocities. The coefficient $\mathscr{K}$ in the product $\mathscr{K} v f$ in the integrand on the left-hand side in (4.3.34) has the same form (4.3.20) as in (4.3.19). Hence, taking into account (4.3.27) and (4.3.30), we see that the determining equations (4.3.34) is reduced to $\beta_{ttt} = 0$, which implies

$$\beta(t) = A_2 + A_5 t + \frac{1}{2} A_6 t^2. \tag{4.3.35}$$

Substitution of (4.3.31), (4.3.33) and (4.3.35) into (4.3.13) yields canonical coordinates that satisfy determining equations (4.3.11), and (4.3.18), and therefore define the sought for group symmetry

$$
\begin{aligned}
\varkappa^1 = & -A_1 f_t - A_2 f_x - A_3 (3f + 2tf_t + xf_x - vf_v) - A_4 (f + xf_x + vf_v) \\
& - A_5 (tf_x + f_v) - A_6 \left(\frac{t^2}{2} f_x + tf_v \right), \\
\varkappa^2 = & -A_1 E_t - A_2 E_x - A_3 (3E + 2tE_t + xE_x) - A_4 (-E + xE_x) \\
& - A_5 tE_x - A_6 \left(\frac{t^2}{2} E_x - \frac{m}{e} \right), \\
\varkappa^3 = & -A_1 j_t - A_2 j_x - A_3 (5j + 2tj_t + xj_x) - A_4 (-j + xj_x) \\
& - A_5 (tj_x - \rho) - A_6 \left(\frac{t^2}{2} j_x - t\rho \right), \\
\varkappa^4 = & -A_l \rho_t - A_2 \rho_x - A_3 (4\rho + 2t\rho_t + x\rho_x) - A_4 x\rho_x \\
& - A_5 t\rho_x - A_6 \left(\frac{t^2}{2} \right) \rho_x .
\end{aligned}
\tag{4.3.36}
$$

Formulas (4.3.36) refer to the following six basic generators, written in a non-canonical form [7]:

$$
\begin{aligned}
& X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \\
& X_3 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - v\frac{\partial}{\partial v} - 3f\frac{\partial}{\partial f} - 3E\frac{\partial}{\partial E} - 5j\frac{\partial}{\partial j} - 4\rho\frac{\partial}{\partial \rho}, \\
& X_4 = x\frac{\partial}{\partial x} + v\frac{\partial}{\partial v} - f\frac{\partial}{\partial f} + E\frac{\partial}{\partial E} + j\frac{\partial}{\partial j}, \quad X_5 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial v} + \rho\frac{\partial}{\partial j}, \\
& X_6 = \frac{t^2}{2}\frac{\partial}{\partial x} + t\frac{\partial}{\partial v} + \frac{m}{e}\frac{\partial}{\partial E} + t\rho\frac{\partial}{\partial j}.
\end{aligned}
\tag{4.3.37}
$$

The set of generators (4.3.37) span the six-dimensional Lie algebra

$$
L_6 = \langle X_1, X_2, \ldots, X_6 \rangle. \tag{4.3.38}
$$

Generators (4.3.37) of the six-parametric continuous point Lie group admitted by the Vlasov–Maxwell equations (4.3.1)–(4.3.3), have clear physical meaning: the operators X_1 and X_2 describe translations along t and x-axes, the generator X_3 and X_4 relate to dilations, which can be easily verified, and the generators X_5 define the Galilean transformations. The finite transformations corresponding to the generator X_6 have the following form for each of six variables (4.3.9):

$$
\begin{aligned}
& \bar{t} = t; \ \bar{x} = x + \frac{at^2}{2}; \ \bar{v} = v + at; \ \bar{f} = f; \\
& \bar{E} = E + \frac{ma}{e}; \ \bar{j} = j + at\rho; \ \bar{\rho} = \rho.
\end{aligned}
\tag{4.3.39}
$$

In mechanics, the one-parameter transformation group with generator X_6 can be interpreted for the first three equations in (4.3.39) as the transformation of variables due to passing into a coordinate system moving linearly with constant acceleration $a = \text{const}$ with respect to the laboratory frame.

4.3.1.3 Including Electron Charge and Electron Mass into Group Transformations

The set of group variables (4.3.9) can be extended by involving the parameters e and m of the Vlasov–Maxwell equations (4.3.1)–(4.3.3) into the group transformations

$$X = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \xi^3 \frac{\partial}{\partial v} + \xi^4 \frac{\partial}{\partial e} + \xi^5 \frac{\partial}{\partial m} + \eta^1 \frac{\partial}{\partial f} + \eta^2 \frac{\partial}{\partial E} + \eta^3 \frac{\partial}{\partial j} + \eta^4 \frac{\partial}{\partial \rho}. \quad (4.3.40)$$

The extension adds two more basis generator to the algebra (4.3.37), (4.3.38). They correspond to the dilations of electron charge and mass:

$$X_7 = e\frac{\partial}{\partial e} + m\frac{\partial}{\partial m} - 2f\frac{\partial}{\partial f}, \quad X_8 = m\frac{\partial}{\partial m} + E\frac{\partial}{\partial E} + j\frac{\partial}{\partial j} + \rho\frac{\partial}{\partial \rho}. \quad (4.3.41)$$

The operators (4.3.41) commute with each other and with all operator (4.3.37), so that the set (4.3.37), (4.3.41) is the eight-dimensional Lie algebra

$$L_8 = \langle X_1, X_2, \ldots, X_6, X_7, X_8 \rangle. \quad (4.3.42)$$

If the electron charge and mass are not invariant, then the general operator of the continuous point Lie group admitted by the Vlasov–Maxwell equations (4.3.1)–(4.3.3) is given by

$$X = \sum_{\alpha=1}^{8} A_\alpha(e, m) X_\alpha. \quad (4.3.43)$$

It corresponds to an infinite group with continual arbitrariness given by eight functions A_α depending on two of the nine variables

$$t, x, v, e, m, f, E, j, \rho \quad (4.3.44)$$

and can be obtained by solving local and nonlocal determining equations for the coordinates of the generator under the conditions

$$f = f(t, x, v, e, m); \;\; E = E(t, x, e, m); \quad j = j(t, x, e, m); \;\; \rho = \rho(t, x, e, m) \quad (4.3.45)$$

in a way similar to that given in the previous sections.

The infiniteness of the Lie group (4.3.44), (4.3.37), (4.3.41) is due to the fact that the parameters e and m that are arbitrary elements of the group classification are included in the set of the group variables (4.3.44). This procedure that looks trivial from the group analysis viewpoint is typical in "classical" renormalization group method in quantum field theory (for details see Sect. 4.7). In the similar manner to take into account the relativistic motion of electrons, we have to introduce a third parameter, namely, the light velocity in vacuum (denoted by c). We can pass to relativistic velocities also in the one-dimensional approximation with the same field equations (4.3.2). In doing so, the one-parameter Galilean group with the generator X_5 from (4.3.37) is transformed into the Lorentz group and is inherited (in the sense of [11, 12]) in an arbitrary order with respect to the parameter v/c, which takes into account the finiteness of the light velocity. This will be demonstrated in Sect. 4.3.2.

4.3.2 Relativistic Electron Gas

The one-dimensional system of self-consistent field equations (4.3.1)–(4.3.3) for charged relativistic electron gas is modified as follows:

$$f_t + vf_x + eEf_p = 0, \quad \rho = e\int \mathrm{d}pf, \quad j = e\int \mathrm{d}pfv. \tag{4.3.46}$$

In contrast to (4.3.1) and (4.3.3), instead of electron velocity v we use moment p, which can be expressed in terms of v by the well-known equality

$$p = mv\gamma \equiv mv\big(1-(v/c)^2\big)^{-1/2}, \tag{4.3.47}$$

where γ is the relativistic factor. Using (4.3.47) and passing from electron moment p to electron velocity v in (4.3.46), we obtain the equations

$$f_t + vf_x + \frac{e}{m}\gamma^{-3}Ef_v = 0, \tag{4.3.48}$$

$$\rho = em\int_{-c}^{+c} \mathrm{d}v\gamma^3 f, \quad j = em\int_{-c}^{+c} \mathrm{d}v\gamma^3 fv, \tag{4.3.49}$$

which differ from (4.3.1) and (4.3.3) in that the relativistic factor $\gamma > 1$ is taken into account. In finding symmetry of the integro-differential system of equations (4.3.48), (4.3.49), (4.3.2), and (4.3.6) we assume that not only time t, coordinate x, and electron velocity v but also charge e, electron mass m, and light velocity c are independent variables.

Omitting the calculations akin to that were done in the previous Sect. 4.3.1 we present the final expression for the group generator in the form of a linear combination of seven basic generators with the coefficients A_α that are arbitrary functions of three variables [7]:

$$\begin{gathered}
X = \sum_{\alpha=1}^{7} A_\alpha(e,m,c)X_\alpha, \\
X_1 = \frac{\partial}{\partial t}, \quad X_2 = c\frac{\partial}{\partial x}, \\
X_3 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - 2f\frac{\partial}{\partial f} - E\frac{\partial}{\partial E} - 2j\frac{\partial}{\partial j} - 2\rho\frac{\partial}{\partial \rho}, \\
X_4 = \frac{1}{c}\left(x\frac{\partial}{\partial t} + c^2 t\frac{\partial}{\partial x} + (c^2 - v^2)\frac{\partial}{\partial v} + \rho c^2\frac{\partial}{\partial j} + j\frac{\partial}{\partial \rho}\right), \\
X_5 = x\frac{\partial}{\partial x} + v\frac{\partial}{\partial v} + c\frac{\partial}{\partial c} - f\frac{\partial}{\partial f} + E\frac{\partial}{\partial E} + j\frac{\partial}{\partial j}, \\
X_6 = m\frac{\partial}{\partial m} + E\frac{\partial}{\partial E} + j\frac{\partial}{\partial j} + \rho\frac{\partial}{\partial \rho}, \\
X_7 = e\frac{\partial}{\partial e} + m\frac{\partial}{\partial m} - 2f\frac{\partial}{\partial f}.
\end{gathered} \tag{4.3.50}$$

The generators (4.3.50) span the seven-dimensional Lie algebra

$$L_7 = \langle X_1, X_2, \dots, X_7 \rangle \tag{4.3.51}$$

with numerical structural constants. The last three generators in (4.3.50), which determine the dilations of electron charge and mass and of light velocity, commute with all remaining generator in (4.3.50) and with one another. The first four generators in (4.3.50) form the four-dimensional subalgebra

$$L_4 = \langle X_1, X_2, X_3, X_4 \rangle. \tag{4.3.52}$$

The finite transformations given by solutions to the Lie equations for the generator X_4 (the Lorentz transformations) correspond to hyperbolic rotations in the planes (ct, x) and (cp, j), and to the linear-fractional transformation of electron velocity v with group parameter a:

$$\begin{gathered}
\bar{t} = t\cosh(ac) + (x/c)\sinh(ac),\ \ \bar{x} = x\cosh(ac) + ct\sinh(ac),\\
\bar{v} = (v + c\tanh(ac))(1 + (v/c)\tanh(ac))^{-1},\\
\bar{\rho} = \rho\cosh(ac) + (j/c)\sinh(ac),\ \ \bar{j} = j\cosh(ac) + c\rho\sinh(ac),\\
\bar{e} = e,\ \bar{m} = m,\ \bar{c} = c,\ \bar{f} = f,\ \bar{E} = E.
\end{gathered} \tag{4.3.53}$$

The generator X_4 from (4.3.50) and its finite transformations in the form (4.3.53) extends the Galilean generator X_5 from (4.3.37) to the relativistic domain of electron velocities. Comparing the algebras (4.3.37) and (4.3.50) of the point symmetry groups we see that transition from non-relativistic to relativistic electron gas deletes the generator X_6 from (4.3.37).

The algebra (4.3.50), (4.3.51) is fairly consistent with the physical ideas on the symmetry of system (4.3.48), (4.3.2) and (4.3.49), developed in the theory of plasma. The characteristic feature of the system is in that the relativistic effects are taken into account for electron motion but the finite value of light velocity c is ignored in the field equations (4.3.2) in one dimensional approximation. However, we can extend the scope of the method by taking into account the three-dimensional relativistic motion of electrons in self-coordinated electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{B}$ obeying the Maxwell equations. This is done in Sect. 4.4.1.

4.3.3 Collisionless Non-relativistic Electron–Ion Plasma

In this section we turn to a model that contains two plasma particle species, namely electrons and ions. It means that the basic system of equations should be supplemented by the kinetic equation for the ion distribution function $\bar{f}$ and the corresponding items in the field equations,

$$\begin{gathered}
f_t + v f_x + \frac{e}{m} E f_v = 0, \quad \bar{f}_t + v\bar{f}_x + \frac{\bar{e}}{\bar{m}} E\bar{f}_v = 0,\\
\rho = \int \mathrm{d}v(emf + \bar{e}\bar{m}\bar{f}), \quad j = \int \mathrm{d}v v(emf + \bar{e}\bar{m}\bar{f}),\\
E_x = 4\pi\rho, \quad E_t + 4\pi j = 0.
\end{gathered} \tag{4.3.54}$$

The solution of the local and nonlocal determining equations for non-relativistic Vlasov–Maxwell equations for electron–ion plasma is fulfilled in the same root as for the electron gas model. The final result is given as the general element of the Lie algebra of point symmetry operators of the Vlasov–Maxwell equations (4.3.54) is determined by the linear combination [8]

$$X = \sum_{\alpha=1}^{9} A_\alpha(e, m, \bar{e}, \bar{m}) X_\alpha, \tag{4.3.55}$$

$$\begin{gathered}
X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \\
X_3 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - v\frac{\partial}{\partial v} - 3f\frac{\partial}{\partial f} - 3\bar{f}\frac{\partial}{\partial \bar{f}} - 3E\frac{\partial}{\partial E} - 5j\frac{\partial}{\partial j} - 4\rho\frac{\partial}{\partial \rho}, \\
X_4 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial v} + \rho\frac{\partial}{\partial j}, \\
X_5 = x\frac{\partial}{\partial x} + v\frac{\partial}{\partial v} - f\frac{\partial}{\partial f} - \bar{f}\frac{\partial}{\partial \bar{f}} + E\frac{\partial}{\partial E} + j\frac{\partial}{\partial j}, \\
X_6 = \frac{1}{em}\frac{\partial}{\partial f} - \frac{1}{\bar{e}\bar{m}}\frac{\partial}{\partial \bar{f}}, \quad X_7 = e\frac{\partial}{\partial e} + m\frac{\partial}{\partial m} - 2f\frac{\partial}{\partial f}, \\
X_8 = m\frac{\partial}{\partial m} + \bar{m}\frac{\partial}{\partial \bar{m}} + E\frac{\partial}{\partial E} + j\frac{\partial}{\partial j} + \rho\frac{\partial}{\partial \rho}, \\
X_9 = \bar{e}\frac{\partial}{\partial \bar{e}} + \bar{m}\frac{\partial}{\partial \bar{m}} - 2\bar{f}\frac{\partial}{\partial \bar{f}}.
\end{gathered} \tag{4.3.56}$$

4.3.4 Collisionless Relativistic Electron–Ion Plasma

This section presents the result of the symmetry group calculation for the relativistic analogue of equations discussed in the previous section:

$$\begin{gathered}
f_t + vf_x + \frac{e}{m\gamma^3}Ef_v = 0, \quad \bar{f}_t + v\bar{f}_x + \frac{\bar{e}}{\bar{m}\gamma^3}E\bar{f}_v = 0, \\
E_x = 4\pi\rho, \quad E_t + 4\pi j = 0, \\
\rho = \int \mathrm{d}v\gamma^3(emf + \bar{e}\bar{m}\bar{f}), \quad j = \int \mathrm{d}v\gamma^3 v(emf + \bar{e}\bar{m}\bar{f}).
\end{gathered} \tag{4.3.57}$$

The Lie group admitted by the Vlasov–Maxwell equations (4.3.57) is a one-dimensional analog of the group with algebra (4.3.50) provided that the parameters e, m, $\bar{e}$, $\bar{m}$ and c are invariant [8]:

$$L_5 = \langle X_1, X_2, X_3, X_4, X_5 \rangle, \tag{4.3.58}$$

$$\begin{gathered}
X_1 = \frac{\partial}{\partial t}, \quad X_2 = c\frac{\partial}{\partial x}, \\
X_3 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - 2f\frac{\partial}{\partial f} - 2\bar{f}\frac{\partial}{\partial \bar{f}} - E\frac{\partial}{\partial E} - 2j\frac{\partial}{\partial j} - 2\rho\frac{\partial}{\partial \rho},
\end{gathered}$$

$$X_4 = \frac{1}{c}\left(x\frac{\partial}{\partial t} + c^2 t\frac{\partial}{\partial x} + (c^2 - v^2)\frac{\partial}{\partial v} + c^2\rho\frac{\partial}{\partial j} + j\frac{\partial}{\partial \rho}\right), \tag{4.3.59}$$

$$X_5 = \frac{1}{em}\frac{\partial}{\partial f} - \frac{1}{\bar{e}\bar{m}}\frac{\partial}{\partial \bar{f}}.$$

Here the first, second, and fourth generators coincide with those of algebra L_4 for the relativistic electron gas. The dilation generator X_3 in (4.3.59) differs from the corresponding generator in (4.3.50) by the term $(-2\bar{f}\partial_{\bar{f}})$ containing the ion partition function $\bar{f}$. The quasi-neutrality operator X_5 in (4.3.59) is new as compared with the four-dimensional "electron" algebra (4.3.52) in Sect. 4.3.2. Taking into consideration transformations of parameters, we obtain the four generators

$$\begin{aligned} X_6 &= c\frac{\partial}{\partial c} + x\frac{\partial}{\partial x} + v\frac{\partial}{\partial v} - f\frac{\partial}{\partial f} - \bar{f}\frac{\partial}{\partial \bar{f}} + E\frac{\partial}{\partial E} + j\frac{\partial}{\partial j}, \\ X_7 &= m\frac{\partial}{\partial m} + \bar{m}\frac{\partial}{\partial \bar{m}} + E\frac{\partial}{\partial E} + j\frac{\partial}{\partial j} + \rho\frac{\partial}{\partial \rho}, \\ X_8 &= \bar{e}\frac{\partial}{\partial \bar{e}} + \bar{m}\frac{\partial}{\partial \bar{m}} - 2\bar{f}\frac{\partial}{\partial \bar{f}}, \\ X_9 &= e\frac{\partial}{\partial e} + m\frac{\partial}{\partial m} - 2f\frac{\partial}{\partial f} \end{aligned} \tag{4.3.60}$$

in addition to the basis (4.3.59).

The general element of the Lie point algebra is a linear combination of nine generators with coefficients that are arbitrary scalar functions of five variables,

$$X = \sum_{\alpha=0}^{9} A_\alpha(e, m, \bar{e}, \bar{m}, c)X_\alpha. \tag{4.3.61}$$

We omit the calculations that lead to (4.3.59)–(4.3.60), since they just reproduce the calculations made above.

Sections 4.3.3 and 4.3.4 demonstrate the point symmetry of kinetic equations of collisionless electron–ion plasma. Two additional Lie groups admitted by the Vlasov–Maxwell equations of quasi-neutral plasma are presented in the next sections. In contrast to present section, these equations correspond to a simplified model of electron plasma, i.e., we consider ions as a positively charged background neutralizing the negative charge of the electron plasma. Thus we omit the kinetic Vlasov equations for the ion distribution function and describe ions by means of "hydrodynamic" parameters.

