Chapter 1
Introduction to Group Analysis of Differential Equations

In this chapter we introduce the basic concepts from Lie group analysis: continuous transformation groups, their generators, Lie equations, groups admitted by differential equations, integration of ordinary differential equations using their symmetries, group classification and invariant solutions of partial differential equations. It contains also an introduction to the theory of Lie–Bäcklund transformations groups and approximate groups. The reader interested in studying more about Lie group methods of integration of differential equations is referred to [8] and to the recent textbook [10].

1.1 One-Parameter Groups

1.1.1 Definition of a Transformation Group

We will consider here only one-parameter groups. Let T_a be an invertible transformation depending on a real parameter a and acting in the (x, y)-plane:

$$\bar{x} = f(x, y, a), \quad \bar{y} = g(x, y, a), \tag{1.1.1}$$

where the functions f and g satisfy the conditions

$$f\big|_{a=0} = x, \quad g\big|_{a=0} = y. \tag{1.1.2}$$

The invertibility is guaranteed if one requires that the Jacobian of f, g with respect to x, y is not zero in a neighborhood of $a = 0$. Further, it is assumed that the functions f and g as well as their derivatives that appear in the subsequent discussion are continuous in x, y, a.

Definition 1.1.1 A set G of transformations (1.1.1) is a *one-parameter transformation group* if it contains the identical transformation $I = T_0$ and includes the inverse T_a^{-1} as well as the composition $T_b T_a$ of all its elements $T_a, T_b \in G$. By a suitable choice of the group parameter a, the main group property $T_b T_a \in G$ can be written

$$T_b T_a = T_{a+b},$$

Y.N. Grigoriev et al., *Symmetries of Integro-Differential Equations,*
Lecture Notes in Physics 806,
DOI 10.1007/978-90-481-3797-8_1,

that is

$$f\big(f(x,y,a),g(x,y,a),b\big)=f(x,y,a+b),\\ g\big(f(x,y,a),g(x,y,a),b\big)=g(x,y,a+b). \tag{1.1.3}$$

In practical applications, the conditions (1.1.3) hold only for sufficiently small values of a and b. Then one arrives at what is called a *local one-parameter group* G. For brevity, local groups are also termed groups.

1.1.2 Generator of a One-Parameter Group

The expansion of the functions f, g into the Taylor series in a near $a = 0$, taking into account the initial condition (1.1.2), yields the *infinitesimal transformation* of the group G (1.1.1):

$$\bar{x} \approx x + \xi(x,y)a, \quad \bar{y} \approx y + \eta(x,y)a, \tag{1.1.4}$$

where

$$\xi(x,y) = \frac{\partial f(x,y,a)}{\partial a}\Big|_{a=0}, \quad \eta(x,y) = \frac{\partial g(x,y,a)}{\partial a}\Big|_{a=0}. \tag{1.1.5}$$

The vector (ξ, η) with components (1.1.5) is the tangent vector (at the point (x, y)) to the curve described by the transformed points $(\bar{x}, \bar{y})$, and is therefore called the *tangent vector field* of the group G.

Example 1.1.1 The group of rotations

$$\bar{x} = x\cos a + y\sin a, \quad \bar{y} = y\cos a - x\sin a$$

has the following infinitesimal transformation:

$$\bar{x} \approx x + ya, \quad \bar{y} \approx y - xa.$$

The tangent vector field (1.1.5) is sometimes also written as a first-order differential operator

$$X = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}, \tag{1.1.6}$$

which behaves as a *scalar* under an arbitrary change of variables, unlike the *vector* (ξ, η). Lie called the operator (1.1.6) the *symbol* of the infinitesimal transformation (1.1.4) or of the corresponding group G. In the current literature, the operator X (1.1.6) is called the *generator* of the group G of transformations (1.1.1).

Example 1.1.2 The generator of the group of rotations from Example 1.1.1 has the form

$$X = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\cdot \tag{1.1.7}$$

1.1.3 Construction of a Group with a Given Generator

Given an infinitesimal transformation (1.1.4), or the generator (1.1.6), the transformations (1.1.1) of the corresponding one-parameter group G are defined by solving the following equations known as the *Lie equations*:

$$\begin{aligned} \frac{df}{da} &= \xi(f,g), \quad f\big|_{a=0} = x, \\ \frac{dg}{da} &= \eta(f,g), \quad g\big|_{a=0} = y. \end{aligned} \tag{1.1.8}$$

We will write (1.1.8) also in the following equivalent form:

$$\begin{aligned} \frac{d\bar{x}}{da} &= \xi(\bar{x},\bar{y}), \quad \bar{x}\big|_{a=0} = x, \\ \frac{d\bar{y}}{da} &= \eta(\bar{x},\bar{y}), \quad \bar{y}\big|_{a=0} = y. \end{aligned} \tag{1.1.9}$$

Example 1.1.3 Consider the infinitesimal transformation

$$\bar{x} \approx x + ax^2, \quad \bar{y} \approx y + axy.$$

The corresponding generator has the form

$$X = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}\cdot \tag{1.1.10}$$

The Lie equations (1.1.9) are written as follows:

$$\begin{aligned} \frac{d\bar{x}}{da} &= \bar{x}^2, \quad \bar{x}\big|_{a=0} = x, \\ \frac{d\bar{y}}{da} &= \bar{x}\bar{y}, \quad \bar{y}\big|_{a=0} = y. \end{aligned}$$

The differential equations of this system are easily solved and yield

$$\bar{x} = -\frac{1}{a + C_1}, \quad \bar{y} = \frac{C_2}{a + C_1}.$$

The initial conditions imply that $C_1 = -1/x$, $C_2 = -y/x$. Consequently we arrive at the following one-parameter group of *projective transformations*:

$$\bar{x} = \frac{x}{1 - ax}, \quad \bar{y} = \frac{y}{1 - ax}\cdot \tag{1.1.11}$$

One can represent the solution to the Lie equations (1.1.9) by means of infinite power series (Taylor series). Then the group transformation (1.1.1) for a generator X (1.1.6) is given by the so-called *exponential map*:

$$\bar{x} = \mathrm{e}^{aX}(x), \quad \bar{y} = \mathrm{e}^{aX}(y), \tag{1.1.12}$$

where

$$\mathrm{e}^{aX} = 1 + \frac{a}{1!}X + \frac{a^2}{2!}X^2 + \cdots + \frac{a^s}{s!}X^s + \cdots. \tag{1.1.13}$$

Example 1.1.4 Consider again the generator (1.1.10) discussed in Example 1.1.3:

$$X = x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}\cdot$$

According to (1.1.12)–(1.1.13), one has to find $X^s(x)$ and $X^s(y)$ for all $s = 1, 2, \ldots$. We calculate several terms, e.g.

$$X(x) = x^2, \quad X^2(x) = X(X(x)) = X(x^2) = 2!x^3, \quad X^3(x) = X(2!x^3) = 3!x^4,$$

and then make a guess:

$$X^s(x) = s!x^{s+1}.$$

The proof of the latter equation is given by induction:

$$X^{s+1}(x) = X(s!x^{s+1}) = (s+1)!x^2x^s = (s+1)!x^{s+2}.$$

Furthermore, one obtains

$$X(y) = xy, \quad X^2(y) = X(xy) = yX(x) + xX(y) = yx^2 + xxy = 2!yx^2,$$
$$X^3(y) = 2![yX(x^2) + x^2X(y)] = 2![y(2x^3) + x^2xy] = 3!yx^3,$$

then makes a guess

$$X^s(y) = s!yx^s$$

and proves it by induction:

$$X^{s+1}(y) = s!X(yx^s) = s![syx^{s+1} + x^s(xy)] = (s+1)!yx^{s+1}.$$

Substitution of the above expressions in the exponential map yields:

$$e^{aX}(x) = x + ax^2 + \cdots + a^s x^{s+1} + \cdots.$$

One can rewrite the right-hand side as $x(1 + ax + \cdots + a^s x^s + \cdots)$. The series in brackets is manifestly the Taylor expansion of the function $1/(1 - ax)$ provided that $|ax| < 1$. Consequently,

$$\bar{x} = e^{aX}(x) = \frac{x}{1 - ax}.$$

Likewise, one obtains

$$e^{aX}(y) = y + ayx + a^2yx^2 + \cdots + a^s yx^s + \cdots$$
$$= y(1 + ax + \cdots + a^s x^s + \cdots).$$

Hence,

$$\bar{y} = e^{aX}(y) = \frac{y}{1 - ax}.$$

Thus, we have arrived at the transformations (1.1.11):

$$\bar{x} = \frac{x}{1 - ax}, \quad \bar{y} = \frac{y}{1 - ax}\cdot$$

1.1.4 Introduction of Canonical Variables

Theorem 1.1.1 *Every one-parameter group of transformations* (1.1.1) *reduces to the group of translations* $\bar{t} = t + a$, $\bar{u} = u$ *with the generator* $X = \frac{\partial}{\partial t}$ *by a suitable change of variables*

$$t = t(x, y), \quad u = u(x, y).$$

The variables t, u *are called canonical variables.*

Proof Under a change of variables the differential operator (1.1.6) transforms according to the formula

$$X = X(t)\frac{\partial}{\partial t} + X(u)\frac{\partial}{\partial u}. \tag{1.1.14}$$

Therefore canonical variables are found from the linear partial differential equations of the first order:

$$\begin{aligned} X(t) &\equiv \xi(x, y)\frac{\partial t(x, y)}{\partial x} + \eta(x, y)\frac{\partial t(x, y)}{\partial y} = 1,\\ X(u) &\equiv \xi(x, y)\frac{\partial u(x, y)}{\partial x} + \eta(x, y)\frac{\partial u(x, y)}{\partial y} = 0. \end{aligned} \tag{1.1.15}$$

□

1.1.5 Invariants (Invariant Functions)

Definition 1.1.2 A function $F(x, y)$ is an invariant of the group G of transformations (1.1.1) if $F(\bar{x}, \bar{y}) = F(x, y)$, i.e.

$$F\big(f(x, y, a), g(x, y, a)\big) = F(x, y) \tag{1.1.16}$$

identically in the variables x, y and the group parameter a.

Theorem 1.1.2 *A function* $F(x, y)$ *is an invariant of the group* G *if and only if it solves the following first-order linear partial differential equation*

$$XF \equiv \xi(x, y)\frac{\partial F}{\partial x} + \eta(x, y)\frac{\partial F}{\partial y} = 0. \tag{1.1.17}$$

Proof Let $F(x, y)$ be an invariant. Let us take the Taylor expansion of $F(f(x, y, a), g(x, y, a))$ with respect to a:

$$F\big(f(x, y, a), g(x, y, a)\big) \approx F(x + a\xi, y + a\eta) \approx F(x, y) + a\Big(\xi\frac{\partial F}{\partial x} + \eta\frac{\partial F}{\partial y}\Big),$$

or

$$F(\bar{x}, \bar{y}) = F(x, y) + aX(F) + o(a),$$

and substitute it in (1.1.16):

$$F(x, y) + aX(F) + o(a) = F(x, y).$$

It follows that $aX(F) + o(a) = 0$, whence $X(F) = 0$, i.e. (1.1.17).

Conversely, let $F(x, y)$ be a solution of (1.1.17). Assuming that the function $F(x, y)$ is analytic and using its Taylor expansion, one can extend the exponential map (1.1.12) to the function $F(x, y)$ as follows:

$$F(\bar{x}, \bar{y}) = \mathrm{e}^{aX} F(x, y) \stackrel{\text{def}}{=} \left(1 + \frac{a}{1!}X + \frac{a^2}{2!}X^2 + \cdots + \frac{a^s}{s!}X^s + \cdots\right) F(x, y).$$

Since $XF(x, y) = 0$, one has $X^2 F = X(XF) = 0, \ldots, X^s F = 0$. We conclude that $F(\bar{x}, \bar{y}) = F(x, y)$, i.e. (1.1.16) thus proving the theorem.

It follows from Theorem 1.1.2 that every one-parameter group of transformations in the plane has one independent invariant, which can be taken to be the left-hand side of any first integral $\psi(x, y) = C$ of the characteristic equation for (1.1.17):

$$\frac{dx}{\xi(x, y)} = \frac{dy}{\eta(x, y)}\cdot \tag{1.1.18}$$

Any other invariant F is then a function of ψ, i.e. $F(x, y) = \Phi(\psi(x, y))$. □

Example 1.1.5 Consider the group with the generator

$$X = x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y}\cdot$$

The characteristic equation (1.1.18) is written

$$\frac{dx}{x} = \frac{dy}{2y}$$

and yields the first integral $\psi = y/x^2$. Hence, the general invariant is given by $F(x, y) = \Phi(y/x^2)$ with an arbitrary function Φ of one variable.

The concepts introduced above can be generalized in an obvious way to the multi-dimensional case by considering groups of transformations

$$\bar{x}^i = f^i(x, a), \quad i = 1, \ldots, n, \tag{1.1.19}$$

in the n-dimensional space R^n of points $x = (x^1, \ldots, x^n)$ instead of transformations (1.1.1) in the (x, y)-plane. The generator of the group of transformations (1.1.19) is written

$$X = \xi^i(x)\frac{\partial}{\partial x^i}, \tag{1.1.20}$$

where

$$\xi^i(x) = \frac{\partial f^i(x, a)}{\partial a}\Big|_{a=0}.$$

The Lie equations (1.1.9) become

$$\frac{d\bar{x}^i}{da} = \xi^i(\bar{x}), \quad \bar{x}^i\big|_{a=0} = x^i. \tag{1.1.21}$$

The exponential map (1.1.12) is written:

$$\bar{x}^i = \mathrm{e}^{aX}(x^i), \quad i = 1, \ldots, n, \tag{1.1.22}$$

where

$$\mathrm{e}^{aX} = 1 + \frac{a}{1!}X + \frac{a^2}{2!}X^2 + \cdots + \frac{a^s}{s!}X^s + \cdots. \tag{1.1.23}$$

The extension of the exponential map to a function $F(x)$ is written

$$F(\bar{x}) = \mathrm{e}^{aX}F(x) \equiv F(x) + aX(F(x)) + \frac{a^2}{2!}X^2(F(x)) + \cdots. \tag{1.1.24}$$

Definition 1.1.2 of invariant functions of several variables remains the same, namely an invariant is defined by the equation $F(\bar{x}) = F(x)$. The invariant test given by Theorem 1.1.2 has the same formulation with the evident replacement of (1.1.17) by its n-dimensional version:

$$\sum_{i=1}^{n} \xi^i(x)\frac{\partial F}{\partial x^i} = 0. \tag{1.1.25}$$

Then $n-1$ functionally independent first integrals $\psi_1(x), \ldots, \psi_{n-1}(x)$ of the characteristic system for (1.1.25):

$$\frac{dx^1}{\xi^1(x)} = \frac{dx^2}{\xi^2(x)} = \cdots = \frac{dx^n}{\xi^n(x)} \tag{1.1.26}$$

provides a basis of invariants. Namely, any invariant $F(x)$ is given by

$$F(x) = \Phi\big(\psi_1(x), \ldots, \psi_{n-1}(x)\big). \tag{1.1.27}$$

1.1.6 Invariant Equations (Manifolds)

Let $x = (x^1, \ldots, x^n) \in R^n$. Consider an $(n-s)$-dimensional manifold $M \subset R^n$ defined by a system of equations[1]

$$F_1(x) = 0, \ldots, F_s(x) = 0, \quad s < n. \tag{1.1.28}$$

It is assumed that

$$\operatorname{rank}\left\|\frac{\partial F_k}{\partial x^i}\right\|_M = s. \tag{1.1.29}$$

[1]Manifolds are treated locally and all functions under consideration are supposed to be continuous and differentiable sufficiently many times.

Definition 1.1.3 The system of equations (1.1.28) is said to be invariant with respect to the group G of transformations (1.1.19),

$$\bar{x}^i = f^i(x, a), \quad i = 1, \dots, n,$$

if each solution $x = (x^1, \dots, x^n)$ of the system (1.1.28) is mapped to a solution $\bar{x} = (\bar{x}^1, \dots, \bar{x}^n)$ of the same system, i.e.

$$F_1(\bar{x}) = 0, \dots, F_s(\bar{x}) = 0. \tag{1.1.30}$$

We also say that (1.1.28) admit the group G. The invariance of (1.1.28) means that the manifold $M \subset R^n$ defined by (1.1.28) is also invariant in the sense that each point x on the surface M is moved by G along the surface M, i.e. $x \in M$ implies that $\bar{x} \in M$.

Theorem 1.1.3 *The system of equations* (1.1.28) *admits the group G of transformations* (1.1.19) *with the generator X* (1.1.20) *if and only if*

$$XF_k\big|_M = 0, \quad k = 1, \dots, s. \tag{1.1.31}$$

Proof (See also [8], Sect. 7.2.) Let the system (1.1.28) be invariant under the group G, i.e. let (1.1.30) hold for every point $x \in M$ and every admissible value of the group parameter a. Taking into account that

$$F_k(\bar{x}) = F_k(x) + aXF_k + o(a), \quad k = 1, \dots, s,$$

and that $F_k(x) = 0$ whenever $x \in M$, one arrives at (1.1.31).

