Mirja Hartimo
Editor

Phenomenology and Mathematics

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CONTRIBUTORS

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Mirja Hartimo received her PhD from Boston University in 2005. After a brief period at Tufts, she has been working as a researcher at the University of Tampere, Finland, and, beginning of 2009, at the University of Helsinki. Currently she has an Academy of Finland post doc position. She is an author of many articles primarily on E. Husserl's philosophy and mathematics, published in journals such as *Synthese*, *Journal of History and Philosophy of Logic*, and *Southern Journal of Philosophy*.

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Jaakko Hintikka was born in Finland in 1929 and introduced into philosophical research by G.H. von Wright. He was a Junior Fellow at Harvard in 1956–59 and has since held academic positions in Finland, where he trained most of a generation of philosophers, and in the United States, most recently at Boston University. Hintikka has been instrumental in developing epistemic logic, possible-worlds semantics, the interrogative model of inquiry, independence-friendly logic, and a reduction of higher-order logic to the first-order logic. He has also studied history of philosophy, including Aristotle, Descartes, Kant, Peirce, Hilbert, and most extensively Wittgenstein.

Dieter Lohmar is Professor of Philosophy at the University of Cologne. He has studied mathematics and philosophy in Cologne, Bonn and Wuppertal. His most important publications include *Phänomenologie der Mathematik* (Kluwer, Dordrecht 1989), *Erfahrung und kategoriales Denken* (Kluwer, Dordrecht 1998), *Edmund Husserls* Formale und transzendentale Logik, (Wissenschaftliche Buchgesellschaft Darmstadt 2000) (commentary), *Phänomenologie der schwachen Phantasie* (Springer, Dordrecht 2008), as well as many articles.

Guillermo E. Rosado Haddock is Professor at the University of Puerto Rico at Río Piedras since 1974. He obtained his PhD from the University of Bonn in 1973 with a thesis on Husserl's philosophy of logic and mathematics. Rosado Haddock has published more than thirty-five papers in more than a dozen international journals with prestigious publishers. He is the co-author with Claire Ortiz Hill of a book on Husserl and Frege (Open Court 2000), as well as the author of a book on Frege (Ashgate 2006) and one on Carnap (Ashgate 2008).

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Olav K. Wiegand (Ph.D., Mainz University, 1995) teaches philosophy at the University of Mainz, Germany. He has been a visiting scholar at the M.I.T. (1998) and New York University (1998 and 1999), there working with Kit Fine. His publications include *Interpretationen der Modallogik* (Kluwer, 1998) and articles on Husserl, formal logic, and Kant. Together with R. Dostal, L. Embree, J.N. Mohanty, and J. Kockelmans, he is an editor of a *Festschrift* for Thomas Seebohm entitled *Phenomenology on Kant, German Idealism, Hermeneutics, and Logic* (Kluwer, 2000). Since 2005, he has worked as healthsystem consultant for a private company, but he still also teaches at Mainz University.

LIST OF ABBREVIATIONS

APS *Analyses Concerning Passive and Active Synthesis: Lectures on Transcendental Logic.* Translated by Anthony J. Steinbock. Kluwer: Dordrecht, Boston, London. 2001.

CM English translation of Hua 1: *Cartesian Meditations*. Translated by Dorion Cairns. The Hague: Nijhoff. 1970.

Crisis English translation of Hua 6: *The Crisis of European Sciences and Transcendental Phenomenology*. Translated by David Carr. Evanston: Northwestern University Press. 1970.

EU English translation of *Erfahrung und Urteil* [1939]: *Experience and Judgment.* Translated by J. Churchill and K. Ameriks. Evanston: Northwestern University Press. 1973.

FTL English translation of Hua 17: *Formal and Transcendental Logic*. Translated by Dorion Cairns. The Hague: Martinus Nijhoff, 1978.

Ideas I English translation of Hua 3: *Ideas*, *General Introduction to Pure Phenomenology*. Translated by Boyce Gibson, New York: Collier Books. 1962.

LI 1-6 English translation of Hua 19: *Logical Investigations* Vols. 1–2. Translated by J. N. Findlay. London: Routledge & Kegan Paul, 1970.

PA English translation of Hua 12: *Philosophy of Arithmetic, Psychological and Logical Investigations with Supplementary Texts from 1887–1901*. Translated by Dallas Willard, Dordrecht: Kluwer, 2003.

Prolegomena English translation of Hua 18: *Logical Investigations* 1. Translated by J. N. Findlay. London: Routledge & Kegan Paul, 1970.

Husserliana volumes referred to

Hua 1 *Cartesianische Meditationen und Pariser Vorträge.* [Cartesian meditations and the Paris lectures.] Edited by S. Strasser. The Hague, Netherlands: Martinus Nijhoff, 1973.

Hua 3 *Ideen zu einer reinen Phänomenologie und phänomenologischen Philosophie.* Erstes Buch: Allgemeine Einführung in die reine Phänomenologie. [Ideas: general introduction to pure phenomenology and to a phenomenological philosophy. First book.] Edited by Walter Biemel. The Hague, Netherlands: Martinus Nijhoff Publishers, 1950.

Hua 7 *Erste Philosophie (1923/4).* Erste Teil: Kritische Ideengeschichte. [First philosophy (1923/24). First part: the critical history of ideas.] Edited by Rudolf Boehm. The Hague, Netherlands: Martinus Nijhoff, 1956.

Hua 9 *Phänomenologische Psychologie.* Vorlesungen Sommersemester. 1925. [Phenomenological psychology. Lectures from the summer semester. 1925.] Edited by Walter Biemel. The Hague, Netherlands: Martinus Nijhoff, 1968.

Hua 10 *Zur Phänomenologie des inneren Zeitbewusstseins (1893–1917).* [The Phenomenology of internal time-consciousness (1893–1917).] Edited by Rudolf Boehm. The Hague, Netherlands: Martinus Nijhoff, 1969.

Hua 12 *Philosophie der Arithmetik.* Mit ergänzenden Texten (1890–1901). [Philosophy of arithmetic. With complementary texts. (1890–1901).] Edited by Lothar Eley. The Hague, Netherlands: Martinus Nijhoff, 1970.

Hua 18 *Logische Untersuchungen.* Erster Teil. Prolegomena zur reinen Logik. Text der 1. und der 2. Auflage. [Logical investigations: first part. Prolegomena to pure logic. Text of the first and second edition.] Halle: 1900, rev. ed. 1913. Edited by Elmar Holenstein. The Hague, Netherlands: Martinus Nijhoff, 1975.

Hua 19 *Logische Untersuchungen.* Zweiter Teil. Untersuchungen zur Phänomenologie und Theorie der Erkenntnis. In zwei Bänden. [Logical investigations. Second part. Investigations concerning phenomenology and the theory of knowledge. In two volumes.] Edited by Ursula Panzer. Halle: 1901; rev. ed. 1922. The Hague, Netherlands: Martinus Nijhoff, 1984.

Hua 20/1 *Logische Untersuchungen. Ergänzungsband.* Erster Teil. Entwürfe zur Umarbeitung der VI. Untersuchung und zur Vorrede für die Neuauflage der Logischen Untersuchungen (Sommer 1913). [Logical investigations. Supplementary volume. Draft plan for the revision of the 6th Logical Investigation and the foreword of the Logical Investigations (Summer 1913).] Edited by Ulrich Melle. The Hague, Netherlands: Kluwer Academic Publishers, 2002.

Hua 21 *Studien zur Arithmetik und Geometrie.* Texte aus dem Nachlass (1886–1901). [Studies on arithmetic and geometry. Texts from the estate (1886–1901)]. Edited by Ingeborg Strohmeyer. The Hague, Netherlands: Martinus Nijhoff, 1983.

Hua 22 *Aufsätze und Rezensionen (1890–1910).* [Articles/essays and reviews (1890–1910).] Edited by B. Rang. The Hague, Netherlands: Martinus Nijhoff, 1979.

Hua 24 *Einleitung in die Logik und Erkenntnistheorie.* Vorlesungen 1906/07. [Introduction to logic and the theory of knowledge. Lectures 1906/07]. Edited by Ullrich Melle. The Hague, Netherlands: Martinus Nijhoff, 1985.

Hua 26 *Vorlesungen über Bedeutungslehre.* Sommersemester 1908. [Lectures on the doctrine of meaning: summer semester 1908.] Edited by Ursula Panzer. The Hague, Netherlands: Martinus Nijhoff, 1987.

Hua 28 *Vorlesungen über Ethik und Wertlehre.* 1908–1914. [Lectures on ethics and value theory, 1908–1914.] Edited by Ullrich Melle. The Hague, Netherlands: Kluwer Academic Publishers, 1988.

Hua 30 *Logik und allgemeine Wissenschaftstheorie.* Vorlesungen 1917/18. Mit ergänzenden Texten aus der ersten Fassung 1910/11. [Logic and general theory of science. Lectures 1917/18, with complementary texts from the first version 1910/11.] Edited by Ursula Panzer. The Hague, Netherlands: Kluwer Academic Publishers, 1995.

Briefwechsel 1–10 Husserl, Edmund. 1994. *Briefwechsel*. [Correspondence.] Edited by Karl Schuhmann. The Hague, Netherlands: Kluwer Academic Publishers.

INTRODUCTION

It is beginning to be a commonplace that Edmund Husserl (1859–1938), the founder of the phenomenological movement, was originally a mathematician who studied with Weierstrass and Kronecker. The roots of the phenomenological tradition are in the nineteenth century mathematics and logic, very much like those of the analytic tradition. As analytic philosophy has grown to view itself as a historically conditioned tradition the relationship between Husserl and other nineteenth century logicians and mathematicians have become a focus of much research. An early pioneer was Dagfinn Føllesdal's classic (1958) that appeared in English in Leila Haaparanta's (1994) collection of essays *Mind, Meaning, and Mathematics*. In it Føllesdal suggested that Frege possibly influenced Husserl to turn away from psychologism. The paper initiated a debate that has continued ever since (e.g., Rosado-Haddock 1973 and together with Claire Hill 2000, J.N. Mohanty 1974 and 1982, see also Chapters 2 and 3 in the present volume).

The publication of thousands of pages of Husserl's writings has been another important factor shaping the research on Husserl. Indeed, of 40 *Husserliana* volumes that had appeared by 2005, 20 has appeared after 1980 and 9 after 2000, the whole *Dokumente* series has appeared after 1980, and the whole *Materialien* series has appeared after 2000. Much of the newly published texts are about mathematics and logic. In particular, almost all of early Husserl's writings on mathematics are available to the general public in *Husserliana* volumes 21 (1983) on arithmetics and geometry, 22 (*Aufsätze und Rezensionen*), and of course 12 (*Philosophy of Arithmetic*). Thanks to Dallas Willard's monumental work, the Husserliana volumes 12 and 21 have appeared also in English as the Collected Works 5 (1994) and 10 (2003). Likewise Husserl's lectures on logic from various years published in the *Materialien* series document the development of Husserl's views on logic from 1896 onwards. The vast amounts of new material available has deepened the scholarship considerably and has been the focus of for example Hill (1991, 2000), Tieszen (1989, 2005), Seebohm (1991), and Lohmar (1989, 2000). Arguably, the growing amount of historical research on Husserl's development has paved the way for overcoming the juxtaposition between the analytic and continental traditions.

Presumably also at least partly due to the interest in the common roots of phenomenological and analytical traditions, a growing amount of research has focused on the development of modern logic, mathematics and physics in the nineteenth and the early twentieth century producing books such as *Die Philosophie und die Mathematik: Oskar Becker in der mathematischen Grundlagendiskussion* (2005) edited by Volker Peckhaus, *The Architecture of Modern Mathematics* (2006) by Jeremy Gray and Jose Ferreiros, *Intuition and the Axiomatic Method* (2006) edited by Emily Carson and Renate Huber, the forthcoming edition of David Hilbert's lectures on the foundations of mathematics and physics 1891–1933 by Michael Hallett and Ulrich Majer (Volume 1 has appeared when writing this introduction), and *The Development of Modern Logic* (2009), edited by Leila Haaparanta. In these works the development of mathematics and logic during Husserl's time is discussed and interest to Husserl's possible contribution to it is shown. Moreover, there are attempts to include discussion of Husserl to the history of the late nineteenth century logic. For example, Gabbay's *Handbook on History of Logic* (2004) has an extensive chapter on Husserl's logic written by Richard Tieszen, not to mention the special issue of *Philosophia Mathematica* on phenomenology and mathematics that appeared in June 2002.

The above mentioned historical approaches to Husserl and mathematics focus particularly on Husserl's early writings on mathematics and logic. Another, more systematic approach comes from the attempts to situate also later Husserl's thoughts on mathematics in the general *Grundlagenstreit* of the early twentieth century discussions on the foundations of mathematics. At this time Husserl's own writings focused on more general questions in phenomenology and his writings on mathematics and logic were rather suggestive. Moreover, Husserl's manuscript A I 35 that is arguably one of the most important texts on mathematics in Husserl's later works and is referred to several times in the present volume is only now being published. Husserlian legacy continued mainly, in more or less faithful way, in the work of Oskar Becker and Hermann Weyl. Thus many studies related to the phenomenology of mathematics focus rather on these figures than on Husserl himself (e.g., Mancosu and Ryckman 2002, 2005). The comparisons of Husserl' s approach to the intuitionists, Brouwer (van Atten), in particular, are closely related to these works as well as now already rather numerous investigations discussing Gödel's background in Husserl (Føllesdal, Van Atten, Kennedy). Upon publication of Heinz-Dieter Ebbinghaus's book on Ernst Zermelo (2007), one hopes to read papers elaborating on Husserl's relationship to Zermelo soon too.