4.3.5 Non-relativistic Electron Plasma Kinetics with a Moving and Stationary Ion Background

The non-relativistic one-dimensional equations of self-consistent fields for electron plasma with moving positive homogeneous ion background neutralizing the charge of electrons read

$$\begin{gathered} f_t + v f_x + \frac{e}{m} E f_v = 0; \quad E_x = 4\pi\rho, \quad E_t + 4\pi j = 0, \\ \rho = em \int \mathrm{d}v f + \bar{e}n, \quad j = em \int \mathrm{d}v v f + \bar{e}nu. \end{gathered} \tag{4.3.62}$$

Here f is the partial function of non-relativistic electrons with charge $e < 0$ and mass m. The parameters $\bar{e}$, n, and u correspond to the ion charge ($\bar{e} > 0$), ion density n, and ion velocity u, respectively. Unlike the case of the electron–ion plasma (Sect. 4.3.3), the ion mass $\bar{m}$ is not involved in (4.3.62) and the ion motion is described by the term $\bar{e}nu$ in the plasma current density j. Group analysis of (4.3.62) give rise to a ten-dimensional Lie algebra L_{10} with numerical structural constants [8]:

$$L_{10} = \langle X_1, X_2, \ldots, X_{10} \rangle, \tag{4.3.63}$$

$$\begin{gathered} X_1 = \frac{1}{\omega}\frac{\partial}{\partial t}, \quad X_2 = \frac{u}{\omega}\frac{\partial}{\partial x}, \\ X_3 = (x - ut)\frac{\partial}{\partial x} + (v - u)\frac{\partial}{\partial v} - f\frac{\partial}{\partial f} + E\frac{\partial}{\partial E} + (j - u\rho)\frac{\partial}{\partial j}, \\ X_4 = \sin(\omega t)\frac{\partial}{\partial x} + \omega\cos(\omega t)\frac{\partial}{\partial v} + 4\pi\bar{e}n\sin(\omega t)\frac{\partial}{\partial E} + \omega\cos(\omega t)(\rho - \bar{e}n)\frac{\partial}{\partial j}, \\ X_5 = \cos(\omega t)\frac{\partial}{\partial x} - \omega\sin(\omega t)\frac{\partial}{\partial v} + 4\pi\bar{e}n\cos(\omega t)\frac{\partial}{\partial E} - \omega\sin(\omega t)(\rho - \bar{e}n)\frac{\partial}{\partial j}, \\ X_6 = ut\frac{\partial}{\partial x} + u\frac{\partial}{\partial v} + u\frac{\partial}{\partial u} + u\rho\frac{\partial}{\partial j}, \\ X_7 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - v\frac{\partial}{\partial v} - 4n\frac{\partial}{\partial n} - u\frac{\partial}{\partial u} - 3f\frac{\partial}{\partial f} - 3E\frac{\partial}{\partial E} - 5j\frac{\partial}{\partial j} - 4\rho\frac{\partial}{\partial \rho}, \\ X_8 = e\frac{\partial}{\partial e} + m\frac{\partial}{\partial m} - 2f\frac{\partial}{\partial f}, \\ X_9 = m\frac{\partial}{\partial m} + n\frac{\partial}{\partial n} + E\frac{\partial}{\partial E} + j\frac{\partial}{\partial j} + \rho\frac{\partial}{\partial \rho}, \quad X_{10} = \bar{e}\frac{\partial}{\partial \bar{e}} - n\frac{\partial}{\partial n}. \end{gathered} \tag{4.3.64}$$

Here ω is the well-known Langmuir electron frequency

$$\omega = \left(-\frac{4\pi e\bar{e}n}{m}\right)^{1/2}.$$

The general element X of the Lie algebra is a linear combination of all generators

$$X = \sum_{\alpha=1}^{10} A_\alpha(e, m, \bar{e}, n, u) X_\alpha, \tag{4.3.65}$$

with coefficients A_α, which are arbitrary functions of the five variables

$$e, m, \bar{e}, n, u. \tag{4.3.66}$$

Parameters (4.3.66) are invariants of the first five generators (4.3.64) and when the ion velocity is zero, $u = 0$, these generators correspond to the result obtained in [5].

The additional terms, which are missing in generators obtained in [5], take into account the transformation of the plasma current density j (which is equal to the electron current in the limit (4.3.62)), while the plasma charge density ρ is invariant. These terms are the prolongation of the group in [5] to the nonlocal variables

$$\rho = em \int \mathrm{d}v f + \bar{e}n, \quad j = em \int \mathrm{d}v v f, \tag{4.3.67}$$

and can be omitted in case we consider the group of transformations in the space of group variables $\{t, x, v, f, E\}$.

The generator X_6 in (4.3.64) is due to the nonzero ion velocity u included in the set of variables (4.3.66) together with all variables involved in group transformations. By doing this we preserve an analog of the Galilean subgroup in the admitted Lie group, which is absent in the five-parameter group [5] (here a is a group parameter):

$$\begin{gathered} t' = t, \quad x' = x + (e^a - 1)ut, \quad v' = v + (e^a - 1)u, \quad e' = e, \quad m' = m, \\ \bar{e}' = \bar{e}, \quad n' = n, \quad u' = ue^a, \quad f' = f, \\ E' = E, \quad \rho' = \rho, \quad j' = j + (e^a - 1)u\rho. \end{gathered} \tag{4.3.68}$$

This example shows the importance of including parameters into group transformations.

4.3.6 Relativistic Electron Plasma Kinetics with a Moving Ion Background

In this section we present the result of the symmetry group calculation for relativistic equations generalizing (4.3.62):

$$\begin{gathered} f_t + v f_x + \frac{e}{m\gamma^3} E f_v = 0; \quad E_x = 4\pi\rho, \quad E_t + 4\pi j = 0, \\ \rho = em \int \mathrm{d}v \gamma^3 f + \bar{e}n, \quad j = em \int \mathrm{d}v \gamma^3 v f + \bar{e}nu, \\ \gamma \equiv \left[1 - (v/c)^2\right]^{-1/2}. \end{gathered} \tag{4.3.69}$$

An infinite symmetry group admitted by (4.3.69) is given by a linear combination of the eight generators [8]

$$X = \sum_{\alpha=1}^{8} A_\alpha(e, m, \bar{e}, n, u, c) X_\alpha, \tag{4.3.70}$$

$$
\begin{aligned}
X_1 &= \frac{\partial}{\partial t}, \quad X_2 = c\frac{\partial}{\partial x}, \\
X_3 &= t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - 2n\frac{\partial}{\partial n} - 2f\frac{\partial}{\partial f} - E\frac{\partial}{\partial E} - 2j\frac{\partial}{\partial j} - 2\rho\frac{\partial}{\partial \rho}, \\
X_4 &= \frac{1}{c}\bigg[x\frac{\partial}{\partial t} + c^2 t\frac{\partial}{\partial x} + (c^2 - v^2)\frac{\partial}{\partial v} + (c^2 - u^2)\frac{\partial}{\partial u} \\
&\quad + un\frac{\partial}{\partial n} + c^2\rho\frac{\partial}{\partial j} - j\frac{\partial}{\partial \rho}\bigg], \\
X_5 &= e\frac{\partial}{\partial e} + m\frac{\partial}{\partial m} - 2f\frac{\partial}{\partial f}, \\
X_6 &= m\frac{\partial}{\partial m} + n\frac{\partial}{\partial n} + E\frac{\partial}{\partial E} + j\frac{\partial}{\partial j} + \rho\frac{\partial}{\partial \rho}, \quad X_7 = \bar{e}\frac{\partial}{\partial \bar{e}} - n\frac{\partial}{\partial n}, \\
X_8 &= c\frac{\partial}{\partial c} + x\frac{\partial}{\partial x} + v\frac{\partial}{\partial v} + u\frac{\partial}{\partial u} - f\frac{\partial}{\partial f} + E\frac{\partial}{\partial E} + j\frac{\partial}{\partial j},
\end{aligned}
\tag{4.3.71}
$$

This example again shows the importance of the inclusion of the parameters into the group transformations: the six generators in (4.3.71) are due to the noninvariance of the parameters.

4.3.7 Non-relativistic Electron–Ion Plasma in Quasi-neutral Approximation

Essential progress in studying dynamics of plasma expansion and acceleration of ions was achieved by use of quasi-neutral approximation [13, 14], suitable for descriptions of plasma flows with characteristic scale of density variation which is large in comparison with Debye length for plasma particles. In this approximation charge and current densities in plasma are set equal to zero, that essentially simplifies the initial model with non-local terms. Thus, instead of the complete system of Vlasov–Maxwell equations (4.1.1)–(4.1.3) with the corresponding material relations here we will only use the kinetic equations for particle distribution functions for various species

$$f_t^\alpha + v f_x^\alpha + (e_\alpha / m_\alpha) E(t, x) f_v^\alpha = 0 \tag{4.3.72}$$

with additional non-local restrictions imposed on them, which arise from vanishing conditions for the current and the charge densities

$$\int \mathrm{d}v \sum_\alpha e_\alpha f^\alpha = 0, \qquad \int \mathrm{d}v v \sum_\alpha e_\alpha f^\alpha = 0. \tag{4.3.73}$$

At that the electric field E is expressed through the moments of distribution functions:

$$E(t,x) = \left(\int \mathrm{d}v v^2 \partial_x \sum_\alpha e_\alpha f^\alpha \right) \left(\int \mathrm{d}v \sum_\alpha \frac{e_\alpha^2}{m_\alpha} f^\alpha \right)^{-1}. \tag{4.3.74}$$

Equations (4.3.72), (4.3.73) describe one-dimensional dynamics of a plasma, which is inhomogeneous upon the coordinate x; thus the distribution functions of particles f^α depend upon t, x and the velocity component v in the directions of plasma inhomogeneity.

The group of point Lie transformations admitted by system (4.3.72) and (4.3.73) is specified by the following set of infinitesimal operators [14]:

$$\begin{gathered} X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = t\frac{\partial}{\partial t} - v\frac{\partial}{\partial v}, \quad X_4 = x\frac{\partial}{\partial x} + v\frac{\partial}{\partial v}, \\ X_5 = \sum_\alpha f^\alpha \frac{\partial}{\partial f^\alpha}, \quad X_6 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial v}, \\ X_7 = t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} + (x - vt)\frac{\partial}{\partial v}, \\ X_\alpha = \frac{1}{Z_{\alpha+1}}\frac{\partial}{\partial f^{\alpha+1}} - \frac{1}{Z_\alpha}\frac{\partial}{\partial f^\alpha} \end{gathered} \tag{4.3.75}$$

with the general element of the algebra represented by their linear combination

$$X = \sum_{j=1}^{7} c_j X_j + \sum_\alpha b_\alpha X_\alpha. \tag{4.3.76}$$

In the operators X_α in system (4.3.75), $Z_\alpha = e_\alpha/|e|$ is the charge number of the particle species α, and the index $\alpha + 1$ denotes the particle species that follows α. The operators X_α exist only in plasma with the number of particle types larger than or equal to two and their number is less than the number of plasma components by one. Transformation of charge and mass of particles are not included in (4.3.75).

The method for calculating the admitted symmetry group used here qualitatively differs from the method used earlier in Sect. 4.3 in that the electric field E is treated not as one of the dependent variables but as an unknown function of the variables t and x, $E(t,x)$. This case of finding the symmetry logically follows from the simpler, quasineutral model of plasma (4.3.72), (4.3.73) in contrast to the complete system of Vlasov–Maxwell equations (4.3.1)–(4.3.3). It is easy to verify that the translation operators X_1 and X_2, the Galilean transformation operator X_6, and the quasineutrality operators X_α are contained in the symmetry group obtained in Sect. 4.3.3 by a different method without assuming that E is an arbitrary function of two variables to be determined. The two dilation generators specified in (4.3.56) are obtained by combining the three expansion operators X_3, X_4, and X_5 from (4.3.75) and by adding the contributions responsible for the dilation transformations of the electric field E, charge density ρ, and electric current density j. The projective group operator X_7 is new among the generators (4.3.75). Since here, in contrast to (4.3.54), we chose a different normalization of the particle distribution functions, the quasineutrality generators, X_α, contrary to (4.3.56) contain factors that do not depend on particle mass.

4.4 Group Analysis of Three Dimensional Collisionless Plasma Kinetic Equations

In this section we calculate the point symmetry of the self-consistent field equations for three dimensional kinetic models of collisionless plasma. In the first subsection the group analysis is fulfilled for the three-dimensional kinetic equations of relativistic electron gas. In the second one the same is done for the model of quasi-neutral multi-species plasma.

4.4.1 Relativistic Electron Gas Kinetics

We start with the equations of kinetic theory for collisionless relativistic electron gas, described by the system of equations (4.1.1)–(4.1.3) where only one particle species, electrons, are taken into account. As in one-dimensional case, (4.1.1)–(4.1.3) should be supplemented by additional differential constraints

$$\boldsymbol{E}_{\boldsymbol{v}} = 0, \quad \boldsymbol{B}_{\boldsymbol{v}} = 0, \quad \boldsymbol{j}_{\boldsymbol{v}} = 0, \quad \boldsymbol{\rho}_{\boldsymbol{v}} = 0, \tag{4.4.1}$$

which explicitly show that electromagnetic fields and momenta of the distribution function do not depend on the electron velocity $\boldsymbol{v}$.

The canonical group generator Y of the continuous point Lie group admitted by system (4.1.1)–(4.1.3), (4.4.1) has the form

$$Y = \varkappa^1 \frac{\partial}{\partial f} + \vec{\varkappa}^{\mathbf{2}} \frac{\partial}{\partial \boldsymbol{E}} + \vec{\varkappa}^{\mathbf{3}} \frac{\partial}{\partial \boldsymbol{B}} + \vec{\varkappa}^{\mathbf{4}} \frac{\partial}{\partial \boldsymbol{j}} + \varkappa^5 \frac{\partial}{\partial \rho}, \tag{4.4.2}$$

where the first term is given by the following three-dimensional relativistic analog of representations (4.3.15) in Sect. 4.3.1:

$$\varkappa^1 \frac{\partial}{\partial f} = \int \mathrm{d}\boldsymbol{v} \varkappa^1(\boldsymbol{v}) \frac{\delta}{\delta f(\boldsymbol{v})}. \tag{4.4.3}$$

As in (4.1.2), the integration domain in this formula is the sphere $|\boldsymbol{v}| < c$ of radius c. The procedure of symmetry group construction is similar to that in the one-dimensional case though calculus are a little bit more tedious here. As a result we get the group that is represented by the following basic generators [6, 8] (for convenience, they are written in a non-canonical form):

$$\begin{aligned} &X_0 = \frac{\partial}{\partial t}, \quad X_i = c\frac{\partial}{\partial x_i}, \\ &Y_i = \frac{1}{c}\Bigg[x_i \frac{\partial}{\partial t} + c^2 t \frac{\partial}{\partial x_i} + (c^2\delta_{is} - v_i v_s)\frac{\partial}{\partial v_s} \\ &\qquad - c e_{isk} B_s \frac{\partial}{\partial E_k} + c e_{isk} E_s \frac{\partial}{\partial B_k} + c^2 \rho \frac{\partial}{\partial j_i} + j_i \frac{\partial}{\partial \rho}\Bigg], \\ &Z_i = e_{isk}\left(x_s \frac{\partial}{\partial x_k} + v_s \frac{\partial}{\partial v_k} + E_s \frac{\partial}{\partial E_k} + B_s \frac{\partial}{\partial B_k} + j_s \frac{\partial}{\partial j_k}\right), \end{aligned} \tag{4.4.4}$$

$$X_4 = t\frac{\partial}{\partial t} + x_s\frac{\partial}{\partial x_s} - 2f\frac{\partial}{\partial f} - E_s\frac{\partial}{\partial E_s} - B_s\frac{\partial}{\partial B_s} - 2j_s\frac{\partial}{\partial j_s} - 2\rho\frac{\partial}{\partial \rho}.$$

Here summation is performed over repeated indices, δ_{is} and e_{isk} are the Kronecker symbols of the second and the third order, $1 \le i, s, k \le 3$.

Generators (4.4.4) form the 11-dimensional Lie algebra

$$L_{11} = \langle X_0, \boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}, X_4\rangle \tag{4.4.5}$$

and any infinitesimal operator in (4.4.4) has a simple physical meaning. The generators X_0 and $\boldsymbol{X}$ correspond to time and space translations, respectively. The operator $\boldsymbol{Y}$ generates Lorentz transformations, which do not involve the distribution function f, e.g., hyperbolic rotations in the planes $(ct, \boldsymbol{x})$ and $(c\rho, \boldsymbol{j})$ and linear-fractional transformations of the electron velocity $\boldsymbol{v}$. Lorentz transformations of vectors $\boldsymbol{E}$ and $\boldsymbol{B}$ correspond to the transformation of the 4-tensor of electromagnetic field (see [15, §22, 23]). The operator $\boldsymbol{Z}$ generates rotations. The operator X_4 generates dilations, and it is the only group transformations in (4.4.4) which involve f.

The 10-dimensional algebra of the Poincaré group,

$$L_{10} = \langle X_0, \boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}\rangle \tag{4.4.6}$$

is included in (4.4.5), $L_{10} \subset L_{11}$, and it also appears in the independent (local) group analysis of the Maxwell equations (4.1.2),

$$X = \xi^1\frac{\partial}{\partial t} + \boldsymbol{\xi}^{\mathbf{2}}\frac{\partial}{\partial \boldsymbol{x}} + \boldsymbol{\eta}^{\mathbf{2}}\frac{\partial}{\partial \boldsymbol{E}} + \boldsymbol{\eta}^{3}\frac{\partial}{\partial \boldsymbol{B}} + \boldsymbol{\eta}^{\mathbf{4}}\frac{\partial}{\partial \boldsymbol{j}} + \eta^5\frac{\partial}{\partial \rho}, \tag{4.4.7}$$

as a subalgebra of the 16-dimensional algebra

$$L_{16} = \langle X_0, \boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}, U_0, \boldsymbol{U}, \bar{X}_4, \bar{X}_5\rangle \tag{4.4.8}$$

of the conformal group admitted by (4.1.2). Here the scalar U_0 and the vector $\boldsymbol{U}$ operators are given by

$$\begin{aligned} U_0 = \frac{1}{c^2}\Big[&\frac{1}{2}(c^2t^2 + \mathbf{x}^2)\frac{\partial}{\partial t} + c^2tx_i\frac{\partial}{\partial x_i} - c(2ctE_i - e_{isk}B_sx_k)\frac{\partial}{\partial E_i} \\ &- c(2ctB_i + e_{isk}E_sx_k)\frac{\partial}{\partial B_i} + c^2(-3tj_i + \rho x_i)\frac{\partial}{\partial j_i} + (-3t\rho c^2 + j_ix_i)\frac{\partial}{\partial \rho}\Big], \\ U_i = \frac{1}{c}\Big[&tx_i\frac{\partial}{\partial t} + \Big(x_ix_s + \frac{1}{2}(c^2t^2 - \mathbf{x}^2)\delta_{is}\Big)\frac{\partial}{\partial x_s} + \Big(x_kE_i - (\boldsymbol{E}\cdot\boldsymbol{x})\delta_{ik} \\ &- 2x_iE_k - cte_{isk}B_s\Big)\frac{\partial}{\partial E_k} + \big(x_kB_i - (\boldsymbol{B}\cdot\boldsymbol{x})\delta_{ik} - 2x_iB_k + cte_{isk}E_s\big)\frac{\partial}{\partial B_k} \\ &+ \big(x_kj_i - (\boldsymbol{j}\cdot\boldsymbol{x})\delta_{ik} - 3x_ij_k + c^2t\rho\delta_{ik}\big)\frac{\partial}{\partial j_k} + (tj_i - 3\rho x_i)\frac{\partial}{\partial \rho}\Big]. \end{aligned} \tag{4.4.9}$$

The two last (scalar) generators in (4.4.8) generate the dilations

$$\bar{X}_4 = t\frac{\partial}{\partial t} + x_i\frac{\partial}{\partial x_i} - 2E_i\frac{\partial}{\partial E_i} - 2B_i\frac{\partial}{\partial B_i} - 3j_i\frac{\partial}{\partial j_i} - 3\rho\frac{\partial}{\partial \rho},$$
$$\bar{X}_5 = E_i\frac{\partial}{\partial E_i} + B_i\frac{\partial}{\partial B_i} + j_i\frac{\partial}{\partial j_i} + \rho\frac{\partial}{\partial \rho}. \qquad (4.4.10)$$

The invariance conditions for the kinetic Vlasov equation (4.1.1) violates the conformal part (4.4.9) of group (4.4.8) when the intermediate group symmetry is taken into account. Adding the dilation operators (4.4.10) and "correcting" the sum by taking into account the dilation of f we obtain a "prototype" of the generator X_4 in the algebra (4.4.5). Thus, using the relation between the algebras L_{10}, L_{11} and L_{16} we can interpret the result (4.4.5) in terms of the group symmetry of the Maxwell equations (4.1.2). The nonlocal determining equations yield the contribution $(-2f\partial_f)$ into the generator X_4 in (4.4.6); this term cannot be obtained from the standard group analysis, but it is easily reproduced from physical considerations.

Including parameters e, m, and c in the set of group variables of the system under consideration we add three scalar generators to (4.4.6) and thereby take into account dilations of the electron charge, mass, and the light velocity c in vacuum:

$$X_5 = m\frac{\partial}{\partial m} - 2f\frac{\partial}{\partial f} + E_s\frac{\partial}{\partial E_s} + B_s\frac{\partial}{\partial B_s} + j_s\frac{\partial}{\partial j_s} + \rho\frac{\partial}{\partial \rho},$$
$$X_6 = e\frac{\partial}{\partial e} + m\frac{\partial}{\partial m} - 4f\frac{\partial}{\partial f}, \qquad (4.4.11)$$
$$X_7 = c\frac{\partial}{\partial c} + x_s\frac{\partial}{\partial x_s} + v_s\frac{\partial}{\partial v_s} - 3f\frac{\partial}{\partial f} + E_s\frac{\partial}{\partial E_s} + B_s\frac{\partial}{\partial B_s} + j_s\frac{\partial}{\partial j_s}.$$

Then the Lie group of the Vlasov–Maxwell equations (4.1.1)–(4.1.3) becomes infinite and the common element X of the operator algebra depending on 14 scalar functions of three variables e, m, and c is given by [8]

$$X = \sum_{\alpha=0}^{7} A_\alpha(e,m,c)X_\alpha + \boldsymbol{b}(e,m,c)\boldsymbol{Y} + \boldsymbol{g}(e,m,c)\boldsymbol{Z}. \qquad (4.4.12)$$

The group analysis of the equations of collisionless electron gas (single-component charged plasma) performed in the present section is supplemented in the next section by the group analysis of kinetic equations of a quasi-neutral multi-component (electron–ion) plasma.