Let us prove now that (1.1.31) imply the invariance of the system (1.1.28), i.e. that (1.1.30) hold for any point $x \in M$. We assume in what follows that the functions $F_k(z)$ and $XF_k(z)$ are analytic in a neighborhood of the manifold M. Then (1.1.31) can be written in the form

$$XF_k(z) = \lambda_k^l(z) F_l(z), \quad k = 1, \dots, s, \tag{1.1.32}$$

where the coefficients $\lambda_k^l(z)$ are bounded in a neighborhood of M. Equations (1.1.32), together with (1.1.24), provide the proof. Indeed, it follows from (1.1.32) that

$$X^2 F_k = X(\lambda_k^l) F_l + \lambda_k^l X(F_l) = \left[X(\lambda_k^p) + \lambda_k^l \lambda_l^p \right] F_p.$$

Iteration and substitution into (1.1.24) yields $F_k(\bar{x}) = \Lambda_k^l(x) F_l(x)$. It follows that (1.1.30) hold, thus completing the proof. □

Remark 1.1.1 The condition (1.1.29) is used for reducing (1.1.28) to the form (1.1.32).

1.1.7 Representation of Regular Invariant Manifolds via Invariants

Definition 1.1.4 Let G be a one-parameter group of transformations (1.1.19) with the generator (1.1.20),

$$X = \xi^i(x)\frac{\partial}{\partial x^i}.$$

An invariant manifold M of the group G is said to be *regular* with respect to G if at least one of the coefficients $\xi^i(x)$ does not vanish on M, and it is *singular* if all coefficients $\xi^i(x)$ of the generator X vanish on M.

Invariant manifolds of a given group G can be equivalently represented by different systems of equations (1.1.28). A general procedure for constructing invariant manifolds is provided by the following theorem on representation of *regular* invariant manifolds by invariant functions (for the proof, see [16], §8.7, or [8], Sect. 7.2.2).

Theorem 1.1.4 *Let G be a group of transformations* (1.1.19). *Any regular $(n-s)$-dimensional manifold $M \subset R^n$ can be represented by a system of equations* (1.1.28) *with invariant functions F_k, i.e.* (*see* (1.1.27))

$$F_k(x) = \varPhi_k\big(\psi_1(x), \ldots, \psi_{n-1}(x)\big), \quad k = 1, \ldots, s, \tag{1.1.33}$$

where $\psi_1(x), \ldots, \psi_{n-1}(x)$ is a basis of invariants of the group G.

Example 1.1.6 Let G be the group of dilations

$$\bar{x} = x\mathrm{e}^a, \quad \bar{y} = y\mathrm{e}^a, \quad \bar{z} = z\mathrm{e}^{2a}$$

in the three-dimensional space R^3. The generator of this group is

$$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2z\frac{\partial}{\partial z}\cdot$$

The characteristic equations (1.1.26) are written

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$$

and yield the following basis of invariants for the group $< G$:

$$\psi_1 = \frac{x^2}{z}, \quad \psi_2 = \frac{y^2}{z}\cdot$$

According to Theorem 1.1.4, any regular two-dimensional invariant manifold (a surface in R^3) is given by $\varPhi(\psi_1, \psi_2) = 0$:

$$\varPhi\Big(\frac{x^2}{z}, \frac{y^2}{z}\Big) = 0.$$

In particular, taking $\Phi(\psi_1, \psi_2) = \psi_1 + \psi_2 - C$ with any constant C we obtain a paraboloid

$$\frac{x^2 + y^2}{z} - C = 0.$$

The left-hand side of this equation is an invariant function with respect to the group G. But if multiply the above equation by z, we represent the same invariant paraboloid by the equation

$$x^2 + y^2 - Cz = 0$$

whose left-hand side is not an invariant function.

1.2 Symmetries and Integration of Ordinary Differential Equations

1.2.1 The Frame of Differential Equations

Any differential equation has two components, namely, the *frame* and the *class of solutions* (see [8]). For example, the frame of a first-order ordinary differential equation

$$F(x, y, y') = 0$$

is the surface $F(x, y, p) = 0$ in the space of three *independent variables* x, y, p. It is obtained by replacing the first derivative y' in the differential equation $F(x, y, y') = 0$ by the variable p.

The class of solutions is defined in accordance with certain "natural" mathematical assumptions or from a physical significance of the differential equations under discussion.

The crucial step in integrating differential equations is a "simplification" of the frame by a suitable change of the variables x, y. The Lie group analysis suggests methods for simplification of the frame by using *symmetry groups* (or *admissible groups*) of differential equations.

Consider, as an example, the following Riccati equation:

$$y' + y^2 - \frac{2}{x^2} = 0. \tag{1.2.1}$$

Its frame is defined by the algebraic equation

$$p + y^2 - \frac{2}{x^2} = 0 \tag{1.2.2}$$

and is a "hyperbolic paraboloid". For the Riccati equation (1.2.1), a one-parameter symmetry group is provided by the following scaling transformations (non-homogeneous dilations) obtained in Sect. 1.2.7:

$$\bar{x} = xe^a, \quad \bar{y} = ye^{-a}. \tag{1.2.3}$$

Indeed, the transformations (1.2.3) after the extension to the first derivative y' and the substitution $y' = p$ are written

$$\bar{x} = xe^{a}, \quad \bar{y} = ye^{-a}, \\ \bar{p} = pe^{-2a}. \tag{1.2.4}$$

One can readily verify that the frame of (1.2.2) is invariant with respect to the transformations (1.2.4). Let us check the infinitesimal invariance condition (1.1.31). The generator (1.1.20) of the group of transformations (1.2.4) has the form

$$X = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} - 2p\frac{\partial}{\partial p}.$$

One can easily check that the invariance condition is satisfied. Indeed:

$$X\Big(p + y^2 - \frac{2}{x^2}\Big) = -2p - 2y^2 + \frac{4}{x^2} = -2\Big(p + y^2 - \frac{2}{x^2}\Big),$$

and hence $X(p + y^2 - \frac{2}{x^2})|_{(1.2.2)} = 0$. For the transformations (1.2.3), the canonical variables are

$$t = \ln x, \quad u = xy. \tag{1.2.5}$$

In the canonical variables (1.2.5), the Riccati equation (1.2.1) becomes:

$$u' + u^2 - u - 2 = 0 \quad (u' = du/dt). \tag{1.2.6}$$

Its frame is obtained by substituting $u' = q$ in (1.2.6) and is given by the following algebraic equation:

$$q + u^2 - u - 2 = 0. \tag{1.2.7}$$

The left-hand side of (1.2.7) does not involve the variable t. Thus the curved frame (1.2.2) has been reduced to a cylindrical surface protracted along the t-axis. Namely it is a "parabolic cylinder". We see that, in integrating differential equations, the decisive step is that of simplifying the frame by converting it into a cylinder. For such purpose, it is sufficient to simplify the symmetry group by introducing canonical variables. In consequence, any first-order ordinary differential equation with a known symmetry reduces to the integrable form $u' = f(u)$ similar to (1.2.6).

Of course, in certain particular examples the equation in question may be solved by other means. For example, it is well-known that the substitution $y = (\ln|u|)'$ reduces (1.2.1) to Euler's equation

$$x^2u'' - 2u = 0$$

having the general solution $u = C_1x^{-1} + C_2x^2$. Hence, the general solution of (1.2.1) has the form

$$y = \frac{d}{dx}\ln\Big|\frac{C_1}{x} + C_2x^2\Big| = \frac{2C_2x^3 - C_1}{x(C_2x^3 + C_1)}.$$

If $C_2 \neq 0$ one has the solution

$$y = \frac{2x^3 - C}{x(x^3 + C)}$$

depending on one arbitrary constant $C = C_1/C_2$. The case $C_2 = 0$ yields the singular solution $y = -1/x$.

1.2.2 *Prolongation of Group Transformations and Their Generators*

The transformation of derivatives $y', y'', \ldots$ under the action of the point transformations (1.1.1), regarded as a change of variables, is well-known from Calculus. It is convenient to write these transformation formulae by using the operator of *total differentiation*:

$$D = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + y''\frac{\partial}{\partial y'} + \cdots .$$

Then the transformation formulae, e.g. for the first and second derivatives are written

$$\bar{y}' \equiv \frac{d\bar{y}}{d\bar{x}} = \frac{Dg}{Df} = \frac{g_x + y'g_y}{f_x + y'f_y} \equiv P(x, y, y', a), \tag{1.2.8}$$

$$\bar{y}'' \equiv \frac{d\bar{y}'}{d\bar{x}} = \frac{DP}{Df} = \frac{P_x + y'P_y + y''P_{y'}}{f_x + y'f_y}. \tag{1.2.9}$$

Starting from the group G of point transformations (1.1.1) and then adding the transformation (1.2.8), one obtains the group $G_{(1)}$, which acts in the space of the three variables (x, y, y'). Further, by adding the transformation (1.2.9) one obtains the group $G_{(2)}$ acting in the space (x, y, y', y'').

Definition 1.2.1 The groups $G_{(1)}$ and $G_{(2)}$ are termed the first and second *prolongations* of G, respectively. The higher prolongations are determined similarly.

Substituting into (1.2.8), (1.2.9) the infinitesimal transformation (1.1.4),

$$\bar{x} \approx x + a\xi, \quad \bar{y} \approx y + a\eta,$$

and neglecting all terms of higher order in a, one obtains the following infinitesimal transformations of derivatives:

$$\begin{aligned}
\bar{y}' = \frac{y' + aD(\eta)}{1 + aD(\xi)} &\approx [y' + aD(\eta)][1 - aD(\xi)] \\
&\approx y' + [D(\eta) - y'D(\xi)]a \equiv y' + a\zeta_1, \\
\bar{y}'' = \frac{y'' + aD(\zeta_1)}{1 + aD(\xi)} &\approx [y'' + aD(\zeta_1)][1 - aD(\xi)] \\
&\approx y'' + [D(\zeta_1) - y''D(\xi)]a \equiv y'' + a\zeta_2.
\end{aligned}$$

Therefore the generators of the prolonged groups $G_{(1)}, G_{(2)}$ are

$$X_{(1)} = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \zeta_1\frac{\partial}{\partial y'}, \quad \zeta_1 = D(\eta) - y'D(\xi), \tag{1.2.10}$$

$$X_{(2)} = X_{(1)} + \zeta_2\frac{\partial}{\partial y''}, \quad \zeta_2 = D(\zeta_1) - y''D(\xi). \tag{1.2.11}$$

These are called the *first* and *second* prolongations of the *infinitesimal operator* (1.1.9). The term *prolongation formulae* is frequently used to denote the expressions for the additional coordinates:

$$\zeta_1 = D(\eta) - y'D(\xi) = \eta_x + (\eta_y - \xi_x)y' - y'^2\xi_y, \tag{1.2.12}$$

$$\begin{aligned}\zeta_2 = D(\zeta_1) - y''D(\xi) &= \eta_{xx} + (2\eta_{xy} - \xi_{xx})y' \\ &+ (\eta_{yy} - 2\xi_{xy})y'^2 - y'^3\xi_{yy} + (\eta_y - 2\xi_x - 3y'\xi_y)y''.\end{aligned} \tag{1.2.13}$$

1.2.3 Group Admitted by Differential Equations

Let G be a group of point transformations and let $G_{(1)}$, $G_{(2)}$ be its first and second prolongations, defined in the previous section.

Definition 1.2.2 We say that a group G of point transformations (1.1.1) is a symmetry group of a first-order ordinary differential equation

$$F(x, y, y') = 0, \tag{1.2.14}$$

or that (1.2.14) admits the group G if (1.2.14) is form invariant under the transformations (1.1.1), or, in other words, if the frame of (1.2.14) is invariant (in the sense of Definition 1.1.3) with respect to the first prolongation $G_{(1)}$ of the group G.

Likewise, an nth order differential equation

$$F(x, y, y', \ldots, y^{(n)}) = 0 \tag{1.2.15}$$

admits a group G if the frame (the surface in the space $x, y, y', \ldots, y^{(n)}$) is invariant with respect to the nth prolongation $G_{(n)}$ of G.

Consider (1.2.15) written in the form solved with respect to the $y^{(n)}$:

$$y^{(n)} = f(x, y, y', \ldots, y^{(n-1)}) \tag{1.2.16}$$

with a smooth function f. The main property of a symmetry group first proved by S. Lie (the proof for first-order equations is given, e.g. in [13], Chap. 16, Sect. 1, Theorem 1) is the following.

Theorem 1.2.1 *A group G is a symmetry group for* (1.2.16) *if and only if G converts any classical solution* (*i.e. n times continuously differentiable*) *of* (1.2.16) *into a classical solution of the same equation.*

1.2.4 Determining Equation for Infinitesimal Symmetries

According to Sect. 1.1.3, it is sufficient to find *infinitesimal symmetries*, i.e. generators (1.1.6) of symmetry groups.

Here, the algorithm of construction of infinitesimal symmetries is discussed for second-order equations

$$F(x, y, y', y'') = 0. \tag{1.2.17}$$

The infinitesimal invariance criterion has the form:

$$X_{(2)} F\big|_{F=0} \equiv (\xi F_x + \eta F_y + \zeta_1 F_{y'} + \zeta_2 F_{y''})\big|_{F=0} = 0, \tag{1.2.18}$$

where ζ_1 and ζ_2 are computed from the prolongation formulae (1.2.12) and (1.2.13). Equation (1.2.18) is called the *determining equation* for the group admitted by the ordinary differential equation (1.2.17).

If the differential equation is written in the explicit form

$$y'' = f(x, y, y'), \tag{1.2.19}$$

the determining equation (1.2.18), after substituting the values of ζ_1, ζ_2 from (1.2.12), (1.2.13) with y'' given by the right-hand side of (1.2.19), assumes the form

$$\begin{aligned}
&\eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})y'^2 \\
&\quad - y'^3\xi_{yy} + (\eta_y - 2\xi_x - 3y'\xi_y) f \\
&\quad - [\eta_x + (\eta_y - \xi_x)y' - y'^2\xi_y] f_{y'} - \xi f_x - \eta f_y = 0.
\end{aligned} \tag{1.2.20}$$

Here $f(x, y, y')$ is a known function (we are dealing with a *given* differential equation (1.2.19) while the coordinates ξ and η of the generator (1.1.6)),

$$X = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y},$$

are unknown functions of x, y. Since the left-hand side of (1.2.20) contains the quantity y' considered as an *independent variable* along with x, y, the determining equation splits into several independent equations, thus becoming an overdetermined system of differential equations for $\xi(x, y)$, $\eta(x, y)$. Solving this system, we find all the infinitesimal symmetries of (1.2.19).

1.2.5 An Example on Calculation of Symmetries

Let us find the operators

$$X = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y}$$

admitted by the second-order equation

$$y'' + \frac{1}{x}y' - e^y = 0. \tag{1.2.21}$$

Here $f = e^y - \frac{1}{x}y'$ and the determining equation (1.2.20) has the form

$$\eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})y'^2$$
$$- y'^3\xi_{yy} + (\eta_y - 2\xi_x - 3y'\xi_y)\left(e^y - \frac{y'}{x}\right)$$
$$+ \frac{1}{x}[\eta_x + (\eta_y - \xi_x)y' - y'^2\xi_y] - \xi\frac{y'}{x^2} - \eta e^y = 0.$$

The left-hand side of this equation is a third-degree polynomial in the variable y'. Therefore the determining equation decomposes into the following four equations, obtained by setting the coefficients of the various powers of y' equal to zero:

$$(y')^3: \quad \xi_{yy} = 0, \tag{1.2.22}$$

$$(y')^2: \quad \eta_{yy} - 2\xi_{xy} + \frac{2}{x}\xi_y = 0, \tag{1.2.23}$$

$$y': \quad 2\eta_{xy} - \xi_{xx} + \left(\frac{\xi}{x}\right)_x - 3\xi_y e^y = 0, \tag{1.2.24}$$

$$(y')^0: \quad \eta_{xx} + \frac{1}{x}\eta_x + (\eta_y - 2\xi_x - \eta)e^y = 0. \tag{1.2.25}$$

Integration of (1.2.22) and (1.2.23) with respect to y yields:

$$\xi = p(x)y + a(x), \quad \eta = \left(p' - \frac{p}{x}\right)y^2 + q(x)y + b(x).$$

Let us substitute these expressions for ξ, η into (1.2.24), (1.2.25). As the dependence of ξ and η on y is polynomial, while the left-hand sides of (1.2.24), (1.2.25) contain e^y, we must have

$$\xi_y = 0, \quad \eta_y - 2\xi_x - \eta = 0.$$

The first of these gives us $p = 0$, that is, the equality $\xi = a(x)$; taking this into account, the second condition can be written in the form

$$q(x) - 2a'(x) - b(x) - q(x)y = 0.$$

Hence $q = 0,\ 2a' + b = 0$. Therefore

$$\xi = a(x), \quad \eta = -2a'(x).$$

Substituting these expressions into (1.2.24), we have

$$\left(a' - \frac{a}{x}\right)' = 0,$$

from which $a = C_1 x \ln x + C_2 x$; here (1.2.25) is satisfied identically.

As a result, we have obtained the general solution of the determining equations (1.2.22)–(1.2.25) in the form

$$\xi = C_1 x \ln x + C_2 x, \quad \eta = -2[C_1(1 + \ln x) + C_2 x]$$

with constant coefficients $C_1,\ C_2$. In view of the linearity of the determining equations, the general solution can be represented as a linear combination of two independent solutions

$$\xi_1 = x\ln x, \quad \eta_1 = -2(1+\ln x);$$
$$\xi_2 = x, \quad \eta_2 = -2.$$

This means that (1.2.21) admits two linearly independent operators

$$X_1 = x\ln x\frac{\partial}{\partial x} - 2(1+\ln x)\frac{\partial}{\partial y}, \quad X_2 = x\frac{\partial}{\partial x} - 2\frac{\partial}{\partial y}, \tag{1.2.26}$$

and that the set of all admissible operators is a two-dimensional vector space with basis (1.2.26).