As becomes clear from the above, Husserl's views in relation to history and philosophy of mathematics and logic provide us an incredibly rich field for research. The *Husserliana* volumes offer enormously material for internal studies on Husserl's development for also those working outside the archives. Husserl wrote at the time when modern logic and mathematics were rapidly developing toward their current outlook. Thus his writings can also be fruitfully compared and contrasted with both nineteenth century figures such as Boole, Schröder and Weierstrass as well as the twentieth century characters like Heyting, Zermelo, and Gödel. Besides the more historical studies, both the internal ones on Husserl alone and the external ones attempting to clarify his role in the more general context of the developing mathematics and logic, the field has also systematic importance. Indeed, one motivation of the present volume is to make sense of Husserl's transcendental idealism in mathematics.

The volume at hand manifests all the above mentioned aspects in which Husserl's views on mathematics are of interest and can be studied. It gathers the contributions of the main scholars of the field into one publication for the first time. Thus it gives an overview of the current debates and themes in the phenomenology of mathematics. The systematic and historical approaches are intertwined in the contributions. Ultimately, the papers chart answers to the question "What kind of philosophy of mathematics is phenomenology?" In the course of answering this question Husserl's philosophy of mathematics is compared and contrasted to the constructivist as well as various kinds of Platonist views of mathematics.

In Chapter 1 “Mathematical Realism and Transcendental Phenomenological Idealism” Richard Tieszen addresses the question whether mathematical realism is compatible with transcendental phenomenological idealism. His answer is that the views are indeed compatible provided that neither “idealism” nor “realism” are understood in their naïve senses. Rather they should be understood in accordance to Husserl’s transcendental phenomenology in which a kind of Platonism is embedded within transcendental idealism. Tieszen calls the view “constituted Platonism” or “constituted realism.” Tieszen first considers the standard simple formulations of realism and idealism (anti-realism) about mathematics. Mathematical realism is the view that there are mind-independent abstract (or “ideal”) mathematical objects or truths; the standard antirealist view is the negation of this view. To Tieszen Husserl’s view about mathematical objects is Platonist rather than Aristotelian realist. Tieszen then goes on to consider transcendental phenomenological idealism. With phenomenological reduction we can accomplish the point of view of reflection and focus on how objects are given to us. Now we can find out that the ideal objects such as objects of mathematics are constituted as transcendent objects, which explains the choice of his term “constituted Platonism.” Tieszen then distinguishes several conceptions of how objects can be considered mind-dependent and mind-independent or immanent and transcendent. Naive mind-independence is metaphysical Platonism. But within the mind-dependent sphere there is also mind-independence, the constituted realism of Husserl. In the end Tieszen raises interesting questions about which mathematical objects are constituted as real as well as a question about the compatibility of Husserl’s view with Putnam’s internal realism.

The next two chapters purport to demonstrate that Husserl was not Brouwerian intuitionist, nor constructivist of any sort. In his “Platonism, Phenomenology and Interderivability,” Guillermo Rosado Haddock defends the view that Husserl is a Platonist rather than a constructivist philosopher. Van Atten has held that this is not so obvious when Husserl’s later texts are taken into consideration. Rosado Haddock, on basis of Husserl’s manuscript A I 35 from the years 1912 and 1920, aligns Husserl with Cantor and Zermelo. According to him, here, in *Formale und transzendentale Logik* (1929) as well as in his lectures from 1906–1907 and of 1917–1918, Husserl propounds essentially the same Platonist philosophy of mathematics as in his earlier texts. Rosado Haddock then examines the interderivability phenomena, i.e., that equivalent mathematical statements can be found in seemingly unrelated parts of mathematics, arguing that while Husserl's Platonism is able to tackle the issue, a constructivist has difficulties in accounting for it. Rosado Haddock then discusses Husserl's concept of situations of affairs (*Sachlage*) as distinguished from his notion of states of affairs (*Sachverhalt*). With this distinction Husserl is thus able to assess the interderivability phenomena (also contrary to Frege). The outcome is that while e.g., the Axiom of Choice and Tychonoff's Theorem refer to different states of affairs, their situation of affairs remains the same.

In her "Husserl on Axiomatization and Arithmetic" Claire Ortiz Hill complements Rosado Haddock's chapter by a more systematic demonstration of the extent to which Husserl rejected Brouwer's intuitionism. She draws from the newly published Husserl's logic courses from 1896 and 1902/03. She explores various aspects of the axiomatic nature of Husserl's logic, ultimately arguing that Husserl's approach is much closer to that of David Hilbert than Brouwer's intuitionism. In so doing, she touches upon Husserl's view of the relationship between mathematics and logic, arguments against psychologism, objectivity of meaning, the law of the excluded middle, axiomatic account of number, Husserl's account of time, Husserl's view of the three levels of logic and the theory of manifolds, and the relationship between mathematics and phenomenology. In all of these respects Husserl's views differ from those of Brouwer. She concludes her contribution raising the question about the systematic importance of Husserl's views, which, she thinks, should be given a try next, once the relationship of his ideas to Frege's, Brouwer's, and Hilbert's theories has been clarified.

Dieter Lohmar's chapter (Chapter 4) then focuses on Husserl's notion of categorial intuition emphasizing the empirical side of Husserl's approach. Dieter Lohmar's paper focuses on Husserl's notion of intuition and evidence in formal contexts. In mathematics we can gain the highest form of evidence which is apodictic evidence. This takes place by means of what Husserl calls *Wesensschau*, which is a special case of categorial intuition. Lohmar's aim is to explain how apodictic evidence is gained in mathematics by means of the method of eidetic variation.

Lohmar starts with an analysis of the sensible givenness of objects, noting that already the sensual perception of an object exceeds what is actually given by our senses. Intentions such as "This book is green" are fulfilled not only with sensuality but also something more that relates back to our thinking activity, namely "synthesis of coincidence." Mathematical knowledge is in most cases independent of sensuality but nevertheless intuitive. Lohmar then goes on to discuss specifically mathematical knowledge where the syntheses of coincidence are more easily structured and more distinct due to the lack of horizonal intentions, which usually accompany the intending of everyday objects. Lohmar then goes on to explain the method of seeing essences, i.e., *Wesensschau* with which a priori insight into the universal structures of consciousness, sounds, colors, as well as, geometry, arithmetic and other parts of mathematics can be obtained. Lohmar summarizes the development of the method in Husserl's texts and settles to detail it on basis of Husserl's final form of the method worked out in the lecture *Phenomenological Psychology*. In it an element of sense that I can freely go on with the variation is added to the process of variation idealizing it so that it can be termed an "infinite variation." Lohmar goes on to explain in detail various stages in the method of eidetic variation, with which phenomenological a priori is obtained. Lohmar then discusses several examples showing how apodictic evidence can be obtained first in material mathematical disciplines and then in formal axiomatic contexts.

In Chapter 5 Jaakko Hintikka compares Husserl's views to e.g., Mach, Russell, Wittgenstein, and Gödel, and points out the importance of Husserl's view of the theory of theories. Hintikka starts his contribution with a discussion of what phenomenology is and settles to the sense of the term that derives from Mach and Hering, and, according to Hintikka, is the sense in which Husserl himself uses it too. Accordingly Hintikka finds Husserl's and Mach's approaches rather similar disregarding their views of mathematics. While mathematics to Mach is tautological, Husserl sought for a much richer approach to mathematics. Hintikka examines Husserl's notion of *Anschauung*, which Hintikka takes Husserl to understand in a minimal sense as "immediate knowledge," a counterpart to Russellian acquaintance. Husserl developed this into *Wesensschau*, which provides us with an access to what Husserl calls "formal ontology" as well as various "regional ontologies." Hintikka compares Husserl's view to Aristotle, and then to Wittgenstein, whom Hintikka takes to be closer to Husserl than is usually thought. Wittgenstein's later criticism of Husserl, Hintikka views as directed at expressibility of the testimony of *Wesensschau*, and not at the possibility of a kind of *Wesensschau*. This is the source for the impossibility of a genuine theory of logical forms to Wittgenstein. Hintikka then uses Husserl's Aristotelianism to explain why Husserl is not a finitist or an intuitionist as Husserl held that the human mind is able to grasp directly abstract structures. Behind this view is Husserl's vision of a universal "structure of all structures" or "model of all models," which Hintikka suggests was tacitly in the minds of many mathematicians contemporary to Husserl. While the comprehensive ideas such as the set of all sets have turned out to be difficult to implement, Husserl's view of logic still lives in the model theory. Hintikka concludes his discussion with a brief remark on Gödel. The comparison between Husserl and Gödel proves favorable for Husserl especially in philosophical respects.

Mirja Hartimo's contribution "Development of mathematics and the birth of phenomenology" ties Husserl's view to the more general development of mathematics in the late nineteenth century. Her aim is to connect Husserl's discovery of categorial intuition to his investigations into the development of mathematics in the late nineteenth century. She focuses in particular on Husserl's Weierstrassian heritage, which shows in Husserl's search for intuitively evident foundations for the basic concepts of mathematics. Following the mainstream view of mathematics Husserl adopts a structural, axiomatic view of mathematics by the turn of the century. Her view of Husserl's term *Definitheit* is that it means roughly the categoricity of an axiomatic system, i.e., that the axioms define the formal domain uniquely, up to isomorphism. However, contrary to Hilbert, Husserl remains Weierstrassian in that he continues to demand intuitively evident foundations for axiomatics. To that extent Husserl developed the notion of categorial intuition. Hartimo then goes on to discuss the consequences of the view to Husserl's approach toward reality. She distinguishes two senses in which Husserl can be said to be a Platonist. The other derives from Lotze and relates to Husserl's objectivist view of the formal objects. The other derives originally from Weiersrass and relates to the search for *justification* for the axiomatic systems.

In his chapter "Beyond Leibniz: Husserl's Vindication of Symbolic Knowledge," Jairo José da Silva gives another account of the development of Husserl's views of symbolic knowledge in mathematics up to his *Doppelvortrag* in 1901. He focuses on how Husserl struggled with the problem of how to explain that we can obtain knowledge by operating "blindly" with symbols according to rules, even when these symbols do not represent anything. The imaginaries are improper representations since they do not represent any object. Nevertheless in calculations they are needed and they pass off as denoting something.

Da Silva follows Husserl's development from his first discussion of the problem in the *Philosophy of Arithmetic* to his Göttingen talks. In the *Philosophy of Arithmetic* the problem had two variants, one concerning justification for arithmetical computations, the other is about the symbols 0 and 1. Husserl's solution to the former problem is that the symbolic system and its isomorphic copy, the system of number concepts, share a common formal structure. Husserl's reason for accepting 0 and 1 as numbers, according to da Silva, is that they are required as necessary completions of arithmetical domains. Husserl's 1891 review of Schröder's *Lectures on the Algebra of Logic* already anticipates Husserl's mature view, which he discusses in his Göttingen talks in 1901. In these lectures Husserl held that the introduction of imaginary elements in a domain is allowed provided: (1) the (formal) theory extending the (formal) theory of the domain in question by means of formal axioms introducing imaginaries in an extended language is consistent, and (2) the formal theory of the domain, written in the restricted language without imaginaries, is complete, in Husserl's terms, definite. The solution tells that imaginary entities can be treated like the real ones. Nonetheless, according to *Logical Investigations* symbolic theories are mere forms of theories. The creation and study of formal theories for their own sake, would amount to a "formalist alienation." Hence Da Silva suggests that Husserl's claim could be understood to be that formal theories are only interesting if they can be applied, thus emphasizing more instrumentalist or pragmatist aspects of Husserl's views. Imaginaries are useful as they enrich the structural milieu so that the problems can be better solved.

The last two chapters elaborate specifically on the systematic value of Husserl's philosophy and relate it to the more general present day discussions, first in philosophy of mathematics, second in metaphysics. In his "Mathematical Truth Regained," Robert Hanna seeks a fully realistic and inescapably anthropocentric conception of mathematical truth and knowledge—*real mathematics for humans*, as he puts it. In so doing, he offers what he calls "a positive Kantian phenomenological solution" to Benacerraf's Dilemma (BD) of reconciling a "standard" (i.e., classical Platonic) semantics with a reasonable epistemology of mathematical knowledge. Hanna's solution is positive because it accepts Benacerraf's preliminary philosophical assumptions about the nature of semantics and knowledge, as well as all the basic premises of BD, and then shows how we can still reject the skeptical conclusion BD and adequately explain mathematical knowledge. The solution relies on mathematical structuralism, i.e., on a view that mathematical entities are not ontologically autonomous or independent objects, but instead are essentially positions-in-a-mathematical-structure (a view not necessarily so far from Husserl's conception either). In particular, Hanna interprets mathematical objects with the role players of the roles determined by the system as a whole. However, Hanna favors specifically Kantian Structuralism which is a *non-reductive* and *ante rem* version of structuralism in contrast to Benacerraf's structuralism, which is reductive and *ante rem*. In Hanna's favorite kind of structuralism time-structure is what binds arithmetic to our world. With this view Hanna solves the BD. To Hanna, Husserl's views prove useful in explaining how mathematical knowledge is possible. He proposes what he calls the Husserl-Wittgenstein Theory of Logical and Mathematical Self-Evidence (the HW theory). The HW theory is based on Husserl's doctrine of categorial intuition and related views in Wittgenstein's *Tractatus*, according to which abstract structures are immediately represented in our non-conceptual conscious awareness.

The volume ends with Olav Wiegand's "On Referring to Gestalts," in which he explores mereological semantics on the basis of Husserl's phenomenology and Gestalt psychology leaning on the work of Aron Gurwitsch. The primary motivation of the paper is to formalize the notion of structured whole. In his paper, Wiegand defines Gestalts as "R-structured wholes" aiming to capture the interconnectedness of all parts of a Gestalt. He then discusses relations from the point of view of mereological semantics. In so doing Wiegand shows the usefulness of Husserl's views to contemporary formal semantics and metaphysics.