4.4.2 Relativistic Electron–Ion Plasma Kinetic Equations

In this Section we point to the distinctive features that arise for the symmetry group of a multi-species electron–ion plasma (with $k > 1$ particle species), as compared to algebra (4.4.5). Starting with Vlasov–Maxwell equations (4.1.1)–(4.1.3) in the most general form with $1 \le \alpha \le k$ and adding (4.4.1) we come after fulfilling the procedure used above to the following $11 + (k-1)$-parameter Lie group [6, 8]

$$L_{11+(k-1)} = \langle X_0, \boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}, X_4, X^{\mu}\rangle, \quad 2 \le \mu \le k, \tag{4.4.13}$$

including the Poincaré group as a subgroup. The first ten (scalar) generators of the algebra (4.4.13) are listed in (4.4.5) and span the algebra (4.4.8). The infinitesimal operator X_4 in (4.4.13), in contrast to X_4 in (4.4.9), includes dilations of all distribution functions f^{α}:

$$\begin{aligned} X_4 = t\frac{\partial}{\partial t} + x_s\frac{\partial}{\partial x_s} - 2\sum_{\mu=1}^{k} f^{\mu}\frac{\partial}{\partial f^{\mu}} - E_s\frac{\partial}{\partial E_s} \\ - B_s\frac{\partial}{\partial B_s} - 2j_s\frac{\partial}{\partial j_s} - 2\rho\frac{\partial}{\partial \rho}. \end{aligned} \tag{4.4.14}$$

The algebra $L_{11+(k-1)}$ contains $k-1$ new operators not included in L_{11} in (4.4.9); these are the "quasi-neutrality operators"

$$X^{\mu} = \frac{1}{e^1(m^1)^3}\frac{\partial}{\partial f^1} - \frac{1}{e^{\mu}(m^{\mu})^3}\frac{\partial}{\partial f^{\mu}}, \quad 2 \le \mu \le k, \tag{4.4.15}$$

typical for the multi-component plasma. The quasi-neutrality generator (4.4.15) determines consistent translation transformations of the distribution functions f^{μ}.

Including $2k+1$ parameters (masses and charges of particles and light velocity in vacuum) of multi-component plasma equations in the set of group variables yields $2k+1$ additional generators of dilations ($1 \le \lambda, \nu \le k$)

$$\begin{aligned} X_5 &= c\frac{\partial}{\partial c} + x_s\frac{\partial}{\partial x_s} + v_s\frac{\partial}{\partial v_s} - 3\sum_{q=1}^{k} f^q\frac{\partial}{\partial f^q} + E_s\frac{\partial}{\partial E_s} + B_s\frac{\partial}{\partial B_s} + j_s\frac{\partial}{\partial j_s}, \\ X_6 &= \sum_{q=1}^{k} m^q\frac{\partial}{\partial m^q} - 2\sum_{q=1}^{k} f^q\frac{\partial}{\partial f^q} + E_s\frac{\partial}{\partial E_s} + B_s\frac{\partial}{\partial B_s} + j_s\frac{\partial}{\partial j_s} + \rho\frac{\partial}{\partial \rho}, \\ X^{\lambda} &= e^{\lambda}\frac{\partial}{\partial e^{\lambda}} + m^{\lambda}\frac{\partial}{\partial m^{\lambda}} - 4f^{\lambda}\frac{\partial}{\partial f^{\lambda}}, \\ X^{\nu} &= e^{\nu}\frac{\partial}{\partial e^{\nu}} + m^{\nu}\frac{\partial}{\partial m^{\nu}} - 4f^{\nu}\frac{\partial}{\partial f^{\nu}}. \end{aligned} \tag{4.4.16}$$

Then the Lie group admitted by the Vlasov–Maxwell equations (4.1.1)–(4.1.3) becomes infinite and its general element X depends on $3k+9$ arbitrary scalar functions of the $2k+1$ group variables e^{α}, m^{α}, and c:

$$\begin{aligned} X = \sum_{\alpha=0}^{6} A_{\alpha}(e^{\alpha}, m^{\alpha}, c)X_{\alpha} + \boldsymbol{b}(e^{\alpha}, m^{\alpha}, c)\boldsymbol{Y} + \boldsymbol{g}(e^{\alpha}, m^{\alpha}, c)\boldsymbol{Z} \\ + \sum_{\mu=2}^{k} A_{\mu}(e^{\alpha}, m^{\alpha}, c)X^{\mu} + \sum_{\lambda=1}^{k} A_{\lambda}(e^{\alpha}, m^{\alpha}, c)X^{\lambda} + \sum_{\nu=1}^{k} A_{\nu}(e^{\alpha}, m^{\alpha}, c)X^{\nu}. \end{aligned} \tag{4.4.17}$$

4.5 Symmetry of Vlasov–Maxwell Equations in Lagrangian Variables

This section is devoted to calculation of the symmetry group for the system of equations (4.1.5)–(4.1.7) that presents the Lagrangian formulation of the known Vlasov–Maxwell equations (4.1.1)–(4.1.3). The infinitesimal operator of the admitted local group of point one-parameter transformations in a standard form

$$X = \xi^1 \frac{\partial}{\partial t} + \boldsymbol{\xi}^{\mathbf{2}} \frac{\partial}{\partial \boldsymbol{r}} + \boldsymbol{\xi}^{\mathbf{3}} \frac{\partial}{\partial \boldsymbol{w}} + \sum_\alpha \eta^{1\alpha} \frac{\partial}{\partial N^\alpha} + \sum_\alpha \eta^{2\alpha} \frac{\partial}{\partial \boldsymbol{V}^\alpha} + \boldsymbol{\eta}^{\mathbf{3}} \frac{\partial}{\partial \boldsymbol{E}} + \boldsymbol{\eta}^{\mathbf{4}} \frac{\partial}{\partial \boldsymbol{B}} + \boldsymbol{\eta}^{\mathbf{5}} \frac{\partial}{\partial \boldsymbol{j}} + \eta^6 \frac{\partial}{\partial \rho}, \tag{4.5.1}$$

where coordinates ξ^i and η^k depend upon t, $\boldsymbol{r}$, $\boldsymbol{w}$, N^α, $\boldsymbol{V}^\alpha$, $\boldsymbol{E}$, $\boldsymbol{B}$, $\boldsymbol{j}$ and ρ. In the canonical form this operator is given as:

$$Y = \sum_\alpha \varkappa^{1\alpha} \frac{\partial}{\partial N^\alpha} + \vec{\mathscr{K}}^{2\alpha} \frac{\partial}{\partial \boldsymbol{V}^\alpha} + \vec{\mathscr{K}}^{\mathbf{3}} \frac{\partial}{\partial \boldsymbol{E}} + \vec{\mathscr{K}}^{\mathbf{4}} \frac{\partial}{\partial \boldsymbol{B}} + \vec{\mathscr{K}}^{\mathbf{5}} \frac{\partial}{\partial \boldsymbol{j}} + \varkappa^6 \frac{\partial}{\partial \rho},$$

$$\varkappa^{1\alpha} = \eta^{1\alpha} - \mathscr{D} N^\alpha, \quad \vec{\mathscr{K}}^{2\alpha} = \eta^{2\alpha} - \mathscr{D} \boldsymbol{V}^\alpha,$$

$$\vec{\mathscr{K}}^{\mathbf{3}} = \boldsymbol{\eta}^{\mathbf{3}} - \mathscr{D} \boldsymbol{E}, \quad \vec{\mathscr{K}}^{\mathbf{4}} = \boldsymbol{\eta}^{\mathbf{4}} - \mathscr{D} \mathbf{B},$$

$$\vec{\mathscr{K}}^{\mathbf{5}} = \boldsymbol{\eta}^{\mathbf{5}} - \mathscr{D} \boldsymbol{j}, \quad \varkappa^6 = \eta^6 - \mathscr{D} \rho,$$

$$\mathscr{D} \equiv \xi^1 \partial_t - (\boldsymbol{\xi}^{\mathbf{2}} \cdot \nabla_{\boldsymbol{r}}) - (\boldsymbol{\xi}^{\mathbf{3}} \cdot \nabla_{\boldsymbol{w}}). \tag{4.5.2}$$

The current and charge densities in (4.1.6) are moments of functions N^α and $\boldsymbol{V}^\alpha$ and, similar to electric and magnetic fields in Maxwell equations (4.1.3), do not depend upon the plasma particles velocity. This lead to additional differential constraints

$$\boldsymbol{E}_{\boldsymbol{w}} = 0; \quad \boldsymbol{B}_{\boldsymbol{w}} = 0; \quad \boldsymbol{j}_{\boldsymbol{w}} = 0; \quad \rho_{\boldsymbol{w}} = 0, \tag{4.5.3}$$

that are obvious from the physical point of view, however essential for calculating symmetries of Vlasov–Maxwell equations.

Following the procedure, fulfilled in the preceding section, we obtain the continuous Lie point transformation group for Vlasov–Maxwell equations (with Lagrangian velocity), which we present in a non-canonical form (compare to (4.4.5), (4.4.13) in Sect. 4.4)

$$\mathrm{P}_0 = i \frac{\partial}{\partial t}, \quad \mathbf{P} = i \frac{\partial}{\partial \boldsymbol{r}},$$

$$\begin{aligned}
\mathbf{B} &= \boldsymbol{r}\frac{\partial}{\partial t} + c^2 t\frac{\partial}{\partial \boldsymbol{r}} - c\left[\boldsymbol{B}\times\frac{\partial}{\partial \boldsymbol{E}}\right] + c\left[\boldsymbol{E}\times\frac{\partial}{\partial \boldsymbol{B}}\right] + c^2\rho\frac{\partial}{\partial \boldsymbol{j}} + \boldsymbol{j}\frac{\partial}{\partial\rho} \\
&\quad + \sum_\alpha\left(N^\alpha \boldsymbol{V}^\alpha \partial_{N^\alpha} + c^2\partial_{\boldsymbol{V}^\alpha} - \boldsymbol{V}^\alpha\left(\boldsymbol{V}^\alpha\cdot\frac{\partial}{\partial \boldsymbol{V}^\alpha}\right)\right), \\
\mathbf{R} &= \left[\boldsymbol{r}\times\frac{\partial}{\partial \boldsymbol{r}}\right] + \left[\boldsymbol{V}^\alpha\times\frac{\partial}{\partial \boldsymbol{V}^\alpha}\right] + \left[\boldsymbol{E}\times\frac{\partial}{\partial \boldsymbol{E}}\right] + \left[\boldsymbol{B}\times\frac{\partial}{\partial \boldsymbol{B}}\right] + \left[\boldsymbol{j}\times\frac{\partial}{\partial \boldsymbol{j}}\right], \\
\mathrm{D} &= t\frac{\partial}{\partial t} + \boldsymbol{r}\frac{\partial}{\partial \boldsymbol{r}} - 2\sum_\alpha N^\alpha\frac{\partial}{\partial N^\alpha} - \boldsymbol{E}\frac{\partial}{\partial \boldsymbol{E}} - \boldsymbol{B}\frac{\partial}{\partial \boldsymbol{B}} - 2\boldsymbol{j}\frac{\partial}{\partial \boldsymbol{j}} - 2\rho\frac{\partial}{\partial\rho}, \\
X_\infty &= \boldsymbol{\xi}\frac{\partial}{\partial \boldsymbol{w}} - \left(5\frac{(\boldsymbol{w}\cdot\boldsymbol{\xi})}{c^2}\gamma^2 + (\nabla_{\boldsymbol{w}}\cdot\boldsymbol{\xi})\right)\sum_\alpha N^\alpha\frac{\partial}{\partial N^\alpha}.
\end{aligned} \tag{4.5.4}$$

The operators in (4.5.4) has a simple physical interpretation: $\mathrm{P}_\mu = (\mathrm{P}_0, \mathbf{P})$, where $\mu = 0, 1, 2, 3$, specify translation in time and translation along the three components of radius-vector $\boldsymbol{r}$, $\mathbf{B}$ defines Lorentz transformations, consisting of hyperbolic rotations (boosts) in the $\{ct, \boldsymbol{r}\}$ and $\{c\rho, \boldsymbol{j}\}$ planes, linear-fractional transformations of the velocity $\boldsymbol{V}^\alpha$, transformations of the density N^α and transformations of components of the 4-tensor of the electromagnetic field (see, e.g., §24, 25 in [15]), while $\mathbf{R}$ specifies circular rotations. These ten (scalar) operators define the Poincaré group:[3]

$$L_{10} = \langle \mathrm{P}_0, \mathbf{P}, \mathbf{B}, \mathbf{R}\rangle.$$

In (4.5.4) this is supplemented by the operator D, specifying dilations, and the operator of the infinite subgroup X_∞ (see also [16] and [17] (p. 419, vol. 2)), specifying the consistent transformations of Lagrangian velocity and the density of the plasma particles. Thus, provided parameters e_α, m_α and c are not involved in transformations the continuous Lie point group, admitted by Vlasov–Maxwell equations with Lagrangian velocity, is defined by the 11-dimensional subalgebra, specified by the algebra L_{10} of the Poincaré group and the one-dimensional algebra with the dilation operator D, and the infinite-dimensional subalgebra with the operator X_∞.

To end of this section we prolong the generators (4.5.4) to the space of Fourier variables for functions, independent of Lagrangian velocity $\boldsymbol{w}$. From a point of initial representation, specifying of the Fourier transformation, say, of a charge density

$$\tilde{\rho}(\omega, \boldsymbol{k}) = \int \mathrm{d}t\, \mathrm{d}\boldsymbol{r}\, \rho(t, \boldsymbol{r}) \exp(i\omega t - i\boldsymbol{k}\boldsymbol{r}), \tag{4.5.5}$$

[3] Frequently the six operators specifying hyperbolic and circular rotations in $(c^2 t, x^k)$ and (x^j, x^k) planes, respectively $(j, k = 1, 2, 3; \boldsymbol{r} = (x^1, x^2, x^3))$, are written in a universal form using the operators $M_{\mu\nu}$, where $M_{0k} = i\mathrm{B}_{0k}$ and $M_{jk} = i\mathrm{R}_{jk}$. The three operators (M_{23}, M_{31}, M_{12}) are components of the vector-operator $\mathbf{M} = [\boldsymbol{r}\times\mathbf{P}]$.

is equivalent to introduction of a new non-local variable. Similar to Sect. 4.2.4 to fulfill the procedure of prolongation of Lie point group operator (4.5.1) on a non-local variable, we rewrite down this operator in the canonical form (4.5.2) and formally prolong it on the non-local variable $\tilde{\rho}(\omega, \boldsymbol{k})$

$$\tilde{Y} \equiv Y + \tilde{\varkappa}^6 \frac{\partial}{\partial \tilde{\rho}}. \tag{4.5.6}$$

The integral relation between $\varkappa^6$ and $\tilde{\varkappa}^6$ results while applying the operator (4.5.2) to (4.5.5). Here we consider it as the definition of the variable $\tilde{\rho}$

$$\tilde{\varkappa}^6 = \int \mathrm{d}t\, \mathrm{d}\boldsymbol{r}\, \varkappa^6 \exp(i\omega t - i\boldsymbol{k}\boldsymbol{r}). \tag{4.5.7}$$

Substituting $\varkappa^6$ from (4.5.5), (4.5.6) into (4.5.7) and calculating the integrals obtained (integrating by parts), we get the desired coordinate $\tilde{\varkappa}^6$. For example, for the operator of time translations P_0 the coordinate $\varkappa^6 = -i\rho_t$ after substitution into (4.5.5) yields the following expression for the coordinate $\tilde{\varkappa}^{1e} = -\omega\tilde{\rho}$ in Fourier variables. Other coordinates of a canonical operator are calculated in a similar way. Inserting these results into (4.5.6), restricting the group to Fourier variables not containing dependencies upon Lagrangian velocity $\boldsymbol{w}$ (i.e. leaving in (4.5.7) only the contributions responsible for transformation of these variables in Fourier representation) and returning back to non-canonical representation, we obtain the following set of operators for 11-parametric Lie point group in $\{\omega, \boldsymbol{k}\}$ representation (see also [16])

$$\begin{gathered}
\tilde{\mathrm{P}}_0 = -\omega\left(\tilde{\boldsymbol{E}}\frac{\partial}{\partial\tilde{\boldsymbol{E}}} + \tilde{\boldsymbol{B}}\frac{\partial}{\partial\tilde{\boldsymbol{B}}} + \tilde{\boldsymbol{j}}\frac{\partial}{\partial\tilde{\boldsymbol{j}}} + \tilde{\rho}\frac{\partial}{\partial\tilde{\rho}}\right);\\
\tilde{\mathbf{P}} = \boldsymbol{k}\left(\tilde{\boldsymbol{E}}\frac{\partial}{\partial\tilde{\boldsymbol{E}}} + \tilde{\boldsymbol{B}}\frac{\partial}{\partial\tilde{\boldsymbol{B}}} + \tilde{\boldsymbol{j}}\frac{\partial}{\partial\tilde{\boldsymbol{j}}} + \tilde{\rho}\frac{\partial}{\partial\tilde{\rho}}\right);\\
\tilde{\mathbf{B}} = c^2\boldsymbol{k}\frac{\partial}{\partial\omega} + \omega\frac{\partial}{\partial\boldsymbol{k}} - c\left[\tilde{\boldsymbol{B}}\times\frac{\partial}{\partial\tilde{\boldsymbol{E}}}\right] + c\left[\tilde{\boldsymbol{E}}\times\frac{\partial}{\partial\tilde{\boldsymbol{B}}}\right] + c^2\tilde{\rho}\frac{\partial}{\partial\tilde{\boldsymbol{j}}} + \tilde{\boldsymbol{j}}\frac{\partial}{\partial\tilde{\rho}};\\
\tilde{\mathbf{R}} = \left[\boldsymbol{k}, \frac{\partial}{\partial\boldsymbol{k}}\right] + \left[\tilde{\boldsymbol{E}}\times\frac{\partial}{\partial\tilde{\boldsymbol{E}}}\right] + \left[\tilde{\boldsymbol{B}}\times\frac{\partial}{\partial\tilde{\boldsymbol{B}}}\right] + \left[\tilde{\boldsymbol{j}}\times\frac{\partial}{\partial\tilde{\boldsymbol{j}}}\right];\\
\tilde{\mathrm{D}} = -\omega\frac{\partial}{\partial\omega} - \boldsymbol{k}\frac{\partial}{\partial\boldsymbol{k}} + 3\tilde{\boldsymbol{E}}\frac{\partial}{\partial\tilde{\boldsymbol{E}}} + 3\tilde{\boldsymbol{B}}\frac{\partial}{\partial\tilde{\boldsymbol{B}}} + 2\tilde{\boldsymbol{j}}\frac{\partial}{\partial\tilde{\boldsymbol{j}}} + 2\tilde{\rho}\frac{\partial}{\partial\tilde{\rho}}.
\end{gathered} \tag{4.5.8}$$

Formulas (4.5.8) supplement the group (4.5.4) by the appropriate transformations of variables in Fourier-space. For example, Lorentz transformations with the operator **B** are supplemented with hyperbolic rotations in $\{\omega, c\boldsymbol{k}\}$ and $\{c\tilde{\rho}, \tilde{\boldsymbol{j}}\}$ planes and transformations of the 4-tensor of the Fourier-components of the electromagnetic field.

4.6 Vlasov-Type Equations: Symmetries of the Benney Equations

4.6.1 Different Forms of the Benney Equations

The Benney equations referred to by the name of the author of a pioneering work [18] appear in long wavelength hydrodynamics of an ideal incompressible fluid of a finite depth in a gravitational field. From the group theoretical point of view they are of particular interest due to the existence of an infinite set of conservation laws obtained in [18]. The latter property of the Benney equations emphasizes their significance that goes far beyond an interesting example of an integrable system of hydrodynamic equations.

In practice, the Benney equations are used in various representation. One of them is the kinetic Benney equation (a kinetic equation with a self-consistent field):

$$f_t + vf_x - A^0_x f_v = 0, \qquad A^0(t, x) = \int_{-\infty}^{+\infty} f(t, x, v)dv. \tag{4.6.1}$$

This equation appears as a unique representative of a set of hierarchy of kinetic equations of Vlasov-type [19]. A detailed study of its group properties will lead to better understanding of the symmetry properties of kinetic equations of collisionless plasma, namely the Vlasov–Maxwell equations.

Another form of the Benney equations is an infinite set of coupled equations

$$A^i_t + A^{i+1}_x + iA^0_x A^{i-1} = 0, \qquad i \geq 0 \tag{4.6.2}$$

for a countable set of functions A^i of two independent variables, time t and the spatial coordinate x. In terms of hydrodynamics these functions appear as averaged values (with respect to the depth) of integer powers $i \geq 0$ of the horizontal component of the liquid flow velocity. The corresponding integrals that describe this averaging are taken over the vertical coordinate in the limits from the flat bottom up to the free liquid surface. Solutions, Hamiltonian structure and conservation laws for (4.6.2) were discussed in details in [20, 21].