1.2.6 Lie Algebras. Specific Property of Determining Equations

Definition 1.2.3 Let X and X' be first-order linear differential operators of the form (1.1.6):

$$X = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}, \quad X' = \xi'(x,y)\frac{\partial}{\partial x} + \eta'(x,y)\frac{\partial}{\partial y}\cdot \tag{1.2.27}$$

Their commutator $[X, X']$ is defined by $[X, X'] = XX' - X'X$. It is a first-order linear differential operator and has the form:

$$[X, X'] = \big(X(\xi') - X'(\xi)\big)\frac{\partial}{\partial x} + \big(X(\eta') - X'(\eta)\big)\frac{\partial}{\partial y}\cdot \tag{1.2.28}$$

Definition 1.2.4 A vector space L of operators (1.1.6) is called a Lie algebra if it is closed under the commutator, i.e. if $[X, X'] \in L$ for any $X, X' \in L$. The Lie algebra is denoted by the same letter L, and its dimension is the dimension of the vector space L.

If a Lie algebra L has the dimension $r < \infty$ it is denoted by L_r. If the vector space L_r is spanned by linearly independent operators $X_1, \ldots, X_r$, then the operators $X_1, \ldots, X_r$ provide a basis of the Lie algebra L_r. The condition that $[X, X'] \in L$ for any $X, X' \in L$ is equivalent to the following:

$$[X_i, X_j] = c_{ij}^k X_k, \quad c_{ij}^k = \text{const.} \quad (i, j, k = 1, \ldots, r). \tag{1.2.29}$$

Definition 1.2.5 Let L_r be a Lie algebra spanned by $X_1, \ldots, X_r$. A subspace K_s $(s < r)$ of the vector space L_r spanned by linearly independent operators $Y_1, \ldots, Y_s \in L_r$ is called a *subalgebra* of L_r if

$$[Y, Y'] \in K_s \quad \text{for any } Y, Y' \in K_s.$$

This condition is equivalent to the following:

$$[Y_i, Y_j] \in K_s, \quad i, j = 1, \ldots, s.$$

Let us return to general properties of determining equations. As can be seen from (1.2.20), a determining equation is a linear partial differential equation with the unknown functions ξ and η of the variables x and y. Therefore the set of its solutions forms a vector space, which was already noted in the previous example. However, a specific property of determining equations is given by the following statement due to S. Lie.

Theorem 1.2.2 *The set of all solutions of any determining equation forms a Lie algebra.*

Investigation of the determining equations for symmetries of second-order ordinary differential equations lead Lie to the following significant result [13] (see also [8]).

Theorem 1.2.3 *For a second-order equation* (1.2.19), *the symmetry Lie algebra L has the dimension $r \leq 8$. The maximal dimension $r = 8$ is attained if and only if* (1.2.19) *either is linear or can be linearized by a change of variables.*

We will discuss below two methods of integration of first-order ordinary differential equations with a known infinitesimal symmetry.

1.2.7 Integration of First-Order Equations: Lie's Integrating Factor

We begin with the method of Lie's integrating factor. Consider a first-order ordinary differential equation written in the form

$$Q(x, y)dx + P(x, y)dy = 0. \tag{1.2.30}$$

Lie [13] showed that if (1.2.30) admits a one-parameter group with the generator (1.1.6)

$$X = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y}$$

and if $\xi Q + \eta P \neq 0$, then the function

$$\mu = \frac{1}{\xi Q + \eta P} \tag{1.2.31}$$

is an integrating factor for (1.2.30).

Example 1.2.1 Consider the Riccati equation (1.2.1):

$$y' + y^2 - \frac{2}{x^2} = 0. \tag{1.2.32}$$

Its symmetry group can be readily found by considering dilations $\bar{x} = ax$, $\bar{y} = by$. Substitution in (1.2.32) yields:

$$\bar{y}' + \bar{y}^2 - \frac{2}{\bar{x}^2} = \frac{b}{a}y' + b^2y^2 - \frac{2}{a^2x^2}.$$

The invariance of (1.2.32) requires $b/a = b^2 = 1/a^2$. Hence $b = 1/a$. Therefore the equation admits a one-parameter group of dilations (which can be written in the form $\bar{x} = xe^a$, $\bar{y} = ye^{-a}$) with the generator

$$X = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}. \tag{1.2.33}$$

Writing (1.2.32) in the form (1.2.30),

$$dy + (y^2 - 2/x^2)dx = 0 \tag{1.2.34}$$

and applying the formula (1.2.31), one obtains the integrating factor

$$\mu = \frac{x}{x^2y^2 - xy - 2}.$$

After multiplication by this factor, (1.2.34) is brought to the following form:

$$\frac{xdy + (xy^2 - 2/x)dx}{x^2y^2 - xy - 2} = \frac{xdy + ydx}{x^2y^2 - xy - 2} + \frac{dx}{x} = d\left(\ln x + \frac{1}{3}\ln\frac{xy - 2}{xy + 1}\right) = 0,$$

whence

$$\frac{xy - 2}{xy + 1} = \frac{C}{x^3} \quad \text{or} \quad y = \frac{2x^3 + C}{x(x^3 - C)}.$$

1.2.8 Integration of First-Order Equations: Method of Canonical Variables

Given a one-parameter symmetry group, one can use the canonical variables introduced in Sect. 1.1.4 for integrating first-order equations. Since the property of invariance of an equation with respect to a group is independent of the choice of variables, introduction of canonical variables reduces the equation in question to an equation which does not depend on one of the variables, and hence can be integrated by quadrature. Consider examples.

Example 1.2.2 Let us solve the Riccati equation (1.2.32),

$$y' + y^2 - \frac{2}{x^2} = 0,$$

by the method of canonical variables using the symmetry (1.2.33):

$$X = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}.$$

The partial differential equations

$$X(t) = x\frac{\partial t}{\partial x} - y\frac{\partial t}{\partial y} = 1, \quad X(u) = x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = 0$$

yield the following canonical variables:

$$t = \ln|x|, \quad u = xy.$$

Let us rewrite (1.2.32) in the canonical variables. We have:

$$\frac{dy}{dx} = \frac{d}{dx}\Big(\frac{u}{x}\Big) = -\frac{u}{x^2} + \frac{1}{x}\frac{du}{dx} = -\frac{u}{x^2} + \frac{1}{x}\frac{du}{dt}\frac{dt}{dx} = -\frac{u}{x^2} + \frac{u'}{x^2}.$$

Therefore, the left-hand side of the equation in question is written as follows:

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y^2 - \frac{2}{x^2} = \frac{u'}{x^2} - \frac{u}{x^2} + \frac{u^2}{x^2} - \frac{2}{x^2} = \frac{1}{x^2}\big(u' + u^2 - u - 2\big) = 0.$$

Thus, the Riccati equation is rewritten in the canonical variables in the following integrable form:

$$\frac{du}{dt} + u^2 - u - 2 = 0.$$

It is integrated by separation of variables:

$$\frac{du}{u^2 - u - 2} = -dt.$$

Decomposing the integrand into elementary fractions:

$$\frac{1}{u^2 - u - 2} = \frac{1}{3}\left[\frac{1}{u-2} - \frac{1}{u+1}\right],$$

we evaluate the integral in elementary functions and obtain:

$$\ln\left(\frac{u-2}{u+1}\right) = -3t + \ln C\cdot$$

Now we solve this equation with respect to u,

$$u = \frac{C + 2\mathrm{e}^{3t}}{\mathrm{e}^{3t} - C},$$

substitute $t = \ln|x|$, $u = xy$ and arrive at the solution of the Riccati equation (cf. Example 1.2.1):

$$y = \frac{2x^3 + C}{x(x^3 - C)}\cdot$$

Example 1.2.3 The equation

$$y' = \frac{y}{x} + \frac{y^2}{x^2} \tag{1.2.35}$$

is homogeneous, i.e. it admits the group of dilations (scaling transformations) $\bar{x} = x\mathrm{e}^a$, $\bar{y} = y\mathrm{e}^a$ with the generator

$$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}. \tag{1.2.36}$$

Canonical variables for the operator (1.2.36) are

$$t = \ln|x|, \quad u = \frac{y}{x}. \tag{1.2.37}$$

In these variables, (1.2.35) is written

$$\frac{du}{dt} = u^2.$$

Whence, upon integration:

$$\frac{1}{u} = C - t.$$

Substituting here $t = \ln|x|$ and $y = xu$, we obtain the solution of the original equation:

$$y = \frac{x}{C - \ln|x|}.$$

Example 1.2.4 The equation

$$y' = \frac{y}{x} + \frac{y^3}{x^4} \tag{1.2.38}$$

admits the group of projective transformations

$$\bar{x} = \frac{x}{1 - ax}, \quad \bar{y} = \frac{y}{1 - ax},$$

with the generator

$$X = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}. \tag{1.2.39}$$

Introducing the canonical variables

$$t = -\frac{1}{x}, \quad u = \frac{y}{x}, \tag{1.2.40}$$

we rewrite (1.2.38) in the form

$$\frac{\mathrm{d}u}{\mathrm{d}t} = u^3.$$

Integration yields

$$u = \pm\frac{1}{\sqrt{C - 2t}},$$

whence, substituting the expressions for t and u, we obtain the following general solution to our equation:

$$y = \pm x\sqrt{\frac{x}{2 + Cx}}.$$

Table 1.2.9.1 Structure and standard forms of L_2

Type	Structure of L_2	Standard form of L_2
I	$[X_1, X_2] = 0,\ \xi_1\eta_2 - \eta_1\xi_2 \neq 0$	$X_1 = \frac{\partial}{\partial t},\ X_2 = \frac{\partial}{\partial u}$
II	$[X_1, X_2] = 0,\ \xi_1\eta_2 - \eta_1\xi_2 = 0$	$X_1 = \frac{\partial}{\partial u},\ X_2 = t\frac{\partial}{\partial u}$
III	$[X_1, X_2] = X_1,\ \xi_1\eta_2 - \eta_1\xi_2 \neq 0$	$X_1 = \frac{\partial}{\partial u},\ X_2 = t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}$
IV	$[X_1, X_2] = X_1,\ \xi_1\eta_2 - \eta_1\xi_2 = 0$	$X_1 = \frac{\partial}{\partial u},\ X_2 = u\frac{\partial}{\partial u}$

Table 1.2.9.2 Four types of second-order equations admitting L_2

Type	Standard form of L_2	Canonical form of the equation
I	$X_1 = \frac{\partial}{\partial t},\ X_2 = \frac{\partial}{\partial u}$	$u'' = f(u')$
II	$X_1 = \frac{\partial}{\partial u},\ X_2 = t\frac{\partial}{\partial u}$	$u'' = f(t)$
III	$X_1 = \frac{\partial}{\partial u},\ X_2 = t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}$	$u'' = \frac{1}{t} f(u')$
IV	$X_1 = \frac{\partial}{\partial u},\ X_2 = u\frac{\partial}{\partial u}$	$u'' = f(t)u'$

1.2.9 Standard Forms of Two-Dimensional Lie Algebras

Lie's method of integration of second-order ordinary differential equations employs *canonical variables* in two-dimensional Lie algebras. Introduction of canonical variables reduces any second-order differential equation admitting a two-dimensional Lie algebra L_2 into an integrable form.

Canonical variables reduce a basis of every two-dimensional Lie algebra L_2 to the simplest form and provide four *standard forms* of second-order equations with two symmetries. The basic statements are as follows.

Theorem 1.2.4 *Any two-dimensional Lie algebra can be transformed, by a proper choice of its basis and suitable variables t, u, called canonical variables, to one of the four non-similar standard forms presented in Table 1.2.9.1.*

Remark 1.2.1 In types III and IV, the condition $[X_1, X_2] = X_1$ can be satisfied by a proper change of the basis in L_2 provided that $[X_1,\ X_2] \neq 0$.

Let a second-order equation

$$y'' = f(x, y, y') \tag{1.2.41}$$

admit two or more symmetries. Let us single out from these symmetries a two-dimensional Lie algebra L_2, determine its type according to Table 1.2.9.1, find canonical variables t, u for L_2, and rewrite (1.2.41) in the variables t, u:

$$u'' = g(t, u, u'). \tag{1.2.42}$$

Theorem 1.2.4 guarantees that (1.2.42) belongs to one of four integrable equations given in Table 1.2.9.2.

1.2.10 Lie's Method of Integration for Second-Order Equations

The method of integration of second-order non-linear differential equations (1.2.41) requires the following calculations. First of all, one needs to find the symmetries of the equation in question. Let the equation have two or more symmetries. We single out from these symmetries a two-dimensional Lie algebra L_2 and determine its type according to the *Structure* column of Table 1.2.9.1. Then we find canonical variables by solving the following equations in accordance with the type:

$$\begin{aligned}
&\textbf{Type I}: && X_1(t)=1,\ X_2(t)=0; && X_1(u)=0,\ X_2(u)=1.\\
&\textbf{Type II}: && X_1(t)=0,\ X_2(t)=0; && X_1(u)=1,\ X_2(u)=t.\\
&\textbf{Type III}: && X_1(t)=0,\ X_2(t)=t; && X_1(u)=1,\ X_2(u)=u.\\
&\textbf{Type IV}: && X_1(t)=0,\ X_2(t)=0; && X_1(u)=1,\ X_2(u)=u.
\end{aligned} \tag{1.2.43}$$

Now we rewrite the differential equation in the canonical variables choosing t as a new independent variable and u as a dependent one. It will have one of the integrable forms given in Table 1.2.9.2. It remains to integrate the resulting equation and rewrite the solution in the original variables x, y. This completes the integration procedure.

Example 1.2.5 Let us apply the integration method to the following non-linear second-order equation:

$$y'' + \mathrm{e}^{3y} y'^4 + y'^2 = 0. \tag{1.2.44}$$

First, we have to find the symmetries of (1.2.44). Here

$$f = -(\mathrm{e}^{3y} y'^4 + y'^2)$$

and the determining equation (1.2.20) is written as follows:

$$\begin{aligned}
&\eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})y'^2 - y'^3\xi_{yy}\\
&\quad + 3\mathrm{e}^{3y}y'^4\eta - (\eta_y - 2\xi_x - 3y'\xi_y)(\mathrm{e}^{3y}y'^4 + y'^2)\\
&\quad + [\eta_x + (\eta_y - \xi_x)y' - y'^2\xi_y](4\mathrm{e}^{3y}y'^3 + 2y') = 0.
\end{aligned}$$

The left-hand side of this equation is a polynomial of fifth degree in y'. Since it should vanish identically in y', we equate to zero the coefficients of y'^5, y'^4, ... and obtain the following four independent equations:

$$\begin{aligned}
&(y')^5 : \xi_y = 0,\\
&(y')^4 : 3(\eta_y + \eta) - 2\xi_x = 0,\\
&(y')^3 : \eta_x = 0,\\
&(y')^1 : \xi_{xx} = 0.
\end{aligned}$$

The coefficients for $(y')^2$ and $(y')^0$ vanish together with the coefficients of $(y')^4$ and $(y')^1$, respectively. The above four differential equations for two unknown functions $\xi(x, y)$ and $\eta(x, y)$ are readily solved and yield:

$$\xi = C_1 + 3C_3 x, \quad \eta = 2C_3 + C_2 \mathrm{e}^{-y}, \quad C_1, C_2, C_3 = \text{const.}$$

Hence, the general form of the operator

$$X = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y}$$

admitted by (1.2.44) is

$$X = C_1 X_1 + C_2 X_2 + C_3 X_3,$$

where

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \mathrm{e}^{-y}\frac{\partial}{\partial y}, \quad X_3 = 3x\frac{\partial}{\partial x} + 2\frac{\partial}{\partial y}. \tag{1.2.45}$$

In other words, (1.2.44) admits the three-dimensional Lie algebra L_3 spanned by the operators (1.2.45).