CHAPTER I

MATHEMATICAL REALISM AND TRANSCENDENTAL PHENOMENOLOGICAL IDEALISM

Richard Tieszen

Abstract. In this paper I investigate the question whether mathematical realism is compatible with Husserl's transcendental phenomenological idealism. The investigation leads to the conclusion that a unique kind of mathematical realism that I call "constituted realism" is compatible with and indeed entailed by transcendental phenomenological idealism. Constituted realism in mathematics is the view that the transcendental ego constitutes the meaning of being of mathematical objects in mathematical practice in a rationally motivated and non-arbitrary manner as abstract or ideal, non-causal, unchanging, non-spatial, and so on. The task is then to investigate which kinds of mathematical objects, e.g., natural numbers, real numbers, particular kinds of functions, transfinite sets, can be constituted in this manner. Various types of founded acts of consciousness are conditions for the possibility of this meaning constitution.

The main question I would like to address in this paper is this: is mathematical realism compatible with transcendental phenomenological idealism or not? In the discussion that follows I will use the expressions "mathematical realism" and "mathematical platonism" interchangeably. In a moment I will be much more specific in speaking about both mathematical realism and transcendental phenomenological idealism but, for now, let me just say that I am mainly interested in forms of mathematical realism that have appeared in the recent literature in the philosophy of mathematics, and that in speaking of transcendental phenomenological idealism I am thinking of the philosophical view that Husserl began to develop around 1907 or so, and that appears especially in works such as *The Idea of Phenomenology*, *Ideas I*, *Cartesian Meditations*, Part II of *Formal and Transcendental Logic*, and portions of the lectures on *The Phenomenology of the Consciousness of Internal Time* (Husserl 1991).

The division between realism and idealism in philosophy has of course a long history. For my purposes in this paper, it is in Kant's transcendental philosophy that we find the most important approach to the realism/idealism debate prior to Husserl. In his *Critique of Pure Reason*

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Kant thought he could reconcile empirical realism with his transcendental idealism. Kant was not what we nowadays think of as a mathematical realist (platonist), and he was in fact critical of platonism in general (see Kant 1973 A4/B8-A6/B10). For Kant there would not have been a question of trying to reconcile mathematical platonism and transcendental idealism. By 1907 or so, however, it is quite possible to read Husserl as attempting to do such a thing. I would say that it is thanks to Kant that we can consider the possibility of reconciling realism with transcendental idealism at all, and it is thanks to Husserl that we can consider the possibility of reconciling mathematical realism and transcendental phenomenological idealism.

In Husserlian phenomenology there is an old division between supporters of realism and supporters of transcendental idealism, and this division has its roots in the changes in Husserl's thinking that, as indicated above, began to take place around 1907. There has been a line of thinking according to which realism in phenomenology is incompatible with transcendental idealism in phenomenology. You must choose one or the other. In my view, however, this issue of the compatibility or incompatibility deserves further study. In particular, the issue has not been explored deeply enough in connection with the Kantian background. It also has not been explored fully enough in the case of mathematics, and especially in connection with developments that occurred in the foundations of mathematics after Husserl began to lose touch with this area of research. Thus, what I would like to do in this paper is to (i) characterize some recent forms of mathematical realism, (ii) present some of the core claims of transcendental phenomenological idealism from *Ideas I* and other writings, and then (iii) examine in more detail some of the issues about the compatibility of mathematical realism and transcendental phenomenological idealism.

The starting positions are these: either you can be a mathematical realist and not a transcendental phenomenological idealist, or you can be a transcendental phenomenological idealist and not a mathematical realist, or you can in some sense be both. You can certainly be neither. There are many philosophers who would embrace neither view. Husserl's own early work (prior to roughly 1900) arguably falls into this latter category (see, e.g., Hua 12; Hua 21; Husserl 1994; also Tieszen 2004). I will not discuss this option here, however, since it puts us outside the circle of ideas in Husserl's later work that I want to discuss.

Let me now give a brief opening characterization of mathematical realism and, on the other hand, of a standard idealism (or anti-realism) about mathematics.

I. Standard Simple Formulations of Realism and Idealism (Anti-Realism) About Mathematics

Mathematical Realism: There are mind-independent abstract (or "ideal") mathematical objects or truths. Notice that I am formulating this specifically for *mathematical* objects or truths. By "mathematical" I just mean the kinds of objects or truths that practicing mathematicians typically take themselves to be thinking about. This includes geometric objects, natural numbers, real numbers, complex or imaginary numbers, functions, groups, sets, or categories, and truths about these objects. I do not necessarily want to exclude other kinds of objects that platonists might take to exist, such as meanings, propositions, properties, concepts, or essences, but I do want to note that mathematicians themselves (unlike some logicians) do not typically take themselves to be talking directly about such things in their theories. Logicians who are platonists are more likely to talk about such things. I do not want to make too much of the difference at the moment but I will note that at least part of what is involved here is that propositions, properties, concepts, essences, and the like are usually thought of as overtly *intensional* objects, whereas this is not typical in the case of standard mathematical objects in classical mathematics. We should note, in any case, that one can be a platonist about extensional objects, intensional objects, or both. Some platonists who recognize both kinds of objects might also prioritize the relationship between the two, holding that one kind of object is derivable from or dependent on the other.

A standard formulation of idealism or anti-realism about mathematics is now very easy to come by. Simply negate the formulation of mathematical realism: It is not the case that there are mind-independent abstract (or "ideal") mathematical objects or truths. Putting this as a positive statement, *Mathematical Idealism (Anti-Realism)*: Mathematical objects (which may be "abstract" in some sense but not eternal or atemporal) are mind-dependent.

On these formulations, mathematical realism and standard mathematical idealism (which is distinct from transcendental phenomenological idealism) are incompatible. They are incompatible at a level of generality that spares us the need to consider any further details.

Now if we had captured the essential features of mathematical realism and transcendental phenomenological idealism in these formulations then we would have an answer to our question and I could conclude this paper. Needless to say, I think we have hardly scratched the surface. Therefore, let us consider mathematical realism in somewhat more detail.

II. Mathematical Realism

The mind-independent abstract (or ideal) mathematical objects that are thought to exist by mathematical realists are usually taken to have the following properties. As the formulation obviously indicates, they are *mind-independent*. This means several things. First, they are not themselves mental entities. They are not the subjective ideas or thoughts or images of human beings. They are not immanent to human consciousness but they are supposed to *transcend* human consciousness. They are not internal to human consciousness but are in some sense external to it. They are supposed to exist whether there are minds in the universe or not. They would exist even if there were no minds or had never been any minds. The properties of "being expressed" or "being thought of" are not essential to mathematical objects. Mathematical objects are external to human consciousness but not in the sense of sensory, physical or material objects. This is what it means to say they are *abstract*. (Note that I'm using the term "abstract" as it is often used in the recent literature in the philosophy of mathematics, not in the sense of Husserl's theory of parts and wholes in which non-independent parts ("moments") of a whole are said to be "abstract.") To say they are abstract is to say that they are not spatial in nature, not involved in causal relations, as material objects presumably are, and not the kinds of objects that can be sensed with one or more of our five senses. "Concrete" objects, however, would have all of these latter properties. Not only are mathematical objects not in (physical) space but they are also not in time. Unlike objects in physical space or even the objects of "inner sense" (i.e., mental processes, thoughts, images, etc.), they do not have a temporal extension. They are not, as Plato would have said, subject to generation and decay. They are, on the contrary, unchanging. Some platonists say they are "eternal" or "timeless." As I will note below, Husserl has interesting things to say about the relationship of abstract objects to time.

Starting in 1900, in the *Logical Investigations*, Husserl draws a sharp distinction between *real* and *ideal* objects. Although this distinction is not widely used in the recent literature on mathematical realism it will be useful to note its relationship to some of the current terminology. The first thing to note is that the "ideal" in this distinction does not refer to "ideas" in a subjective sense. It does not refer to mental entities. It is rather a platonic use of "idealism" that is operative in this case and not, in spite of the potentially confusing language, the use involved in the realism/idealism division. The real/ideal distinction can be drawn in terms of the temporality of objects. Real objects are objects in time. They have temporal duration. This applies to the objects of "inner sense," i.e., thoughts, mental processes, and the like, but also to objects of "outer sense," i.e., objects in space and in external time. Ideal objects are not in time in the same sense. They do not come into being and pass away. Much of what I have said about abstract objects applies directly to the ideal objects that Husserl introduces in the *Logical Investigations*. Mathematical objects, as ideal in Husserl's sense, are not, as I indicated, abstract parts (moments) of real objects. Non-independent parts of real objects are just real parts even though we can speak and think of them in isolation from the wholes of which they are parts. This does not mean, however, that they can *exist* in isolation from the wholes of which they are parts. If Husserl is to be a mathematical realist (platonist) in the sense described above then mathematical objects, as ideal, could not depend for their existence on underlying real wholes. They must exist independently of real objects. Otherwise, Husserl's view would be closer to an Aristotelian realism. There are many remarks in the *Logical Investigations*, especially Investigations II and VI, and elsewhere in Husserl's later writings that indicate that he is not in this sense an Aristotelian realist about mathematical objects. § 52 of Investigation VI, for example, is entitled "Universal objects and their self-constitution in universal intuitions." In this section he contrasts the kind of abstraction involved in setting into relief a non-independent moment of a sensible object with *ideational abstraction* in which an *idea* or universal, not a non-independent moment, is brought to consciousness.

I briefly mention one other feature of the real/ideal distinction that is not always salient in the distinction between the concrete and the abstract. The real/ideal distinction embodies the difference between the inexact and the exact, or the imperfect and the perfect. This feature of Husserl's distinction has a distinctively platonic pedigree that is omitted from some modern versions of mathematical realism. Plato's forms were supposed to be perfect in relation to their imperfect or inexact instantiations in the material world. In relation to logic and mathematics, the idea is that logical or mathematical objects are exact and "perfect" in a way that instantiations of, expressions for, or thoughts about such objects cannot be. In Euclidean geometry, for example, the lines, triangles, circles, and so on, are supposed to be perfect or exact in a way in which drawings of circles and the like, which we can perceive visually, could never be. A globe, which we hold in our hands, could never be exact and perfect in the way that a sphere in Euclidean space is conceived to be perfect or exact. The instantiations can only approximate the ideal.

This will be enough for now about the general properties that mathematical objects are supposed to possess for the mathematical realist. Further specifications along different lines are possible, and I would now like to mention one such set of specifications that is, I think, quite important. Mathematical realists could agree with everything that has been said thus far about their realism and yet disagree about *which* mind-independent abstract or ideal mathematical objects exist. Among the types of mathematical objects about which one might be a realist are geometric objects of different kinds, natural numbers, real numbers, complex or imaginary numbers, sets of different kinds, functions of different kinds, groups, or categories. One might be a reductionist or eliminativist about some of the items on this list. For example, one might adopt a realist view about natural numbers but not about real numbers or imaginary numbers. One popular strategy has been to recognize the existence only of sets and then to define some of the other objects on the list in terms of sets.

Modern set theory is of special interest in connection with mathematical realism for a number of reasons. One of the principal reasons is that it compels philosophers to confront a distinctive and relatively new set of epistemological and ontological issues about mathematical realism. These are issues, by the way, which either emerged after Husserl's time or to which Husserl himself devoted very little if any attention. Modern set theory forces the mind-independence issue in a striking way. Human minds are finite and have finite capacities. Objects such as natural numbers are finite objects. Even if the human mind cannot actually grasp or form very large natural numbers we can *idealize* the notion of *finite capacity* to cover the grasp or formation of each natural number, thus imagining that there could be a *complete* grasp of each natural number. In modern set theory, however, we are faced with existence statements about huge transfinite sets. Suppose, for example, that we consider some of the existence axioms in Zermelo-Fraenkel set theory with the axiom of choice; in particular, the axioms of infinity, power set, and replacement. These latter three axioms allow us to show rather quickly that very large transfinite sets exist. Not only will denumerably infinite sets exist but also non-denumerably infinite sets will exist, and then power sets of non-denumerably infinite sets, and so on. There is a significant disanalogy with the case of natural numbers: we cannot idealize the *finite* mind or *finite* capacities in such a way as to cover the grasp or formation of such transfinite objects. Transfinite sets transcend the possibility of being known on the basis of acquaintance with all of their members. A much more substantial idealization has to be involved. If we return to our simple formulation of mathematical realism then, in connection with set theory, we should ask whether there are mind-independent abstract *infinite* objects. In particular, are there *actual, complete infinite sets*?

Many additional details come into focus once the question of realism about set theory emerges. There are of course the traditional worries about the axiom of choice. Furthermore, with the replacement axiom we also have impredicative specification of sets. Should we therefore hold as part of our mathematical realism that impredicatively specified transfinite sets exist or not? Should we recognize only the existence of predicatively specified sets and hold to only a predicative set theory?

Some philosophers and mathematicians, such as Gödel in his later work, are prepared to be realists about full impredicative set theory with the axiom of choice. Indeed, they might be prepared to adopt a realism that goes beyond the existential commitments of a theory such as ZFC, arguing for the need for new axioms to express more of what already exists in the universe of abstract, mind-independent transfinite sets. Gödel suggests that the search for new axioms depends on sharpening or clarifying our intuition of the concepts concerning this existing realm of objects or truths (see Gödel [1964] 1990; and Wang 1974, 189).

Now let me make some comments about transcendental phenomenological idealism.

III. Transcendental Phenomenological Idealism

What has been called transcendental phenomenological idealism emerges in the writings of Husserl in which he introduces the phenomenological reduction or *epoché*, starting around 1907. *The Idea of Phenomenology* (1907) (Husserl 1964) is a particularly interesting text because Husserl says in it that the way to solve the old, vexing philosophical problem of how we can be related to transcendent objects is through the phenomenological reduction. The only way to solve this problem is from within the reduction. This is perhaps the reason why Gödel refers to *The Idea of Phenomenology* as a "momentous lecture" (see Item 050120.1 in the Gödel *Nachlass*, Firestone Library, Princeton University).