From the kinetic point of view the system (4.6.2) can be treated as a system of equations for moments of the distribution function f that obeys the kinetic Benney equation (4.6.1)

$$A^i(t, x) = \int_{-\infty}^{+\infty} v^i f dv, \qquad i \geq 0. \tag{4.6.3}$$

This fact with the explicit formulation of the Benney equation (4.6.1) was first stated independently in [22, 23]. The Lagrangian change of the Euler velocity v,

$$v = V(t, x, u) \tag{4.6.4}$$

yields one more representation for Benney equations (4.6.1):

$$f_t + Vf_x = 0, \quad V_t + VV_x = -A^0_x, \quad A^0(t, x) = \int V_u f(t, x, u)du. \tag{4.6.5}$$

Equations (4.6.3) are readily converted into the hydrodynamic-type form

$$n_t + (nV)_x = 0, \quad V_t + VV_x = -A^0_x, \quad A^0 = \int n(t,x,u)du, \tag{4.6.6}$$

if one employs the "density" n depending on the Lagrangian velocity u:

$$n = f(t,x,u)V_u. \tag{4.6.7}$$

Using the form (4.6.6) of the Benney equations an infinite set of conservation laws were constructed in [22] with the densities regarded as functions of the Lagrangian velocity u.

The knowledge of the complete Lie–Bäcklund symmetry for the Benney equations in different representations (4.6.1)–(4.6.6) can clarify the question of structure of solutions and conservation laws for these equations. This statement is partially confirmed by the fact that one of the main results of the works [20, 21], namely the higher order Benney equations, can be re-formulated in terms of the first order Lie–Bäcklund group, admitted by the system (4.6.2). Unfortunately, the complete description of the Lie–Bäcklund symmetry for (4.6.2) is not available in the literature. This section is devoted to calculating an infinite (countable) part of the Lie point symmetries of the moment equations (4.6.2).

4.6.2 Lie Subgroup and Lie–Bäcklund Group: Statement of the Problem

A Lie subgroup, admitted by the kinetic Benney equation (4.6.1) in the space of four variables

$$t,\ x,\ v,\ f \tag{4.6.8}$$

is defined by five basic infinitesimal operators

$$\begin{gathered} X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial v}, \\ X_4 = t\frac{\partial}{\partial t} - v\frac{\partial}{\partial v} - f\frac{\partial}{\partial f}, \quad X_5 = x\frac{\partial}{\partial x} + v\frac{\partial}{\partial v} + f\frac{\partial}{\partial f}. \end{gathered} \tag{4.6.9}$$

With the less computation difficulties this group can be obtained using the approach developed in Sect. 4.3.1.

Prolongation of infinitesimal operators (4.6.9) on nonlocal variables (4.6.3) extends the set of variables (4.6.8) up to a countable set

$$t,\ x,\ v,\ f,\ A^0,\ \ldots,\ A^i,\ \ldots. \tag{4.6.10}$$

In the latter case infinitesimal operators (4.6.9) rewritten in the canonical form and restricted on the sub-manifold

$$t,\ x,\ A^0,\ \ldots,\ A^i,\ \ldots \tag{4.6.11}$$

are given by the following expressions

$$X_1 = \sum_{i=0}^{\infty}\big(A_x^{i+1} + iA^{i-1}A_x^0\big)\frac{\partial}{\partial A^i}; \quad X_2 = \sum_{i=0}^{\infty} A_x^i \frac{\partial}{\partial A^i};$$
$$X_3 = \sum_{i=0}^{\infty}\big(iA^{i-1} - tA_x^i\big)\frac{\partial}{\partial A^i};$$
$$X_4 = \sum_{i=0}^{\infty}\big[(i+2)A^i - t(A_x^{i+1} + iA^{i-1}A_x^0)\big]\frac{\partial}{\partial A^i};$$
$$X_5 = \sum_{i=0}^{\infty}\big[(i+2)A^i - xA_x^i\big]\frac{\partial}{\partial A^i}. \tag{4.6.12}$$

It can be easily checked that infinitesimal operators (4.6.12) are admitted by Benney equations (4.6.2) and it goes without saying that they directly result from the group analysis of Benney equations (4.6.2). Just in this way (i.e., using the method of moments) infinitesimal operators (4.6.9) were first obtained in [19] by using non-canonical form of infinitesimal operators (4.6.12) with the subsequent passage to the representation (4.6.9) in the space of variables (4.6.10).

4.6.3 Incompleteness of the Point Group: Statement of the Problem

It is evident, however, that the subgroup (4.6.12) does not exhaust the complete group symmetry of Benney equations (4.6.2). The incompleteness of the result (4.6.12) is obvious form many points of view. Here we shall only point on the non-conformity of finite dimension of the algebra (4.6.12) to the infinite set of conservation laws for Benney equations, and on the infinite extension of the point symmetry group for Benney equations in the form of (4.6.5), (4.6.6) with Lagrangian velocity. Here of principle significance for us is the following statement [24]: *the group* (4.6.12) *is incomplete not only from the standpoint of Lie–Bäcklund symmetry for Benney equations but also from the standpoint of the Lie point symmetry*. The validity of the statement can be proved by direct solving of determining equations for the first order Lie–Bäcklund group (contact group, that is not reduced to point one)

$$D_t(\varkappa^i) + D_x(\varkappa^{i+1}) + iA^{i-1}D_x(\varkappa^0) + iA_x^0\varkappa^{i-1} = 0, \quad i \geq 0, \tag{4.6.13}$$

where coordinates $\varkappa^i$ of canonical operator

$$X = \sum_{i=0}^{\infty}\varkappa^i\frac{\partial}{\partial A^i}, \tag{4.6.14}$$

depend upon the countered set of group variables

$$t,\ x;\ A^0,\ \ldots,\ A^j,\ \ldots;\ A_x^0,\ \ldots,\ A_x^j, \ldots; \quad j \geq 0. \tag{4.6.15}$$

To prove the above statement one can consider only partial solutions of determining equations (4.6.12)

$$\varkappa^i = \eta^i(A^0, \ldots, A^j, \ldots); \quad i, j \geq 0, \tag{4.6.16}$$

that depend upon moments A^j, $j \geq 0$, and does not depend upon t, x. It appears that thanks to these infinitesimal operators (4.6.14), (4.6.16) an infinite extension of the group (4.6.12) takes place. Now the problem is to find these operators.

4.6.4 Determining Equations and Their Solution

Before proceeding further we write determining equations of first-order Lie–Bäcklund group, admitted by a more infinite system of coupling equations for functions $A^i(t,x)$ with the arbitrary element $\varphi(A^0)$

$$A_t^i + A_x^{i+1} + iA^{i-1}\big[\varphi(A^0)\big]_x = 0, \qquad i \geq 0. \tag{4.6.17}$$

For the coordinates $\varkappa^i$ of canonical infinitesimal operator (4.6.14) the following chains of determining equations are valid which result from splitting (4.6.17) with respect to second derivatives:

$$\begin{gathered}
\varkappa^{i+1}_{A_x^0} + i\varphi_1 A^{i-1}\varkappa^0_{A_x^0} = \sum_{j=0}^{\infty} j\varphi_1 A^{j-1}\varkappa^i_{A_x^j}, \quad i \geq 0; \\
\varkappa^{i+1}_{A_x^j} + i\varphi_1 A^{i-1}\varkappa^0_{A_x^j} = \varkappa^i_{A_x^{j-1}}; \quad i \geq 0,\ j \geq 1, \\
\varkappa_t^i + \varkappa_x^{i+1} + i\varphi_1 A^{i-1}\varkappa_x^0 + A_x^0\big(i\varphi_1\varkappa^{i-1} + i\varphi_2 A^{i-1}\varkappa^0\big) \\
+ \sum_{j=0}^{\infty}\big[i\varphi_1 A^{i-1}A_x^j\varkappa^0_{A^j} - \big(A_x^{j+1} + j\varphi_1 A_x^0 A^{j-1}\big)\varkappa^i_{A^j} + A_x^j\varkappa^{i+1}_{A^j}\big] \\
- \sum_{j=0}^{\infty} jA_x^0\big(\varphi_1 A_x^{j-1} + \varphi_2 A_x^0 A^{j-1}\big)\varkappa^i_{A_x^j} = 0, \quad i \geq 0.
\end{gathered} \tag{4.6.18}$$

Here φ_1 and φ_2 are the first and the second derivatives of the function φ with respect to its argument. From the various standpoints at list three distinct values of the function φ are specified. In case $\varphi(A^0) = A^0$ we come to kinetic Benney equations (4.6.2), whereas for $\varphi = a(A^0)^2$ extension of the admitted point group takes place thanks to projective transformations in t, x-plane (see [19]). For $\varphi = a \ln A^0$ the corresponding kinetic equation

$$f_t + vf_x - a\frac{A_x^0}{A^0}f_v = 0, \quad A^0 = \int_{-\infty}^{+\infty} dv f, \tag{4.6.19}$$

that gives rise to the discussed system of equations for moments, is of special interest in plasma theory. It appears as the equation for the distribution function of plasma ions, while electrons obey the Boltzmann distribution. More complicated dependencies of $\varphi(A^0)$ upon A^0 can also be of interest in plasma physics for non-Boltzmann distribution functions for hot electrons. Equation (4.6.19) was studied in details in [25].

For the Benney equations (4.6.2) the determining equations (4.6.18) are rewritten in the following form

$$\varkappa^{i+1}_{A^0_x} + iA^{i-1}\varkappa^0_{A^0_x} - \sum_{j=0}^{\infty} jA^{j-1}\varkappa^i_{A^j_x} = 0, \quad i \geq 0,$$
$$\varkappa^{i+1}_{A^{j+1}_x} - \varkappa^i_{A^j_x} + iA^{i-1}\varkappa^0_{A^{j+1}_x} = 0, \quad i \geq 0,\ j \geq 0.$$
$$\varkappa^i_t + \varkappa^{i+1}_x + iA^{i-1}\varkappa^0_x + A^0_x\left(i\varkappa^{i-1} - \sum_{j=0}^{\infty} jA^{j-1}\varkappa^i_{A^j} - \sum_{j=0}^{\infty}(j+1)A^j_x\varkappa^i_{A^{j+1}_x}\right)$$
$$+ iA^{i-1}\sum_{j=0}^{\infty} A^j_x\varkappa^0_{A^j} + \sum_{j=0}^{\infty} A^j_x\varkappa^{i+1}_{A^j} - \sum_{j=0}^{\infty} A^{j+1}_x\varkappa^i_{A^j} = 0, \quad i \geq 0. \tag{4.6.20}$$

Under conditions (4.6.16) the determining equations (4.6.20) are split and reduced to two infinite chains of equalities, namely one-dimensional (vector) and two-dimensional (tensor):

$$\begin{gathered}\eta^{i+1}_{A^0} - \sum_{j=0}^{\infty} jA^{j-1}\eta^i_{A^j} + iA^{i-1}\eta^0_{A^0} + i\eta^{i-1} = 0, \quad i \geq 0;\\ \eta^{i+1}_{A^{k+1}} - \eta^i_{A^k} + iA^{i-1}\eta^0_{A^{k+1}} = 0, \quad i \geq 0,\ k \geq 0.\end{gathered} \tag{4.6.21}$$

The apparent difficulty in analytical solving of the given system of determining equations (4.6.21) is due to a "nonlocal" nature of the second term in the vector chain in the form of an infinite sum with respect to index $j \geq 0$. The measure of this non-locality is characterized by a number of nonzero components of tensor η^i_j. But in fact in case of an overdetermined system (4.6.21) we obtain a finite upper value of the summation index $j < \infty$, which depends upon the other index i of this tensor.[4] Then we come to a much more simplified (but equivalent) formulation of the system (4.6.21)

$$\begin{gathered}\eta^{i+1}_{A^0} - \sum_{j=0}^{i-2} jA^{j-1}\eta^i_{A^j} + i\eta^{i-1} = 0, \quad \eta^{i+1}_{A^i} = 0, \quad i \geq 0;\\ \eta^{i+1}_{A^{k+1}} = \eta^i_{A^k}, \quad \eta^i_{A^{i+k}} = 0, \quad i \geq 0,\ k \geq 0.\end{gathered} \tag{4.6.22}$$

Before proceeding to enumerating all solutions of the system of determining equations (4.6.22), we present here yet another form of the chain in (4.6.22)

$$\eta^{i+1}_{A^0} - \sum_{j=0}^{i-2} jA^{j-1}\eta^{i-j}_{A^0} + i\eta^{i-1} = 0, \quad i \geq 0. \tag{4.6.23}$$

This form can be employed to clarify the general structure of these solutions on basis of the corresponding generating functions.

[4]For more details we refer the reader to [24].

4.6.5 Discussion of the Solution of the Determining Equations

The integrability procedure in itself for determining equations (4.6.22) is of no difficulties. For example the first six coordinates η^i $(0 \le i \le 5)$ of the desired infinitesimal operator (4.6.14), (4.6.16) are given by the following formulas for the general solutions of determining equations (4.6.22) that depend upon six arbitrary constants C^j $(0 \le j \le 5)$ and are described by polynomials in moments A^l

$$\eta^0 = C^0, \quad \eta^1 = C^1, \quad \eta^2 = C^2 - C^0 A^0, \quad \eta^3 = C^3 - 2C^1 A^0 - C^0 A^1,$$
$$\eta^4 = C^4 - 3C^2 A^0 - 2C^1 A^1 + C^0\big[-A^2 + (A^0)^2\big], \tag{4.6.24}$$
$$\eta^5 = C^5 - 4C^3 A^0 - 3C^2 A^1 + C^1\big[-2A^2 + 3(A^0)^2\big] + C^0\big(-A^3 + 2A^0 A^1\big).$$

It appears that the polynomial dependence of any solution η^i of determining equations (4.6.22) upon moments A^j is a general property of components of the vector η^i for any $i \ge 0$. The example (4.6.24) demonstrates that the procedure of obtaining solutions of determining equations (4.6.22) is reduced to their enumeration. To be concrete, we assume the following scheme of indicating of the k-th basic solution η^i_k of determining equations (4.6.22) for the coordinate η^i:

$$\eta^i_k = \begin{cases} 0, & \text{i < k;} \\ 1, & \text{i = k;} \\ 0, & \text{i = k+1;} \end{cases} \qquad \big[\eta^i_k\big] = i - k, \; i \ge k+2; \quad i, k \ge 0. \tag{4.6.25}$$

In the solutions (4.6.25) this scheme demands quit definite choice of values of integration constants C^j in the form of Kronecker symbols

$$C^j = \delta_{jk}; \quad j, k \ge 0. \tag{4.6.26}$$

The last of the four equalities for η^i_k in (4.6.25) (in square brackets) indicates the homogeneity degree $(i - k)$ of the polynomial "tail" of the solution η^i for $i \ge k + 2$ in accordance with the attributed to any of the moments A^i of the order i the homogeneity degree, which is equal to positive number $(i + 2)$ (see e.g. [20])

$$\big[A^i\big] = i + 2, \quad i \ge 0. \tag{4.6.27}$$

For instance, the component η^5_1 of the basis solution η^i_1 of determining equations (4.6.22) in accordance with (4.6.24), (4.6.25) and (4.6.26) has the homogeneity degree equal to four

$$\eta^5_1 = -2A^2 + 3(A^0)^2; \quad \big[\eta^5_1\big] = 4. \tag{4.6.28}$$

The indexing of the presented infinite (countable) vectors η^i by one more integral number $k \ge 0$ yields the desired representation of all linear independent solutions of determining equations (4.6.22) in the form of tensor of the second rank (matrix) η^i_k, in which the lower index $k \ge 0$ indicates the index of the basis infinitesimal operator in the general element of an infinite Lie algebra under consideration

$$X = \sum_{i,k=0}^{\infty} C^k \eta^i_k \frac{\partial}{\partial A^i}. \tag{4.6.29}$$

Under the conditions (4.6.25) the integration of determining equations (4.6.22) for the given basis vector η_k^i for a fixed value $k \geq 0$ is carried out with boundary conditions, that are imposed by requirements (4.6.25) in a single way.

The representation of matrix η_k^i for different lines are as follows (i is the column number, k is the line number)

$$\eta_k^i = \{0, \dots, 0, 1, 0, -(k+1)A^0, -(k+1)A^1, \dots\}. \tag{4.6.30}$$

Here zeroes preceding unity describe matrix elements, which exist only for $i < k$, i.e. which are located below the principle diagonal $i = k$, that contains only units. The first nearest upper off-diagonal $i = k + 1$ also contains only zeroes. Expressions for elements from the second $i = k + 2$ and the third $i = k + 3$ upper off-diagonals are given in (4.6.30) explicitly: they contain monomials, the homogeneity degree of which is equal to 2 and 3 respectively, while the numerical coefficient $(k + 1)$ is defined by the line number.

In general, any one of the nonzero off-diagonals $i = k + s$ with the number $s \geq 2$ is presented by polynomials with the homogeneity degree equal to s. This "line scheme" (4.6.30) is readily illustrated by a pictorial rendition of elements of the high left block of the discussed matrix ($0 \leq i \leq 5$, $0 \leq k \leq 3$)

$$\eta_k^i = \begin{pmatrix} 1 & 0 & -A^0 & -A^1 & -A^2 + (A^0)^2 & -A^3 + 2A^0A^1 & \dots \\ 0 & 1 & 0 & -2A^0 & -2A^1 & -2A^2 + 3(A^0)^2 & \dots \\ 0 & 0 & 1 & 0 & -3A^0 & -3A^1 & \dots \\ 0 & 0 & 0 & 1 & 0 & -4A^0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}. \tag{4.6.31}$$

As a more illustrative example we present here the element η_1^i of the matrix (4.6.30) with sufficiently high column number $i = 10$ and the homogeneity degree 9, that is located in the line with $k = 1$ (the second from above)

$$\begin{aligned} \eta_1^{10} = &-2A^7 + 6A^5A^0 + 6A^4A^1 + 6A^3A^2 - 12A^3(A^0)^2 \\ &- 24A^2A^1A^0 - 4(A^1)^3 + 20A^1(A^0)^3. \end{aligned} \tag{4.6.32}$$

4.6.6 Illustrative Example for Matrix Elements

A much more comprehensive idea of definite expressions of matrix elements η_k^i is given by the following list of elements (with the previous result included) of the first 11 columns ($0 \leq i \leq 10$) and 4 lines ($0 \leq k \leq 3$) of matrix η_k^i, which define the k-th basic solution of determining equations (4.6.22) for vectors η_k^i of the canonical infinitesimal operator (4.6.14), (4.6.16). The lower index "k" is omitted for simplicity.