The operators X_1 and X_2 span a two-dimensional subalgebra $L_2 \subset L_3$ and has the type I. Canonical variables t and u are obtained by solving (1.2.43) for type I, i.e. the following equations:

$$\frac{\partial t}{\partial x} = 1, \quad \mathrm{e}^{-y}\frac{\partial t}{\partial y} = 0; \qquad \frac{\partial u}{\partial x} = 0, \quad \mathrm{e}^{-y}\frac{\partial u}{\partial y} = 1.$$

We take the following solutions to this system:

$$t = x, \quad u = \mathrm{e}^y.$$

Thus, we set $u = u(t)$ and rewrite the equation in question in the new variables to obtain

$$u'' + u'^4 = 0.$$

The standard substitution $u' = v$ reduces it to the first-order equation $v' + v^4 = 0$, whence

$$v = \frac{1}{\sqrt[3]{3x + C_1}}.$$

Now we integrate the equation

$$\frac{du}{dx} = \frac{1}{\sqrt[3]{3x + C_1}}$$

and obtain:

$$u = \frac{1}{2}\Big[\sqrt[3]{(3x + C_1)^2} + C_2\Big].$$

Substitution of the expressions for t, u yields the solution to (1.2.44):

$$y = \ln\Big|\sqrt[3]{(3x + C_1)^2} + C_2\Big| - \ln 2.$$

Example 1.2.6 Integrate the non-linear equation

$$y'' + 2\Big(y' - \frac{y}{x}\Big)^3 = 0 \tag{1.2.46}$$

which admits the algebra L_2 of type II spanned by

$$X_1 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}, \quad X_2 = xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}\cdot \tag{1.2.47}$$

Solution. The equations $X_1(t) = 0$, $X_1(u) = 1$; $X_2(t) = 0$, $X_2(u) = t$ provide the canonical variables

$$t = \frac{y}{x}, \quad u = -\frac{1}{x}\cdot \tag{1.2.48}$$

Since the variable t involves the dependent variable y, t can be a new independent variable only if one excludes the *singular* solutions of (1.2.46) along which t is identically constant. These singular solutions are the straight lines:

$$y = Kx, \quad K = \text{const.}$$

In the variables (1.2.48) the equation (1.2.46) becomes

$$u'' = 2$$

and yields $u = t^2 + C_1 t + C_2$. Substituting the expressions for t and u, we obtain:

$$y^2 + C_1 xy + C_2 x^2 + x = 0.$$

Solving this equation with respect to y and introducing the new constants $A = -C_1/2$, $B = A^2 - C_2$, we obtain the solution to (1.2.46):

$$y = Kx, \quad y = Ax \pm \sqrt{Bx^2 - x}. \tag{1.2.49}$$

1.3 Symmetries and Invariant Solutions of Partial Differential Equations

1.3.1 Discussion of Symmetries for Evolution Equations

Consider evolutionary partial differential equations of the second order with one spatial variable x:

$$u_t = F(t, x, u, u_x, u_{xx}), \quad \partial F/\partial u_{xx} \neq 0. \tag{1.3.1}$$

Definition 1.3.1 A one-parameter group G of transformations (1.1.19) of the variables t, x, u:

$$\bar{t} = f(t, x, u, a), \quad \bar{x} = g(t, x, u, a), \quad \bar{u} = h(t, x, u, a) \tag{1.3.2}$$

is called a *group admitted* by (1.3.1), or a *symmetry group* of (1.3.1), if (1.3.1) has the same form in the new variables $\bar{t}, \bar{x}, \bar{u}$:

$$\bar{u}_{\bar{t}} = F(\bar{t}, \bar{x}, \bar{u}, \bar{u}_{\bar{x}}, \bar{u}_{\bar{x}\bar{x}}). \tag{1.3.3}$$

The function F has the same form in both (1.3.1) and (1.3.3).

According to this definition, the transformations (1.3.2) of the group G map every solution $u = u(t,x)$ of (1.3.1) into a solution $\bar{u} = \bar{u}(\bar{t},\bar{x})$ of (1.3.3). Since (1.3.3) is identical with (1.3.1), the definition of an admitted group can be formulated as follows.

Definition 1.3.2 A one-parameter group G of transformations (1.3.2) is called a *group admitted* by (1.3.1) if the transformations (1.3.2) map any solution of (1.3.1) into a solution of the same equation.

The *infinitesimal transformations* of the group G of transformations (1.3.2) are written

$$\bar{t} \approx t + a\tau(t,x,u), \quad \bar{x} \approx x + a\xi(t,x,u), \quad \bar{u} \approx u + a\eta(t,x,u) \tag{1.3.4}$$

and provide the following generator of the group G:

$$X = \tau(t,x,u)\frac{\partial}{\partial t} + \xi(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u} \tag{1.3.5}$$

acting on any differentiable function $J(t,x,u)$ as follows:

$$X(J) = \tau(t,x,u)\frac{\partial J}{\partial t} + \xi(t,x,u)\frac{\partial J}{\partial x} + \eta(t,x,u)\frac{\partial J}{\partial u}\cdot$$

The generator (1.3.5) of a group G admitted by (1.3.1) is known as an *infinitesimal symmetry* of (1.3.1).

The transformations (1.3.2) of the group with the generator (1.3.5) are found by solving the *Lie equations*

$$\frac{d\bar{t}}{da} = \tau(\bar{t},\bar{x},\bar{u}), \quad \frac{d\bar{x}}{da} = \xi(\bar{t},\bar{x},\bar{u}), \quad \frac{d\bar{u}}{da} = \eta(\bar{t},\bar{x},\bar{u}), \tag{1.3.6}$$

with the initial conditions:

$$\bar{t}\big|_{a=0} = t, \quad \bar{x}\big|_{a=0} = x, \quad \bar{u}\big|_{a=0} = u. \tag{1.3.7}$$

Let us turn now to (1.3.3). The quantities $\bar{u}_{\bar{t}}$, $\bar{u}_{\bar{x}}$ and $\bar{u}_{\bar{x}\bar{x}}$ involved in (1.3.3) are obtained via the usual rule of change of derivatives by treating (1.3.2) as a change of variables. Then, expanding the resulting expressions for $\bar{u}_{\bar{t}}$, $\bar{u}_{\bar{x}}$, $\bar{u}_{\bar{x}\bar{x}}$ into Taylor series with respect to the parameter a and keeping only the terms linear in a, one obtains the infinitesimal form of these expressions:

$$\begin{aligned} \bar{u}_{\bar{t}} &\approx u_t + a\zeta_0(t,x,u,u_t,u_x), \\ \bar{u}_{\bar{x}} &\approx u_x + a\zeta_1(t,x,u,u_t,u_x), \\ \bar{u}_{\bar{x}\bar{x}} &\approx u_{xx} + a\,\zeta_2(t,x,u,u_t,u_x,u_{tx},u_{xx}), \end{aligned} \tag{1.3.8}$$

where $\zeta_0, \zeta_1, \zeta_2$ are given by the following *prolongation formulae*:

$$\begin{aligned} \zeta_0 &= D_t(\eta) - u_t D_t(\tau) - u_x D_t(\xi), \\ \zeta_1 &= D_x(\eta) - u_t D_x(\tau) - u_x D_x(\xi), \\ \zeta_2 &= D_x(\zeta_1) - u_{tx} D_x(\tau) - u_{xx} D_x(\xi). \end{aligned} \tag{1.3.9}$$

Here D_t and D_x denote the *total differentiations* with respect to t and x:

$$D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tt} \frac{\partial}{\partial u_t} + u_{tx} \frac{\partial}{\partial u_x},$$
$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{tx} \frac{\partial}{\partial u_t} + u_{xx} \frac{\partial}{\partial u_x}.$$

Substitution of (1.3.4) and (1.3.8) in (1.3.3) yields:

$$\begin{aligned}
&\bar{u}_{\bar{t}} - F(\bar{t}, \bar{x}, \bar{u}, \bar{u}_{\bar{x}}, \bar{u}_{\bar{x}\bar{x}}) \\
&\quad\approx u_t - F(t, x, u, u_x, u_{xx}) \\
&\qquad + a\Big(\zeta_0 - \frac{\partial F}{\partial u_{xx}}\zeta_2 - \frac{\partial F}{\partial u_x}\zeta_1 - \frac{\partial F}{\partial u}\eta - \frac{\partial F}{\partial x}\xi - \frac{\partial F}{\partial t}\tau\Big).
\end{aligned}$$

Therefore, by virtue of (1.3.1), the equation (1.3.3) yields

$$\zeta_0 - \frac{\partial F}{\partial u_{xx}}\zeta_2 - \frac{\partial F}{\partial u_x}\zeta_1 - \frac{\partial F}{\partial u}\eta - \frac{\partial F}{\partial x}\xi - \frac{\partial F}{\partial t}\tau = 0, \tag{1.3.10}$$

where u_t is replaced by $F(t, x, u, u_x, u_{xx})$ in $\zeta_0, \zeta_1, \zeta_2$.

Equation (1.3.10) determines all infinitesimal symmetries of (1.3.1) and therefore it is called the *determining equation*. Conventionally, it is written in the compact form

$$X\big(u_t - F(t, x, u, u_x, u_{xx})\big)\big|_{u_t=F} = 0, \tag{1.3.11}$$

where the *prolongation* of the operator X (1.3.5) to the first and second order derivatives is understood:

$$X = \tau\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial u} + \zeta_0\frac{\partial}{\partial u_t} + \zeta_1\frac{\partial}{\partial u_x} + \zeta_2\frac{\partial}{\partial u_{xx}},$$

and the symbol $|_{u_t=F}$ means that u_t is replaced by $F(t, x, u, u_x, u_{xx})$.

The determining equation (1.3.10) (or its equivalent (1.3.11)) is a linear homogeneous partial differential equation of the second order for unknown functions $\tau(t, x, u)$, $\xi(t, x, u)$, $\eta(t, x, u)$. In consequence, the set of all solutions to the determining equation is a vector space L. Furthermore, the determining equation possesses the following significant and less evident property. The vector space L is a *Lie algebra*, i.e. it is closed with respect to the *commutator*. In other words, L contains, together with any operators X_1, X_2, their commutator $[X_1, X_2]$ defined by

$$[X_1, X_2] = X_1X_2 - X_2X_1.$$

In particular, if $L = L_r$ is finite-dimensional and has a basis $X_1, \ldots, X_r$, then the Lie algebra condition is written in the form

$$[X_\alpha, X_\beta] = c^\gamma_{\alpha\beta} X_\gamma$$

with constant coefficients $c^\gamma_{\alpha\beta}$ known as the *structure constants* of L_r.

Note that (1.3.10) should be satisfied identically with respect to all the variables involved, the variables $t, x, u, u_x, u_{xx}, u_{tx}$ are treated as five independent variables. Consequently, the determining equation decomposes into a system of several equations. As a rule, this is an over-determined system since it contains more equations than three unknown functions τ, ξ and η. Therefore, in practical applications, the determining equation can be readily solved. The following statement due to Lie [12] simplifies the calculation of the symmetries of evolution equations.[2]

Lemma 1.3.1 *The symmetry transformations* (1.3.2) *of* (1.3.1) *have the form*

$$\bar{t} = f(t,a), \quad \bar{x} = g(t,x,u,a), \quad \bar{u} = h(t,x,u,a). \tag{1.3.12}$$

It means that one can search the infinitesimal symmetries in the form

$$X = \tau(t)\frac{\partial}{\partial t} + \xi(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u}\cdot \tag{1.3.13}$$

For the operators (1.3.13), the prolongation formulae (1.3.9) are written as follows:

$$\begin{aligned} &\zeta_0 = D_t(\eta) - u_x D_t(\xi) - \tau'(t)u_t, \quad \zeta_1 = D_x(\eta) - u_x D_x(\xi), \\ &\zeta_2 = D_x(\zeta_1) - u_{xx} D_x(\xi) = D_x^2(\eta) - u_x D_x^2(\xi) - 2u_{xx} D_x(\xi). \end{aligned} \tag{1.3.14}$$

1.3.2 Calculation of Symmetries for Burgers' Equation

Let us find the symmetries of the Burgers equation

$$u_t = u_{xx} + uu_x. \tag{1.3.15}$$

According to Lemma 1.3.1, the infinitesimal symmetries have the form (1.3.13). For the Burgers equation, the determining equation (1.3.10) has the form

$$\zeta_0 - \zeta_2 - u\zeta_1 - \eta u_x = 0, \tag{1.3.16}$$

where ζ_0, ζ_1 and ζ_2 are given by (1.3.14). Let us single out and annul the terms with u_{xx}. Bearing in mind that u_t has to be replaced by $u_{xx} + uu_x$ and substituting in ζ_2 the expressions

$$\begin{aligned} D_x^2(\xi) &= D_x(\xi_x + \xi_u u_x) = \xi_u u_{xx} + \xi_{uu} u_x^2 + 2\xi_{xu} u_x + \xi_{xx}, \\ D_x^2(\eta) &= D_x(\eta_x + \eta_u u_x) = \eta_u u_{xx} + \eta_{uu} u_x^2 + 2\eta_{xu} u_x + \eta_{xx} \end{aligned} \tag{1.3.17}$$

we arrive at the following equation:

$$2\xi_u u_x + 2\xi_x - \tau'(t) = 0.$$

It splits into two equations, namely $\xi_u = 0$ and $2\xi_x - \tau'(t) = 0$. The first equation shows that ξ depends only on t, x, and integration of the second equation yields

$$\xi = \frac{1}{2}\tau'(t)x + p(t). \tag{1.3.18}$$

[2]In [12], Sect. III, Lie proves a more general statement about contact transformations of parabolic equations.

It follows from (1.3.18) that $D_x^2(\xi) = 0$. Now the determining equation (1.3.16) reduces to the form

$$u_x^2\eta_{uu} + \left[\frac{1}{2}\tau'(t)u + \frac{1}{2}\tau''(t)x + p'(t) + 2\eta_{xu} + \eta\right]u_x + u\eta_x + \eta_{xx} - \eta_t = 0$$

and splits into three equations:

$$\begin{gathered}\eta_{uu} = 0,\\ \frac{1}{2}\Big(\tau'(t)u + \tau''(t)x\Big) + p'(t) + 2\eta_{xu} + \eta = 0,\\ u\eta_x + \eta_{xx} - \eta_t = 0.\end{gathered} \tag{1.3.19}$$

The first equation (1.3.19) yields $\eta = \sigma(t,x)u + \mu(t,x)$, and the second equation (1.3.19) becomes:

$$\left(\frac{1}{2}\tau'(t) + \sigma\right)u + \frac{1}{2}\tau''(t)x + p'(t) + 2\sigma_x + \mu = 0,$$

whence

$$\sigma = -\frac{1}{2}\tau'(t), \quad \mu = -\frac{1}{2}\tau''(t)x - p'(t).$$

Thus, we have

$$\eta = -\frac{1}{2}\tau'(t)u - \frac{1}{2}\tau''(t)x - p'(t). \tag{1.3.20}$$

Finally, substitution of (1.3.20) in the third equation (1.3.19) yields

$$\frac{1}{2}\tau'''(t)x + p''(t) = 0,$$

whence $\tau'''(t) = 0$, $p''(t) = 0$, and hence

$$\tau(t) = C_1t^2 + 2C_2t + C_3, \quad p(t) = C_4t + C_5.$$

Invoking (1.3.18) and (1.3.20), we ultimately arrive at the following general solution of the determining equation (1.3.16):

$$\begin{gathered}\tau(t) = C_1t^2 + 2C_2t + C_3,\\ \xi = C_1tx + C_2x + C_4t + C_5,\\ \eta = -(C_1t + C_2)u - C_1x - C_4.\end{gathered}$$

It contains five arbitrary constants C_i. Hence, the infinitesimal symmetries of the Burgers equation (1.3.15) form the five-dimensional Lie algebra L_5 spanned by the following linearly independent operators:

$$\begin{gathered}X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u},\\ X_4 = t\frac{\partial}{\partial x} - \frac{\partial}{\partial u}, \quad X_5 = t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} - (x + tu)\frac{\partial}{\partial u}.\end{gathered} \tag{1.3.21}$$

Let G be a group admitted by (1.3.1). Then every transformation (1.3.2) belonging to the group G carries over any solution of the differential equation (1.3.1) into a solution of the same equation. It means that the solutions of a partial differential equation are permuted among themselves under the action of a symmetry group. The solutions may also be individually unaltered, then they are called *invariant solutions*. Accordingly, group analysis provides two basic ways for construction of exact solutions: *group transformations* of known solutions and construction of *invariant solutions*.

1.3.3 Invariant Solutions and Their Calculation

If a group transformation maps a solution into itself, we arrive at what is called a *self-similar* or *group invariant solution*. According to Theorem 1.1.4 on invariant representation of invariant manifolds, the invariant solutions under a one-parameter group with a generator X are obtained as follows.

Let X be a given infinitesimal symmetry (1.3.5) of (1.3.1). One calculates two independent *invariants* $J_1 = \lambda(t,x)$ and $J_2 = \mu(t,x,u)$ by solving the first-order linear partial differential equation

$$X(J) \equiv \tau(t,x,u)\frac{\partial J}{\partial t} + \xi(t,x,u)\frac{\partial J}{\partial x} + \eta(t,x,u)\frac{\partial J}{\partial u} = 0,$$

or its characteristic system:

$$\frac{dt}{\tau(t,x,u)} = \frac{dx}{\xi(t,x,u)} = \frac{du}{\eta(t,x,u)}. \tag{1.3.22}$$

Then one designates one of the invariants as a function of the other, e.g.

$$\mu = \phi(\lambda), \tag{1.3.23}$$

and solves (1.3.23) with respect to u. Finally, one substitutes the expression for u in (1.3.1) and obtains an ordinary differential equation for the unknown function $\phi(\lambda)$ of one variable. This procedure reduces the number of independent variables by one.

Example 1.3.1 Let us find the solutions of the Burgers equation that are invariant under the time translations generated by the operator X_1 from (1.3.21). The invariance condition leads to the stationary solutions

$$u = \Phi(x)$$

for which the Burgers equation is written

$$\Phi'' + \Phi\Phi' = 0. \tag{1.3.24}$$

Integrating once, we obtains

$$\Phi' + \frac{\Phi^2}{2} = C_1.$$

We integrate now this first-order equation by setting $C_1 = 0$, $C_1 = \nu^2 > 0$, and $C_1 = -\omega^2 < 0$ and obtain:

$$\begin{aligned}\Phi(x) &= \frac{2}{x+C},\\ \Phi(x) &= \nu \mathrm{th}\Big(C + \frac{\nu}{2}x\Big),\\ \Phi(x) &= \omega \mathrm{tg}\Big(C - \frac{\omega}{2}x\Big).\end{aligned} \tag{1.3.25}$$

1.3.4 Group Transformations of Solutions

Let (1.3.2) be an admitted group for (1.3.1), and let a function

$$u = \Phi(t, x)$$

solve (1.3.1). Since (1.3.2) is a symmetry transformation, the above solution can be also written in the new variables:

$$\bar{u} = \Phi(\bar{t}, \bar{x}).$$

Replacing here $\bar{u}, \bar{t}, \bar{x}$ from (1.3.2), we get

$$h(t, x, u, a) = \Phi\big(f(t, x, u, a), g(t, x, u, a)\big). \tag{1.3.26}$$

Solving (1.3.26) with respect to u one obtains a one-parameter family (with the parameter a) of new solutions to (1.3.1).