What the reduction shows us, to a first approximation, is how to restrict ourselves in a non-naturalistic manner to the sphere of appearances, to what is immanent and absolute. How does it do this? In *Ideas I* the epoché is motivated by way of some comparisons with Descartes' method of doubt. This Cartesian approach to explicating the phenomenological reduction can be contrasted with other paths to the reduction in later writings, such as the path indicated in the *Crisis of the European Sciences and Transcendental Phenomenology (Crisis)*. In *Ideas I* Husserl notes that the reduction is not the same thing as the Cartesian method of doubt but the Cartesian method, even though it was intended for different purposes, can get us into the neighborhood of what he wishes to obtain. The epoché, for example, plays no role in establishing substance dualism but is used instead to make us aware in a non-naturalistic way of mental phenomena as phenomena. As Husserl says in the *Cartesian Meditations*, Descartes did not make the transcendental turn (§ 10). As he says in *Formal and Transcendental Logic* (§ 100), Kant did make the transcendental turn but he neglected to carry it out with respect to the ideal objects of logic and mathematics. In the Introduction to *Formal and Transcendental Logic* Husserl says that mathematics and logic are positive sciences that require a foundation in transcendental phenomenology. What the modern sciences lack is a true logic, i.e., a transcendental logic that investigates the cognition behind science and thereby makes science understandable in all its activities. This logic does not intend to be a mere pure and formal logic, a *mathesis universalis*, for while *mathesis* may be a science of logical idealities it is still only a "positive" science. Transcendental phenomenology should bring to light the system of transcendental principles that gives to the sciences the possible sense of genuine science. The positive sciences are completely in the dark about the true sense of their fundamental principles. Transcendental phenomenology is supposed to make it understandable how the positive sciences can bring about only a relative, one-sided rationality.

What can be accomplished with the phenomenological reduction, which is fundamental for transcendental phenomenology, is this: as we attempt to doubt everything we notice that in fact not everything is doubtful (see *The Idea of Phenomenology*, Hua 2, 23; *Ideas I*, § 31). If I think that everything is doubtful then while I am thinking that everything is doubtful it is indubitable that I am so thinking. In every case of a definite doubt it is indubitable that I am having this doubt. The same is in fact true of every instance of cognition. If I am perceiving or judging, for example, then whether these activities are veridical or not, whether they have objects that exist or not, it is nonetheless clear that I am *perceiving* this or that, or *judging* this or that. The awareness that I am perceiving or judging implies that I have the capacity to *reflect* on my cognitive activities. In this reflection something is given to me that I cannot doubt. It is given, Husserl says, "absolutely" and with certainty. In this manner we are able to find a way to focus on what appears to us, just as it appears. If we are conscious we cannot doubt that something or other appears to us in our cognitive activities but of course we can very well doubt that what appears in the appearing is actually the case. In this manner, we can affect a "suspension" or "bracketing" of the (natural) world and everything in it. This means that we also bracket the natural, psychophysical ego or self, the self that is the object of natural science. The ego that is directed toward objects after the reduction is the "transcendental ego."

The method is thus to restrict ourselves to what is "immanent," to disengage from the natural attitude in which we naively and without reflection take ourselves to be experiencing transcendent objects. In the phenomenological attitude, obtained by the reduction, we experience, on the basis of reflection, the immanent. Husserl then goes on to say that the immanent is absolute, while the transcendent is not. What is transcendent is always relative to consciousness.

Many passages in *Ideas I* express the new transcendental idealism that results from taking the epoché seriously. In § 46, for example, Husserl argues that any physical thing that is given "in person" can be non-existent but that no mental process which is given "in person" can be non-existent. The non-existence of the world is conceivable but the existence of what is immanent—the absolute being of mental processes—would in no respect be altered thereby. In fact, there is a distinct manner, in which mental processes would always remain presupposed in any effort to doubt the existence of various phenomena. Consider the case, which is certainly possible, in which a perception is corrected by a subsequent perception. Now imagine that this process of correction continues to occur. In § 49 of *Ideas I* Husserl says that it is conceivable, due to such conflicts, that experience might dissolve into illusion not only in detail but globally. In this case no natural world would be constituted in our experience. There would be no experience of a natural world but in all of this there would still be consciousness. Consciousness would indeed be necessarily modified by the "annihilation" of the world of physical things but its own existence would not be touched. The absolute being of the mental processes would in no way be altered thereby. Thus, in § 47 Husserl says that "no limits check us in the process of conceiving of the destruction of the Objectivity of something physical—as the correlate of experimental consciousness." Whatever things are, they are as experienceable things. It is experience alone that prescribes their sense. We must not let ourselves be deceived by speaking of the thing naively as something that transcends consciousness and exists in itself, apart from any possible relation to consciousness. The genuine concept of transcendence can only be derived from the contents of our experience itself. "*An object existing in itself is never one with which consciousness or the ego pertaining to consciousness has* *nothing to do*" (§ 47). In § 49 Husserl says that the whole spatiotemporal world and each of its constituents is thus, according to its sense, a merely intentional being. It is a being posited by consciousness in its experiences. Each constituent of the world, of essential necessity, can be determined and intuited only as something identical through motivated multiplicities of appearances. It is something invariant for consciousness through a manifold of appearances. Beyond that it is nothing.

This sphere of absolute consciousness that remains as a residuum after the conceivable annihilation of the world is what provides the subject matter for pure phenomenology. From this point of view, Husserl says, we think of all reality as existent by virtue of a *sense-bestowing* consciousness which, for its part, exists absolutely and not by virtue of another sense-bestowal. Consciousness *constitutes* the sense of objectivity. Although this is a form of idealism it is not, Husserl says, a Berkeleyan subjective idealism. Rather, it is transcendental-phenomenological idealism. It recognizes that not everything is constituted as a mental phenomenon and it also recognizes the role of the overlapping horizons of different egos in the constitution of a common, objective world.

In Part II of *Formal and Transcendental Logic*, in the context of his investigations of logic, Husserl says similar things. Transcendental phenomenological idealism is represented in *FTL* as the view that it is only in our own experience that things are "there" for us, given as what they are, with the whole content and mode of being that experience attributes to these things. In § 94 of this work Husserl says that "nothing exists for me otherwise than by virtue of the actual and potential performance of my own consciousness." Whatever I encounter as an existing object is something that has received its whole sense of being from my intentionality. Illusion also receives its sense from me. Experience teaches me that the "object" could be an illusion. Objects can be thought of as intentional *poles of identity* through the manifold activities of consciousness. There is no conceivable place where the life of consciousness could be broken through so that we might come upon a transcendent object that had any other sense than that of an intentional unity making its appearance in the subjectivity of consciousness. Thus, if what is experienced has the sense of *transcendent being* then it is experience itself that constitutes this sense. If an experience is "imperfect" in the sense that an object is given only partially, then it is only experience that teaches me this.

One of the most interesting features of this transcendental-phenomenological idealism is that it does not deny that there is objectivity or objective truth but rather it makes of objectivity a problem that is to be grasped from what is absolutely given. It enjoins us to investigate how consciousness constitutes the sense of objectivity. We must now engage in constitutional analysis. We must do this, furthermore, for any kind of objectivity. It is not the case that some objects are supposed to escape the phenomenological reduction. Thus, not only are we supposed to analyze the constitution of the sense (meaning) of the being of objects of ordinary founding acts of sensory perception but we are also supposed to analyze the constitution of the sense of the being of objects of founded forms of consciousness which are based on acts of abstraction of different types, acts of generalization, reflection, and idealization. In particular, we are supposed to analyze the constitution of the sense of being of categorial objects, of ideal objects, and of mathematical objects in particular.

If we start with the ordinary physical objects given to us in founding acts of sensory perception then we see that they are given to us only partially and as transcendent, as objects that are in space and external (world) time. They are not given to us as subjective or as mental entities. We do not need to hold, as we noted, that everything is a mental phenomenon or a subjective idea. We can recognize that physical objects transcend mental phenomena (as do mathematical objects) only now we say they are constituted by consciousness in this manner. That is, the meaning of the being of physical objects is constituted by consciousness in such a manner that physical objects are not mental entities. They are not meant as mental entities. They are constituted as external objects, as objects that are in space and in external time. We are led, in this sense, to a kind of realism about physical objects. This is different, however, from a naive realism. It is, rather, a phenomenological or "constituted" realism that has its origins in transcendental subjectivity itself. Thus, starting with physical objects we can say that it is only *naïve* forms of realism about the natural world that take physical objects to somehow exist in themselves, totally independently of consciousness. If we are operating from the position of transcendental-phenomenological idealism then, for the reasons discussed above, we cannot be naïve realists. We also cannot be crude empiricists, naïve naturalists, or positivists.

Taking our lead from Husserl's comments in *The Idea of Phenomenology*, *Ideas I*, *Formal and Transcendental Logic*, and elsewhere, we can say that ideal objects are also constituted as such by consciousness. Let us apply Husserl's words from these texts to mathematical objects in particular: Whatever things are, mathematical objects included, they are as experienceable things. It is experience alone that prescribes their sense. The genuine concept of the transcendence of mathematical objects can only be derived from the contents of mathematical experience itself. Nothing exists for me otherwise than by the actual and potential performance of my own consciousness. Whatever is given as an existing object in mathematics is something that has received its whole sense of being from my intentionality. There is no conceivable place where the life of consciousness could be broken through so that we might come upon a transcendent mathematical object that had any other sense than that of an intentional unity making its appearance in the subjectivity of consciousness. We need to explicitly note the new twist here: If what is experienced has the sense of "transcendent being" then it is experience itself that constitutes this sense. If what is experienced has the sense of being "ideal," "non-mental," "acausal," "unchanging," "non-spatial," (possibly "partially given") and "non-material" then it must be experience itself that, in a non-arbitrary manner, constitutes this sense. If mathematical objects are considered to be objects that existed before we became aware of them and that would exist even if there were no human subjects then it must be the case that this sense of mathematical objects is constituted in a motivated and non-arbitrary manner.

If we consider all of the general features of mathematical realism that we outlined at the beginning of § 2 then we can now say that mathematical objects possess these features except that we must add the crucial qualification that they are constituted non-arbitrarily in this manner in the consciousness of the transcendental subject. One feature that we must now modify, however, concerns the temporality of mathematical objects. Since we are within the sphere of possible experience for transcendental subjects we are within the sphere of temporality. This means that mathematical objects are also objects that must be in time, only now we will say that they exist at all times. Thus, instead of saying that mathematical objects are atemporal or eternal or timeless—somehow outside of time (and all possible experience) altogether—we will now say that they are omnitemporal (see *EU*, § 64). As transcendental phenomenological idealists we cannot speak about the existence of objects that are somehow outside of all possible appearance or outside of all possible consciousness, and hence outside of all possible time.

We thus appear to arrive at a wholly unique kind of "platonism" about mathematics, which I will call "*constituted platonism*." This is, as it were, a platonism embedded within transcendental idealism. In a remarkable new twist in the age-old debate about platonism, we look to the transcendental ego as the source (origin) of platonism about logic and mathematics, where logic and mathematics are built up non-arbitrarily through acts of abstraction, idealization, reflection, and so on. Just as the "realism" about physical objects is not a naïve realism, so this unique kind of platonism about mathematical objects is not a naïve platonism.

IV. Mind-Independence and Mind-Dependence in Formulations of Mathematical Realism

Since mathematical realism and mathematical idealism are viewpoints expressed in terms of mind-independence or mind-dependence, I would now like to single out these characteristics in order to arrive at an explicit formulation of how a form of mathematical realism might be compatible with transcendental phenomenological idealism. As I mentioned above, Husserl says in *The Idea of Phenomenology* that the only way to solve the problem of how we can be related to transcendent (or mind-independent) objects is from within the phenomenological reduction. Once we restrict ourselves to the sphere of appearances, to what is immanent, on the basis of the epoché, we see that consciousness exhibits intentionality. We find that (transcendental) subjects are directed by the contents (or noemata) of their acts toward objects that *transcend* these very subjects. In the language of *Ideas I*, we find the noetic-noematic-hyletic structure at work in our experience of ordinary sensory objects. In the case of the founded pure categorial or ideal objects this same structure will be present, without the constraints of sensory hylé but not without grammatical, formal, meaning-theoretic, and other structural constraints. In other words, within the sphere of the immanent and absolute that we obtain after the reduction we can draw a new distinction between the immanent and transcendent, i.e., we can distinguish *what appears as immanent* from *what appears as transcendent*. We can distinguish *what appears as mind-dependent* from *what appears as mind-independent*. Of course the terms "immanent" and "transcendent," or "mind-dependent" and "mind-independent" in this context will have a sense different from their sense prior to the reduction. Similarly, if we view the reduction as depending on a distinction between appearance and reality (in the naïve sense of the natural attitude) and then restricting us to the sphere of appearances, then we find that within the sphere of appearances we can still distinguish appearance from "reality." This will be true in ordinary sensory experience but also in the case of our experience in mathematics and logic.