(0) $k = 0;\ \eta^0 = 1,\ \eta^1 = 0,\ [\eta^i] = i,\ i \geq 2.$

$\eta^2 = -A^0,$

$\eta^3 = -A^1,$

$$\eta^4 = -A^2 + (A^0)^2,$$
$$\eta^5 = -A^3 + 2A^0A^1,$$
$$\eta^6 = -A^4 + 2A^0A^2 + (A^1)^2 - (A^0)^3,$$
$$\eta^7 = -A^5 + 2A^0A^3 + 2A^2A^1 - 3A^1(A^0)^2,$$
$$\eta^8 = -A^6 + 2A^0A^4 + 2A^3A^1 + (A^2)^2 - 3A^2(A^0)^2 - 3A^0(A^1)^2 + (A^0)^4,$$
$$\eta^9 = -A^7 + 2A^0A^5 + 2A^4A^1 + 2A^3A^2 - 3A^3(A^0)^2 - 6A^0A^1A^2 - (A^1)^3 + 4A^1(A^0)^3,$$
$$\eta^{10} = -A^8 + 2A^0A^6 + 2A^5A^1 + A^4[2A^2 - 3(A^0)^2] + A^3[A^3 - 6A^0A^1] + A^2[-3(A^1)^2 - 3A^0A^2 + 4(A^0)^3] + 6(A^1)^2(A^0)^2 - (A^0)^5.$$

(1) $k = 1;\ \eta^0 = 0,\ \eta^1 = 1,\ \eta^2 = 0,\ [\eta^i] = i - 1,\ i \geq 3.$

$$\eta^3 = -2A^0,$$
$$\eta^4 = -2A^1,$$
$$\eta^5 = -2A^2 + 3(A^0)^2,$$
$$\eta^6 = -2A^3 + 6A^0A^1,$$
$$\eta^7 = -2A^4 + 6A^0A^2 + 3(A^1)^2 - 4(A^0)^3,$$
$$\eta^8 = -2A^5 + 6A^0A^3 + 6A^2A^1 - 12A^1(A^0)^2,$$
$$\eta^9 = -2A^6 + 6A^0A^4 + 6A^3A^1 + A^2[3A^2 - 12(A^0)^2] - 12A^0(A^1)^2 + 5(A^0)^4,$$
$$\eta^{10} = -2A^7 + 6A^0A^5 + 6A^4A^1 + 6A^3[A^2 - 2(A^0)^2] - 24A^0A^1A^2 + A^1[-4(A^1)^2 + 20(A^0)^3].$$

(2) $k = 2;\ \eta^0 = 0,\ \eta^1 = 0,\ \eta^2 = 1, \eta^3 = 0,\ [\eta^i] = i - 2,\ i \geq 4.$

$$\eta^4 = -3A^0,$$
$$\eta^5 = -3A^1,$$
$$\eta^6 = -3A^2 + 6(A^0)^2,$$
$$\eta^7 = -3A^3 + 12A^0A^1,$$
$$\eta^8 = -3A^4 + 12A^0A^2 + 6(A^1)^2 - 10(A^0)^3,$$
$$\eta^9 = -3A^5 + 12A^0A^3 + 12A^2A^1 - 30A^1(A^0)^2,$$
$$\eta^{10} = -3A^6 + 12A^0A^4 + 12A^3A^1 + 6(A^2)^2 - 30A^0(A^1)^2 + 15(A^0)^4 - 30A^2(A^0)^2.$$

(3) $k = 3;\ \eta^0 = 0,\ \eta^1 = 0,\ \eta^2 = 0,\ \eta^3 = 1,\ \eta^4 = 0,\ [\eta^i] = i - 3,\ i \geq 5.$

$$\eta^5 = -4A^0,$$
$$\eta^6 = -4A^1,$$
$$\eta^7 = -4A^2 + 10(A^0)^2,$$

$$\eta^8 = -4A^3 + 20A^0A^1,$$
$$\eta^9 = -4A^4 + 20A^0A^2 + 10(A^1)^2 - 20(A^0)^3,$$
$$\eta^{10} = -4A^5 + 20A^0A^3 + 20A^2A^1 - 60A^1(A^0)^2.$$

To conclude, we present a result of calculation of the infinite (countable) part of Lie point group admitted by the system of Benney equations — moment equations (4.6.2). In standard (non-canonical representation) the point Lie group of Benney equations (4.6.2) is described by the infinitesimal operator

$$X = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \sum_{i=0}^{\infty} \eta^i \frac{\partial}{\partial A^i}, \tag{4.6.33}$$

where coordinates ξ and η obey the system of determining equations

$$\begin{aligned} &\eta^{i+1}_{A^0} - \sum_{j=0}^{\infty} jA^{j-1}\eta^i_{A^j_x} + i\eta^{i-1} + iA^{i-1}\left(\eta^0_{A^0} + \xi^1_t - \xi^2_x\right) \\ &+ (i+1)A^i\xi^1_x - \xi^2_t\delta_{i,0} = 0, \\ &\quad \eta^{i+1}_{A^{k+1}} - \eta^i_{A^k} + iA^{i-1}\left(\eta^0_{A^{k+1}} + \xi^1_x\delta_{0,k}\right) \\ &\qquad + (\xi^1_t - \xi^2_x)\delta_{i,k} + \xi^1_x\delta_{i+1,k} - \xi^2_t\delta_{i,k+1} = 0, \\ &\qquad \eta^i_t + \eta^{i+1}_x + iA^{i-1}\eta^0_x = 0, \qquad i,\ k \geq 0. \end{aligned} \tag{4.6.34}$$

Determining equations (4.6.34) result from (4.6.20) in account of relationships between coordinates of infinitesimal operators (4.6.33) and (4.6.14)

$$\varkappa^i = \eta^i + \xi^1(A^{i+1}_x + iA^{i-1}A^0_x) - \xi^2A^i_x. \tag{4.6.35}$$

Infinitesimal operators (4.6.12), that were presented above, gives rise to the following coordinates

$$\begin{aligned} \xi^1 = K^4 + K^5t, \quad \xi^2 = K^1 + K^2t + K^3x, \\ \eta^i = iA^{i-1}K^2 + (i+2)A^i(K^3 - K^5). \end{aligned} \tag{4.6.36}$$

The problem of finding coordinates of the operator (4.6.33) was first treated in [19], where only these solutions, namely (4.6.9), (4.6.12) and (4.6.36), were described. The main result described in Sect. 4.6 is that point symmetries of Benney equations (4.6.2) are exhausted by formulas (4.6.12) and solutions of determining equations (4.6.22), i.e. determining equations (4.6.34) do not have any other solutions. Solutions of determining equations (4.6.22) which are responsible for the infinite part of the point group probably have not been known so far [24].

As a next step it seems intriguing to generalize the result (4.6.35), i.e. to find the first order Lie–Bäcklund group admitted by Benney equations (4.6.2) with coordinates $\varkappa^i$ of the canonical infinitesimal operator (4.6.14), that has the linear form

$$\varkappa^i = \eta^i + \sum_{j=0}^{\infty} \eta^{i,j}A^j_x, \qquad i \geq 0. \tag{4.6.37}$$

Though the unique existence of the linear form (4.6.37) as well as the complete solution of determining equations[5] for the tensor $\eta^{i,j}$ has not yet been obtained, all known facts are in agreement with this linear form. In particular, results of [20, 21] mentioned above are consistent with the following expression for the tensor $\eta^{i,j}$ of the linear form

$$\eta_s^{i,j} = \sum_{k=0}^{\infty} k H^s_{A^k} \delta_{i+k,j+1} + s \sum_{k=0}^{s-j-2} (i+k) A^{i+k-1} H^{s-1}_{A^{j+k+1}}; \quad i,\ j,\ s \geq 0. \tag{4.6.38}$$

Here s is the number of the basis solution (similar to that used for η^i in (4.6.28)), H^s is a polynomial of the homogeneity degree $(s+2)$ in moments A^i. Compatibility conditions for determining equations for the tensor $\eta^{i,j}$ give rise to many relationships for H^s, for example

$$\sum_{j=0}^{\infty} j A^{j-1} H^s_{A^j} = s H^{s-1}, \quad s \geq 0. \tag{4.6.39}$$

An explicit form for the polynomial H^7 is presented below just to illustrate the aforesaid

$$\begin{aligned} H^7 = {} & A^7 + 7A^5A^0 + 7A^4A^1 + 7A^3A^2 + 21A^3(A^0)^2 + 42A^2A^1A^0 \\ & + 7(A^1)^3 + 35A^1(A^0)^3. \end{aligned} \tag{4.6.40}$$

Comparison between formulas (4.6.32) and (4.6.40) shows that they differ only in numerical values (and signs) of coefficients. The generating function for polynomials H^s is given in [20, 21]. So constructing of a recursion operator, which relates solutions of determining equations (4.6.22) for the point group to the solutions of the determining equations for the first order Lie–Bäcklund symmetry defined by the linear form (4.6.37) with coefficients given by (4.6.38) in particular is of principal interest.

4.7 Symmetries in Application to Plasma Kinetic Theory. Renormalization Group Symmetries for Boundary Value Problems and Solution Functionals

The above Sects. 4.3–4.6 deal with calculating symmetries for systems of integro-differential (nonlocal) equations while this section gives illustrations of symmetry applications to problems of mathematical physics with nonlocal equations.

[5] For simplicity these equations are omitted here.

4.7.1 Introduction to Renormgroup Symmetries

In mathematical physics a solution of a physical problem usually appears as a solution of some boundary value problem. Note that the symmetry of boundary value problem solutions is closely related to RenormGroup (RG) symmetry, introduced in mathematical physics in the beginning of the 1990s [26, 27] (see also reviews [28, 29, 32]). As for the notion of Renormalization Group, or briefly RenormGroup, this was imported to mathematical physics from the most complicated part of theoretical physics, quantum field theory. Recall that the (Lie transformation) group structure discovered by Stueckelberg and Petermann in the early 1950s in calculation results in renormalized quantum field theory and the exact symmetry of solutions related to this structure were used in 1955 by Bogoliubov and Shirkov to develop a regular method for improving approximate solutions of quantum field problems, the RG method. This method is based on the use of the infinitesimal form of the exact group property of a solution to improve a perturbative (that is, obtained by means of the perturbation theory) representation of this solution. The improvement of the approximation properties of a solution turns out to be most efficient in the presence of a singularity, because the correct structure of the singularity is then recovered.

In extending the RG conceptions in quantum field theory to boundary value problems of classical mathematical physics the main achievement was the development of a regular algorithm (see the reviews [28–32]) for finding symmetries of the RG type by means of the modern group analysis. The existence of such an algorithm eliminates the usual deficiency of the RG approach beyond the scope of quantum field theory problems: finding the group property of solutions requires using special-purpose methods of analysis, usually nonstandard, in each particular case. The new algorithm has the same aim of finding an improved solution (in comparison with the initial approximate solution) as the algorithm of Bogoliubov's RG method, but in finding symmetries of a solution of a boundary value problem it uses a scheme of calculations similar to that of the modern group analysis. The attribute 'renormalization group' thus points to similarities existing between these symmetries and the symmetries in quantum field theory related to the operation of renormalization of masses and charges (coupling constants).

Initially [26, 28, 29], applying the RG algorithm was mainly limited to problems based on differential equations, although this algorithm can be used formally in any problem for which a regular way of calculating symmetries for the basic equations can be specified. Hence, transition to such objects, which until recently were not a subject of group analysis, in particular, to integral and integro-differential equations, essentially expands the area of the RG symmetry applications [30–32].

In problems with involved equations, e.g., in transfer theory with integro-differential Boltzmann equation or in quantum field theory with an infinite chain of coupled integro-differential Dyson–Schwinger equations, only some solution components or their integrated characteristics satisfy a sufficiently simple symmetry. Thus, in the one-velocity plane transfer problem, the RG property is related [33] to the asymptotics of the "density of particles, moving deep into the medium" $n_+(x)$,

$x \to \infty$, not entering the Boltzmann equation.[6] In such problems, integral relations form the problem skeleton. But they can appear as some independent objects for applying the RG symmetry constructed for solutions of differential equations. Frequently, not the solution itself in its entire range of the variables and parameters but rather some integral characteristic, a solution functional, is of physical interest. This characteristic can appear, for example, as a result of averaging (integrating) over one of the independent variables or of transition to a new integral representation, for example, a Fourier representation.

This section is structured as follows. In Sect. 4.7.2, one finds an introductory example of the RG algorithm in mathematical physics, illustrated by a solution of a simple boundary value problem. In Sect. 4.7.3, a general scheme for constructing the RG algorithm, valid for models with both local (differential) and nonlocal terms, including integral and integro-differential equations, is described. Section 4.7.4 gives several examples of application of the RG algorithm.

4.7.2 RG Symmetry: An Idea of Construction and Its Simple Realization

We preface the description of the RG algorithm with the following simple argument.

Let the Lie group G with generator

$$X = \xi^t \frac{\partial}{\partial t} + \xi^x \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} \tag{4.7.1}$$

be defined for the system of the first-order partial differential equations

$$y_t = F(t, x, y, y_x). \tag{4.7.2}$$

The typical boundary value problem for (4.7.2) is the Cauchy problem with boundary manifold defined by

$$t = 0, \qquad y = \psi(x). \tag{4.7.3}$$

Solution of this Cauchy problem is the G-invariant solution iff for any generator (4.7.1), function ψ satisfies the equation [34, §29]

$$\eta(0, x, \psi) - \xi^x(0, x, \psi)\psi_x - \xi^t(0, x, \psi)F(0, x, \psi, \psi_x) = 0. \tag{4.7.4}$$

The solution of Cauchy problem (4.7.2), (4.7.3) coincides with orbit of the group G, and the boundary manifold is *not* the invariant manifold of the group.

This example gives an instructive idea for constructing generators of RG symmetries. The milestones here are (a) considering the boundary value problem in the extended space of group variables that involve parameters of boundary conditions in group transformations, (b) calculating the admitted group using the infinitesimal

[6] This is representable as the integral $\int_0^1 n(x, \vartheta)\, \mathrm{d}\cos\vartheta$ of the kinetic equation solution $n(x, \vartheta)$.

approach, (c) checking the invariance condition akin to (4.7.4) to find the symmetry group with the orbit that coincides with the boundary value problem solution, and (d) using the RG symmetry to find the improved (renormalized) solution of the boundary value problem.

The complete algorithm [28, 29, 31, 32] will be described in detail in the next section; here we only give a general grasp of the problem using a trivial example, the boundary value problem for the Hopf equation

$$v_t + v v_x = 0, \quad v(0,x) = \varepsilon U(x), \tag{4.7.5}$$

where U is an invertible function of x and the parameter ε defines the initial amplitude at the boundary $t = 0$. For small values of $t \ll 1/\varepsilon$, i.e., near the boundary, $t \to 0$, a perturbation theory (PT) solution of (4.7.5) has the form of a truncated power series in εt,

$$v = \varepsilon U - \varepsilon^2 t U U_x + O\big(t^2\big). \tag{4.7.6}$$

It is obvious that this solution is invalid for large distances from the boundary, when $\varepsilon t U_x \backsimeq 1$. The RG symmetry gives a way to improve the perturbation theory result and restore the correct structure of the boundary value problem solution in the vicinity of a singularity (in the event that such singularity appears for some finite value of t).

It is convenient to introduce the new function $u = v/\varepsilon$ and rewrite (4.7.5) in the form

$$u_t + \varepsilon u u_x = 0, \quad u(0,x) = U(x). \tag{4.7.7}$$

In order to calculate the renormgroup symmetries, we add the parameter ε to the list of the independent variables and consider the manifold (termed in general the *basic manifold*) given by (4.7.7) in the space of variables $\{t, x, \varepsilon, u, u_t, u_x\}$. Then we calculate the generator

$$X = \xi^t \frac{\partial}{\partial t} + \xi^x \frac{\partial}{\partial x} + \xi^\varepsilon \frac{\partial}{\partial \varepsilon} + \eta \frac{\partial}{\partial u} \tag{4.7.8}$$

of the group admitted by the first equation in (4.7.7) and obtain the following coordinates of the generator (4.7.8):

$$\xi^t = \psi^1, \quad \xi^x = \varepsilon u \psi^1 + \psi^2 + x(\psi^3 + \psi^4), \quad \xi^\varepsilon = \varepsilon \psi^4, \quad \eta = u \psi^3, \tag{4.7.9}$$

where ψ^i, $i = 2, 3, 4$, are arbitrary functions of ε, u, and $x - \varepsilon u t$ and ψ^1 being an arbitrary function of all the group variables. These formulas define an infinite-dimensional Lie algebra with four generators

$$\begin{aligned} X_1 &= \psi^1 \left(\frac{\partial}{\partial t} + \varepsilon u \frac{\partial}{\partial x} \right), \quad X_2 = \psi^2 \frac{\partial}{\partial x}, \\ X_3 &= \psi^3 \left(x \frac{\partial}{\partial x} + u \frac{\partial}{\partial u} \right), \quad X_4 = \psi^4 \left(\varepsilon \frac{\partial}{\partial \varepsilon} + x \frac{\partial}{\partial x} \right). \end{aligned} \tag{4.7.10}$$

Suppose that a particular solution of boundary value problem (4.7.7),

$$S \equiv u - W(t, x, \varepsilon) = 0,$$

which defines an invariant manifold of group (4.7.8), (4.7.9) is known. The corresponding invariance condition evaluated on frame S is similar to (4.7.4):

$$XS_{|[S]} \equiv (W - xW_x)\psi^3 - W_x\psi^2 - (\varepsilon W_\varepsilon + xW_x)\psi^4 = 0. \tag{4.7.11}$$

The term with ψ^1 does not give any input in (4.7.11) since it is proportional to $W_t + \varepsilon W W_x$ and vanishes on the solutions of (4.7.7). Equation (4.7.11) is valid for all t. Hence, it remains valid for $t \to 0$, when W is replaced with approximate solution, which follows from (4.7.6),

$$W = U - \varepsilon t U U_x + O\left(t^2\right). \tag{4.7.12}$$

In this limit, $t \to 0$, condition (4.7.11) gives a relation between the ψ^i, $i = 2, 3, 4$ (no restrictions are imposed on ψ^1), that can be easily prolonged on $t \neq 0$,

$$\psi^2 = -\chi(\psi^3 + \psi^4) + (u/U_\chi)\psi^3, \quad \chi = x - \varepsilon u t, \tag{4.7.13}$$

where the derivative U_χ should be expressed, due to the boundary condition, either in terms of χ or u. By substituting (4.7.13) in (4.7.9), we obtain a group of a smaller dimension with generators

$$\begin{aligned} R_1 &= \psi^1 \left(\frac{\partial}{\partial t} + \varepsilon u \frac{\partial}{\partial x}\right), \\ R_2 &= u\psi^3 \left[\left(\varepsilon t + 1/U_\chi\right)\frac{\partial}{\partial x} + \frac{\partial}{\partial u}\right], \\ R_3 &= \varepsilon\psi^4 \left(tu\frac{\partial}{\partial x} + \frac{\partial}{\partial \varepsilon}\right). \end{aligned} \tag{4.7.14}$$

The above procedure, which transforms (4.7.10) to (4.7.14), is the *restriction of the group* (4.7.8) *on a particular solution*.

The boundary value problem solution defines a manifold, that, by construction, turns to be invariant for any generator R_i. Hence, (4.7.14) defines the desired RG symmetries. This means that the boundary value problem solution can be constructed by use any of generators in (4.7.14), the generator R_3 for example. Without loss of generality, we choose $\varepsilon\psi^4 = 1$ and obtain the finite RG transformations (a is a group parameter)

$$x' = x + atu, \quad \varepsilon' = \varepsilon + a, \quad t' = t, \quad u' = u, \tag{4.7.15}$$

where t and u are invariants of the RG transformations while the transformations of ε and x are translations, which also depend on t and u in the case of x. For $\varepsilon = 0$, in view of (4.7.6), we have $x = H(u)$, where $H(u)$ is a function inverse to $U(x)$. Eliminating a, t, u from (4.7.15) and omitting the primes on variables, we obtain the desired solution of boundary value problem (4.7.7) in the implicit form

$$x - \varepsilon t u = H(u). \tag{4.7.16}$$

This in fact is the improved perturbation theory solution (4.7.6), which is valid not only for small $\varepsilon t \ll 1$, provided dependence (4.7.16) can be resolved uniquely. Depending upon $H(u)$ it gives either proper singular behavior at some finite $t \to t_{sing}$ or correct asymptotic behavior at $t \to \infty$.

Example 4.7.1 One example of the first option is the solution of the boundary value problem for the linear function $U(x) = x$. This yields the solution $v = \varepsilon x(1+\varepsilon t)^{-1}$, which remains finite as $t \to \infty$.

Example 4.7.2 For the second option, we can select, for instance, a sine wave $U(x) = -\sin x$ at the boundary. Then solution (4.7.16) describes the well-known distortion of the initial profile of a sine wave, transforming it into a saw-tooth shape [35, Chap. 6, §1], with a singularity forming at a finite distance $t_{sing} = 1/\varepsilon$ from the boundary.

We note that for finding solution (4.7.16) of the boundary value problem we use *only* the known symmetry of the solution and the corresponding perturbation theory (PT).

The peculiarity of the procedure for constructing RG symmetries is the multi-choice first step, which depends on how the boundary conditions are formulated and the form in which the admitted group is calculated. For example, instead of calculating the Lie point symmetry group, we can consider the Lie–Bäcklund symmetries (see Sect. 1.5 in Chap. 1) with the canonical generator $R = \varkappa\partial_u$, where $\varkappa$ depends not only on t, x, ε, and u but also on higher-order derivatives of u. We can seek $\varkappa$ in the form of a power series in ε, and invariance condition (4.7.11) is formulated as vanishing of $\varkappa$ at $t = 0$. Depending on the choice of the zeroth-order term representation, we obtain either an infinite or a truncated power series for $\varkappa$, for example, a form linear in ε,

$$R = \varkappa\frac{\partial}{\partial u}, \quad \varkappa = 1 - \frac{u_x}{U_x(u)} - \varepsilon t u_x. \tag{4.7.17}$$

This RG generator (4.7.17) is equivalent to the Lie point generator R_2 in (4.7.14) and therefore gives the same result.

Another possibility for calculating RG symmetries for boundary value problem (4.7.7) is offered by taking some additional differential constraints consistent with boundary conditions and input equations into account. For example, when the boundary condition in (4.7.7) is linear in its argument, $U(x) = x$, the differential constraint can be chosen as $u_{xx} = 0$; this equality reflects the invariance of the original equation with respect to the second-order Lie–Bäcklund symmetry group. Calculating the Lie point symmetry group for the joint system of this constraint and the Hopf equation gives another way to find RG symmetries for boundary value problem (4.7.7).

The above example demonstrates the key features of the RG algorithm in mathematical physics. The details of the general approach are discussed in the next section.

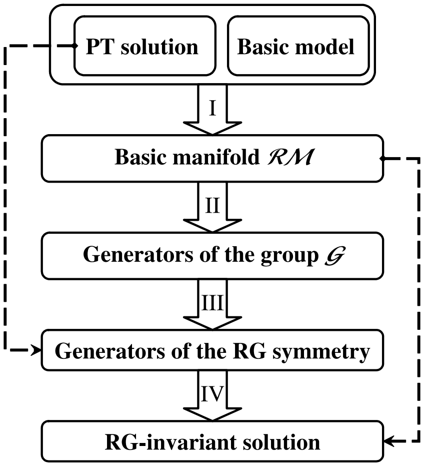

Fig. 4.1 Scheme of RG algorithm

4.7.3 Renormgroup Algorithm

The general construction scheme of the RG algorithm (shown in Fig. 4.1) is given as four consecutive steps [28–32]:

I. constructing the basic *manifold* $\mathscr{RM}$,
II. calculating the admitted (*symmetry*) group $\mathscr{G}$,
III. *restricting* it on the particular boundary value problem solution and constructing $\mathscr{RG}$, and
IV. seeking an *analytic solution*.