Example 1.3.2 Consider the Burgers equation (1.3.15),

$$u_t = u_{xx} + uu_x,$$

and apply the above procedure to the admitted one-parameter group generated by the operator X_5 from (1.3.21):

$$X_5 = t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} - (x + tu)\frac{\partial}{\partial u}.$$

The one-parameter group generated by X_5 has the form

$$\bar{t} = \frac{t}{1-at}, \quad \bar{x} = \frac{C_2}{1-at}, \quad \bar{u} = (1-at)u - ax. \tag{1.3.27}$$

Using the transformations (1.3.27) and applying (1.3.26) to any known solution $u = \Phi(t, x)$ of the Burgers equation, one obtains the following one-parameter set of new solutions:

$$u = \frac{ax}{1-at} + \frac{1}{1-at}\,\Phi\Big(\frac{t}{1-at}, \frac{x}{1-at}\Big). \tag{1.3.28}$$

Let us apply the transformation (1.3.28), e.g. to the first stationary solution (1.3.25):

$$\Phi(x) = \frac{2}{x+C}, \quad C = \text{const.},$$

one obtains the new non-stationary solutions

$$u = \frac{ax}{1 - at} + \frac{2}{x + C(1 - at)}$$

depending on the parameter a.

1.3.5 Optimal Systems of Subalgebras

The concept of optimal systems of subalgebras of a given Lie algebra was used by Ovsyannikov [16] for describing essentially different invariant solutions. This concept is useful in dealing with nonlinear mathematical models. A simple method of construction of an optimal system is illustrated in this section by of means of the five-dimensional Lie algebra L_5 spanned by the symmetries (1.3.21) of the Burgers equation. The result is used in the next section for describing all invariant solutions of the Burgers equation.

The symmetry Lie algebra L_5 spanned by the operators (1.3.21) allows one to construct invariant solutions of the Burgers equation (1.3.15),

$$u_t = u_{xx} + uu_x,$$

by using any one-dimensional subalgebra of the algebra L_5, i.e. on any operator $X \in L_5$. However, there are infinite number of one-dimensional subalgebras of L_5 since an arbitrary operator from L_5 is written

$$X = l^1 X_1 + \cdots + l^5 X_5, \tag{1.3.29}$$

and hence depends on five arbitrary constants $l^1, \ldots, l^5$. Ovsyannikov [16] has noticed, however, that if two subalgebras are *similar*, i.e. connected with each other by a transformation of the symmetry group, then their corresponding invariant solutions are connected with each other by the same transformation. Consequently, it is sufficient to deal with an optimal system of subalgebras obtained in our case as follows. We put into one class all similar operators $X \in L_5$ and select a representative of each class. The set of the representatives of all these classes is an *optimal system of one-dimensional subalgebras*.

An optimal system of one-dimensional subalgebras of the Lie algebra L_5 is constructed as follows [11]. The transformations of the symmetry group with the Lie algebra L_5 provide the 5-parameter group of linear transformations of the operators $X \in L_5$ or, equivalently, linear transformations of the vector

$$l = (l^1, \ldots, l^5), \tag{1.3.30}$$

where $l^1, \ldots, l^5$ are taken from (1.3.29). To find these linear transformations, we use their generators (see, e.g. Sect. 1.4 in [6])

$$E_\mu = c^\lambda_{\mu\nu} l^\nu \frac{\partial}{\partial l^\lambda}, \quad \mu = 1, \ldots, 5, \tag{1.3.31}$$

where $c^{\lambda}_{\mu\nu}$ are the structure constants of the Lie algebra L_5 defined by

$$[X_\mu, X_\nu] = c^{\lambda}_{\mu\nu} X_\lambda.$$

For computing the operators (1.3.31) it is convenient to use the following commutator table of the operators (1.3.21):

	X_1	X_2	X_3	X_4	X_5
X_1	0	0	$2X_1$	X_2	X_3
X_2	0	0	X_2	0	X_4
X_3	$-2X_1$	$-X_2$	0	X_4	$2X_5$
X_4	$-X_2$	0	$-X_4$	0	0
X_5	$-X_3$	$-X_4$	$-2X_5$	0	0

(1.3.32)

Let us find, e.g. the operator E_1. According to (1.3.31), it is written

$$E_1 = c^{\lambda}_{1\nu} l^{\nu} \frac{\partial}{\partial l^{\lambda}},$$

where $c^{\lambda}_{1\nu}$ are defined by the commutators $[X_1, X_\nu] = c^{\lambda}_{\mu\nu} X_\lambda$, i.e. by the first raw in table (1.3.32). Namely, the non-vanishing $c^{\lambda}_{\mu\nu}$ are

$$c^{1}_{1\,3} = 2, \quad c^{2}_{1\,4} = 1, \quad c^{2}_{1\,5} = 1.$$

Therefore we have:

$$E_1 = 2l^3 \frac{\partial}{\partial l^1} + l^4 \frac{\partial}{\partial l^2} + l^5 \frac{\partial}{\partial l^3}.$$

Substituting in (1.3.31) all structure constants given by table (1.3.32) we obtain:

$$\begin{gathered} E_1 = 2l^3 \frac{\partial}{\partial l^1} + l^4 \frac{\partial}{\partial l^2} + l^5 \frac{\partial}{\partial l^3}, \quad E_2 = l^3 \frac{\partial}{\partial l^2} + l^5 \frac{\partial}{\partial l^4}, \\ E_3 = -2l^1 \frac{\partial}{\partial l^1} - l^2 \frac{\partial}{\partial l^2} + l^4 \frac{\partial}{\partial l^4} + 2l^5 \frac{\partial}{\partial l^5}, \\ E_4 = -l^1 \frac{\partial}{\partial l^2} - l^3 \frac{\partial}{\partial l^4}, \quad E_5 = -l^1 \frac{\partial}{\partial l^3} - l^2 \frac{\partial}{\partial l^4} - 2l^3 \frac{\partial}{\partial l^5}. \end{gathered} \tag{1.3.33}$$

Let us find the transformations provided by the generators (1.3.33). For the generator E_1, the Lie equations with the parameter a_1 are written

$$\frac{d\tilde{l}^1}{da_1} = 2\tilde{l}^3, \quad \frac{d\tilde{l}^2}{da_1} = \tilde{l}^4, \quad \frac{d\tilde{l}^3}{da_1} = \tilde{l}^5, \quad \frac{d\tilde{l}^4}{da_1} = 0, \quad \frac{d\tilde{l}^5}{da_1} = 0.$$

Integrating these equations and using the initial condition $\tilde{l}|_{a_1=0} = l$, we obtain:

$$\begin{aligned} E_1: \qquad & \tilde{l}^1 = l^1 + 2a_1 l^3 + a_1^2 l^5, \quad \tilde{l}^2 = l^2 + a_1 l^4, \\ & \tilde{l}^3 = l^3 + a_1 l^5, \quad \tilde{l}^4 = l^4, \quad \tilde{l}^5 = l^5. \end{aligned} \tag{1.3.34}$$

Taking the other operators (1.3.33) we obtain the following transformations:

$$E_2: \quad \tilde{l}^1 = l^1,\ \tilde{l}^2 = l^2 + a_2 l^3,\ \tilde{l}^3 = l^3,\ \tilde{l}^4 = l^4 + a_2 l^5,\ \tilde{l}^5 = l^5, \tag{1.3.35}$$

$$E_3: \quad \tilde{l}^1 = a_3^{-2} l^1,\ l^2 = a_3^{-1} l^2,\ \tilde{l}^3 = l^3,\ \tilde{l}^4 = a_3 l^4,\ \tilde{l}^5 = a_3^2 l^5, \tag{1.3.36}$$

where $a_3 > 0$ since the integration of the Lie equations yields, e.g.

$$l^4 = l^4 \mathrm{e}^{\tilde{a}_3} = a_3 l^4.$$

$$E_4: \quad \tilde{l}^1 = l^1,\ \tilde{l}^2 = l^2 - a_4 l^1,\ \tilde{l}^3 = l^3,\ \tilde{l}^4 = l^4 - a_4 l^3,\ \tilde{l}^5 = l^5. \tag{1.3.37}$$

$$\begin{aligned} E_5: \qquad \tilde{l}^1 = l^1, \quad \tilde{l}^2 = l^2, \quad \tilde{l}^3 = l^3 - a_5 l^1,\\ \tilde{l}^4 = l^4 - a_5 l^2, \quad \tilde{l}^5 = l^5 - 2a_5 l^3 + a_5^2 l^1. \end{aligned} \tag{1.3.38}$$

Note that the transformations (1.3.34)–(1.3.38) map the vector $X \in L_5$ given by (1.3.29) to the vector $\tilde{X} \in L_5$ given by the following formula:

$$\tilde{X} = \tilde{l}^1 X_1 + \cdots + \tilde{l}^5 X_5. \tag{1.3.39}$$

Now we can prove the following statement on an optimal system of one-dimensional subalgebras of symmetry algebra for the Burgers equation.

Theorem 1.3.1 *The following operators provide an optimal system of one-dimensional subalgebras of the Lie algebra L_5 with the basis* (1.3.21):

$$\begin{aligned} &X_1, \quad X_2, \quad X_3, \quad X_4, \quad X_1 + X_4, \quad X_1 - X_4,\\ &X_5, \quad X_1 + X_5, \quad X_2 + X_5, \quad X_2 - X_5, \end{aligned} \tag{1.3.40}$$

where k is an arbitrary parameter.

Proof We first clarify if the transformations (1.3.34)–(1.3.38) have invariants $J(l^1, \ldots, l^5)$. The reckoning shows that the 5×5 matrix $\|c^\lambda_{\mu\nu} l^\nu\|$ of the coefficients of the operators (1.3.33) has the rank four. It means that the transformations (1.3.34)–(1.3.38) have precisely one functionally independent invariant. The integration of the equations

$$E_\mu(J) = 0, \quad \mu = 1, \ldots, 5,$$

shows that the invariant is

$$J = (l^3)^2 - l^1 l^5. \tag{1.3.41}$$

Knowledge of the invariant (1.3.41) simplifies further calculations significantly.

The last equation in (1.3.38) shows that if $l^1 \neq 0$, we get $\bar{l}_5 = 0$ by solving the quadratic equation $l^5 - 2a_5 l^3 + a_5^2 l^1 = 0$ for a_5, i.e. by taking

$$a_5 = \frac{l^3 \pm \sqrt{J}}{l^1}, \tag{1.3.42}$$

where J is the invariant (1.3.41). We can use (1.3.42) only if $J \geq 0$.

Now we begin the construction of the optimal system. The method requires a simplification of the general vector (1.3.30) by means of the transformations (1.3.34)–(1.3.38). As a result, we will find the simplest representatives of each class of similar vectors (1.3.30). Substituting these representatives in (1.3.29), we will obtain the optimal system of one-dimensional subalgebras of L_5. We will divide the construction to several cases. □

1.3.5.1 The Case $l^1 = 0$

I will divide this case into the following two subcases.

1°. $l^3 \neq 0$. In other words, we consider the vectors (1.3.30) of the form

$$(0, l^2, l^3, l^4, l^5), \quad l^3 \neq 0.$$

First we take $a_5 = l^5/(2l^3)$ in (1.3.38) and reduce the above vector to the form

$$(0, l^2, l^3, l^4, 0).$$

Then we subject the latter vector to the transformation (1.3.37) with $a_4 = l^4/l^3$ and obtain the vector

$$(0, l^2, l^3, 0, 0).$$

Since the operator X is defined up to a constant factor and $l^3 \neq 0$, we divide the above vector by l^3 and transform it using (1.3.35) to the form

$$(0, 0, 1, 0, 0).$$

Substituting it in (1.3.29), we obtain the operator

$$X_3. \tag{1.3.43}$$

2°. $l^3 = 0$. Thus, we consider the vectors (1.3.30) of the form

$$(0, l^2, 0, l^4, l^5).$$

2°(1). If $l^2 \neq 0$, we can assume $l^2 = 1$ (see above), use the transformation (1.3.38) with $a_5 = Al^4$ and get the vector

$$(0, 1, 0, 0, l^5).$$

If $l^5 \neq 0$ we can make $l^5 = \pm 1$ by the transformation (1.3.36). Thus, taking into account the possibility $l^5 = 0$, we obtain the following representatives for the optimal system:

$$X_2, \quad X_2 + X_5, \quad X_2 - X_5. \tag{1.3.44}$$

2°(2). Let $l^2 = 0$. If $l^5 \neq 0$ we can set $l^5 = 1$. Now we apply the transformation (1.3.35) with $a_2 = -l^4$ and obtain the vector $(0, 0, 0, 0, 1)$. If $l^5 = 0$ we get the vector $(0, 0, 0, 1, 0)$. Thus, the case $l^2 = 0$ provides the operators

$$X_4, \quad X_5. \tag{1.3.45}$$

1.3.5.2 The Case $l^1 \neq 0$, $J > 0$

Now we can define a_5 by (1.3.42) and annul $\bar{l}^5$ by the transformation (1.3.38). Thus, we will deal with the vector

$$(l^1, l^2, l^3, l^4, 0), \quad l^1 \neq 0.$$

Since J is invariant under the transformations (1.3.34)–(1.3.38), the condition $J > 0$ yields that in the above vector we have $l^3 \neq 0$. Therefore we can use the transformation (1.3.37) with $a_4 = l^4/l^3$ and get $\bar{l}^4 = 0$. Then we apply the transformation (1.3.34) with $a_1 = -l^1/(2l^3)$ and obtain $\bar{l}^1 = 0$, thus arriving at the vector $(0, l^2, l^3, 0, 0)$, and hence at the previous operator (1.3.43). Hence, this case contributes no additional subalgebras to the optimal system.

1.3.5.3 The Case $l^1 \neq 0$, $J = 0$

In this case (1.3.42) reduces to $a_5 = l^3/l^1$.

If $l^3 \neq 0$, we use the transformation (1.3.38) with $a_5 = l^3/l^1$ and obtain $\bar{l}^5 = 0$. Due to the invariance of J we conclude that the equation $J = 0$ yields $(\bar{l}^3)^2 - \bar{l}^1\bar{l}^5 = 0$. Since $\bar{l}^5 = 0$, it follows that $\bar{l}^3 = 0$. Thus we can deal with the vectors of the form

$$(l^1, l^2, 0, l^4, 0), \quad l^1 \neq 0. \tag{1.3.46}$$

Furthermore, if $l^3 = 0$, we have $J = -l^1 l^5$, and the equation $J = 0$ yields $l^5 = 0$ since $l^1 \neq 0$. Therefore we again have the vectors of the form (1.3.46) where we can assume $l^1 = 1$. Subjecting the vector (1.3.46) with $l^1 = 1$ to the transformation (1.3.37) with $a_4 = l^2$ we obtain $\bar{l}^2 = 0$, and hence map the vector (1.3.46) to the form

$$(1, 0, 0, l^4, 0).$$

If $l^4 \neq 0$, we use the transformation (1.3.36) with an appropriately chosen a_3 and obtain $l^4 = \pm 1$. taking into account the possibility $l^4 = 0$, we see that this case contributes the following operators:

$$X_1, \quad X_1 + X_4, \quad X_1 - X_4. \tag{1.3.47}$$

1.3.5.4 The Case $l^1 \neq 0$, $J < 0$

It is obvious from the condition $J = (l^3)^2 - l^1 l^5 < 0$ that $l^5 \neq 0$. Therefore we successively apply the transformations (1.3.38), (1.3.37) and (1.3.35) with $a_5 = l^3/l^1$, $a_4 = l^2/l^1$ and $a_2 = -l^4/l^5$, respectively and obtain $\bar{l}^3 = \bar{l}^2 = \bar{l}^4 = 0$. The components l^1 and l^5 of the resulting vector

$$(l^1, 0, 0, 0, l^5)$$

have the common sign since the condition $J < 0$ yields $l^1 l^5 > 0$. Therefore using the transformation (1.3.36) with an appropriate value of the parameter a_3 and invoking that we can multiply the vector l by any constant, we obtain $l^1 = l^5 = 1$, i.e. the operator

$$X_1 + X_5. \tag{1.3.48}$$

Finally, collecting the operators (1.3.43), (1.3.44), (1.3.45), (1.3.47) and (1.3.48), we arrive at the optimal system (1.3.40), thus completing the proof.

1.3.6 All Invariant Solutions of the Burgers Equation

Constructing the invariant solution for each operator from the optimal system of subalgebras (1.3.40), we obtain the following *optimal system of invariant solutions* [11].