How does this work? We can start with an example in sensory experience. Suppose that at a certain stage of your experience you perceive a snake lying in a garden. At a later stage, however, you perceive that it is not really a snake lying in a garden but a coiled garden hose. Now what usually happens in situations such as this is that our experience settles down so that we do not have a continuous series of misperceptions of this sort. Instead, there is typically a more or less harmonious course of experience involving transcendent objects. This opens up the possibility of making an appearance/reality distinction after the epoché. Looking back on the experience, we can say that there was merely an appearance of a snake at the earlier stage in the perception and that what we have "in reality" is a coiled garden hose. We cannot simply say that "to be is to be perceived" because in a case such as this subsequent experience shows that there was no snake. It is not the case that in fact I was perceiving a snake at the earlier stage. It only appeared that I was. What seems to be mind-independent, given the evidence thus far, is the coiled garden hose. The perception of the coiled garden hose, however, could itself be overturned in future experience. That is, there might be evidence (experience) in the future that would show us that it is also not a coiled garden hose. Its being a coiled garden hose is *not absolute* even if the coiled garden hose is given as what is "real" and mind-independent in accordance with all of our evidence thus far. Our evidence that the coiled garden hose is mind-independent is in this sense presumptive.

What this shows is that from within the epoché everything is indeed understood as appearance or phenomenon and that appearances are corrected or verified only by further appearances. Within the sphere of appearance, however, we can still distinguish the "real," the transcendent, or the mind-independent from the "merely apparent," the immanent, or the mind-dependent, on the basis of what stabilizes or becomes invariant in our experience. This is a key idea of transcendental phenomenology, and it holds for both sensory experience and mathematical experience. There are illusions and corrections and refinements in mathematical experience just as there are in sensory experience. We cannot somehow get outside of appearances to an appearance-independent thing-in-itself. The "real" will be that for which we have evidence across places, times, and persons. This will not hold for the "merely apparent." Rational justification depends on evidence. Imagine a form of experience in which nothing ever stabilizes or becomes invariant. This would be a form of experience that is without reason. It would be experience in which there is no order and no rational connection among the contents of consciousness. We are nonetheless not entitled to say that what is stable or invariant is the final, absolute reality. We cannot have a realism that recognizes an appearance-independent absolute reality. At best, the notion of "absolute reality" might be preserved as an infinite ideal. Thus, transcendental phenomenology recognizes an appearance/reality distinction after the reduction that allows for a kind of realism, only it is not naive or absolute realism. It is also not a naive idealism for the same reason: it makes an appearance/reality distinction after the epoché.

These considerations show us that there are weak and strong senses of "appearance-independence" or "mind-independence." There could not be mind-independent objects in the strong or absolute sense of lying outside of all possible experience (or appearance). We simply cannot say anything about the possibility of such radically independent things-in-themselves. On the other hand, there are objects that are mind-independent in a weaker sense according to which objects are invariants in a manifold of appearances. We could be mistaken about objects in our experience, so that we could at some later stage come to see that we had been under an illusion, that we had mere appearances at an earlier stage. To say that there are weak and strong senses of "mind-independence" or "mind-dependence" will then affect the formulations of mathematical realism and mathematical idealism. In transcendental phenomenology we must set aside the strong (or naïve) sense of mind-independence. The weaker sense, however, will allow us to preserve important insights of realism.

V. Compatibility or Incompatibility?

To summarize the discussion thus far we can say that we need to index our conceptions of the mind-dependent and mind-independent, of the immanent and transcendent, and of appearance and reality.

With the phenomenological reduction we turn to phenomena, to appearances, to the immanent. So we first distinguish between appearances (or the immanent), and the naïve view of appearance-independent reality (or of the transcendent). On the one side of this distinction we have appearances, the immanent and mind-dependent, and on the other side of the distinction we have appearance-independent reality or the transcendent as mind-independent. Now, restricting ourselves to the sphere of phenomena, to the immanent and absolute, we find that consciousness exhibits intentionality. Transcendental subjects are directed toward objects that transcend subjects. We find the noesis-noema-object structure, minus sensory hylé in the case of mathematical objects. Intentionality in pure mathematics is not constrained by sensory hylé but there are still grammatical, formal, meaning-theoretic, and other structural constraints on it. One of the marks of objectivity in both sensory and mathematical experience is that we find our awareness to be constrained in certain ways. It is not possible to will objects or states of affairs in either sensory or mathematical experience to be just anything we want them to be. We find all of these moments of experience after the epoché. Within the sphere of appearances we can then draw a distinction between the immanent and the transcendent. Here we introduce a new distinction between the immanent and transcendent, the mind-dependent and mind-independent, between appearance and reality. Some things *appear to us* as immanent, some as transcendent.

We can depict the situation in the following diagram, which I will formulate for the mind-dependent/mind-independent distinction, since the issue of mathematical realism is typically described in these terms:

Mathematical objects are

mind-dependent$_1$ mind-independent$_1$

$\Box$ $\Diamond$

/   \

mind-dependent$_2$ mind-independent$_2$

$\Box$ $*$

It is inconsistent to say that abstract mathematical objects are mind-dependent$_1$ and mind-independent$_1$. Formulated in this way, mathematical realism and mathematical idealism are incompatible. Most of the debate about realism and idealism, including recent debate, seems to take place at this level. It is also inconsistent to say that abstract mathematical objects are mind-dependent$_2$ and mind-independent$_2$. Formulated in this way, mathematical realism and mathematical idealism are still incompatible. It is not inconsistent, however, to say that abstract or ideal mathematical objects are mind-dependent$_1$ and mind-independent$_2$. Indeed, mind-independence$_2$ falls under mind-dependence$_1$. What this means is that mathematical realism, in this sense, is compatible with transcendental phenomenological idealism. Mathematical realism in this sense, which we can call "*constituted mathematical realism*" or "*constituted platonism*," is concerned with non-arbitrarily or rationally motivated constituted mind-independence.

What we are now to investigate is the *constitution of the sense* of mind-independence from within the epoché. We need to investigate the rationally motivated *constitution of the sense of the existence of ideal mind-independent mathematical objects*.

As we look back from this viewpoint, we can say that the standard positions of mathematical realism and mathematical idealism that we set out in our initial formulations are too simple. They are ambiguous. If we make the distinctions just indicated then the assertion that mathematical objects are mind-independent$_1$ is naïve (or pre-critical) mathematical realism and is untenable. The assertion that mathematical objects are mind-dependent$_1$, with no further qualification, is naïve (or pre-critical) mathematical idealism and is untenable. The third position that we outlined combines a transcendental phenomenological idealism and a mathematical realism in which neither the realism nor idealism is any longer naïve. We have left naïve metaphysics behind. It also follows that transcendental phenomenological idealism is not compatible with naïve mathematical realism.

Once we make these distinctions then existence claims, whether in ordinary perception or in the case of mathematics, will have to be understood accordingly. There are of course important disanalogies between sensory objects and mathematical objects but in either case the existence of mind-independent objects will now have to be understood in the sense of mind-independence$_2$. If these phenomenological considerations are correct then what other sense could we legitimately give to existence claims?

Constituted platonism, unlike naive metaphysical platonism, does not cut off the possibility of knowledge of mathematical objects. Knowledge involves intentionality. Mathematical knowledge is to be spelled out in terms of intentional directedness toward ideal or abstract objects, where the objects are to be thought of as (founded) invariants in mathematical experience. What we are describing here is a position about mathematical *experience*. Note how different this is, for example, from a position that starts with neuroscience and then asks how the brain could be related to abstract objects. How could brains be causally related to abstract objects? From my point of view, this is the wrong question. There is a reason for wanting to suspend or bracket natural sciences of the mind such as neuroscience. The reason is not to avoid neuroscience in particular or natural sciences of cognition in general. We of course need such important sciences. The reason is rather to avoid a reductionistic, eliminativist, and one-sided philosophy of mind that leaves out consciousness and intentionality. Such sciences abstract away from *experience*. There is much more to say about these matters but further discussion will have to wait for another occasion (see, however, Tieszen 2005a, 2005b, 2006).

On the kind of realism described above objects can be mind-dependent$_1$ and mind-independent$_2$. Before concluding this section I would like to note that such a realism bears more than a passing resemblance to Hilary Putnam's "internal realism" (see, e.g., Putnam 1981, 1987). Although it is not possible to do so here, it would be worthwhile to compare the views in some detail. I am arguing, for example, that the notions of mind-dependence$_1$ and mind-independence$_2$ can be applied in the case of *mathematical* objects or states of affairs. Does Putnam apply his internal realism to the question of mathematical realism? Does he have anything like the idea of constituted mathematical platonism? There are certainly developments in the phenomenological analyses that would not be found in Putnam's internal realism but there might also be many points on which the views in fact reinforce one another.

VI. Brief Interlude: Where to Place Gödel, Brouwer, and Other Mathematical Realists and Idealists in our Schematization?

It is of some interest to consider where various philosophers, logicians, and mathematicians would fall within the set of distinctions I have drawn. In terms of the diagram above, I have suggested that the later Husserl is at the position marked by "∗," or at least that the principles I have discussed would lead him there, even if he did not explicitly analyze all of the consequences of the position. Brouwer and Gödel are frequently regarded as antipodes on the mathematical realism/mathematical idealism issue. It seems that Brouwer would be at the position marked by "□" because he does not have the distinction between weak and strong senses of mind-independence. The idea that mathematical objects could be non-mental and yet not be appearance-independent does not seem to be part of his view. There are very interesting connections between ideas of Brouwer and Husserl but if it is not part of Brouwer's view that mathematical objects could be non-arbitrarily constituted by subjects as non-mental then Brouwer would still be what we have called a naive idealist. But perhaps Brouwer's position could or should be modified. On the other hand, there are various philosophers would who place Gödel at the position marked by "◊." Perhaps that is where he belongs, in which case his position will be subject to all of the problems associated with naive metaphysical platonism. We know, however, that Gödel was interested in aspects of Kant's transcendental idealism and that from 1954 to 1959 he corresponded with Gotthard Günther at some length about transcendental philosophy (see Gödel [1954–59] 2003). Starting in 1959, he became an avid reader of Husserl's philosophy, and was especially interested in transcendental phenomenology (see, e.g., Gödel [∗1961/?] 1995). Thus, we should perhaps put him in the position marked by "∗," or at least regard him as groping toward such a position (see also Tieszen, Introduction and Part II of 2005b).

VII. A Conclusion and an Introduction

If it is therefore possible to see how a form of mathematical realism is compatible with transcendental phenomenological idealism we can still ask about which kind of mathematical realism can be supported from this point of view. About which kinds of mathematical objects and states of affairs, that is, can we be constituted realists? This is now a question of meaning constitution. What kinds of mathematical objects, e.g., geometric objects, natural numbers, real numbers, complex or imaginary numbers, functions of different types, sets of different types, and the like, can the mind constitute?

Here things are much more complicated. This is where the real work of constitutional analysis in the case of mathematics must begin. It could be argued that Husserl would hold that natural numbers and the geometric objects of Euclidean geometry could in principle be constituted as particular objects. Which other (alleged) mathematical objects on our list could be constituted? Husserl himself does not have much to say about the constitution of such objects. He does present some ideas on the origin and constitution of sets in *Experience and Judgment* and other places but, relative to modern set theory, they do not take us very far. They leave a lot undetermined. It is not clear that they would lead to the existence of any sets beyond those that constructivists would be prepared to recognize.

There are many questions about the constitution of mathematical objects and the constitution of generalities about mathematical objects that need to be considered. To mention just one question of this type, for example, it might be asked whether it is possible to constitute generalities about mathematical objects even if we cannot constitute such objects individually. Could there even be a kind of objectivity in mathematics without objects? I will not try to address the many questions that could be asked here. What I have tried to do in this paper is to show how, at a *general* level, a form of mathematical realism can be compatible with transcendental phenomenological idealism. One can then enter into issues about constituted realism in the case of different kinds of mathematical objects.[1]

[1] I would like to thank participants in the Phenomenology and Mathematics conference at Tampere for questions and comments. Work on this paper was partially supported by a National Endowment for the Humanities (NEH) fellowship, which support I hereby gratefully acknowledge.

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CHAPTER II

PLATONISM, PHENOMENOLOGY, AND INTERDERIVABILITY

Guillermo E. Rosado Haddock

Abstract. In this paper I try to offer a definitive answer to the question of the relation of Husserl's phenomenology to mathematical Platonism and constructivism of the Brouwerian sort. The controversial issue of Frege's presumed influence on Husserl is also considered and it is briefly argued against such an influence. In the second part of the paper Husserl's semantics of sense and objectuality (or referent) is discussed, and it is shown that it is much more adequate for mathematics than Frege's semantics. Finally, a possible theory of degrees of extensionality is briefly sketched.

I. Introduction

Recently, a great interest has surged in finding links between Husserlian phenomenology and Brouwer's intuitionism, or maybe in trying to rescue the latter through a sort of foundation on the former. Although such developments could be fruitful, it is time to set matters straight by (i) making it clear that Husserl's views on mathematics are very distant from those of Brouwer, since the former never considered in print the abandonment of any part of classical mathematics, (ii) that Husserl's Platonistic views on mathematics were complemented by especially deep semantic insights and a still little known but very interesting epistemology of mathematics, and (iii) that both Husserl's semantics and epistemology of mathematics can shred some light on important current philosophical debates. It would take us too far to consider here all these issues at length. As a sort of guide to our discussion, I will use some critical observations made in a review of my and Claire O. Hill's book *Husserl or Frege?: Meaning, Objectivity and Mathematics* (2000) by Mark van Atten, one of the more staunch propounders of the above mentioned trend (van Atten 2003).