4.7.3.1 Basic Manifold $\mathscr{RM}$

The initial issue is to construct the RG symmetry and appropriate transformations that involve the parameters of partial solution. Therefore, the purpose of step **I** is to include all the parameters, both from the equations and from the boundary conditions on which a particular solution depends, in group transformations in one or another way. This purpose is achieved by constructing a special manifold $\mathscr{RM}$ given by a system that consists of s kth-order differential equations and q nonlocal relations

$$F_\sigma(z, u, u_{(1)}, \ldots, u_{(k)}) = 0, \quad \sigma = 1, \ldots, s, \tag{4.7.18}$$

$$F_\sigma(z, u, u_{(1)}, \ldots, u_{(r)}, J(u)) = 0, \quad \sigma = 1 + s, \ldots, q + s. \tag{4.7.19}$$

The nonlocal variables $J(u)$ here are introduced by integral objects,

$$J(u) = \int \mathscr{F}(u(z))\,\mathrm{d}z. \tag{4.7.20}$$

The presence of relations (4.7.19) in the system determining $\mathscr{RM}$ characterizes the basic difference between the case of a nonlocal problem and the case of a boundary value problem for differential equations, for which $\mathscr{RM}$ is a differential manifold.

4.7.3.2 Admitted Group $\mathscr{G}$

Step **II** is to calculate the widest admitted group $\mathscr{G}$ for system (4.7.18), (4.7.19). In application to an $\mathscr{RM}$ defined only by system of differential equations (4.7.18), the question is about a local group of transformations in a space of differential functions $\mathscr{A}$, for which system (4.7.18) remains unchanged. This group is defined by the generator of form (4.7.8) prolonged on all higher-order derivatives,

$$X = \xi^i \frac{\partial}{\partial z^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \zeta_i^\alpha \frac{\partial}{\partial u_i^\alpha} + \zeta_{i_1 i_2}^\alpha \frac{\partial}{\partial u_{i_1 i_2}^\alpha} + \cdots, \tag{4.7.21}$$

where $\xi^i([z,u]), \eta^\alpha([z,u]) \in \mathscr{A}$ and

$$\zeta_i^\alpha = D_i(\varkappa^\alpha) + \xi^j u_{ij}^\alpha, \quad \zeta_{i_1 i_2}^\alpha = D_{i_1} D_{i_2}(\eta^\alpha - \xi^i u_i^\alpha) + \xi^j u_{j i_1 i_2}^\alpha.$$

Meanwhile, the classical Lie algorithm using the infinitesimal approach seems to be inapplicable to a manifold $\mathscr{RM}$ set by system (4.7.18), (4.7.19). The issue is that the $\mathscr{RM}$ in this case is not determined *locally* in the space of differential functions. Therefore, the main advantage of the Lie computational algorithm, namely, representation of the determining equations as an over-determined system of equations is not realized here. Furthermore, the procedure for prolongation the group operator of point transformations on nonlocal variables is not defined in the framework of classical group analysis.

In modifying the RG algorithm, we rely on the direct method for calculating symmetries described in Chaps. 2 and 4. Therefore, constructing the symmetries for the nonlocal equations also appears as an algorithmic procedure. This is the generalization of the second step of the algorithm to the case where $\mathscr{RM}$ is an integral or integro-differential manifold.

4.7.3.3 Restriction of the Admitted Group on Solutions

The group $\mathscr{G}$ found in step **II** and determined by operators (4.7.21) is generally wider than the RG of interest, which is related to a particular solution of a boundary value problem. Hence, to obtain the RG symmetry, we need step **III**, *restricting* the group $\mathscr{G}$ on a manifold determined by this particular solution. From the mathematical standpoint, this procedure consists in checking the vanishing conditions for a linear combination of coordinates $\varkappa_j^\alpha$ of a canonical operator equivalent to (4.7.21) on some particular boundary value problem solution $U^\alpha(z)$,

$$\left\{ \sum_j A^j \varkappa_j^\alpha \equiv \sum_j A^j \left(\eta_j^\alpha - \xi_j^i u_i^\alpha \right) \right\}_{|u^\alpha = U^\alpha(z)} = 0. \tag{4.7.22}$$

The form of the condition set by relation (4.7.22) is common for any solution of the boundary value problem, but how the restriction procedure of a group is realized may differ in each partial case. In the general scheme (given at the beginning of the section), it is related to the dashed arrow connecting the "initial object" (a perturbation theory solution of a particular boundary value problem) to the object arising as a result of step **III**.

In calculating combination (4.7.22) on a particular solution $U^{\alpha}(z)$, the latter is transformed from a system of differential equations for group invariants to algebraic relations. Note two consequences of step **III**. First, the restriction procedure results in a set of relations between A^j and thus "links" the coordinates of various group operators X_j admitted by $\mathscr{RM}$ (4.7.18), (4.7.19). Second, it (partially or completely) eliminates an arbitrariness that can arise in the values of the coordinates ξ^i and η^{α} in the case of an infinite group $\mathscr{G}$.

As a rule, the procedure of restricting the group $\mathscr{G}$ reduces its dimension. After performing this procedure a general element (4.7.21) of a new group $\mathscr{RG}$ is represented by a linear combination of new generators R_i with coordinates $\hat{\xi}^i$ and $\hat{\eta}^{\alpha}$ and arbitrary constants B^j:

$$X \Rightarrow R = \sum_j B^j R_j, \quad R_j = \hat{\xi}^i_j \frac{\partial}{\partial x^i} + \hat{\eta}^{\alpha}_j \frac{\partial}{\partial u^{\alpha}}. \tag{4.7.23}$$

The set of operators R_j, each containing the required solution of a problem in the invariant manifold, defines a group of transformations $\mathscr{RG}$, which we also call RenormGroup.

4.7.3.4 Renormgroup Invariant Solutions

The three steps described above completely form the regular algorithm for constructing the RG symmetry, but to finish a final step is needed. This step **IV** uses the RG symmetry operators to find analytic expressions for new, improved boundary value problem solutions (compared with the input perturbative solution).

From the mathematical standpoint, realizing this step involves use of *RG-invariance* conditions set by a *joint* system of equations (4.7.18) and (4.7.19) and the vanishing conditions for a linear combination of the coordinates $\hat{\varkappa}^{\alpha}_j$ of the canonical operator equivalent to (4.7.23),

$$\sum_j R^j \hat{\varkappa}^{\alpha}_j \equiv \sum_j B^j \left(\hat{\eta}^{\alpha}_j - \hat{\xi}^i_j u^{\alpha}_i\right) = 0. \tag{4.7.24}$$

The need to use $\mathscr{RM}$ in constructing the boundary value problem solution is shown in the scheme by the dashed arrow connecting these objects.

Specification of step **IV** concludes the description of the regular algorithm of RG symmetries construction for models with integro-differential equations. We note that last the two steps are basically the same as for models with differential equations. The next sections contains a set of examples showing the ability of the upgraded RG algorithm.

4.7.4 Examples of RG Symmetries in Plasma Theory

4.7.4.1 Nonlinear Dielectric Permittivity of Plasma

Nonlinearity of electrodynamics of real medium is due to nonlinear relation between the induced current and charge density inside the medium and the electromagnetic field. This relation, named the material equation, originates from a dependence of an electric induction vector upon the electromagnetic field (see [36], p. 48). The induction vector $\boldsymbol{D}(t, \boldsymbol{r})$ is related to the electric field $\boldsymbol{E}(t, \boldsymbol{r})$ and the current density $\boldsymbol{j}(t, \boldsymbol{r})$ via an equality, which in Fourier representation has the following form (here variables "with tildes" are used to distinguish the Fourier representation from the usual space-time representation):

$$\tilde{\boldsymbol{D}}(\omega, \boldsymbol{k}) = \tilde{\boldsymbol{E}}(\omega, \boldsymbol{k}) + i\frac{4\pi}{\omega}\tilde{\boldsymbol{j}}(\omega, \boldsymbol{k}). \tag{4.7.25}$$

In an effort to describe weak-turbulent plasma, processes of particle-wave scattering, parametric instabilities, generation of harmonics, and etc., the material equation is represented as a series in positive powers of electromagnetic fields. Hence, the current density $\tilde{\boldsymbol{j}}(\omega, \boldsymbol{k})$ is expressed as a sum

$$\tilde{\boldsymbol{j}}(\omega, \boldsymbol{k}) = \sum_l \tilde{\boldsymbol{j}}^{(l)}(\omega, \boldsymbol{k}), \quad \tilde{\boldsymbol{j}}^{(l)}(\omega, \boldsymbol{k}) \backsim O(\tilde{\boldsymbol{E}}^l). \tag{4.7.26}$$

In view of time and spatial dispersion the relation between the induced current and the field appears as integral, nonlocal, that results in the material equation which in Fourier representation has the following form [36]:[7]

$$\begin{aligned}\tilde{D}_i(\omega, \boldsymbol{k}) = \varepsilon_{ij}(\omega, \boldsymbol{k})\tilde{E}_j(\omega, \boldsymbol{k}) + \sum_{n=2}^{\infty}\int \delta(\omega - \omega_1 - \cdots - \omega_n)\\ \times \delta(\boldsymbol{k} - \boldsymbol{k}_1 - \cdots - \boldsymbol{k}_n)\varepsilon_{ij_1\ldots j_n}(\omega_1, \boldsymbol{k}_1; \ldots; \omega_n, \boldsymbol{k}_n)\\ \times \tilde{E}_{j_1}(\omega_1, \boldsymbol{k}_1)\ldots\tilde{E}_{j_n}(\omega_n, \boldsymbol{k}_n)\,\mathrm{d}\omega_1\,\mathrm{d}\boldsymbol{k}_1\ldots\mathrm{d}\omega_n\,\mathrm{d}\boldsymbol{k}_n.\end{aligned} \tag{4.7.27}$$

We compare (4.7.26) and (4.7.25) with (4.7.27) to establish a relation between the current density $\tilde{\boldsymbol{j}}^{(l)}$ of the appropriate order $l \geq 2$ and multi-index tensors of nonlinear dielectric permittivity of plasma $\varepsilon_{ij_1\ldots j_n}$, which are kernels of nonlinear (with respect to electromagnetic field) integral terms in series (4.7.27).

Usually, without use of the RG algorithm, the nonlinear dielectric permittivity for hot plasma is obtained by iterating the Vlasov kinetic equation for the distribution function of particles $f(t, \boldsymbol{r}, \boldsymbol{v})$ (4.1.1) with a stationary and homogeneous in coordi-

[7] Here the bottom index specifies on a corresponding tensor component, instead of designating a derivative.

nate $\boldsymbol{r}$ background distributions $f_0(\boldsymbol{v})$ in powers of a self-consistent electromagnetic field (here we omit an index of particles):

$$f(t,\boldsymbol{r},\boldsymbol{v}) = f_0(\boldsymbol{v}) + \sum_{l\geq 1} f^{(l)}(t,\boldsymbol{r},\boldsymbol{v}), \quad f^{(l)} \backsim O(\boldsymbol{E}^l),$$
$$\boldsymbol{j}^{(l)}(t,\boldsymbol{r}) = em^3 \int f^{(l)}\gamma^5 \boldsymbol{v}\,\mathrm{d}\boldsymbol{v}. \tag{4.7.28}$$

As for the nonlinear dielectric permittivity for cold plasma it is usually obtained by iterations of more simple equations of collisionless hydrodynamics for density $N(t,\boldsymbol{r})$ and velocity $\boldsymbol{V}(t,\boldsymbol{r})$ of particles (written down here for one sort of particles in non-relativistic approach)

$$N_t + \mathrm{div}(N\boldsymbol{V}) = 0, \quad \boldsymbol{V}_t + (\boldsymbol{V}\cdot\nabla)\boldsymbol{V} = \frac{e}{m}\left\{\boldsymbol{E} + \frac{1}{c}[\boldsymbol{V}\times\boldsymbol{B}]\right\}, \tag{4.7.29}$$

in which the electric $\boldsymbol{E}$ and the magnetic field $\boldsymbol{B}$ obey Maxwell equations (4.1.3), and charge ρ and current $\boldsymbol{j}$ densities have the form

$$\rho = eN, \quad \boldsymbol{j} = eN\boldsymbol{V}. \tag{4.7.30}$$

In the right-hand part of (4.7.30) summation upon various species of plasma particles is implied, however for simplification of notations the index of species is omitted and only one sort of particles, for example electrons is underlined further.

It is commonly accepted, that formulas for the nonlinear dielectric permittivity in hot plasma are more general, than in cold (see, for example, [36], Chap. 2) and they are reduced to the last in that specific case, when the distribution function of plasma particles upon momentum in the initial equilibrium state is represented by the Dirac delta-function, $f_0(\boldsymbol{v}) = \delta(\boldsymbol{v})$. With growth of the order of nonlinearity ($l \geq 4$) an algebraic procedure of symmetrization for nonlinear dielectric permittivity tensors becomes more cumbersome in hot plasma, than in cold. The use of RG algorithm allows to establish a one-to-one correspondence between tensors of the nonlinear dielectric permittivity in cold and hot plasma in any order of nonlinearity l and also specifies a way of obtaining expressions for tensors of the nonlinear dielectric permittivity in hot plasma from appropriate "cold" expressions.

For this purpose we present a current density of the given order $\tilde{\boldsymbol{j}}^{(l)}(\omega,\boldsymbol{k})$ in hot plasma as a convolution of two functions, the partial current density $\hat{\boldsymbol{j}}^{(l)}(\omega,\boldsymbol{k},\boldsymbol{w})$, which depends on the Lagrangian velocity of particles $\boldsymbol{w}$, and an equilibrium velocity distribution function of particles in absence of electromagnetic fields $f_0(\boldsymbol{w})$,

$$\tilde{\boldsymbol{j}}^{(l)}(\omega,\boldsymbol{k}) = \int f_0(\boldsymbol{w})\hat{\boldsymbol{j}}^{(l)}(\omega,\boldsymbol{k},\boldsymbol{w})\,\mathrm{d}\boldsymbol{w}. \tag{4.7.31}$$

An expression for the partial current density for $f_0(\boldsymbol{w}) = \delta(\boldsymbol{w})$, i.e. in cold plasma ($\boldsymbol{w} = 0$), is obtained by iterating (4.7.29), (4.7.30) with respect to the self-consistent field, while a transition from $\hat{\boldsymbol{j}}^{(l)}(\omega,\boldsymbol{k},0)$ to $\hat{\boldsymbol{j}}^{(l)}(\omega,\boldsymbol{k},\boldsymbol{w})$ with arbitrary $\boldsymbol{w} \neq 0$ is carried out with the help of group of transformations, defined by the appropriate RG symmetry operator.

Since the procedure of construction of the multi-index nonlinear dielectric permittivity tensor in hot plasma from the appropriate expressions in cold plasma is identical for the permittivity tensor of any order we illustrate it by using linear with respect to a self-consistent electric field $\boldsymbol{E}$ material relations in non-relativistic plasma. In cold plasma Fourier-components of the partial current $\hat{\boldsymbol{j}}^{(1)}(\omega, \boldsymbol{k}, 0)$ and charge $\hat{\varrho}^{(1)}(\omega, \boldsymbol{k}, 0)$ densities, which are linear in the field $\tilde{\boldsymbol{E}}(\omega, \boldsymbol{k})$, are obtained by linearizing (4.7.29), (4.7.30) on a background of the homogeneous and equilibrium electron density n_{e0} and are determined by well-known relations

$$\hat{\boldsymbol{j}}^{(1)}(\omega, \boldsymbol{k}, 0) = i\frac{e^2 n_{e0}}{m\omega}\tilde{\boldsymbol{E}}; \quad \hat{\varrho}^{(1)}(\omega, \boldsymbol{k}, 0) = i\frac{e^2 n_{e0}}{m\omega^2}(\boldsymbol{k}\cdot\tilde{\boldsymbol{E}}). \tag{4.7.32}$$

The use of the latter in (4.7.25) gives a scalar dielectric permittivity for cold homogeneous non-relativistic plasma,

$$\varepsilon(\omega, \boldsymbol{k}) = 1 - \frac{4\pi e^2 n_e}{m\omega^2}. \tag{4.7.33}$$

Expressions (4.7.32) define zero-order terms in expansion of the partial current density $\hat{\boldsymbol{j}}^{(l)}(\omega, \boldsymbol{k}, \boldsymbol{w})$ in powers of plasma particles velocity $\boldsymbol{w}$. For obtaining the next terms of this series one should use the kinetic description of plasma. Here it appears more convenient to use instead of Vlasov equations (4.1.1) with the Euler velocity $\boldsymbol{v}$ the non-relativistic hydrodynamic analogue (4.1.5) of Vlasov equations with Lagrangian velocity $\boldsymbol{w}$ and the equilibrium distribution function $f_0(\boldsymbol{w})$. Such (Lagrangian) formulation of the kinetic description of plasma results from a non-relativistic limit of (4.1.5), and coincides in the form with (4.7.29), with that, however, an essential difference, that as against (4.7.29) the density $N(t, \boldsymbol{r}, \boldsymbol{w})$ and the velocity $\boldsymbol{V}(t, \boldsymbol{r}, \boldsymbol{w})$ now depend upon Lagrangian velocity as well and in the homogeneous non-perturbed plasma state obey the "initial" conditions at $t = t_0 = -\infty$

$$\begin{gathered} N(t_0, \boldsymbol{r}, \boldsymbol{w}) = n_{e0} f_0(\boldsymbol{w}), \quad \boldsymbol{V}(t_0, \boldsymbol{r}, \boldsymbol{w}) = \boldsymbol{w}; \\ \boldsymbol{E}(t_0, \boldsymbol{r}) = \boldsymbol{B}(t_0, \boldsymbol{r}) = 0, \quad \int f_0 \, \mathrm{d}\boldsymbol{w} = 1. \end{gathered} \tag{4.7.34}$$

In a non-relativistic limit material relations (4.1.6) also become simpler (we use different normalization for the distribution function here, hence material relations do not contain mass multipliers)

$$\rho(t, \boldsymbol{r}) = e\int N \, \mathrm{d}\boldsymbol{w}, \quad \boldsymbol{j}(t, \boldsymbol{r}) = e\int N\boldsymbol{V} \, \mathrm{d}\boldsymbol{w}. \tag{4.7.35}$$

Linearizing the equations of plasma kinetics in Lagrangian variables on the background of the basic state (4.7.34) results to the following formulas for corrections to the partial current density for small values of $\boldsymbol{w}$:

$$\begin{aligned} &\hat{\boldsymbol{j}}^{(1)}(\omega, \boldsymbol{k}, \boldsymbol{w}) \\ &\quad = i\frac{e^2 n_{e0}}{m\omega}\left\{\tilde{\boldsymbol{E}} + \frac{1}{\omega}\big(\boldsymbol{w}(\boldsymbol{k}\cdot\tilde{\boldsymbol{E}}) + \boldsymbol{k}(\boldsymbol{w}\cdot\tilde{\boldsymbol{E}})\big)\right\} + O(\boldsymbol{w}^2). \end{aligned} \tag{4.7.36}$$

To prolong this formula on any nonzero values of $\boldsymbol{w}$ we employ the RG symmetry operator which is constructed from the Lie group of point transformations (4.5.4), admitted by plasma kinetic equations. Two operators of the admitted group are of interest for us, namely, the operator of translations in Lagrangian velocity, which results from the operator X_∞, and the operator of Galilean transformations, which is a non-relativistic analogue of the operator of Lorentz transformations **B** in the set (4.5.4),

$$\boldsymbol{Z_1} = \frac{\partial}{\partial \boldsymbol{w}}, \quad \boldsymbol{Z_2} = t\frac{\partial}{\partial \boldsymbol{r}} + \frac{\partial}{\partial \boldsymbol{V}} - \frac{1}{c}\left[\boldsymbol{B} \times \frac{\partial}{\partial \boldsymbol{E}}\right] + \rho\frac{\partial}{\partial \boldsymbol{j}}. \tag{4.7.37}$$

Let us proceed in the operator $\boldsymbol{Z_2}$ from the velocity $\boldsymbol{V}$ and the density N to the partial current and charge densities, $\hat{\boldsymbol{j}}$ and $\hat{\varrho}$, prolong the operator obtained on Fourier variables and combine it with the operator of translations $\boldsymbol{Z_1}$. As a result we get the operator that leaves the partial current density (4.7.36) invariant at $\boldsymbol{w} \to 0$, i.e. the required RG symmetry operator

$$\boldsymbol{R} = \boldsymbol{k}\frac{\partial}{\partial \omega} + \frac{\partial}{\partial \boldsymbol{w}} - \frac{1}{c}\left[\tilde{\boldsymbol{B}} \times \frac{\partial}{\partial \tilde{\boldsymbol{E}}}\right] + \hat{\varrho}\frac{\partial}{\partial \hat{\boldsymbol{j}}}. \tag{4.7.38}$$

The operator (4.7.38) is related to a three-parameter group with the vector parameter $\boldsymbol{w}$, and its final transformations (the variables with primes here correspond to transformed variables)

$$\begin{gathered}\omega' = \omega + \boldsymbol{k}\boldsymbol{w}; \quad (\beta'_{is}/\omega')\tilde{E}'_s = (1/\omega)\tilde{E}_i; \quad \hat{\varrho}' = \hat{\varrho}; \quad \hat{j}'_i = \beta'_{si}\hat{j}_s; \\ \boldsymbol{k}' = \boldsymbol{k}; \quad \tilde{\boldsymbol{B}}' = \tilde{\boldsymbol{B}} = (c/\omega)[\boldsymbol{k} \times \tilde{\boldsymbol{E}}]; \quad \beta_{is} = \delta_{is} + k_i w_s/(\omega - \boldsymbol{k}\boldsymbol{w}),\end{gathered} \tag{4.7.39}$$

give the required relationship between the value of the partial current density $\hat{\boldsymbol{j}}(\omega, \boldsymbol{k}, 0)$ at $\boldsymbol{w} = 0$ (in cold plasma) and the analogous value of the partial current density $\hat{\boldsymbol{j}}(\omega, \boldsymbol{k}, \boldsymbol{w})$ with any $\boldsymbol{w} \neq 0$. When integrating over velocity $\boldsymbol{w}$ with the "weight" $f_0(\boldsymbol{w})$, following (4.7.31), we get an expression for a current density of the given order in hot plasma which defines the appropriate multi-index nonlinear dielectric permittivity tensor of plasma.