Theorem 1.3.2 *An optimal system of invariant solutions for the Burgers equation is provided by the following solutions, where* σ, γ *and* K *are arbitrary constants.*

$$X_1: \quad \text{(i)} \quad u = \frac{2}{x+\gamma};$$
$$\text{(ii)} \quad u = \sigma \frac{\gamma e^{\sigma x}+1}{\gamma e^{\sigma x}-1} \equiv \tilde{\sigma} \tanh\Big(\tilde{\gamma} + \frac{\tilde{\sigma}}{2}x\Big); \tag{1.3.49}$$
$$\text{(iii)} \quad u = \sigma \tan\Big(\gamma - \frac{\sigma}{2}x\Big).$$

$$X_3: \quad u = \frac{\varphi(\lambda)}{\sqrt{t}}, \quad \lambda = \frac{x}{\sqrt{t}}, \quad \text{where} \quad \varphi' + \frac{1}{2}\varphi^2 + \frac{1}{2}\lambda\varphi = K. \tag{1.3.50}$$

$$X_4: \quad u = \frac{K-x}{t}. \tag{1.3.51}$$

$$X_1 + X_4: \quad u = \varphi(\lambda) - t, \quad \lambda = x - \frac{t^2}{2}, \quad \text{where} \quad \varphi' + \frac{1}{2}\varphi^2 + \lambda = K. \tag{1.3.52}$$

$$X_1 - X_4: \quad u = t + \varphi(\lambda), \quad \lambda = x + \frac{t^2}{2}, \quad \text{where} \quad \varphi' + \frac{1}{2}\varphi^2 - \lambda = K. \tag{1.3.53}$$

$$X_5: \quad u = -\lambda + \frac{\varphi(\lambda)}{t}, \quad \lambda = \frac{x}{t}, \quad \text{where}$$
$$\text{(i)} \quad \varphi(\lambda) = \sigma \frac{\gamma e^{\sigma\lambda}-1}{\gamma e^{\sigma\lambda}+1}, \quad |\varphi| < \sigma;$$
$$\text{(ii)} \quad \varphi(\lambda) = \sigma \frac{\gamma e^{\sigma\lambda}+1}{\gamma e^{\sigma\lambda}-1}, \quad |\varphi| > \sigma; \tag{1.3.54}$$
$$\text{(iii)} \quad \varphi(\lambda) = \sigma \tan\Big(\gamma - \frac{\sigma}{2}\lambda\Big).$$

$$X_1 + X_5: \quad u = -\frac{tx}{1+t^2} + \frac{\varphi(\lambda)}{\sqrt{1+t^2}}, \quad \lambda = \frac{x}{\sqrt{1+t^2}}, \quad \text{where} \quad \varphi' + \frac{1}{2}\varphi^2 + \frac{1}{2}\lambda^2 = K. \tag{1.3.55}$$

$$X_2 + X_5: \quad u = -\frac{x}{t} - \frac{1}{t^2} + \frac{\varphi(\lambda)}{t}, \quad \lambda = \frac{x}{t} + \frac{1}{2t^2}, \quad \text{where} \quad \varphi' + \frac{1}{2}\varphi^2 - \lambda = K. \tag{1.3.56}$$

$$X_2 - X_5: \qquad u = -\frac{x}{t} + \frac{1}{t^2} + \frac{\varphi(\lambda)}{t}, \quad \lambda = \frac{x}{t} - \frac{1}{2t^2}, \qquad \text{where} \quad \varphi' + \frac{1}{2}\varphi^2 + \lambda = K. \tag{1.3.57}$$

If one subjects each solution from the optimal system of invariant solutions (1.3.49)–(1.3.57) to all transformations of the group admitted by the Burgers equation, one obtains *all invariant solutions* of the Burgers equation. We see that the invariant solutions of the Burgers equation are given either by elementary functions or by solving a Riccati equation. Furthermore, one can verify that the set of all invariant solutions involves 76 parameters.

We see that the invariant solutions of the Burgers equation are given either by elementary functions or by solving a Riccati equation.

Furthermore, the solutions to the Riccati equations describing the solutions (1.3.52), (1.3.53), (1.3.56) and (1.3.57) can be represented by special functions. Namely, setting $\varphi = \sqrt{2}\psi$, $\mu = \lambda + K$ and using the substitution

$$\psi = \frac{d \ln |z|}{d\mu} \equiv \frac{z'}{z}$$

we reduce the Riccati equation in (1.3.53) and (1.3.56) to the *Airy equation*

$$\frac{d^2 z}{d\mu^2} - \mu z = 0. \tag{1.3.58}$$

The general solution to (1.3.58) is the linear combination

$$z = C_1 \mathrm{Ai}(\mu) + C_2 \mathrm{Bi}(\mu), \qquad C_1, C_2 = \text{const.}, \tag{1.3.59}$$

of the *Airy functions* (see, e.g. [14], [17])

$$\mathrm{Ai}(\mu) = \frac{1}{\pi}\int_0^\infty \cos\Big(\mu\tau + \frac{1}{3}\tau^3\Big) d\tau, \qquad \mathrm{Bi}(\mu) = \frac{1}{\pi}\int_0^\infty \Big[\exp\Big(\mu\tau - \frac{1}{3}\tau^3\Big) + \sin\Big(\mu\tau + \frac{1}{3}\tau^3\Big)\Big] d\tau. \tag{1.3.60}$$

Hence, the function $\varphi(\lambda)$ in the solutions (1.3.53) and (1.3.56) is given by

$$\varphi(\lambda) = \sqrt{2}\frac{d}{d\lambda} \ln \big| C_1 \mathrm{Ai}(\lambda + K) + C_2 \mathrm{Bi}(\lambda + K) \big|. \tag{1.3.61}$$

One can obtain likewise that $\varphi(\lambda)$ in (1.3.52) and (1.3.57) is given by

$$\varphi(\lambda) = \sqrt{2}\frac{d}{d\lambda} \ln \big| C_1 \mathrm{Ai}(K - \lambda) + C_2 \mathrm{Bi}(K - \lambda) \big|. \tag{1.3.62}$$

Finally, it is worth noting that the optimal system of subalgebras is not unique, it depends on the choice of a representative in each class of similar operators. Consequently, the form of the solutions included in an optimal system of invariant solutions depends on the choice of representatives. However, this choice does not affect the amount of the optimal system of invariant solutions since the number of the classes of similar operators does not depend on the choice of representatives. Moreover, this choice does not affect the final form of the 76-parameter set of all invariant solutions obtained from an optimal system of invariant solutions by the transformations of the general group admitted by the Burgers equation.

1.4 General Definitions of Symmetry Groups

1.4.1 Differential Variables and Function

We will use the following notation. Consider the algebraically independent variables

$$x = \{x^i\}, \quad u = \{u^\alpha\}, \quad u_{(1)} = \{u_i^\alpha\}, \quad u_{(2)} = \{u_{ij}^\alpha\}, \ldots, \tag{1.4.1}$$

where $\alpha = 1, \ldots, m$, and $i, j = 1, \ldots, n$. The variables $u_{ij}^\alpha, \ldots$ are assumed to be symmetric in subscripts, i.e. $u_{ij}^\alpha = u_{ji}^\alpha$. The operator

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \cdots \quad (i = 1, \ldots, n), \tag{1.4.2}$$

is called the *total differentiation* with respect to x^i. The operator D_i is a formal sum of an infinite number of terms. However, it truncates when acting on any function of a finite number of the variables $x, u, u_{(1)}, \ldots$. In consequence, the total differentiations D_i are well defined on the set of all functions depending on a finite number of $x, u, u_{(1)}, \ldots$.

Though the variables (1.4.1) are assumed to be *algebraically independent*, they are connected by the following *differential relations*:

$$u_i^\alpha = D_i(u^\alpha), \quad u_{ij}^\alpha = D_j(u_i^\alpha) = D_j D_i(u^\alpha). \tag{1.4.3}$$

The variables x^i are called independent variables, and the variables u^α are known as *differential* (or dependent) *variables* with the successive derivatives $u_{(1)}$, $u_{(2)}$, etc. The universal space of modern group analysis is the space $\mathscr{A}$ of differential functions introduced by Ibragimov [3] (see also [4], Sect. 19) as a generalization of differential polynomials considered by J.F. Ritt in the 1950s.

Definition 1.4.1 A locally analytic function (i.e., locally expandable in a Taylor series with respect to all arguments) of a finite number of variables (1.4.1) is called a *differential function.* The highest order of derivatives appearing in the differential function is called the order of this function. The set of all differential functions of all finite orders is denoted by $\mathscr{A}$. This set is a vector space with respect to the usual addition of functions and becomes an associative algebra if multiplication is defined by the usual multiplication of functions. A significant property of the space $\mathscr{A}$ is that it is closed under the action of total derivatives (1.4.2).

Definition 1.4.2 A group G of transformations of the form

$$\bar{x}^i = f^i(x, u, a), \quad f^i|_{a=0} = x^i, \tag{1.4.4}$$

$$\bar{u}^\alpha = \varphi^\alpha(x, u, a), \quad \varphi^\alpha|_{a=0} = u^\alpha, \tag{1.4.5}$$

is called a group of point transformations in the space of dependent and independent variables. The generator of the group G is

$$X = \xi^i(x, u)\frac{\partial}{\partial x^i} + \eta^\alpha(x, u)\frac{\partial}{\partial u^\alpha}, \tag{1.4.6}$$

where

$$\xi^i = \frac{\partial f^i}{\partial a}\Big|_{a=0}, \quad \eta^\alpha = \frac{\partial \varphi^\alpha}{\partial a}\Big|_{a=0}. \tag{1.4.7}$$

Let $F_k \in \mathscr{A}$ be any differential functions and let p be the maximum of orders of the differential functions F_k, $k = 1, \ldots, s$. Consider the system of equations

$$F_k(x, u, u_{(1)}, \ldots, u_{(p)}) = 0, \quad k = 1, \ldots, s. \tag{1.4.8}$$

If one treats the variables u^α as functions of x so that

$$u^\alpha = u^\alpha(x), \quad u_i^\alpha = \frac{\partial u^\alpha(x)}{\partial x^i}, \ldots,$$

then one arrives at the usual concept of a *system of differential equations* (1.4.8) of order p.

1.4.2 Frame and Extended Frame

Recall the definitions of the *frame* and *extended frame* of differential equations given in [5] (see also [6], Chap. 1).

Definition 1.4.3 Let us treat $x, u, u_{(1)}, \ldots$ as functionally independent variables connected only by the differential relations (1.4.3). Then (1.4.8) determine a surface in the space of the independent variables $x, u, u_{(1)}, \ldots, u_{(p)}$. This surface is called the *frame* (or *skeleton*) of the system of differential equations (1.4.8).

Definition 1.4.4 Consider the frame equation (1.4.8) together with its differential consequences,

$$F_k = 0, \quad D_i F_k = 0, \quad D_i D_j F_k = 0, \ldots. \tag{1.4.9}$$

The totality of points $(x, u, u_{(1)}, \ldots)$ satisfying (1.4.9) is called the *extended frame* of the system of differential equations (1.4.8) and is denoted by $[F]$.

We will assume that

$$\text{rank}\left\| \frac{\partial F_k}{\partial x^i}, \frac{\partial F_k}{\partial u^\alpha}, \frac{\partial F_k}{\partial u_i^\alpha}, \ldots \right\| = s$$

on the frame of the differential equations under consideration.

1.4.3 Definition Using Solutions

The first definition of a symmetry group of an arbitrary system of differential equations coincides with Definition 1.3.2 for a single evolution equation.

Definition 1.4.5 The system of differential equations (1.4.8) is said to be invariant under the group G of transformations (1.4.4), (1.4.5) if the transformations (1.4.4), (1.4.5) convert every solution of the system (1.4.8) into a solution of the same system. Here the solutions of differential equations are considered as classical ones, i.e., are assumed to be smooth functions $u^\alpha = u^\alpha(x)$. If the system of equations (1.4.8) is invariant under the group G then G is also known as a symmetry group for the system (1.4.8) or a group admitted by this system.

1.4.4 Definition Using the Frame

Though the first definition is conceptually simple, it depends upon knowledge of solutions. Therefore, in practical calculation of symmetries the following second, geometric definition is more efficient.

Definition 1.4.6 The system of differential equations (1.4.8) is said to be invariant under the group G if the frame of the system is an invariant surface with respect to the prolongation of the transformations (1.4.4), (1.4.5) of the group G to the derivatives $u_{(1)}, \ldots, u_{(p)}$.

According to this definition and the invariance test of equations given by Theorem 1.1.3, one obtains the following infinitesimal test for obtaining symmetries of differential equations.

Theorem 1.4.1 *The group G with the generator X is admitted by the system of differential equations* (1.4.8) *if and only if*

$$X_{(p)} F_k \big|_{(1.4.8)} = 0, \quad k = 1, \ldots, s, \tag{1.4.10}$$

where $X_{(p)}$ is the p-th prolongation of X and $|_{(1.4.8)}$ means evaluated on the frame the system of differential equations (1.4.8). *Equations* (1.4.10) *are the determining equations.*

Let $z_0 = (x_0, u_0, \ldots, u_{0(p)})$ be a point on the frame of the system (1.4.8), i.e. $F_k(x_0, u_0, \ldots, u_{0(p)}) = 0$ $(k = 1, \ldots, s)$. The system of differential equations (1.4.8) is said to be locally solvable at z_0 if there is a solution passing through this point, i.e., there exist a solution $u = h(x)$ of differential equations (1.4.8) defined in a neighborhood of the point x_0 such that $u_0 = h(x_0), \ldots, u_{0(p)} = \partial p_h / \partial x_p (x_0)$. The system (1.4.8) is said to be *locally solvable* if it has this property at every generic point of the frame.

It can be shown that for locally solvable systems the first and the second definitions of the symmetry group are equivalent, i.e. that Definition 1.4.5 and Definition 1.4.6 provide exactly the same symmetry group. A discussion of this equivalence is to be found in Lie [13], Chap. 6, Sect. 1, and Ovsyannikov [16], Sect. 15.1. See also Olver [15], Sect. 2.6, for a modern treatment of this subject.

1.4.5 Definition Using the Extended Frame

If the system (1.4.8) is not locally solvable, e.g. if the system (1.4.8) is over-determined, it may happen (see further Example 1.4.1) that Definition 1.4.6 provides only a subgroup of the symmetry group given by Definition 1.4.5. Therefore, Ibragimov proposed ([4], Sect. 17.1, see also [6], Chap. 1) the following third definition and proved the appropriate infinitesimal test for the invariance of over-determined systems of differential equations.

Definition 1.4.7 The system of differential equations (1.4.8) is said to be invariant under the group G if the extended frame $[F]$ is invariant with respect to the infinite-order prolongation of G.

The infinitesimal test for this invariance is written as follows (see [4], Theorem 17.1).

Theorem 1.4.2 *Let X be the generator of a group G. The system of differential equations* (1.4.8) *are invariant under the group G in the sense of Definition* 1.4.7 *if and only if the following equations are satisfied*:

$$X_{(p)}F_k\big|_{[F]} = 0, \quad k = 1, \dots, s. \tag{1.4.11}$$

Equations (1.4.11) *are also called determining equations.*

Remark 1.4.1 According to Theorem 1.4.2, the invariance test does not involve all the differential consequences (1.4.9) of the differential equations (1.4.8). In fact, it can be easily shown that it suffices to consider only a finite number of the differential consequences (1.4.9) such that they form a system in involution. It is also worth noting that we do not need to take into account the additional equations such as $X_{(p)}(D_iF_k) = 0$ since they are satisfied identically due to (1.4.11).

For locally solvable systems, all three definitions of symmetry groups are equivalent. For over-determined systems, the first and third definitions are equivalent, whereas the second definition provides, in general, only a subgroup of the symmetry group given by the third definition.

Example 1.4.1 Consider the over-determined system ([6], Sect. 1.3.10)

$$u_t = (u_x)^{-4/3}u_{xx}, \quad v_t = -3(u_x)^{-1/3}, \quad v_x = u. \tag{1.4.12}$$

This is a system of three equations for two dependent variables u and v. The maximal order of equations involved in the system is $p = 2$. Let us first solve the determining equations (1.4.10). The left-hand side of (1.4.10) depends upon the variables $x, t, u, v, u_x, u_{xx}, u_{xt}$, and v_{xx} in accordance with the prolongation formulae. The solution of the determining equations yields the 6-dimensional Lie algebra spanned by

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial t}, \quad X_3 = \frac{\partial}{\partial v}, \quad X_4 = \frac{\partial}{\partial u} + x\frac{\partial}{\partial v},$$
$$X_5 = 4t\frac{\partial}{\partial t} + 3u\frac{\partial}{\partial u} + 3v\frac{\partial}{\partial v}, \quad X_6 = 2x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}\cdot \tag{1.4.13}$$

This is the Lie algebra of the maximal symmetry group for (1.4.12) obtained by the second definition (Definition 1.4.6).

Consider now the determining equations (1.4.11). Differentiation of the third equation (1.4.12) yields $v_{xx} = u_x$. Therefore, we replace v_{xx} in the determining equation by u_x. Then the left-hand side of (1.4.11) involves only the variables x, t, u, v, u_x, u_{xx} and u_{xt}. Solving the determining equations (1.4.11), one obtains the 7-dimensional Lie algebra spanned by the operators (1.4.13) and by

$$X_7 = x^2\frac{\partial}{\partial x} + xv\frac{\partial}{\partial v} + (v - xu)\frac{\partial}{\partial u}\cdot \tag{1.4.14}$$

Thus, the third definition (Definition 1.4.7) provides a more general symmetry group than the second definition.

1.5 Lie–Bäcklund Transformation Groups

This is section provides an introduction to the theory of Lie–Bäcklund transformation groups and contains the basic definitions, theorems and algorithms used for computation of Lie–Bäcklund symmetries of differential equations. The space $\mathscr{A}$ of differential functions introduced in Sect. 1.4.1 play a central role in this theory.

1.5.1 Lie–Bäcklund Operators

Geometrically, Lie–Bäcklund transformations appear in attempting to find a higher-order generalization of the classical contact (first-order tangent) transformations (see Bäcklund's paper [1], its English translation is available in [9]) and are identified with infinite-order tangent transformations. A historical survey of the development of this branch of group analysis and a detailed discussion of the modern theory with many applications are to be found in [4] (see also [7], Chap. 1). We will use here a shortcut to the theory of Lie–Bäcklund transformation groups by using a generalization of infinitesimal generators of point and contact transformation groups. The generalization is known as a Lie–Bäcklund operator and is defined as follows.