First of all, since the myth of Frege's influence on Husserl, and the corresponding prejudice concerning Husserl's views on logic, mathematics and related issues has not vanished, let us stress that Husserl obtained the

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distinction between sense and reference not later than 1890[1] with complete independence of Frege, and this fact was acknowledged by Frege himself in a letter to Husserl of May 1891 (Hua 12, 340–373). See on this issue my paper (2000a). Moreover, as already argued in my dissertation (Rosado Haddock 1973) and elsewhere, Frege did not exert any decisive influence on Husserl's abandonment of his mild psychologism nor on Husserl's views on mathematics in *Logische Untersuchungen*. On these two issues many of the writings of Claire O. Hill, and a paper and a book by Mohanty (1974) are especially relevant. Thus, it has been clearly shown that the supposed influence of Frege on Husserl's abandonment of psychologism, as well as his supposed influence on Husserl's distinction between sense and reference, are untenable. Frege's review of Husserl's *Philosophie der Arithmetik* (1891) simply came too late,[2] since the process of abandoning his psychologistic leanings in that early work began more or less simultaneously with its publication, and that is the principal reason why Husserl did not publish the second volume, even if it would have dealt with logical—not with psychological—foundations of arithmetic. Although Frege's *Die Grundlagen der Arithmetik* (1884) and the first volume of his *Grundgesetze der Arithmetik* (1893 and 1903) may have played some marginal role, there is no reason to doubt what Husserl says, namely, that the study of Leibniz, Bolzano, Lotze and Hume played the decisive role (1975, 36–38). Moreover, Husserl's mature views on logic—which date at least from 1894—have clearly more affinities with Bolzano's than with Frege's, and his views on mathematics were decisively influenced by Riemann—who seems to have been totally foreign to Frege,[3] and have more affinities with the Bourbaki school than with any other views on mathematics. By the way, it is not excluded that there was a marginal influence in the other direction, since there is a famous passage in "Der Gedanke"([1918] 1967, esp. 360)—in which Frege says that there is something non sensorial present both in our perceptual knowledge of physical entities and in that of entities belonging to the third

[1]See his paper "Zur Logik der Zeichen," posthumously published as an Appendix to Hua 12, 340-373, especially pp. 343-344.

[2]Frege's review is from 1894. (See Frege 1990, 179-192.)

[3]In a recent paper, Jamie Tappenden (2006) has argued on behalf of a Riemannian influence on Frege. For a brief rebbutal, see the Appendix to this paper.

realm—, which is perfectly compatible with Husserl's views on the role of categorial elements in sense perception and in our purely categorial acquaintance of formal entities and structures.

Another point that I want to set straight concerns the defense of constructivism made by van Atten on p. 241 of his review against my surely not original criticism of constructivism (2000e). Van Atten says:

> First, as an argument against constructivism (and intuitionism in particular), it is not sufficient to point out that among its adherents "there has been some disagreement concerning what should be considered mathematically sound" (p. 244); the same kind of disagreement is found among classical mathematicians (with regard to certain large cardinals, or the status of the continuum hypothesis). (Van Atten 2003, 242)

Van Atten's examples of disagreements in classical mathematics are totally dissimilar to the situation among constructivists. In the frontiers of any science there usually are disagreements among researchers. Large cardinals and the continuum hypothesis belong in this moment to the frontier of our set-theoretical knowledge. Thus, it is clear that there will be disagreements on such issues. As is well known, Gödel himself once believed in the truth of the continuum hypothesis, and even attempted to prove it, whereas later he became convinced that it was false. Einstein and Bohr, as well as other leading physicists, disagreed on whether quantum mechanics, with its Heisenbergian uncertainty principle, was a definitive theory, or just a preliminary stage until a more deterministic physical theory of the subatomic world could be obtained. Moreover, almost immediately after Einstein's first presentation of his general theory of relativity, Hilbert and Weyl argued for a sort of unified field theory. First, Einstein rejected the idea of a unified field theory, but later tried to develop one himself. All those disagreements, even between views of one researcher at different times, are normal at the frontier of science. But what I have pointed out about disagreements among constructivists is at the very basis of constructivism. Bishop's or Markov's constructivisms are, so far as I can see, not only very different in its very foundations from Brouwer's,—for example, they are not based on such metaphysical emptiness of the "empty two-ity," which, according to van Atten's review (p. 243) could help explain the "deep structure" of mathematics alluded to at the end of my (2000e)—, but also their notions of *constructivity* are stricter than Brouwer's and also different from each other's (and, of course, from Kant's). Moreover, even in the intuitionistic camp, there have been critiques by G. F. C. Griss, who considered Brouwer's notion of *constructivism* too wide. (See Beth 1965, 437–439.) Thus, the disagreement among constructivists lies at the heart of their views, namely, in the definition and understanding of their most basic notion. A comparison with the disagreements in some areas of traditional philosophy and the social sciences would be much more adequate than with the disagreements at the frontier of research in set theory.

II. Phenomenology, Constructivism and Platonism

On p. 243 of his review, van Atten asserts that "...it is taken for granted that phenomenology singles out one of the traditional philosophies of mathematics as the correct one, and that this philosophy is Platonism. It is not immediate that the latter Husserl would have agreed." First of all, I have never taken for granted that phenomenology singles out Platonism as the correct philosophy of mathematics. Superficially seen, it should be exactly the opposite. After the transcendental turn, which many followers of Husserl considered a rapprochement to Kant, one should have expected a radical modification of Husserl's views on mathematics in *Logische Untersuchungen*. Constructivism would seem more compatible with transcendental phenomenology, as interpreted by official phenomenologists, than Platonism. This could at least partially explain Oskar Becker's leanings towards constructivism. Thus, Mark van Atten (2002), Richard Tieszen (1989) and others (see also van Atten et al. 2002) are without doubt in good company when trying to assimilate Husserl to Brouwer's constructivism, and their interpretation seems to fit in better with (official) transcendental phenomenology than Platonism.

However, the fact of the matter is different. If you compare Husserl's views on mathematics in *Logische Untersuchungen* with his views after the transcendental turn, you don't see any essential difference. In the posthumously published *Einleitung in die Logik und Erkenntnistheorie,* which dates from 1906 to 1907, i.e., precisely from the years of the transcendental turn, in *Logik und allgemeine Wissenschaftstheorie*, which dates from 1917 to 1918, and in his final exposition of his views on logic and mathematics, namely, his 1929 *Formale und transzendentale Logik*, Husserl's views on mathematics—as well as those on logic and on their relation—are essentially the same as before the transcendental turn: he still propounds a Platonist philosophy of mathematics. Thus, from 1894, when he conceived Chapter XI of his *opus magnum*, on, Husserl's views on mathematics were Platonist. This Platonism, however, was of a very different sort than Frege's, since Husserl was no reductionist and, in particular, no logicist, and since his Platonism was a sort of structuralist Platonism, clearly influenced by Riemann's notion of a manifold, and to a lesser extent by Cantor's views, and—as we already mentioned above—with affinities to the views on mathematics much later propounded by the Bourbaki school. For Husserl mathematics was nothing else than a formal ontology, based on the concept of something whatsoever, or something in general (*Etwas überhaupt*). The fundamental concepts of mathematics are, for Husserl, sorts of variations of the something whatsoever. Those concepts, like set, relation, cardinal and ordinal number, or whole and part, are the basis of the fundamental mathematical structures, the mother structures in Bourbaki's terminology, and the remaining mathematical structures are obtained from the fundamental ones either by adding new axioms, by specialization or by combination of two or more structures. Logic, on the other hand, is not a foundation of mathematics but a non-ontological sister discipline of mathematics, which unites with it to form Husserl's version of the *mathesis universalis*, the latter being crowned by a theory of all possible forms of theories (correlatively, of all possible forms of manifolds).

At first sight, it would seem as if there were some tension between Husserl's views on mathematics and his transcendental phenomenology. But this apparent tension is present only if we interpret the transcendental turn, as it has been traditionally done, namely, as a rapprochement to Kant. However, if the phenomenological reduction is seen, as I see it, and as was also interpreted by Husserl's secret student Rudolf Carnap [4] in *Der logische Aufbau der Welt*, i.e., as a methodological device (Carnap 1928, §64), the tension disappears. Moreover, if the official interpretation of the transcendental turn as a rapprochement to Kant were correct, Husserl

[4] Although Carnap never acknowleged having been a student of Husserl, he was his student at least during three semesters after having completed his dissertation. See on this issue (Schuhmann 1977, 281), as well as, my recent book (Rosado Haddock 2008).

would seem to fall under his own criticism of Kantianism and of other sorts of specific relativism in Chapter VII of the first volume of *Logische Untersuchungen*. However Husserl never felt the need to defend his later views against such criticism. Moreover, there should be little doubt that Brouwerian intuitionism fits, in the best of cases, under the title specific relativism.

It should be said, however, that although Husserl's philosophy of mathematics from 1894 on was Platonistic, that does not mean that he was unaware of the paradoxes of set theory. Already on April 16, 1902, Zermelo communicated Husserl the so-called Russell Paradox—which Zermelo had discovered before Russell (Schuhmann 1977, 71; Husserl 1979, 399)—and they remained very near during their stay in Göttingen. (See also Schuhmann 1977, 158.)[5] Thus, Husserl seemed to have been well-informed on the issue of the paradoxes of set theory and the crisis in the foundations of mathematics. Moreover, in a double manuscript with the inscription A I 35 in the Husserl Archives in Köln (still unpublished), part (α), which dates from 1912 and part (β), which dates from 1920, Husserl discusses the Zermelo-Russell Paradox—as it should rightly be called. In my dissertation I interpreted both parts of the manuscript as momentary constructivist leanings of Husserl. Although we cannot dwell too long on this issue here,[6] I now consider the part (α) perfectly compatible with Husserl's epistemology of mathematics, as presented in his 6th Logical Investigation. Husserl offers there an iterative constitution—not construction—of mathematical entities in categorial intuition, and is very conscious of the possibility of paradoxes. Such iterative constitution has clear affinities with Cantor's procedures and with Zermelo's later iterative hierarchy of sets. In (2000d) I offered a reconstruction of Husserl's epistemology of mathematics and showed in Appendix II that neither the Zermelo-Russell Paradox nor Cantor's Paradox could be obtained in the Husserlian hierarchy. Before we discuss, however, the second part of this very extensive manuscript, I would like to mention that in part (α) (p.11) Husserl anticipates the Lesniewski-Tarski's hierarchy of metalanguages,

[5]Zermelo, who was already teaching at the university, belonged to Hilbert's circle, with which Husserl was in constant contact during his Göttingen years. Some years later they were both in Freiburg.

[6]I have discussed the manuscript more thoroughly in the Appendix to my recent paper "Husserl's Philosophy of Mathematics: its Origin and Relevance."

used by the latter two authors to solve the semantic paradoxes, when he asserts that we should distinguish between different levels of language. Thus, he says: "When I say the name name, then I have a new name, which is distinct from the name 'name,' a name of second level, and so forth."[7]

Part (β) dates from 1920, thus, two years after his friend and former student Hermann Weyl had published his monograph *Das Kontinuum*[8]—by the way, still another moderate version of constructivism. Husserl was perfectly aware of Weyl's views, and it seems very probable that this part of manuscript A I 35 bears its influence. Thus, in part (β) Husserl asserts that the way to avoid the set-theoretical paradoxes consists in a constructive axiomatization of set theory. Moreover, he says that ". . . 'manifold' here [i.e., in his doctrine of forms of possible manifolds] must mean a formal as constructively (definite) characterized region of objects, or region of an upper genre concept M, which remains undetermined [and] whose objects are constructible by determinately formed operations that can be iterated into infinity. The axioms must be so chosen as to found a priori the constructability" (A I 35, 47–48). Later in the manuscript, and with special reference to the Zermelo-Russell Paradox, Husserl adds: "Mathematics must establish an existence proof of each merely imagined totality." Thus, in the second part of A I 35, Husserl, by speaking of constructible manifolds, constructible axiomatization of set theory, and even requiring an existence proof for every mathematical totality, seems to have considered some restrictions to mathematics. At the end of the manuscript (p. 96), he goes so far as to question the set-theoretical foundation of mathematics of his friends Cantor and Zermelo, and recommends a reflection on the accomplishments of that discipline both as an area of mathematics in its own right and as a foundation for mathematics.[9]

[7]All quotations of manuscript A I 35 are my translations into English of quotations from the manuscript, which appear in my dissertation. See note 3.

[8]For Weyl's relation to Husserl, see the excellent paper (Mancosu and Ryckman 2002, 130–202). See also Ryckman 2005.

[9]I have always wondered both why this important manuscript has still not been published, as well as why scholars like van Atten and Tieszen have never, so far as I know, made any reference to it.

Nonetheless, the fact of the matter is that in his *Formale und transzendentale Logik*, published nine years after Husserl wrote part (β) of A I 35, essentially the same Platonist philosophy of mathematics is propounded as in *Logische Untersuchungen* and in the lectures from 1906 to 1907 and of 1917–1918 published in the last two decades and referred to above. The more plausible explanation is that Husserl thought that his views were not affected by the paradoxes, that his iterative hierarchy of mathematical entities constituted in categorial intuition was enough to block the paradoxes. This possible explanation receives some support from the fact that in the first quotation above from part (β) of A I 35 Husserl uses Zermelo's expression "definite" (in German: *definit*) as if it were synonymous with the expression "constructible." Moreover, Husserlian mathematics is based not on the notion of set but on the notion of something whatsoever—of which the notion of set is just a particularization —, and as Husserl remarks in part (α) (p. 17) of the manuscript, the something whatsoever is not a set, i.e., such a concept does not have any extension.

In the remainder of this paper I will consider two issues that are both more substantial and concern more directly the present author's research, namely the interderivability of seemingly unrelated mathematical statements and the application of the Husserlian distinction between states of affairs and situations of affairs to the semantics of mathematics.

III. Interderivability

I suppose that all well trained mathematicians and logicians have at least once in their life heard about the (meta)mathematical fact that the Axiom of Choice has many different mathematically equivalent statements in very different areas of mathematics. I also suppose that many philosophers working in our area are also conscious about that fact. The interesting issue is what philosophers of mathematics have to say about it. As I have argued before (2000e, 2000f), in my opinion, it is a *sine qua non* of a philosophy of mathematics to try to assess the interderivability phenomena. My argument with respect to the interderivability phenomena—which I presented briefly for the first time in a conference in Pittsburgh in 1985 and developed more thoroughly in a conference in Mexico in 1988—is similar to an argument of Gödel, which appears in his *Collected Works III* (Gödel 1995, 304–323) and which concerns the undecidability results, namely, that non-Platonist philosophies of mathematics are not capable of philosophically assessing the interderivability phenomena.