Example 4.7.3 In particular, in the linear in the electric field approximation the use of (4.7.32) leads to the relationship

$$\hat{j}_i^{(1)}(\omega, \boldsymbol{k}, \boldsymbol{w}) = \frac{ie^2 n_{e0}}{m\omega}\beta_{si}\beta_{sa}\tilde{E}_a(\omega, \boldsymbol{k}). \tag{4.7.40}$$

Substitution of (4.7.40) into (4.7.31) and the further use of $\tilde{j}_i^{(1)}(\omega, \boldsymbol{k})$ in (4.7.25) gives the required expression for the tensor of the linear dielectric permittivity for hot homogeneous non-relativistic plasma in the absence of external fields with the equilibrium distribution function $f_0(\boldsymbol{w})$

$$\varepsilon_{ab}(\omega, \boldsymbol{k}) = \delta_{ab} - \frac{4\pi e^2 n_{e0}}{m\omega^2}\int f_0(\boldsymbol{w})\beta_{sa}\beta_{sb}\,\mathrm{d}\boldsymbol{w}. \tag{4.7.41}$$

Formula (4.7.41), which arises from the scalar equality (4.7.33) as a result of application of RG transformations to partial current density in cold plasma with the subsequent integration over the group parameter, illustrates an opportunity of obtaining a tensor of dielectric permittivity of hot plasma from the appropriate "cold" expression [27].

Example 4.7.4 RG symmetry generator (4.7.38) results from symmetry operators admitted by the plasma kinetic equations after their subsequent prolongation on solution functionals, partial current and a charge densities in Fourier representation. Thus a linear in the electromagnetic field approximation used above is not an essential restriction, as relations between transformed (primed) and non-transformed partial current and a charge density remains linear under group transformations (4.7.39). It means, that it is also possible to apply transformations (4.7.39) to partial current and a charge densities of any order l, i.e. the offered RG scheme allows to build nonlinear dielectric permittivity tensors of any order in hot plasma proceeding from the appropriate "cold" expressions for the nonlinear dielectric permittivity. Omitting intermediate calculations, we present a result of such construction

$$\begin{aligned}\varepsilon_{ij_1\dots j_n}(\omega_1,\boldsymbol{k}_1;\dots;\omega_n,\boldsymbol{k}_n) &= \int f_0(\boldsymbol{w})\bar{\varepsilon}_{ab_1\dots b_n}(\varOmega_1,\boldsymbol{k}_1;\dots;\varOmega_n,\boldsymbol{k}_n)\\ &\times\frac{\varOmega\varOmega_1\dots\varOmega_n}{\omega\omega_1\dots\omega_n}\beta_{ai}(\omega,\boldsymbol{k})\beta_{b_1j_1}(\omega_1,\boldsymbol{k}_1)\dots\beta_{b_nj_n}(\omega_n,\boldsymbol{k}_n)\,\mathrm{d}\boldsymbol{w};\\ &n\ge 2;\\ \omega=\omega_1+\dots+\omega_n;&\ \ \boldsymbol{k}=\boldsymbol{k}_1+\dots+\boldsymbol{k}_n;\\ \varOmega\equiv(\omega-\boldsymbol{k}\boldsymbol{w}),&\ \ \varOmega_i\equiv(\omega_i-\boldsymbol{k}_i\boldsymbol{w}),\ \ i=1,\dots,n.\end{aligned} \tag{4.7.42}$$

Here $\bar{\varepsilon}$ corresponds to the nonlinear dielectric permittivity tensor in cold collisionless plasma without external fields. For example, for the nonlinearity of the second order it is determined by the formula

$$\begin{aligned}&\bar{\varepsilon}_{isj}(\varOmega_1,\boldsymbol{k}_1;\varOmega_2,\boldsymbol{k}_2)\\ &\quad=-\frac{4\pi i e^3 n_{e0}}{2!m^2\varOmega\varOmega_1\varOmega_2}\left(\frac{k_i}{\varOmega}\delta_{js}+\frac{k_{1s}}{\varOmega_1}\delta_{ij}+\frac{k_{2j}}{\varOmega_2}\delta_{is}\right).\end{aligned} \tag{4.7.43}$$

The similar result can be obtained and for relativistic plasma, however thus it is necessary to use not the three-parameter group of Galilean transformations, but the six-parameter group including Lorentz transformations and rotations.

4.7.4.2 Adiabatic Expansion of Plasma Bunches

Here RG algorithm is applied to the problem of expansion of plasma bunches and related generation of the accelerated particles. The mechanisms and characteristics of ions triggered by the interaction of a short-laser-pulse with plasma are of current interest because of their possible applications to the novel-neutron-source development and isotope production. In the near future ultra-intense laser pulses will be used for ion beam generation with energies useful for proton therapy, fast ignition inertial confinement fusion, radiography, neutron-sources.

The commonly recognized effect responsible for ion acceleration is charge separation in the plasma due to high-energy electrons, driven by the laser inside the target. During the plasma expansion, the kinetic energy of the fast electrons transforms into the energy of electrostatic field, which accelerates ions and their energy is expected to be at the level of the hot-electron energy. The mathematical model describing this phenomenon is based on plasma kinetic equations with a self-consistent field (4.1.1)–(4.1.3), which is rather complicated for analytical treatment. However, to describe plasma flows with characteristic scale of density variation large compared to Debye length for plasma particles, the quasi-neutral approximation is used. In this approximation charge and current densities in plasma are set equal to zero, that essentially simplifies the initial model with nonlocal terms. Instead of the system of Vlasov–Maxwell equations (4.1.1), (4.1.3) with the corresponding material equations here we use only the kinetic equations for particle distribution functions for various species (4.3.72) with additional nonlocal restrictions imposed on them, which arise from vanishing conditions for the current and the charge densities (4.3.73). Initial conditions for solutions of (4.3.72) and (4.3.73) correspond to distribution functions for electrons and ions, specified at $t = 0$

$$f^{\alpha}\big|_{t=0} = f_0^{\alpha}(x, v). \tag{4.7.44}$$

Equations (4.3.72), (4.3.73) describe one-dimensional dynamics of a plasma bunch, which is inhomogeneous upon the coordinate x; thus the distribution functions of particles f^{α} depend upon t, x and the velocity component v in the directions of plasma inhomogeneity. Analytical study of such yet simplified model represents the essential difficulties, but due to application of RG algorithm it is possible not only to construct solution at various initial particle distribution functions but also to find the law of variation of particles density without calculations of distribution functions for particles in an explicit form [14, 32].

For construction of RG symmetries we consider (step **I**) a set of local (4.3.72) and nonlocal (4.3.73) equations as $\mathscr{RM}$, in which the electric field $E(t, x)$ appears as some arbitrary function to be found of its variables. Calculating the Lie group of point transformations admitted by this manifold (step **II**) is given by (4.3.75), and in particular contains the generator of time translations and the projective group generator. Precisely these operators enables to construct a class of exact solutions to the initial problem that are of interest, as a linear combination of the operator of time translations and the operator of the projective group leaves the approximate perturbation theory solution of the initial value problem $f^{\alpha} = f_0^{\alpha}(x, v) + O(t)$ invariant at $t \to 0$, i.e. it is the RG symmetry operator,

$$R = (1 + \Omega^2 t^2)\frac{\partial}{\partial t} + \Omega^2 t x \frac{\partial}{\partial x} + \Omega^2 (x - vt)\frac{\partial}{\partial v}, \tag{4.7.45}$$

which results from the group restriction procedure (step **III**), for spatially symmetric initial distribution functions with the zero average velocity. It is possible to treat the constant Ω as the ratio of a characteristic sound velocity c_s to initial inhomogeneity scale of the density of electrons, L_0.

Invariants of the RG generator (4.7.45) are two combinations, $x/\sqrt{1 + \Omega^2 t^2}$ and $v^2 + \Omega^2 (x - vt)^2$, and particle distribution functions f^{α}. Hence, solutions of initial value problem at any time $t \neq 0$ (step **IV**) are expressed via these invariants in terms of initial values (4.7.44),

$$f^{\alpha} = f_0^{\alpha}(I^{(\alpha)}),\ I^{(\alpha)} = \frac{1}{2}\big(v^2 + \Omega^2(x - vt)^2\big) + \frac{e_\alpha}{m_\alpha}\Phi_0(x'). \tag{4.7.46}$$

Here the dependence of Φ_0 upon the variable $x' = x/\sqrt{1+\Omega^2 t^2}$ is defined by quasi-neutral conditions (4.3.73), and the electric field $E = -\Phi_x$ is found with the help of the potential

$$\Phi(t, x) = \Phi_0(x')\big(1 + \Omega^2 t^2\big)^{-1}. \tag{4.7.47}$$

Formulas (4.7.46) give the solution to the initial value problem (4.3.72), (4.3.73). However, for practical applications we need frequently more rough characteristic of plasma dynamics, for example, a density of particles (ions) of the given species $n^q(t, x)$ which can be calculated using the appropriate distribution function:

$$n^q(t, x) = \int_{-\infty}^{\infty} f^q(t, x, v)\,\mathrm{d}v. \tag{4.7.48}$$

In view of the complex dependence upon the invariant $I^{(\alpha)}$ it is not always possible to carry out direct integration of a distribution function over velocity in the analytical form, therefore here the procedure of prolongation of the operator on solution functionals described in Sect. 4.2.1.4 comes to the aid. As the density $n^q(t, x)$ is a linear functional of f^q, the prolongation of the operator (4.7.45) on the functional of the solution (4.7.48) in the narrowed space of variables $\{t, x, n^q\}$ gives the following RG operator

$$R = (1 + \Omega^2 t^2)\frac{\partial}{\partial t} + \Omega^2 tx\frac{\partial}{\partial x} - \Omega^2 tn^q\frac{\partial}{\partial n^q}. \tag{4.7.49}$$

The solution of Lie equations for the operator R in view of initial conditions (4.7.44) gives relations between invariants of this operator, namely one of the combinations $J = x/\sqrt{1+\Omega^2 t^2}$ already given for the operator (4.7.45) and the product $J^q = n^q\sqrt{1+\Omega^2 t^2}$ for arbitrary $t \neq 0$ with their values at $t = 0$: $J_{|t=0} = x'$, $J^q{}_{|t=0} = \mathcal{N}_q(x')$. This relationship immediately leads to the formulas that characterize spatial-temporal distribution of the density of ions of a given species in terms of the initial density distribution

$$n^q = \frac{1}{\sqrt{1+\Omega^2 t^2}}\mathcal{N}_q\left(\frac{x}{\sqrt{1+\Omega^2 t^2}}\right), \qquad \mathcal{N}_q(x') = \int_{-\infty}^{\infty} f_0^q(I^{(q)})\,\mathrm{d}v. \tag{4.7.50}$$

Example 4.7.5 We illustrate general results with reference to expansion of a plasma slab, consisting of cold ($\alpha = c$) and hot ($\alpha = h$) electrons and of two ion species ($q = 1, 2$). Let initially (at $t = 0$) ions are characterized by Maxwellian distribution functions with densities $n_{10}, n_{20} \ll n_{10}$ and temperatures T_1, T_2, and the distribution function of electrons looks like two-temperature Maxwellian distribution with the appropriate densities n_{c0} and $n_{h0} \ll n_{c0}$ ($n_{c0} + n_{h0} = Z_1 n_{10} + Z_2 n_{20}$) and temperatures T_c and $T_h \gg T_c$ of hot and cold components. From the physical point of view such choice of initial conditions refer to an expansion of the target consisting of heavy ions with a small impurity of light ions adsorbed on a surface (for example, protons) which preliminary was heated quickly by a short pulse of laser radiation with formation of a group of hot electrons. Then the solution of the initial problem (4.7.46) is represented as:

$$f^e = \frac{n_{c0}}{\sqrt{2\pi}\, v_{Tc}} \exp\left(-\frac{I^{(c)}}{v_{Tc}^2}\right) + \frac{n_{h0}}{\sqrt{2\pi}\, v_{Th}} \exp\left(-\frac{I^{(h)}}{v_{Th}^2}\right),$$
$$f^q = \frac{n_{q0}}{\sqrt{2\pi}\, v_{Tq}} \exp\left(-\frac{I^{(q)}}{v_{Tq}^2}\right), \quad v_{T\alpha}^2 = \frac{T_\alpha}{m_\alpha}, \quad q = 1, 2, \tag{4.7.51}$$

where invariants $I^{(\alpha)}$ are given by relations:

$$\frac{I^{(c)}}{v_{Tc}^2} = \mathscr{E} + \frac{(1 + \Omega^2 t^2)}{2v_{Tc}^2}(v - u)^2, \quad \frac{I^{(h)}}{v_{T_h}^2} = \mathscr{E}\frac{T_c}{T_h} + \frac{(1 + \Omega^2 t^2)}{2v_{Th}^2}(v - u)^2,$$
$$\frac{I^{(q)}}{v_{Tq}^2} = -\mathscr{E}\left(\frac{Z_q T_{c0}}{T_{q0}}\right) + \frac{U^2}{2v_{T_q}^2}\left(1 + \frac{Z_q m_e}{m_q}\right) + \frac{(1 + \Omega^2 t^2)}{2v_{T_q}^2}(v - u)^2. \tag{4.7.52}$$

Here $u = xt\Omega^2/(1 + \Omega^2 t^2)$ is a local velocity of plasma particles, $U = x\Omega/\sqrt{1 + \Omega^2 t^2}$, and a potential Φ is expressed via the function $\mathscr{E}$,

$$\mathscr{E} = \frac{e\Phi}{T_c}(1 + \Omega^2 t^2) + \frac{U^2}{2v_{Tc}^2}, \tag{4.7.53}$$

that is obtained from the transcendental equation,

$$n_{c0} = \sum_{q=1,2} Z_q n_{q0} \exp\left[\left(1 + \frac{Z_q T_c}{T_q}\right)\mathscr{E} - \frac{U^2}{2v_{Tq}^2}\left(1 + \frac{Z_q m_e}{m_q}\right)\right]$$
$$- n_{h0} \exp\left[\left(1 - \frac{T_c}{T_h}\right)\mathscr{E}\right]. \tag{4.7.54}$$

Formulas (4.7.51)–(4.7.54) completely define the behavior of distribution functions of all particle species considered in the given example when studying the expansion of a plasma slab. At that the space-temporal distribution of the ion density of the given species is determined by formulas (4.7.50), in which the ion density $\mathscr{N}_q$ for the initial distribution functions specified above has the form

$$\mathscr{N}_q = n_{q0} \exp\left[\mathscr{E}\left(\frac{Z_q T_{c0}}{T_{q0}}\right) - \frac{U^2}{2v_{T_q}^2}\left(1 + \frac{Z_q m_e}{m_q}\right)\right], \quad q = 1, 2, \tag{4.7.55}$$

where the relation between the function $\mathscr{E}$ with the variable U still is from (4.7.54).

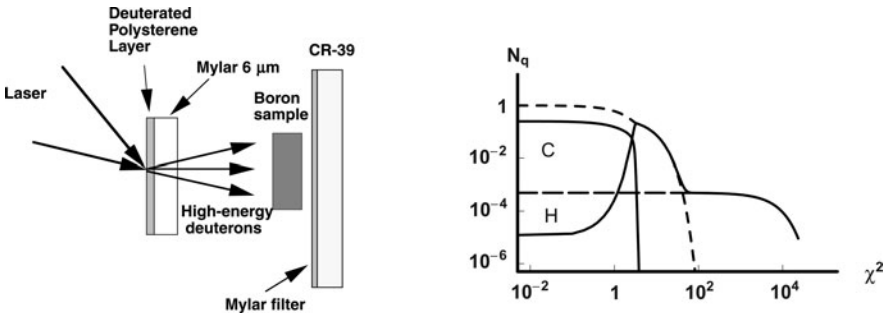

Fig. 4.2 *Left panel*: typical experimental setup for registration of fast ions from the foil under ultra short laser pulses (from Ref. [37]). *Right panel*: the "universal" density N_q of plasma ions — carbon ions (curves (C)) and protons (curves (H)) — versus the dimensionless "coordinate" $\chi^2 = (x/L_0)^2/(1+\Omega^2 t^2)$. *Dotted curves* with *short* and *long strokes* show the dependencies of a dimensionless density for hot and cold electrons

On Fig. 4.2 we illustrate the typical "density" distribution (4.7.55) for a plasma slab, consisting of cold and hot electrons and two ions species: carbon ions C^{+4} $(q=1)$ and protons H^{+1} $(q=2)$. Block curves show dependence of a dimensionless "universal" density of plasma ions $N_q = (n_{q0}/n_{c0})\mathcal{N}_q$, referred to the maximal density of cold electrons, upon the dimensionless "coordinate" $\chi^2 = (x/L_0)^2/(1+\Omega^2 t^2)$, referred to the characteristic initial density scale of ions L_0. "Universality" of this dependencies is the direct consequence of a relation which exists between invariants of the RG operator (4.7.49). Dotted curves give the distribution of the dimensionless density of cold electrons (short strokes), $(n_c/n_{c0})\sqrt{1+\Omega^2 t^2}$ and hot electrons (long strokes), $(n_h/n_{c0})\sqrt{1+\Omega^2 t^2}$, respectively.

Similar results are obtained for more complex distribution functions [14] and beyond the scope of the model used for the one-dimensional expansion, for example for spherically-symmetric expansion of a plasma bunch [38].

4.7.4.3 Coulomb Explosion of a Cluster in Ultra-short Laser Pulses

In this section we apply RG symmetry to the model that is used in a plasma kinetic theory for describing the Coulomb explosion of sub-micron plasmas in the field of multi-terrawatt femto-second laser pulses. Recent developments in this field have enabled examination of the fundamental physics of Coulomb explosion of nanoscale targets and ion acceleration at multi-MeV energies in different geometries of laser-plasma interaction experiments [39–41]. The mechanisms and characteristics of ions triggered by the interaction of a short-laser-pulse with plasma are of current interest because of their possible applications to the novel-neutron-source development, x-ray source, proton radiography, and isotope production.