Definition 1.5.1 Let $\xi^i, \eta^\alpha \in \mathscr{A}$ be differential functions depending on any finite number of variables $x, u, u_{(1)}, u_{(2)}, \ldots$. A differential operator

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \zeta_i^\alpha \frac{\partial}{\partial u_i^\alpha} + \zeta_{i_1 i_2}^\alpha \frac{\partial}{\partial u_{i_1 i_2}^\alpha} + \cdots, \tag{1.5.1}$$

where

$$\begin{aligned} \zeta_i^\alpha &= D_i(\eta^\alpha - \xi^j u_j^\alpha) + \xi^j u_{ij}^\alpha, \\ \zeta_{i_1 i_2}^\alpha &= D_{i_1} D_{i_2}(\eta^\alpha - \xi^j u_j^\alpha) + \xi^j u_{j i_1 i_2}^\alpha, \ldots \end{aligned} \tag{1.5.2}$$

is called a *Lie–Bäcklund operator*. The Lie–Bäcklund operator (1.5.1) is often written in the abbreviated form

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \cdots, \tag{1.5.3}$$

where the prolongation given by (1.5.1)–(1.5.2) is understood.

The operator (1.5.1) is formally an infinite sum. However, it truncates when acting on any differential function. Hence, *the action of Lie–Bäcklund operators is well defined on the space* $\mathscr{A}$.

Consider two Lie–Bäcklund operators

$$X_\nu = \xi_\nu^i \frac{\partial}{\partial x^i} + \eta_\nu^\alpha \frac{\partial}{\partial u^\alpha} + \cdots, \quad \nu = 1, 2,$$

and define their commutator by the usual formula:

$$[X_1, X_2] = X_1 X_2 - X_2 X_1.$$

Theorem 1.5.1 *The commutator* $[X_1, X_2]$ *is identical with the Lie–Bäcklund operator given by*

$$[X_1, X_2] = \big(X_1(\xi_2^i) - X_2(\xi_1^i)\big)\frac{\partial}{\partial x^i} + \big(X_1(\eta_2^\alpha) - X_2(\eta_1^\alpha)\big)\frac{\partial}{\partial u^\alpha} + \cdots, \tag{1.5.4}$$

where the terms denoted by dots are obtained by prolonging the coefficients of $\partial/\partial x^i$ *and* $\partial/\partial u^\alpha$ *in accordance with* (1.5.1) *and* (1.5.2).

According to Theorem 1.5.1, the set of all Lie–Bäcklund operators is an infinite dimensional Lie algebra with respect to the commutator (1.5.4). It is called the *Lie–Bäcklund algebra* and denoted by $L_{\mathscr{B}}$. The Lie–Bäcklund algebra is endowed with the following properties (see [4]).

I. $D_i \in L_{\mathscr{B}}$. In other words, the total differentiation (1.4.2) is a Lie–Bäcklund operator. Furthermore,

$$X_* = \xi_*^i D_i \in L_{\mathscr{B}} \tag{1.5.5}$$

for any $\xi_*^i \in \mathscr{A}$.

II. Let L_* be the set of all Lie–Bäcklund operators of the form (1.5.5). Then L_* is an ideal of $L_{\mathscr{B}}$, i.e., $[X, X_*] \in L_*$ for any $X \in L_{\mathscr{B}}$. Indeed,

$$[X, X_*] = \big(X(\xi_*^i) - X_*(\xi^i)\big) D_i \in L_*.$$

III. In accordance with property II, two operators $X_1, X_2 \in L_{\mathscr{B}}$ are said to be *equivalent* (i.e., $X_1 \sim X_2$) if $X_1 - X_2 \in L_*$. In particular, every operator $X \in L_{\mathscr{B}}$ is equivalent to an operator (1.5.1) with $\xi^i = 0, i = 1, \dots, n$. Namely, , $X \sim \tilde{X}$ where

$$\tilde{X} = X - \xi^i D_i = (\eta^\alpha - \xi^i u_i^\alpha)\frac{\partial}{\partial u^\alpha} + \cdots. \tag{1.5.6}$$

Definition 1.5.2 The operators of the form

$$X = \eta^\alpha \frac{\partial}{\partial u^\alpha} + \cdots, \quad \eta^\alpha \in \mathscr{A}, \tag{1.5.7}$$

are called *canonical Lie–Bäcklund operators*.

Using this definition, we can formulate the property III as follows.

Theorem 1.5.2 *Any operator $X \in L_{\mathscr{B}}$ is equivalent to a canonical Lie–Bäcklund operator.*

Example 1.5.1 Let us take $n = m = 1$ and denote $u_1 = u_x$. The generator of the group of translations along the x-axis and its canonical Lie–Bäcklund form (1.5.6) are written as follows:

$$X = \frac{\partial}{\partial x} \sim \tilde{X} = u_x \frac{\partial}{\partial u} + \cdots.$$

Example 1.5.2 Let x, y be the independent variables, and $k, c =$ const. The generator of non-homogeneous dilations and its canonical Lie–Bäcklund form (1.5.6) are written:

$$X = x\frac{\partial}{\partial x} + ky\frac{\partial}{\partial y} + cu\frac{\partial}{\partial u} \sim \tilde{X} = (cu - xu_x - kyu_y)\frac{\partial}{\partial u} + \cdots.$$

Example 1.5.3 Let t, x be the independent variables. The generator of the Galilean boost and its canonical Lie–Bäcklund form (1.5.6) are written:

$$X = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u} \sim \tilde{X} = (1 - tu_x)\frac{\partial}{\partial u} + \cdots.$$

The canonical operators leave invariant the independent variables x^i. Therefore, the use of the canonical form is convenient, e.g., for investigating symmetries of integro-differential equations.

IV. The following statements describe all Lie–Bäcklund operators equivalent to generators of Lie point and Lie contact transformation groups.

Theorem 1.5.3 *The Lie–Bäcklund operator* (1.5.1) *is equivalent to the infinitesimal operator of a one-parameter point transformation group if and only if its coordinates assume the form*

$$\xi^i = \xi_1^i(x, u) + \xi_*^i, \quad \eta^\alpha = \eta_1^\alpha(x, u) + \big(\xi_2^i(x, u) + \xi_*^i\big)u_i^\alpha,$$

where $\xi_*^i \in \mathscr{A}$ *is an arbitrary differential function, and* $\xi_1^i, \xi_2^i, \eta_1^\alpha$ *are arbitrary functions of* x *and* u.

Theorem 1.5.4 *Let* $m = 1$. *Then the operator* (1.5.1) *is equivalent to the infinitesimal operator of a one-parameter contact transformation group if and only if its coordinates assume the form*

$$\xi^i = \xi_1^i(x, u, u_{(1)}) + \xi_*^i, \quad \eta = \eta_1(x, u, u_{(1)}) + \xi_*^i u_i,$$

where $\xi_*^i \in \mathscr{A}$ *is an arbitrary differential function, and* ξ_1^i, η_1 *are arbitrary first-order differential functions, i.e. depend upon* x, u *and* $u_{(1)}$.

1.5.2 Integration of Lie–Bäcklund Equations

Consider the sequence

$$z = (x, u, u_{(1)}, u_{(2)}, \ldots) \tag{1.5.8}$$

with the elements z^ν, $\nu \geq 1$, were

$$z^i = x^i, 1 \leq i \leq n, \quad z^{n+\alpha} = u^\alpha, 1 \leq \alpha \leq m.$$

Denote by $[z]$ any finite subsequence of z. Then elements of the space $\mathscr{A}$ of differential functions are written as $f([z])$.

Definition 1.5.3 Given an operator (1.5.1), the following infinite system is called *Lie–Bäcklund equations*:

$$\begin{gathered} \frac{d}{da}\bar{x}^i = \xi^i([\bar{z}]), \quad \frac{d}{da}\bar{u}^\alpha = \eta^\alpha([\bar{z}]), \\ \frac{d}{da}\bar{u}_i^\alpha = \zeta_i^\alpha([\bar{z}]), \quad \frac{d}{da}\bar{u}_{ij}^\alpha = \zeta_{ij}^\alpha([\bar{z}]), \ldots, \end{gathered} \tag{1.5.9}$$

where $\alpha = 1, \ldots, m$ and $i, j, \ldots = 1, \ldots, n$.

In the case of canonical operators (1.5.7), the infinite system of equations (1.5.9) can be replaced by the finite system

$$\frac{d}{da}\bar{u}^\alpha = \eta^\alpha([\bar{z}]), \quad \alpha = 1, \ldots, m. \tag{1.5.10}$$

Indeed, upon solving the system (1.5.10), the transformations of the successive derivatives are obtained by the total differentiation:

$$\bar{u}_i^\alpha = D_i(\bar{u}^\alpha), \quad \bar{u}_{ij}^\alpha = D_i D_j(\bar{u}^\alpha), \ldots. \tag{1.5.11}$$

We will use the abbreviated form (1.5.3) of Lie–Bäcklund operators and write the system (1.5.9), together with the initial conditions, as follows:

$$\frac{d}{da}\bar{x}^i = \xi^i([\bar{z}]), \quad \bar{x}^i\big|_{a=0} = x^i,$$
$$\frac{d}{da}\bar{u}^\alpha = \eta^\alpha([\bar{z}]), \quad \bar{u}^\alpha\big|_{a=0} = u^\alpha, \tag{1.5.12}$$
$$\cdots\cdots\cdots\cdots\cdots$$

The formal integrability of the infinite system (1.5.12) has been proved by Ibragimov (see, e.g. [4], Sect. 15.1; it is also discussed in [6]). For the convenience of the reader, we formulate here the existence theorem. The following notation is convenient for formulating and proving the theorem.

Let f and g be formal power series in one symbol a with coefficients from the space $\mathscr{A}$, i.e. let

$$f(z,a) = \sum_{k=0}^{\infty} f_k([z])a^k, \quad f_k([z]) \in \mathscr{A}, \tag{1.5.13}$$

and

$$g(z,a) = \sum_{k=0}^{\infty} g_k([z])a^k, \quad g_k([z]) \in \mathscr{A}.$$

Their linear combination $\lambda f([z]) + \mu g([z])$ with constant coefficients λ, μ and product $f([z]) \cdot g([z])$ are defined by

$$\lambda \sum_{k=0}^{\infty} f_k([z])a^k + \mu \sum_{k=0}^{\infty} g_k([z])a^k = \sum_{k=0}^{\infty}\big(\lambda f_k([z]) + \mu g_k([z])\big)a^k, \tag{1.5.14}$$

and

$$\Big(\sum_{p=0}^{\infty} f_p([z])a^p\Big) \cdot \Big(\sum_{q=0}^{\infty} g_q([z])a^q\Big) = \sum_{k=0}^{\infty}\Big(\sum_{p+q=k} f_p([z])g_q([z])\Big)a^k, \tag{1.5.15}$$

respectively. The space of all formal power series (1.5.13) endowed with the addition (1.5.14) and the multiplication (1.5.15) is denoted by $[[\mathscr{A}]]$.

Lie point and Lie contact transformations, together with their prolongations of all orders, are represented by elements of the space $[[\mathscr{A}]]$. Moreover, the utilization of this space is necessary in the theory of Lie–Bäcklund transformation groups. Therefore, $[[\mathscr{A}]]$ is called *the representation space of modern group analysis* ([6], Sect. 1.2).

The existence theorem is formulated as follows.

Theorem 1.5.5 *The Lie–Bäcklund equations* (1.4.5) *have a solution in the space* $[[\mathscr{A}]]$. *The solution is unique. It is given by formal power series*

$$\bar{x}^i = x^i + \sum_{k=0}^{\infty} A^i_k([z])a^k, \quad A^i_k([z]) \in \mathscr{A},$$
$$\bar{u}^\alpha = u^\alpha + \sum_{k=0}^{\infty} B^\alpha_k([z])a^k, \quad B^\alpha_k([z]) \in \mathscr{A}, \tag{1.5.16}$$
$$\cdots\cdots\cdots\cdots\cdots$$

and satisfies the group property.

Definition 1.5.4 The group of formal transformations (1.5.16) is called a one-parameter Lie–Bäcklund transformation group.

Recall that a point transformation group acting in the finite dimensional space of variables $x = (x^1, \ldots, x^n)$ and generated by an operator X can be represented by the exponential map (1.1.22):

$$\bar{x}^i = \exp(aX)(x^i), \quad i = 1, \ldots, n, \tag{1.5.17}$$

where

$$\exp(aX) = 1 + aX + \frac{a^2}{2!}X^2 + \frac{a^3}{3!}X^3 + \cdots. \tag{1.5.18}$$

Likewise, the solution (1.5.16) to the Lie–Bäcklund equations (1.5.12) can be represented by the exponential map

$$\bar{x}^i = \exp(aX)(x^i), \ \bar{u}^\alpha = \exp(aX)(u^\alpha), \ \bar{u}_i^\alpha = \exp(aX)(u_i^\alpha), \ldots, \tag{1.5.19}$$

where X is a Lie–Bäcklund operator (1.5.1) and $\exp(aX)$ is given by (1.5.18).

If we consider canonical operators (1.5.7) then (1.5.12) reduce to the finite system of equations (1.5.10) supplemented by the initial conditions, i.e. by the system

$$\frac{d}{da}\bar{u}^\alpha = \eta^\alpha([\bar{z}]), \quad \bar{u}^\alpha\big|_{a=0} = u^\alpha. \tag{1.5.20}$$

Consequently, Lie–Bäcklund transformation groups can be constructed by virtue of the following theorem.

Theorem 1.5.6 *Given a canonical Lie–Bäcklund operator,*

$$X = \eta^\alpha \frac{\partial}{\partial u^\alpha} + \cdots,$$

the corresponding formal one-parameter group is represented by the series

$$\bar{u}^\alpha = u^\alpha + a\eta^\alpha + \frac{a^2}{2!}X(\eta^\alpha) + \cdots + \frac{a^n}{n!}X^{n-1}(\eta^\alpha) + \cdots \tag{1.5.21}$$

together with its differential consequences:

$$\bar{u}_i^\alpha = u_i^\alpha + aD_i(\eta^\alpha) + \frac{a^2}{2!}X(D_i(\eta^\alpha)) + \cdots + \frac{a^n}{n!}X^{n-1}(D_i(\eta^\alpha)) + \cdots,$$

$$\bar{u}_{i_1\cdots i_s}^\alpha = u_{i_1\cdots i_s}^\alpha + aD_{i_1}\cdots D_{i_s}(\eta^\alpha) + \cdots + \frac{a^n}{n!}X^{n-1}(D_{i_1}\cdots D_{i_s}(\eta^\alpha)) + \cdots.$$

Example 1.5.4 Let

$$X = u_1\frac{\partial}{\partial u} + u_2\frac{\partial}{\partial u_1} + \cdots.$$

Here $\eta = u_1$ and therefore

$$X(\eta) = u_2, \ X^2(\eta) = u_3, \ \ldots, \ X^{n-1}(\eta) = u_n.$$

Hence, the transformation (1.5.21) has the form

$$\bar{u} = u + \sum_{n=1}^{\infty} \frac{a^n}{n!} u_n.$$

Example 1.5.5 Let

$$X = u_2 \frac{\partial}{\partial u} + u_3 \frac{\partial}{\partial u_1} + u_4 \frac{\partial}{\partial u_2} + \cdots.$$

Here, $\eta = u_2$ and

$$X(\eta) = u_4,\ X^2(\eta) = u_6,\ \ldots,\ X^{n-1}(\eta) = u_{2n}.$$

Hence, the transformation (1.5.21) is given by the power series

$$\bar{u} = u + \sum_{n=1}^{\infty} \frac{a^n}{n!} u_{2n}.$$

1.5.3 Lie–Bäcklund Symmetries

Lie–Bäcklund symmetries of differential equations are given by Definition 1.4.7 from Sect. 1.4.5. Thus, we use the following definition.

Definition 1.5.5 Let G be a Lie–Bäcklund transformation group generated by a Lie–Bäcklund operator (1.5.1),

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \zeta_i^\alpha \frac{\partial}{\partial u_i^\alpha} + \zeta_{i_1 i_2}^\alpha \frac{\partial}{\partial u_{i_1 i_2}^\alpha} + \cdots. \tag{1.5.1}$$

The group G is called a group of *Lie–Bäcklund symmetries* of a system of differential equations

$$F_k(x, u, u_{(1)}, \ldots, u_{(p)}) = 0, \quad k = 1, \ldots, s, \tag{1.5.22}$$

if the extended frame of (1.5.22) defined by (see Definition 1.4.4)

$$[F]: \qquad F_k = 0, \quad D_i F_k = 0, \quad D_i D_j F_k = 0, \ldots \tag{1.5.23}$$

is invariant under G. The operator X (1.5.1) is called an *infinitesimal Lie–Bäcklund symmetry* for (1.5.22).

The infinitesimal invariance criteria proved in [4] is formulated in the following statements.

Theorem 1.5.7 *The operator* (1.5.1) *is an infinitesimal Lie–Bäcklund symmetry for* (1.5.22) *if and only if*

$$XF_k\big|_{[F]} = 0, \quad XD_i(F_k)\big|_{[F]} = 0, \quad XD_iD_j(F_k)\big|_{[F]} = 0, \ldots \quad (k = 1, \ldots, s).$$

Theorem 1.5.7 contains an infinite number of equations. However, it can be simplified and reduced to a finite number of equations by means of the following result.