That the Axiom of Choice is mathematically equivalent to Tychonoff's Theorem—which says that the product of a family of compact topological spaces is a compact topological space—, or that Tarski's Ultrafilter Theorem is mathematically equivalent to the restriction of Tychonoff's theorem to Hausdorff spaces, are (meta)mathematical facts that cannot be explained either by referring to the formalisms, or as a result of some conventions made by mathematicians. Not even Platonism of the Fregean sort is capable of adequately philosophically assessing the interderivability phenomena, due to the insufficiency of a semantics for which the referents of statements are truth values and which, thus, obviates much finer distinctions. The statements "2 + 2 = 4" and "Paris is the capital of France in 2007" have the same truth value as the above mentioned four mathematical statements, but are not interderivable with any of them or with each other. Constructivism does not fare better.

Let us suppose for a moment that constructivism were not only homogeneous but also true, and let us even forget its difficulties to deal with infinities above that of the cardinality of the natural numbers. Thus, a constructivist mathematician constructs the relevant entities—for example, filters—in the case of the Ultrafilter Theorem, and constructs the mathematical fact that every filter can be extended to a maximal filter. Let us suppose that the same or another mathematician constructs the relevant entities—for example, compact Hausdorff spaces and (unrestricted) families of them—and constructs the mathematical fact that the product of a family of compact Hausdorff spaces is a compact Hausdorff space. *Prima facie* those two mathematical facts seem to be as unrelated with one another as the following empirical facts: "Siberian tigers are the biggest and strongest felines on earth at the beginning of the twenty first century" and "The Amazon River is in South America." Let us even suppose that the constructivist mathematician is able to construct a proof of the interderivability of the Ultrafilter Theorem and Tychonoff's Theorem restricted to Hausdorff spaces. Even after all such assumptions, the constructivist could not adequately philosophically assess the interderivability of the two statements. If he were to assert that when he first "constructed" his proof of the Ultrafilter Theorem and his constructivist colleague "constructed" the proof of Tychonoff's Theorem restricted to Hausdorff spaces, they both already had in mind the proof of the interderivability of the two mathematical facts, he would not be telling the truth and could not even be taken seriously. And if they did not have the interderivability in mind when establishing the two mathematical facts, they would have to explain why those two seemingly unrelated mathematical facts are equivalent but not equivalent to other mathematical facts "constructed" by them, for example, to "There exist infinitely many prime numbers." Moreover, the constructivist would have to explain why there are so many thematically unrelated mathematical statements in so many different areas of mathematics equivalent to the Axiom of Choice, but also infinitely many mathematical statements, some of them thematically related to the Axiom of Choice, which are not mathematically equivalent to it. Hence, no matter if Brouwer, Markov, Bishop or any other constructivist was conscious or not of the importance of the interderivability phenomena for the philosophy of mathematics, the constructivist can only remain in awe in front of such (meta)mathematical facts. Moreover, the excuse that, for example, the Axiom of Choice is not constructible is no excuse at all. Even if all statements equivalent to the Axiom of Choice, as well as the Ultrafilter Theorem and all its equivalents in different areas of mathematics were not constructible, the possibility of seemingly unrelated but interderivable statements in mathematics would remain, and the constructivists would have to be able to philosophically assess it. However, as well as Fregean Platonism, conventionalism, formalism, nominalism, or Fieldian fictionalism, constructivism does not have the tools for dealing with the problem posed by the interderivability of seemingly unrelated mathematical statements.

In (2000g) and in my review of Anastasio Alemán's interesting book *Lógica, Matemáticas y Realidad* I have presented another argument against Fieldian fictionalism and conventionalism, respectively, namely that they have consequences incompatible with a result in classical model theory, namely, Robinson's Test for Model Completeness (Rosado Haddock 2003). According to this model-theoretic classical result, for any existential statement, there is a universal statement interderivable with it.

However, it is a consequence of Field's views that all mathematical existential statements are true and all mathematical universal statements vacuously false, whereas it is a consequence of mathematical conventionalism that we are free to assign truth or falsehood to any existential or universal mathematical statements. Both contentions are contradicted by Robinson's Model Completeness Test. So far as I can see, the scope of this argument is more restricted than that of the interderivability argument, but it is in no way less conclusive. It is unnecessary to mention here my more specific arguments offered in (2000g) against Benacerraf's and others' views. I just want to finish this part of the paper with a paraphrasis of Hilbert, namely, that nobody should throw us away from the paradise that Tarski and others built on the shoulders of Cantor.

IV. Situations of Affairs: Historical Preliminaries

In §§2 and 3 of his *Begriffsschrift* Frege (1879) introduced the notions of conceptual content and judgeable content, respectively. As expressed in the Preface of *Grundgesetze der Arithmetik*, the notion of judgeable content lies at the origin of the distinction between the sense and the referent of statements,[10] though clearly the judgeable content is much nearer to Frege's notion of thought than to that of truth value. On the other hand, the conceptual content of a statement was characterized by Frege in §2 of his early work in such a way that two statements had the same conceptual content if they had the same deductive power, i.e., if exactly the same statements could be derived from each of them—the rest of the logical resources remaining fixed throughout. This amounts essentially to the interderivability of two statements with the same conceptual content. The example given by Frege of two statements with the same conceptual content is that of a statement in the active mode and its counterpart in the passive mode. Interestingly enough, most Fregean scholars have preferred to ignore any distinction between conceptual content and judgeable content. It should, however, be clear to all except such Fregean scholars that a

[10]I use the word statement as synonymous with (or an abbreviation of) declarative sentence; and I prefer the word "referent" to that of "reference" as a translation of Frege's unusual use of the German word *Bedeutung*.

mathematician and logician so conscious of the requirement of precision and rigour in logical discussions as Frege would have never introduced in two subsequent pages of a logical treatise, and with different characterizations, two different expressions for the same notion and, moreover, without explicitly saying that they were synonymous, or at least had the same "content." Thus, one can safely conclude that the two notions are different and were intended by Frege as different.

Frege's notion of conceptual content, which is a forerunner of Husserl's notion of situation of affairs, was never mentioned explicitly by that name in Frege's later writings. In *Die Grundlagen der Arithmetik* Frege used in a somewhat unclear and ambiguous way the word "content." However, when he characterizes the two sides of his contextual second attempt at defining the notion of number, namely:

(Γ) The number that corresponds to the concept F is the same as the number that corresponds to the concept G if and only if the concept F is equinumerous to the concept G, as having the same content, what he means by "content" is conceptual content (Frege 1884, §§64–65). Moreover, although in Frege's writings after 1890 the notion of content of a statement was seldom used, and only to mean the sense of the statement, i.e., the thought, plus some rhetorical inessential components (Frege [1918] 1990, 342–362, esp. 347), the old notion of conceptual content resurfaced in some of his attempts to expound the sense of statements. Thus, in "*Der Gedanke*" Frege offers as examples of statements expressing the same thought a statement in the active mode and its counterpart in the passive mode, as well as a similar pair of statements, in which one was obtained from the other by replacing the verb "to give" by the verb "to receive," and interchanging subject and indirect object (with other minor grammatical adjustments) (Frege [1918] 1990, 348). Moreover, in Frege's letters to Husserl of 1906 he states that two statements express the same thought when they have the same deductive power, i.e., when any statement derivable from one of the two statements is derivable from the other (maintaining fixed the remaining logical resources) (Frege 1974, 101–106). Thus, in those two characterizations of sense, sense is basically identified with Frege's old notion of conceptual concept. Hence, as I have argued in "On Frege's Two Notions of Sense" (Rosado Haddock 2000b, see also Rosado Haddock 2006, Ch 4), Frege worked with two notions of sense, the official one of "Über Sinn und Bedeutung" (Frege

[1892] 1967, 143–162) and *Grundgesetze der Arithmetik*, obtained from his old notion of judgeable content, and an unofficial notion, which is essentially his old notion of conceptual content. But, as I have stated before, the fact that Frege lacked a notion of state of affairs did not allow him to prevent the two notions from collapsing in a two-headed notion of sense (Rosado Haddock 2000c). This ambivalence in his use of the words "sense" and "thought" explains the inconsistencies, which are present in Frege's elucidation of the notorious Axiom V of *Grundgesetze der Arithmetik*, namely

(Δ) The course of values of the function F is identical to the course of values of the function G if and only if for all objects x, F(x) if and only if G(x).

Thus, in "Funktion und Begriff" he says that the two sides of that axiom have the same sense (Frege [1891] 1990, 130), whereas in *Grundgesetze der Arithmetik* he says that they have the same referent (Frege [1893] 1903, §3), hence, since both sides are statements, the same truth value. I have argued in (2000b and 2006) that, if "sense" were to mean official sense, both statements of Frege are untenable: the first would be clearly false and the second too vague. What both sides of Frege's Axiom V have in common is once more the conceptual content. Moreover, a similar ambivalence is present, for example, in Frege's paper written in 1914 "Logik in der Mathematik" (Frege 1983, 219–270). In the first part of that paper, Frege requires from definitions that the *definiens* and the *definiendum* have the same sense. In later parts of the same paper, he states as a requirement for definitions that the *definiens* and the *definiendum* have the same referent (See Chapter 6 of Rosado Haddock 2006).

It was Husserl who clearly distinguished the notion of a situation of affairs (*Sachlage*) both from the thought or proposition expressed by a statement and from the state of affairs, which is for him the referent of the statement. In *Logische Untersuchungen* Husserl formulates in a detailed way the distinction between sense and referent he had obtained around 1890. The main differences with Frege were (i) that for Husserl the sense of a predicate was a concept, whereas the extension was the referent, and (ii) that the referent of statements was not a truth value, but a complex objectuality, which was intended to be a state of affairs. Thus, the statements "The Morning Star is a planet" and "The Evening Star is a planet" refer not to a truth value but to the state of affairs that Venus is a planet.

It must, however, be made clear that in the First Logical Investigation, which dates from 1896, Husserl had still not distinguished between state of affairs and situation of affairs, and when he mentioned there states of affairs as a possible referent of statements and tried to offer an example of two statements which express different senses but have the same referent, the example was inadequate, because what the two statements had in common was not the same state of affairs but the situation of affairs. In the second edition he corrected the situation not by changing the example but by replacing the word "*Sachverhalt*"—i.e., state of affairs—by the word "*Sachlage*," that is, situation of affairs.[11] However, in any case, in the 4th Logical Investigation (Hua 19, Investigation IV, §11) he made it clear that states of affairs are the referents of statements and in the Sixth Logical Investigation—completed some four years later than the first—Husserl already clearly distinguished between state of affairs and situation of affairs (ibid., Investigation VI, §48). The distinction is discussed in his very late *Erfahrung und Urteil* (Husserl [1939] 1976, §§59f), on which he was working with his assistant Ludwig Landgrebe at the time of his death, and in his posthumously published *Vorlesungen über Bedeutungslehre* (Hua 26, 29–30), based on a 1908 course. The situation of affairs is a sort of reference basis (or substratum) of states of affairs.[12]

Hence, in comparison with Frege's scheme, between thought and truth value, in the Husserlian scheme there are two intermediate stages, first the state of affairs to which the statement refers, and then the situation of affairs. Let us give two elementary examples, an arithmetical and a non-arithmetical one. Let us suppose that Charles is the father of Mary, and consider the following three statements: (i) "The youngest son of Charles is taller than the eldest son of Charles," (ii) "The youngest brother of Mary is taller than the eldest brother of Mary," and (iii) "The eldest son

[11] (See Hua 19, Investigation I, §12.) In (Rosado Haddock 1997), I already mentioned this confusion of Husserl .

[12] For Husserl specialists: In contrast to states of affairs, situations of affairs are pre-categorial abstract objectualities, whereas states of affairs are categorial objectualities, that is, categorially constituted objectualities, or objectualities of the understanding. On categorial objectualities and categorial intuition, see (Rosado Haddock 2000d).

of Charles is shorter than the youngest son of Charles." (ii) is obtained from (i) by the same procedure by which "The Morning Star is a planet" is obtained from "The Evening Star is a planet," namely by the replacement of an expression by another having a different sense but the same referent. Thus, (i) and (ii)—for Husserl, as for Frege—express different thoughts (or propositions), but refer to the same state of affairs—not to a truth value (as for Frege). (iii), however, cannot be obtained in that way from (i)—or from (ii). The transformation from (i) to (iii) is of a different sort, not one consisting of a replacement of expressions with different sense but the same referent, but a more objectual one. The same happens with the following arithmetical example. Consider the following three statements: (i′) "5 + 3 >7," (ii′) "9 – 1>7," and (iii′) "7 < 5 + 3." (ii′) is obtained from (i′) by the replacement of a name of the number 8 by a name having a different sense but the same referent, namely, the number 8. (iii′), however, cannot be obtained in that way from (i′)—and, of course, not from (ii′). The transformation from (i′) to (iii′) does not consist of the replacement of an expression by another expression with different sense but the same referent, but is a more objectual one.