The macroscopic state of cluster particles is governed by distribution functions f (for cluster ions with mass M and charge Ze), that dependents on time t, a coordinate x of a particle, and its velocity v (for simplicity we consider the one-dimensional plane geometry). Evolution of distribution functions is described by the solution to the Cauchy problem to the Vlasov kinetic equation with the corresponding initial condition $f|_{t=0} = f_0(x, v)$, supplemented by the Poisson equation for the electric field E (similar to (4.3.1)),

$$f_t + v f_x + (Ze/M) E f_v = 0, \quad E_x - 4\pi Ze \int \mathrm{d}v f = 0, \quad f\big|_{t=0} = f_0(x, v). \tag{4.7.56}$$

Analytical study of such yet simplified model represents the essential difficulties, but due to application of RG algorithm it is possible to obtain solution at various initial particle distribution functions and find particles density, mean velocity and energy spectra. To construct RG symmetries we consider a set of local and non-local equations in (4.7.56) and the evident constraint $E_v = 0$ as $\mathscr{RM}$. The Lie group of point transformations admitted by this manifold consists of six generators

$$\begin{gathered} X_0 = \frac{\partial}{\partial t}; \quad X_1 = \frac{\partial}{\partial x}; \quad X_2 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial v}; \\ X_3 = x\frac{\partial}{\partial x} + v\frac{\partial}{\partial v} - f\frac{\partial}{\partial f} + E\frac{\partial}{\partial E}; \\ X_4 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - v\frac{\partial}{\partial v} - 3f\frac{\partial}{\partial f} - 2E\frac{\partial}{\partial E}; \\ X_5 = (t^2/2)\frac{\partial}{\partial x} + t\frac{\partial}{\partial v} + (M/Ze)\frac{\partial}{\partial E}, \end{gathered} \tag{4.7.57}$$

describing time and space translations, X_0 and X_1, Galilean boosts, X_2, dilations, X_3 and X_4, and the generator X_5. Finite transformations defined by X_5 correspond to passing into a coordinate system moving linearly with constant acceleration with respect to the laboratory coordinate system. Two commutating generators in the above list (4.7.57), namely generator of Galilean boosts and generator of the transition to a uniformly accelerated frame, appear as the required RG symmetry generators [31],

$$R_1 = (t^2/2)\frac{\partial}{\partial x} + t\frac{\partial}{\partial v} + (M/Ze)\frac{\partial}{\partial E}, \quad R_2 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial v}. \tag{4.7.58}$$

Successive application of finite transformations defined by theses generators shifts initial coordinates h and velocities ν for any particle in the phase space to new values,

$$R(t, h, \nu) = h + \nu t + (Ze/2M) E(h) t^2, \quad U(t, h, \nu) = \nu + (Ze/M) E(h) t, \tag{4.7.59}$$

and the function $E(h)$ is defined by initial conditions (we assume the electric field to vanish at $x = 0$)

$$E(h) = 4\pi Ze \int_0^h \mathrm{d}y \int_{-\infty}^{\infty} \mathrm{d}v f_0(y, v). \tag{4.7.60}$$

The "partial" distribution function, specified by values h and ν, is the invariant of RG symmetry generators (4.7.58). Hence, the distribution function which is the solution to (4.7.56) is obtained by integrating this "partial" distribution function over all initial parameters, i.e. initial velocities and coordinates of plasma particles,

$$f(t,x,v) = \int_{-\infty}^{\infty} \mathrm{d}\nu \int_{-\infty}^{\infty} \mathrm{d}h f_0(\nu,h)\delta(x - R(t,h,\nu))\delta(v - U(t,h,\nu)). \tag{4.7.61}$$

For "cold" cluster particles, $f_0 \propto \delta(\nu)$, we need only one RG generator, R_1, to construct the solution of a boundary value problem. The zero and the first moments of the distribution function yield the density and the mean velocity distributions of the cluster ions, which enable to estimate the maximum energy of the accelerated ions, the ion energy spectrum and the relation between this spectrum and the initial ion density distribution [39, 40]. The similar approach to the spherical geometry [41] shows that the inhomogeneity of the initial cluster density distribution leads to the solution singularity at finite time interval even for initially immovable ions.

4.7.4.4 Renormgroup Algorithm Using Functionals

We consider some boundary value problem for local equations and assume that we are interested in an integral characteristic of the solution, given by a linear functional of this solution $J(u)$, say by (4.7.20). We also assume that for a particular solution u of this boundary value problem, the RG algorithm has been used to find an RG symmetry with a generator R. To find RG symmetry generator for the functional $J(u)$, we prolong the RG symmetry operator R on nonlocal variable (4.7.20) in much the same way as in Sect. 4.2.4. Considering the prolonged operator R in the narrowed space of the variables defining the solution functional, we obtain the required infinitesimal RG symmetry operator for integral characteristic $J(u)$.

To demonstrate how formulas (4.7.20) and (4.2.22) actually work for functionals of solutions we consider a boundary value problems for a system of two nonlinear first-order partial differential equations for functions v and $n > 0$:

$$\begin{gathered} v_t + vv_x = \alpha\varphi(n)n_x, \quad n_t + vn_x + nv_x = 0, \\ v(0,x) = \alpha W(x), \quad n(0,x) = N(x), \end{gathered} \tag{4.7.62}$$

with constant α and a nonlinearity function φ of the variable n. Depending on the sign of $\alpha\varphi(n)$, these equations are of either the hyperbolic ($\alpha\varphi(n) < 0$) or the elliptic ($\alpha\varphi(n) > 0$) type. In the first case, (4.7.62) corresponds to the standard equations of gas dynamics for one-dimensional planar isentropic motion of gas with the density n and velocity v. The second case relates to equations of quasi-Chaplygin media.[8]

[8]The term 'quasi-Chaplygin media' is used in the discussion of nonlinear phenomena developing in accordance with the mathematical scenario for the Chaplygin gas, i.e., the gas with a negative

To calculate the RG symmetries for (4.7.62) it appears convenient to rewrite these equations in the hodograph variables $\tau = nt$ and $\chi = x - vt$,

$$\tau_v - \psi(n)\chi_n = 0, \quad \chi_v + \tau_n = 0, \quad \psi = n/\alpha\varphi. \tag{4.7.63}$$

Then the RG symmetry is given by the canonical Lie–Bäcklund operator [42]

$$R = f\frac{\partial}{\partial \tau} + g\frac{\partial}{\partial \chi}, \tag{4.7.64}$$

with coordinates f and g that are linear functions of variables τ and χ and their derivatives with respect to n up to a fixed order s. Following the RG algorithm one should add the invariance conditions $f = 0$, and $g = 0$, to the basic manifold (4.7.63) and solve the resulting system of equations to get the solution to the boundary value problem (4.7.62). This procedure may appear complicated in the case of cumbersome formulas for coordinates f and g of RG symmetry generator (4.7.64).

In analyzing (4.7.62) for various physical problems such as a light beam behavior in a nonlinear medium the appearance of a solution singularity on the axis $x = 0$ represents the most attracting physical effect. This effect can be understood without knowledge of a complete solution by applying the RG algorithm to a functional of the solution, $n^0(t) \equiv n(t, 0)$, the value of the variable n on the axis $x = 0$. As the RG symmetry generator (4.7.64) is defined in the space of hodograph variables it is convenient to use another functional of the solution introduced by a formal relationship

$$\tau^0 = \int \delta(v)\tau(v, n)\,\mathrm{d}v. \tag{4.7.65}$$

Using (4.7.65) in (4.2.22) gives the coordinate f^0 of the canonical RG generator for the functional τ^0. Here we present two simple illustrations.

Example 4.7.6 Consider a solution of the boundary value problem for (4.7.62) with $\alpha = 1$, $\varphi(n) = 1$ for $W(x) = 0$ and $N(x) = \cosh^{-2}(x)$. The RG symmetry generator for this boundary value problem is defined by (4.7.64) in which coordinates f and g are given as

$$\begin{aligned} f &= 2n(1-n)\tau_{nn} - n\tau_n - 2nv(\chi_n + n\chi_{nn}) + nv^2\tau_{nn}/2, \\ g &= 2n(1-n)\chi_{nn} + (2-3n)\chi_n + v\,(2n\tau_{nn} + \tau_n) + (v^2/2)\,(n\chi_{nn} + \chi_n)\,. \end{aligned} \tag{4.7.66}$$

For RG symmetry (4.7.66), a solution exists on a finite interval $0 \le t \le t_{sing}$, until a singularity occurs on the axis $x = 0$ at $t = t_{sing} = 1/2$, when $v_x(t_{sing}, 0) \to \infty$ and the value of n remains finite, $n(t_{sing}, 0) = 2$:

$$v = -2nt\tanh(x - vt), \quad n^2t^2 = n\cosh^2(x - vt) - 1. \tag{4.7.67}$$

---

adiabatic exponent. At first glance, such a model looks like the standard model of gas dynamics, but it corresponds to the negative first derivative of the 'pressure' with respect to the 'density.' A characteristic feature of quasi-Chaplygin media is a universal mathematical form of various nonlinear effects accompanying the development of an instability.

From the physical standpoint, solution (4.7.67), which was previously obtained in [43], describes the evolution of a planar light beam in a medium with a cubic nonlinearity (a quasi-Chaplygin medium) for the boundary condition $N(x) = \cosh^{-2}(x)$. The quantities n and v define the intensity and the eikonal derivative of the beam.

Prolongation of the RG symmetry generator (4.7.64), (4.7.66) on functional (4.7.65) gives the generator in the space $\{n, \tau^0\}$

$$R = f^0 \frac{\partial}{\partial \tau^0}, \tag{4.7.68}$$

with the coordinate

$$f^0 = 2n(1-n)\tau_{nn} - n\tau_n. \tag{4.7.69}$$

The RG invariance condition $f^0 = 0$ for operator (4.7.68) leads to an ordinary second-order differential equation for the function $\tau^0(n)$, which must be solved with initial conditions $\tau^0(1) = 0$, and $\tau_n^0 \sqrt{n-1}|_{n\to 1} = 1/2$ that follows from the original equations (4.7.63) for $v = 0$. This solution,

$$\tau^0 = \sqrt{n-1}, \tag{4.7.70}$$

results from (4.7.67) as well, but the method is simpler and solution (4.7.67) is not explicitly required.

Example 4.7.7 Turn now to a solution of the boundary value problem for (4.7.62) with $\alpha = -1$, $\varphi(n) = 1/n$ for $W(x) = 0$ and $N(x) = \exp(-x^2)$. The RG symmetry generator for this boundary value problem is defined by (4.7.64) with the following coordinates f and g

$$\begin{aligned} f &= -n^2 \ln n \tau_{nn} - (n/2)\tau_n + \tau/2 + v(n^3\chi_{nn} + (3/2)n^2\chi_n), \\ g &= -n^2 \ln n \chi_{nn} + (n/2)(1 + 4\ln n)\chi_n + \chi/2 + v\,(n\tau_{nn} + \tau_n/2)\,. \end{aligned} \tag{4.7.71}$$

For RG symmetry (4.7.71), the solution describes a monotonic evolution (decrease) with time t of the density $n \geq 0$, while the particle velocity continues to be linearly dependent on the coordinate:

$$\begin{gathered} v = x\sqrt{2}q\mathrm{e}^{-q^2/2}, \quad n = \mathrm{e}^{-q^2/2}\exp\left(-x^2\mathrm{e}^{-q^2}\right), \\ t = (\sqrt{\pi}/2)\mathrm{erfi}\left(q/\sqrt{2}\right). \end{gathered} \tag{4.7.72}$$

Solution (4.7.72), which was discussed in [44], describes an expanding plasma layer with the initial density distribution $N(x) = \exp(-x^2)$.

Prolongation of the RG symmetry generator (4.7.64), (4.7.71) on functional (4.7.65) gives the generator (4.7.68) in the space $\{n, \tau^0\}$ though with a different coordinate

$$f^0 = -n^2 \ln n \tau_{nn} - (n/2)\tau_n + \tau/2. \tag{4.7.73}$$

On account of (4.7.73) the RG invariance condition $f^0 = 0$ for operator (4.7.68) leads to an ordinary second-order differential equation for the function $\tau^0(n)$, which must be solved with initial conditions $\tau^0(1) = 0$, and $\tau^0_n \sqrt{1-n}\big|_{n\to 1} = -1/2$ that follows from the original equations (4.7.63) for $v = 0$. This solution,

$$\tau^0 = \frac{\sqrt{\pi}}{2} n \operatorname{erfi}\left(\sqrt{\ln \frac{1}{n}}\right), \tag{4.7.74}$$

correlates with (4.7.72) for $v = 0$.

In conclusion we notice that expressions (4.7.70) and (4.7.74) result from the complete solutions as well. However, the RG algorithm for functionals presents here an elegant way of obtaining these formulas without calculating the complete solutions to boundary value problems.

References

1. Vlasov, A.A.: The vibrational properties of an electron gas. J. Exp. Theor. Phys. **8**(3) (1938) 291–317 (in Russian); see also Sov. Phys. Usp. **10**, 721–733 (1968)
2. Lewak, G.J.: More-uniform perturbation theory of the Vlasov equation. J. Plasma Phys. **3**, 243–253 (1969)
3. Pustovalov, V.V., Chernikov, A.A.: Functional averaging and kinetics of plasma in Lagrangean variables. Preprint No. 171, P.N. Lebedev Physical Institute, AN USSR (1980) (in Russian)
4. Pustovalov, V.V., Romanov, A.B., Savchenko, M.A., Silin, V.P., Chernikov, A.A.: One method for solving the Vlasov kinetic equation. Sov. Phys., Lebedev Inst. Rep. **12**, 28–32 (1976)
5. Taranov, V.B.: On the symmetry of one-dimensional high frequency motions of a collisionless plasma. Sov. J. Tech. Phys. **21**, 720–726 (1976)
6. Kovalev, V.F., Krivenko, S.V., Pustovalov, V.V.: Group symmetry of the kinetic equations of a collisionless plasma. JETP Lett. **55**(4), 256–259 (1992)
7. Kovalev, V.F., Krivenko, S.V., Pustovalov, V.V.: Group analysis of the Vlasov kinetic equation, I. Differ. Equ. **29**(10), 1568–1578 (1993)
8. Kovalev, V.F., Krivenko, S.V., Pustovalov, V.V.: Group analysis of the Vlasov kinetic equation, II. Differ. Equ. **29**(11), 1712–1721 (1993)
9. Grigor'ev, Yu.N., Meleshko, S.V.: Group analysis of integro-differential Boltzmann equation. Sov. Phys. Dokl. **32**, 874–876 (1987)
10. Volterra, V.: Theory of Functional and of Integral and Integro-Differential Equations. Blackie, London (1929). Edited by Fantappie, L. Translated by Long, M. Also available as: Volterra, V.: Theory of Functionals and of Integral and Integro-Differential Equations. Dover, New York (1959). Russian translation: Nauka, Moscow (1982)
11. Baikov, V.A., Gazizov, R.K., Ibragimov, N.H.: Perturbation methods in group analysis. J. Sov. Math. **55**(1), 1450 (1991)
12. Baikov, V.A., Gazizov, R.K., Ibragimov, N.Kh.: Perturbation methods in group analysis. In: Itogi Nauki i Tekhniki. Ser. Sovrem. Probl. Mat. Nov. Dostizh., vol. 34, pp. 85–147. VINITI, Moscow (1989) (in Russian). J. Sov. Math. **55**(1), 1450–1490 (1991)
13. Dorozhkina, D.S., Semenov, V.E.: Exact solution of Vlasov equations for quasineutral expansion of plasma bunch into vacuum. Phys. Rev. Lett. **81**, 2691–2694 (1998)
14. Kovalev, V.F., Bychenkov, V.Yu., Tikhonchuk, V.T.: Particle dynamics during adiabatic expansion of a plasma bunch. JETP **95**(2), 226–241 (2002)
15. Landau, L.D., Livshitz, E.M.: Course of Theoretical Physics, vol. 2, The Classical Theory of Fields. Nauka, Moscow (1973)
16. Kovalev, V.F., Krivenko, S.V., Pustovalov, V.V.: Symmetry group of Vlasov–Maxwell equations in plasma theory. In: Proceedings of the International Conference "Symmetry in Nonlinear Mathematical Physics", July 3–8, 1995, Kiev, Ukraina, V. 2. J. Nonlinear Math. Phys. **3**(1–2), 175–180 (1996).

17. Ibragimov, N.H. (ed.): CRC Handbook of Lie Group Analysis of Differential Equations, vol. 1: Symmetries, Exact Solutions and Conservation Laws (1994); vol. 2: Applications in Engineering and Physical Sciences (1995); vol. 3: New Trends in Theoretical Developments and Computational Methods (1996). CRC Press, Boca Raton
18. Benney, D.J.: Some properties of long nonlinear waves. Stud. Appl. Math. **L11**(1), 45–50 (1973)
19. Krasnoslobodtzev, A.V.: Gas dynamic and kinetic analogies in the theory of vertically inhomogeneous shallow water. Trans. Inst. Gen. Phys. USSR Acad. Sci. **18**, 33–71 (1989) (in Russian)
20. Kupershmidt, B.A., Manin, Yu.I.: Long-wave equation with free boundary. I. Conservation laws and solutions. Funct. Anal. Appl. **11**(3), 188–197 (1977)
21. Kupershmidt, B.A., Manin, Yu.I.: Long-wave equation with free boundary. II. Hamiltonian structure and higher equations. Funct. Anal. Appl. **12**(1), 20–29 (1978)
22. Zakharov, V.E.: Benney equation and quasi-classical approximation in the method of the inverse problem. Funct. Anal. Appl. **14**(2), 89–98 (1980)
23. Gibbons, J.: Collisionless Boltzmann equations and integrable moment equations. Physica D3 **3**(3), 503–511 (1981)
24. Ibragimov, N.H., Kovalev, V.F., Pustovalov, V.V.: Symmetries of integro-differential equations: a survey of methods illustrated by the Benney equations. Nonlinear Dyn. **28**, 135–153 (2002). Preprint math-ph/0109012
25. Gurevich, A.V., Pitaevski, L.P.: Nonlinear dynamics of a rarefied plasmas and ionospheric aerodynamics. In: Problems of Plasma Theory, vol. 10, pp. 3–87. Nauka, Moscow (1980) (in Russian). Reviews of Plasma Physics, vol. 10. Edited by Acad. Leontovich, M.A. Translated from Russian by Glebov, O. Translation editor: ter Haar, D., Department of Theoretical Physics, University of Oxford, Oxford, England. Published by Consultants Bureau, New York (1986)
26. Kovalev, V.F., Pustovalov, V.V.: Functional self-similarity in a problem of plasma theory with electron nonlinearity. Theor. Math. Phys. **81**, 1060–1071 (1990)
27. Shirkov, D.V.: Several topics on renorm-group theory. In: Shirkov, D.V., Priezzhev, V.B. (eds.) Renormalization Group '91, Proc. of Second Intern. Conf., Sept. 1991, Dubna, USSR, pp. 1–10. World Scientific, Singapore (1992)
Kovalev, V.F., Krivenko, S.V., Pustovalov, V.V.: The Renormalization group, method based on group analysis. In: Shirkov, D.V., Priezzhev, V.B. (eds.) Renormalization Group '91, Proc. of Second Intern. Conf., Sept. 1991, Dubna, USSR, pp. 300–314. World Scientific, Singapore (1992)
28. Kovalev, V.F., Pustovalov, V.V., Shirkov, D.V.: Group analysis and renormgroup symmetries. J. Math. Phys. **39**, 1170–1188 (1998). Preprint hep-th/9706056
29. Kovalev, V.F., Shirkov, D.V.: Bogoliubov renormalization group and symmetry of solution in mathematical physics. Phys. Rep. **352**(4–6), 219 (2001). hep-th/0001210
30. Kovalev, V.F., Shirkov, D.V.: The renormalization group symmetry for solution of integral equations. In: Nikitin, A.G. (ed.) Proc. of the 5th Intern. Conf. on Symmetry in Nonlinear Mathematical Physics, Kii'v, Ukraine, June 23–29, 2003. Proc. of the Inst. of Math. of the Natl. Acad. Sci. of Ukraine. Math. and its Appl., vol. 50, Pt. 2, pp. 850–861. Inst. of Math. of NAS Ukraine, Kiïv (2004)
31. Kovalev, V.F., Shirkov, D.V.: Renormgroup symmetry for functionals of boundary value problem solutions. J. Phys. A, Math. Gen. **39**, 8061–8073 (2006)
32. Kovalev, V.F., Shirkov, D.V.: Renormalization-group symmetries for solutions of nonlinear boundary value problems. Phys.-Usp. **51**(8), 815–830 (2008). Preprint arXiv:0812.4821 [math-ph]
33. Shirkov, D.V.: Renormalization group, invariance principle and functional self-similarity. Sov. Phys. Dokl. **27**, 197 (1982)
34. Ovsyannikov, L.V.: Group Analysis of Differential Equations. Academic Press, New York (1982)
35. Rudenko, O.V., Soluyan, S.I.: Theoretical Foundations of Nonlinear Acoustics. Consultants Bureau, New York (1977)

36. Pustovalov, V.V., Silin, V.P.: Nonlinear theory of the interaction of waves in a plasma. In: Proceedings of P.N. Lebedev Physical Institute, AN USSR, vol. 61, pp. 42–283. Nauka, Moscow (1972). English translation in: Skobel'tsyn, D.V. (ed.) Theory of Plasmas. Consultants Bureau, New York (1975)
37. Maksimchuk, A., Flippo, K., Krause, H., et al.: High-energy ion generation by short laser pulses. Plasma Phys. Rep. **30**(6), 473–495 (2004)
38. Kovalev, V.F., Bychenkov, V.Yu: Analytic solutions to the Vlasov equations for expanding plasmas. Phys. Rev. Lett. **90**(18), 185004 (2003) (4 pages)
39. Bychenkov, V.Yu, Kovalev, V.F.: Coulomb explosion in a cluster plasma. Plas. Phys. Rep. **31**(2), 178–183 (2005)
40. Bychenkov V.Yu., Kovalev, V.F.: On the maximum energy of ions in a disintegrating ultrathin foil irradiated by a high-power ultrashort laser pulse. Quantum Electron. **35**(12), 1143–1145 (2005)
41. Kovalev, V.F., Popov, K.I., Bychenkov, V.Yu., Rozmus, W.: Laser triggered Coulomb explosion of nanoscale symmetric targets. Phys. Plasmas **14**, 053103 (2007) (10 pages)
42. Kovalev, V.F., Pustovalov, V.V.: Group and renormgroup symmetry of a simple model for nonlinear phenomena in optics, gas dynamics and plasma theory. Math. Comput. Model. **25**, 165–179 (1997)
43. Akhmanov, S.A., Sukhorukov, A.P., Khokhlov, R.V.: On the self-focusing and self-chanelling of intense laser beams in nonlinear medium. Sov. Phys. JETP **23**(6), 1025–1033 (1966)
44. Murakami, M., Kang, Y.-G., Nishihara, K., et al.: Ion energy spectrum of expanding laser-plasma with limited mass. Phys. Plasmas **12**, 062706 (2005) (8 pages)