Lemma 1.5.1 *The equations*

$$XF_k\big|_{[F]} = 0$$

yield the infinite series of equations

$$XD_i(F_k)\big|_{[F]} = 0, \quad XD_iD_j(F_k)\big|_{[F]} = 0, \ldots.$$

Thus, one arrives at the following finite test for calculating Lie–Bäcklund symmetries of differential equations.

Theorem 1.5.8 *The operator* (1.5.1) *is an infinitesimal Lie–Bäcklund symmetry for* (1.5.22) *if and only if the following equations hold*:

$$XF_k\big|_{[F]} = 0, \quad k = 1, \ldots, s. \tag{1.5.24}$$

Equations (1.5.24) *are the determining equations for Lie–Bäcklund symmetries.*

Remark 1.5.1 Every operator of the form (1.5.5), i.e. $X_* = \xi_*^i D_i \in L_{\mathscr{B}}$ is an infinitesimal Lie–Bäcklund symmetry for any system of differential equations. Furthermore all operators (1.5.1) satisfying the conditions

$$\xi^i\big|_{[F]} = 0, \quad \eta^\alpha\big|_{[F]} = 0 \tag{1.5.25}$$

solve the determining equations (1.5.24). All operators $X_* \in L_{\mathscr{B}}$ and the operators obeying the conditions (1.5.25) are termed *trivial Lie–Bäcklund symmetries* ([7], Sect. 1.3.2).

Example 1.5.6 The equations of motion of a planet (Kepler's problem):

$$m\frac{d^2x^k}{dt^2} = \mu\frac{x^k}{r^3}, \quad k = 1, 2, 3,$$

have the following three nontrivial infinitesimal Lie–Bäcklund symmetries different from Lie point and contact symmetries (see [4]):

$$X_i = \Big(2x^i v^k - x^k v^i - (\boldsymbol{x}\cdot\boldsymbol{v})\delta_i^k\Big)\frac{\partial}{\partial x^k}, \quad i = 1, 2, 3.$$

Here the independent variable is time t, the dependent variables are the coordinates of the position vector $\boldsymbol{x} = (x^1, x^2, x^3)$ of the planet. The vector $\boldsymbol{v} = (v^1, v^2, v^3)$ is the velocity of the planet, i.e. $\boldsymbol{v} = d\boldsymbol{x}/dt$.

1.6 Approximate Transformation Groups

A detailed discussion of the material presented here as well as of the theory of multi-parameter approximate groups can be found in [2].

1.6.1 Approximate Transformations and Generators

In what follows, functions $f(x,\varepsilon)$ of n variables $x=(x^1,\ldots,x^n)$ and a parameter ε are considered locally in a neighborhood of $\varepsilon=0$. These functions are continuous in the x's and ε, as are also their derivatives to as high an order as enters in the subsequent discussion.

If a function $f(x,\varepsilon)$ satisfies the condition

$$\lim_{\varepsilon\to 0}\frac{f(x,\varepsilon)}{\varepsilon^p}=0,$$

it is written $f(x,\varepsilon)=o(\varepsilon^p)$ and f is said to be *of order less than* ε^p. If

$$f(x,\varepsilon)-g(x,\varepsilon)=o(\varepsilon^p),$$

the functions f and g are said to be *approximately equal* (with an error $o(\varepsilon^p)$) and written

$$f(x,\varepsilon)=g(x,\varepsilon)+o(\varepsilon^p),$$

or, briefly $f\approx g$ when there is no ambiguity.

The approximate equality defines an equivalence relation, and we join functions into equivalence classes by letting $f(x,\varepsilon)$ and $g(x,\varepsilon)$ to be members of the same class if and only if $f\approx g$.

Given a function $f(x,\varepsilon)$, let

$$f_0(x)+\varepsilon f_1(x)+\cdots+\varepsilon^p f_p(x)$$

be the approximating polynomial of degree p in ε obtained via the Taylor series expansion of $f(x,\varepsilon)$ in powers of ε about $\varepsilon=0$. Then any function $g\approx f$ (in particular, the function f itself) has the form

$$g(x,\varepsilon)\approx f_0(x)+\varepsilon f_1(x)+\cdots+\varepsilon^p f_p(x)+o(\varepsilon^p).$$

Consequently the function

$$f_0(x)+\varepsilon f_1(x)+\cdots+\varepsilon^p f_p(x)$$

is called a *canonical representative* of the equivalence class of functions containing f.

Thus, the equivalence class of functions $g(x,\varepsilon)\approx f(x,\varepsilon)$ is determined by the ordered set of $p+1$ functions

$$f_0(x),\ f_1(x),\ \ldots,\ f_p(x).$$

In the theory of approximate transformation groups, one considers ordered sets of smooth vector-functions depending on x's and a group parameter a:

$$f_0(x,a),\ f_1(x,a),\ \ldots,\ f_p(x,a)$$

with coordinates

$$f_0^i(x,a),\ f_1^i(x,a),\ \ldots,\ f_p^i(x,a),\quad i=1,\ldots,n.$$

Let us define the one-parameter family G of *approximate transformations*

$$\bar{x}^i \approx f_0^i(x,a) + \varepsilon f_1^i(x,a) + \cdots + \varepsilon^p f_p^i(x,a), \quad i = 1,\dots,n, \tag{1.6.1}$$

of points $x = (x^1,\dots,x^n) \in R^n$ into points $\bar{x} = (\bar{x}^1,\dots,\bar{x}^n) \in R^n$ as the class of invertible transformations

$$\bar{x} = f(x,a,\varepsilon) \tag{1.6.2}$$

with vector-functions $f = (f^1,\dots,f^n)$ such that

$$f^i(x,a,\varepsilon) \approx f_0^i(x,a) + \varepsilon f_1^i(x,a) + \cdots + \varepsilon^p f_p^i(x,a).$$

Here a is a real parameter, and the following condition is imposed:

$$f(x,0,\varepsilon) \approx x.$$

Furthermore, it is assumed that the transformation (1.3.2) is defined for any value of a from a small neighborhood of $a = 0$, and that, in this neighborhood, the equation $f(x,a,\varepsilon) \approx x$ yields $a = 0$.

Definition 1.6.1 The set of transformations (1.6.1) is called a one-parameter approximate transformation group if

$$f(f(x,a,\varepsilon),b,\varepsilon) \approx f(x,a+b,\varepsilon)$$

for all transformations (1.6.2).

Remark 1.6.1 Here, unlike the classical Lie group theory, f does not necessarily denote the same function at each occurrence. It can be replaced by any function $g \approx f$ (see the next example).

Example 1.6.1 Let us take $n = 1$ and consider the functions

$$f(x,a,\varepsilon) = x + a\left(1 + \varepsilon x + \frac{1}{2}\varepsilon a\right)$$

and

$$g(x,a,\varepsilon) = x + a(1+\varepsilon x)\left(1 + \frac{1}{2}\varepsilon a\right).$$

They are equal in the first order of precision, namely:

$$g(x,a,\varepsilon) = f(x,a,\varepsilon) + \varepsilon^2 \varphi(x,a), \quad \varphi(x,a) = \frac{1}{2}a^2 x,$$

and satisfy the approximate group property. Indeed,

$$f(g(x,a,\varepsilon),b,\varepsilon) = f(x,a+b,\varepsilon) + \varepsilon^2 \phi(x,a,b,\varepsilon),$$

where

$$\phi(x,a,b,\varepsilon) = \frac{1}{2}a(ax + ab + 2bx + \varepsilon abx).$$

The *generator of an approximate transformation group G* given by (1.6.2) is the class of first-order linear differential operators

$$X = \xi^i(x, \varepsilon)\frac{\partial}{\partial x^i} \tag{1.6.3}$$

such that

$$\xi^i(x, \varepsilon) \approx \xi_0^i(x) + \varepsilon\xi_1^i(x) + \cdots + \varepsilon^p\xi_p^i(x),$$

where the vector fields $\xi_0, \xi_1, \ldots, \xi_p$ are given by

$$\xi_\nu^i(x) = \left.\frac{\partial f_\nu^i(x, a)}{\partial a}\right|_{a=0}, \quad \nu = 0, \ldots, p;\ i = 1, \ldots, n.$$

In what follows, an approximate group generator

$$X \approx \big(\xi_0^i(x) + \varepsilon\xi_1^i(x) + \cdots + \varepsilon^p\xi_p^i(x)\big)\frac{\partial}{\partial x^i}$$

is written simply

$$X = \big(\xi_0^i(x) + \varepsilon\xi_1^i(x) + \cdots + \varepsilon^p\xi_p^i(x)\big)\frac{\partial}{\partial x^i}. \tag{1.6.4}$$

In theoretical discussions, approximate equalities are considered with an error $o(\varepsilon^p)$ of an arbitrary order $p \geq 1$. However, in the most of applications the theory is simplified by letting $p = 1$.

1.6.2 Approximate Lie Equations

Consider one-parameter approximate transformation groups in the first order of precision. Let

$$X = X_0 + \varepsilon X_1 \tag{1.6.5}$$

be a given approximate operator, where

$$X_0 = \xi_0^i(x)\frac{\partial}{\partial x^i}, \quad X_1 = \xi_1^i(x)\frac{\partial}{\partial x^i}.$$

The corresponding approximate transformation group of points x into points $\bar{x} = \bar{x}_0 + \varepsilon\bar{x}_1$ with the coordinates

$$\bar{x}^i = \bar{x}_0^i + \varepsilon\bar{x}_1^i \tag{1.6.6}$$

is determined by the following equations:

$$\frac{d\bar{x}_0^i}{da} = \xi_0^i(\bar{x}_0), \quad \bar{x}_0^i\big|_{a=0} = x^i, \quad i = 1, \ldots, n, \tag{1.6.7}$$

$$\frac{d\bar{x}_1^i}{da} = \sum_{k=1}^n \left.\frac{\partial\xi_0^i(x)}{\partial x^k}\right|_{x=\bar{x}_0} \bar{x}_1^k + \xi_1^i(\bar{x}_0), \quad \bar{x}_1^i\big|_{a=0} = 0. \tag{1.6.8}$$

The equations (1.6.7)–(1.6.8) are called the *approximate Lie equations*.

Example 1.6.2 Let $n = 1$ and let

$$X = (1 + \varepsilon x)\frac{\partial}{\partial x}.$$

Here $\xi_0(x) = 1$, $\xi_1(x) = x$, and equations (1.6.7)–(1.6.8) are written:

$$\frac{d\bar{x}_0}{da} = 1, \quad \bar{x}_0\big|_{a=0} = x,$$
$$\frac{d\bar{x}_1}{da} = \bar{x}_0, \quad \bar{x}_1\big|_{a=0} = 0.$$

Its solution has the form

$$\bar{x}_0 = x + a, \quad \bar{x}_1 = ax + \frac{a^2}{2}.$$

Hence, the approximate transformation group is given by

$$\bar{x} \approx x + a + \varepsilon\Big(ax + \frac{a^2}{2}\Big).$$

Example 1.6.3 Let $n = 2$ and let

$$X = (1 + \varepsilon x^2)\frac{\partial}{\partial x} + \varepsilon xy\frac{\partial}{\partial y}.$$

Here $\xi_0(x, y) = (1, 0)$, $\xi_1(x, y) = (x^2, xy)$, and (1.6.7)–(1.6.8) are written:

$$\frac{d\bar{x}_0}{da} = 1, \quad \frac{d\bar{y}_0}{da} = 0, \quad \bar{x}_0\big|_{a=0} = x, \quad \bar{y}_0\big|_{a=0} = y,$$
$$\frac{d\bar{x}_1}{da} = (\bar{x}_0)^2, \quad \frac{d\bar{y}_1}{da} = \bar{x}_0\bar{y}_0, \quad \bar{x}_1\big|_{a=0} = 0, \quad \bar{y}_1\big|_{a=0} = 0.$$

The integration gives the following approximate transformation group:

$$\bar{x} \approx x + a + \varepsilon\Big(ax^2 + a^2x + \frac{a^3}{3}\Big), \quad \bar{y} \approx y + \varepsilon\Big(axy + \frac{a^2}{2}y\Big).$$

1.6.3 Approximate Symmetries

Let G be a one-parameter approximate transformation group given by

$$\bar{z}^i \approx f(z, a, \varepsilon) \equiv f_0^i(z, a) + \varepsilon f_1^i(z, a), \quad i = 1, \ldots, N. \qquad (1.6.9)$$

An approximate equation

$$F(z, \varepsilon) \equiv F_0(z) + \varepsilon F_1(z) \approx 0 \qquad (1.6.10)$$

is said to be *approximately invariant* with respect to G if

$$F(\bar{z}, \varepsilon) \approx (F(f(z, a, \varepsilon), \varepsilon) = o(\varepsilon)$$

whenever $z = (z^1, \ldots, z^N)$ satisfies (1.6.10).

If $z = (x, u, u_{(1)}, \ldots, u_{(k)})$, then (1.6.10) becomes an approximate differential equation of order k, and G is an *approximate symmetry group* of this differential equation.

For example, the second-order equation

$$y'' - x - \varepsilon y^2 = 0 \tag{1.6.11}$$

has no exact point symmetries if $\varepsilon \neq 0$ is regarded as a constant coefficient, and hence cannot be integrated by the Lie method. Moreover, this equation cannot be integrated by quadrature. However, it possesses approximate symmetries if ε is treated as a small parameter, e.g.

$$\begin{aligned} X_1 &= \frac{\partial}{\partial y} + \frac{\varepsilon}{3}\left[2x^3\frac{\partial}{\partial x} + \left(3yx^2 + \frac{11}{20}x^5\right)\frac{\partial}{\partial y}\right],\\ X_2 &= x\frac{\partial}{\partial y} + \frac{\varepsilon}{6}\left[x^4\frac{\partial}{\partial x} + \left(2yx^3 + \frac{7}{30}x^6\right)\frac{\partial}{\partial y}\right]. \end{aligned} \tag{1.6.12}$$

The operators (1.6.12) span a two-dimensional approximate Lie algebra and can be used for consecutive integration of (1.6.11) (see [8], Sect. 12.4).

For a detailed discussion of *approximate symmetries* of differential equations with a small parameter as well as numerous examples we refer the reader to [7], Chaps. 2 and 9, and to the references therein.

References

1. Bäcklund, A.V.: Ueber Flächentransformationen. Math. Ann. **IX**, 297–320 (1876)
2. Baikov, V.A., Gazizov, R.K., Ibragimov, N.H.: Approximate symmetries. Math. Sb. **136(178)**(3), 435–450 (1988). English transl.: Math. USSR Sb. **64**(2), (1989)
3. Ibragimov, N.H.: Sur l'équivalence des équations d'évolution, qui admettent une algébre de Lie–Bäcklund infinie. C. R. Acad. Sci. Paris Sér. I **293**, 657–660 (1981)
4. Ibragimov, N.H.: Transformation Groups in Mathematical Physics. Nauka, Moscow (1983). English transl.: Transformation Groups Applied to Mathematical Physics. Riedel, Dordrecht (1985)
5. Ibragimov, N.H.: Group analysis of ordinary differential equations and the invariance principle in mathematical physics (for the 150th anniversary of Sophus Lie). Usp. Mat. Nauk **47**(4), 83–144 (1992) English transl.: Russ. Math. Surv. **47**(2), 89–156 (1992)
6. Ibragimov, N.H. (ed.): CRC Handbook of Lie Group Analysis of Differential Equations, vol. 2: Applications in Engineering and Physical Sciences. CRC Press, Boca Raton (1995)
7. Ibragimov, N.H. (ed.): CRC Handbook of Lie Group Analysis of Differential Equations, vol. 3: New Trends in Theoretical Developments and Computational Methods. CRC Press, Boca Raton (1996)
8. Ibragimov, N.H.: Elementary Lie Group Analysis and Ordinary Differential Equations. Wiley, Chichester (1999)
9. Ibragimov, N.H. (ed.): Lie Group Analysis: Classical Heritage. ALGA Publications, Karlskrona (2004)
10. Ibragimov, N.H.: A Practical Course in Differential Equations and Mathematical Modelling, 3rd edn. ALGA Publications, Karlskrona (2006)
11. Ibragimov, N.H.: Optimal system of invariant solutions for the Burgers equation. In: Lecture at the MOGRAN-12 Conference, Porto, July 28–31, 2008

12. Lie, S.: Über die Integration durch bestimmte Integrale von einer Klasse linearer partieller Differentialgleichungen. Arch. Math. **6**(3), 328–368 (1881). English transl. in: Ibragimov, N.H. (ed.) CRC Handbook of Lie Group Analysis of Differential Equations, vol. 2: Applications in Engineering and Physical Sciences. CRC Press, Boca Raton (1995). Reprinted also in: Ibragimov, N.H. (ed.) Lie Group Analysis: Classical Heritage. ALGA Publications, Karlskrona (2004)
13. Lie, S.: Vorlesungen über Differentialgleichungen mit bekannten infinitesimalen Transformationen. Bearbeited und herausgegeben von Dr. G. Scheffers. Teubner, Leipzig (1891)
14. Olver, F.W.J.: Asymptotics and Special Functions. Academic Press, San Diego (1974)
15. Olver, P.J.: Applications of Lie Groups to Differential Equations, Springer, New York (1986). 2nd edn. (1993)
16. Ovsyannikov, L.V.: Group Properties of Differential Equations. Siberian Branch, USSR Academy of Sciences, Novosibirsk (1962) (in Russian)
17. Polyanin, A.D., Zaitsev, V.F.: Handbook of Exact Solutions for Ordinary Differential Equations. CRC Press, Boca Raton (1995)