The transformation from (i′) to (ii′) preserves the state of affairs—and, of course the situation of affairs and the truth value—, but does not preserve the thought. The transformation from (i′) to (iii′) does not preserve the state of affairs or the thought, but preserves the situation of affairs—and, of course, the truth value. Finally, the transformation from (i′), (ii′), or (iii′) to the statement "Paris is the capital of France in 2007" preserves the truth value, but not the situation of affairs—and, of course, not the state of affairs or the thought. I have argued that each of these different sorts of transformations builds a group, and that the transformation groups of statements are so related that the transformation group which preserves thought is a proper subgroup of the transformation group that preserves states of affairs, which is a proper subgroup of the transformation group that preserves situations of affairs, which is a proper subgroup of the transformation group that only preserves truth value (2000a). Using these distinctions, it is clear that Frege's argument to show that only the truth value is preserved when we transform a statement like (i') into one like (ii'), or a statement like (i) in a statement like (ii) is fallacious. In (2000c) I showed that Church's argument with the same purpose in the Introduction to his *Introduction to Mathematical Logic* (Church 1956, 25) is also fallacious.[13]

By the way, it is interesting to observe that, as a consequence of the above distinctions between groups of transformations of statements, it becomes clear that states of affairs are extensional with respect to thoughts—the state of affairs that Venus is a planet is not affected by someone's believing or not that the evening star is a planet but the morning star is not—, situations of affairs are extensional with respect to states of affairs, and truth values are extensional with respect to situations of affairs. I consider that the coarse distinction between extensional and non-extensional contexts, which still causes so many problems to philosophical logicians, could be very fruitfully replaced by the degrees of extensionality determined by each of the four transformation groups. Thus, we would have (i) extensionality$_{G}$—also called "intensionality"—("G" for *Gedanke*, German word for "thought"), (ii) extensionality$_{SV}$ ("SV" for *Sachverhalt*, i.e., state of affairs), (iii) extensionality$_{SL}$ ("SL" for *Sachlage*, i.e., situation of affairs), and (iv) extensionality$_{WW}$ ("WW" for *Wahrheitswert*, i.e., truth value). Moreover, our discussion will show that it is the degree of extensionality determined by situations of affairs, that is, for which situations of affairs remain invariant, which is most important for mathematics. A semantics—like Frege's official one—which ignores both that notion and the notion of state of affairs—is totally unsuited for mathematics. Furthermore, not even logic is, as Frege believed, the science of the laws of truth. What interests logic, and any other science, is, as Husserl clearly stated (Hua 28, Ch. XI, §§62, 63), the structuration of truths in theories, in which statements are connected by relations of derivability, which preserve truth. And logic is concerned, among many other things, with such relations, not simply with truths.

V. Situations of Affairs: Systematic Treatment

So far as I know, Husserl never got further in the application of his notion of situation of affairs to mathematics than the use of examples like (i′)–(iii′) above. However, he did consider the possibility of applying

[13]A much more thorough refutation of the different versions of the so-called '"slingshot argument"' has been recently given by the distinguished Brazilian philosopher Oswaldo Chateaubriand in the first volume of his (Chateaubriand 2001 & 2005).

his notion to physics. In his *Vorlesungen über Bedeutungslehre* he observed that when two different so-called formalisms in physics are equivalent in the sense of corresponding to the same physical law, the situation of affairs in both cases is the same (Hua 26, 101–102). Thus, he would very probably have interpreted the equivalence between Schrödinger's wave mechanics and Heisenberg's matrix mechanics as identity of the situation of affairs, though not only the thoughts expressed but the states of affairs referred to were different.

I have tried to apply Husserlian semantics to mathematics (Rosado Haddock 1996, 2000b, 2000f, see also 2000c). First of all, it is perfectly clear that Fregean semantics is totally inadequate for mathematical statements. Even the most simple arithmetical equations and inequalities, for example, the pairs of statements {"2 + 3 = 5," "6 – 1 = 5"} and {"5 + 3 > 7," "9 – 1 > 7"}, composed of statements obtained from each other by a transformation, which changes the sense of component expressions but preserves the referent, require of the notion of state of affairs to be correctly interpreted. The members of each pair are clearly related by such a transformation, but not so the members of different pairs. Of course, the four statements have the same truth value, which is the same of, for example, Gödel's Completeness Theorem for first-order predicate logic and of "Paris is the capital of France in 2007," but the members of any of the two pairs have much more in common between them than with any of the remaining four statements.

Examples like those of statements (i′)–(iii′) given above show, however, that for a semantics of mathematical statements states of affairs (and truth values) are not enough. The requirement of a notion like that of Husserlian situation of affairs seems inevitable. Let us consider now the phenomena of duality, very common in some areas of mathematics. Dual statements, like (i″) (Tarski's Ultrafilter Theorem): "Every filter can be extended to an ultrafilter" and (ii″) (The Maximal Ideal Theorem): "Every ideal can be extended to a maximal ideal," are more strictly related to each other than to other mathematical statements, like "2 + 3 = 5," but one cannot be transformed into the other by a transformation that replaces an expression by another expression with different sense but the same referent. What can be very well obtained from (i″) by such a transformation is (iii″) "Every dual ideal can be extended to a maximal dual ideal." (i″) and (iii″) refer to the same state of affairs, but (ii″) does not. What (ii″) has in common with (i″) and (iii″), but not with “2 + 3 = 5,” is the situation of affairs.

In the two mentioned papers and in “On the Semantics of Mathematical Statements” I have extended the application of the notion of situation of affairs to the interderivability phenomena. Thus, although the Axiom of Choice and Tychonoff's Theorem clearly refer to different states of affairs, their reference basis is the same, that is, the situation of affairs is the same, and it is different from that of the infinitely many mathematical statements not interderivable with the Axiom of Choice.[14] Moreover, when a statement S is derivable from another statement S*, but S* is not derivable from S, the situation of affairs of S is properly included in the situation of affairs of S*. Hence, the situation of affairs common to the Ultrafilter Theorem, the Maximal Ideal Theorem and Tychonoff's Theorem restricted to Hausdorff spaces is properly included in the situation of affairs common to the Axiom of Choice, Zorn's Lemma and Tychonoff's Theorem.

In my paper “On the Semantics of Mathematical Statements” I tried to make precise some of these ideas about situations of affairs. Following Tarski's intuitive motivation for his semantics[15] I tried to extend Tarskian semantics in a Husserlian fashion. Thus, I identified states of affairs with the results of evaluations and then introduced situations of affairs as equivalence classes of states of affairs. For such a purpose, it was necessary to consider only many sorted languages, since it is clear that a language for mathematics adequate to express interderivability results between seemingly unrelated mathematical statements must be capable of dealing simultaneously with mathematical structures of different sorts. I tried to show, first for a first-order many sorted language and then for a more adequate second-order many sorted language that, under such suitable definition of situations of affairs, two statements are interderivable if

[14]The requirement of including states of affairs and situations of affairs in a semantics adequate for mathematics is a minimum requirement. The possibility of finer distinctions is an open possibility, but they should be made only if needed.

[15]See his epoch-making ”The Concept of Truth in Formalised Languages,” in (Tarski [1956] 1983, see also Tarski [1969] 1986).

and only if they have the same situation of affairs. The field is open for younger or more able logicians to follow.

VI. Conclusion

I agree with the general form, although not with the specificities, of Benacerraf's requirement (Benacerraf 1973) that a philosophy of mathematics has to be bound to an acceptable semantics and to a reasonable epistemology. Of course, I reject any rendering of reasonableness as linked to any sort of causal theory.[16] Moreover, I take as a requirement of any philosophy of mathematics that it be capable of adequately assessing the interderivability phenomena. Husserl's philosophy of mathematics is, so far as I can see, the only philosophy of mathematics which (1) is coupled with an adequate semantics of sense and referent for mathematical statements and, moreover, this semantics is perfectly compatible with Tarskian semantics; (2) with the help of this semantics, one can adequately assess the interderivability phenomena; and (3) it is complemented by an epistemology of mathematics based on the—in no way mysterious and so often neglected—categorial intuition, expounded by Husserl in the Sixth Logical Investigation.[17] I cannot dwell on this last point here, but would only like to say that I am convinced that Husserl's views can offer an alternative to Quine's (de)naturalized epistemology in a more general setting.

Appendix

On Tappenden, Frege and Fregean Mythologies

In his recent paper "The Riemannian Background to Frege's Philosophy" Jamie Tappenden (2006) argues on behalf of a presumed allegiance of Frege to the Riemannian tradition. According to Tappenden,

[16]For my criticism of Benacerraf's and, in general anti-Platonist dogmas and prejudices, see (2000g).

[17]On this issue, see also (Rosado Haddock 2000d) for an exposition of Husserl's categorial intuition.

there were two irreconcilable trends in nineteenth century German mathematics, the school of Weierstrass and Kronecker in Berlin and the Riemannian school in Göttingen. Tappenden begins his discussion by considering two presumed myths in Fregean scholarship, namely, (i) that which he associates with Russell, according to which Frege followed the Weierstrassian program of reducing analysis to the real number system, which was then reduced to the arithmetic of natural numbers and then, finally, thanks to Frege, to logic; and (ii) that which he associates with Kitcher, Currie, Wagner, and Weiner, for which Frege's motivations were purely philosophical and unrelated to Weierstrass' mathematical concerns. Tappenden proposes a third myth, namely, that since there were only two mathematical traditions in the German universities of his time, and Frege studied in Göttingen, learned complex analysis in the Riemannian tradition from Clebsch and geometry from Schering, he must have been immersed in the Riemannian tradition all the way. Moreover, Tappenden argues that when teaching complex analysis, Frege did it following the footsteps of his teacher, thus, in the Riemannian fashion. That is, however, an incredible historiographical simplification. Firstly, it is perfectly understandable that if you are not a researcher in a particular area of your discipline—and Frege certainly was not a reasercher in complex analysis—, but have to teach courses in that area—remember that the mathematics department in Jena was rather small and that Frege was not a full professor—it is perfectly natural that you tend to follow, possibly with some small modifications, what you learned from your teachers in that area. Hence, it was much easier for Frege to teach complex analysis in the way he learnt it from Clebsch than to try to immerse himself in the Weierstrassian tradition or to try to build a version of his own.

On the other hand, it is not uncommon of men of Frege's stature that in the points that interest them most they distance themselves from their teachers. Thus, for example, Dedekind came from the Gauss-Riemann school, was the last (official) doctoral student of Gauss and a personal friend of Riemann, but followed the reductive path of Weierstrass and was a logicist, as Frege was. Indeed, when Frege criticizes Dedekind at the beginning of *Grundgesetze I* it is for his lack of rigour, not for having taken a wrong reductive path. Moreover, Cantor and Husserl were students of Weierstrass and Kronecker—as was also Minkowski—, and Husserl was even Weierstrass's assistant. Nonetheless, Cantor's set theory had little to do with Weierstrass's or Kronecker's research, and, on the contrary, only served to awaken ire in the latter, who used all his academic influence to make life miserable to Cantor. In Husserl's case, his mature philosophy of mathematics—from 1894 until the end of his life—is based on a sort of generalization of Riemann's notion of a manifold, has nothing in common with the Weierstrassian school, and also very little in common with Frege's views, except for being both Platonisms, though different sorts of Platonism. Furthermore, Frege almost never refers to Riemann, even in his posthumous writings. On this issue, a comparison with Husserl is especially pertinent. On the other hand, in *Die Grundlagen der Arithmetik* Frege propounds the view that geometrical statements are synthetic a priori, that geometry is based on intuition, that such an intuition is Euclidean, and that Non-Euclidean geometries are merely objects of thought. Although Frege's arguments differ from those of Kant, their views on geometry are very similar. In fact, Frege's views on geometry were pre-Riemannian. On this specific point, Frege and Husserl also differed, and Husserl was once more much nearer to Riemann, but I cannot dwell on this issue here.

Concerning the different myths in Fregean scholarship, I want to make a few comments. I consider Frege essentially a philosopher of mathematics, certainly not a philosopher of language and by no means an epistemologist. His concerns with semantics are derivative, and his concerns with epistemological issues are not only derivative but also restricted to particular issues related to the nature of mathematics and logic. I consider correct the view that sees Frege's logicist program of reducing arithmetic to logic as an extension of the Weierstrassian program, and as a culmination of what Weierstrass and Dedekind had achieved, even though Frege was, by no means, a Weierstrassian, and was very critical of Weierstrass's lack of rigour. But without the reduction of analysis to the real number system and then, further, to the arithmetic of natural numbers, Frege's logicist program would not make much sense. That sort of mathematical reduction to logic, however, had nothing to do with Russell's empiricist and nominalist tendencies, which Frege would have clearly rejected. Frege was an avowed Platonist and a rationalist, and would never had accepted the use of logical analysis to defend nominalism or empiricism. On the other hand, the tendency—predominant in Anglo-American Fregean scholarship—to interpret Frege as a sort of Neo-Kantian philosopher and as an avowed epistemologist is certainly unfounded. It basically originates in clear misinterpretations of Fregean writings, usually isolating a single sentence from its whole context in order to fit the prejudices of the so-called Fregean scholars. The papers by Sluga, Beaney, Reck and Macbeth included in a recent book edited by Dirk Greimann (2007) are foremost examples of such and other sorts of misinterpretations. Finally, it should be pointed out that Frege, who was such a revolutionary in logic and a most interesting and important philosopher of mathematics, was usually not capable of understanding or at least appreciating other revolutionary advances in mathematics, as attested by his lack of understanding of Cantor exemplified at the end of *Die Grundlagen der Arithmetik*, and precisely by his adherence to a pre-Riemannian, Kantian conception of geometry.

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CHAPTER III

HUSSERL ON AXIOMATIZATION AND ARITHMETIC

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Abstract. Material from Husserl's logic courses is used to piece together a picture of his theory of axiomatization and arithmetic that can be used to lay the groundwork for the study of its ramifications and implications for philosophy of logic and mathematics. It is argued that this shows that Husserl's theory is close to Hilbert's and belies claims of kinship with Brouwerian Intuitionism understood as the view that the collection of natural numbers and all of pure mathematics develops out of the self-unfolding of "the fundamental intellectual phenomenon of the falling apart of a moment of life into qualitatively different things, of which one is experienced as giving way to the other and yet is retained by an act of memory." It is contended that it needs to be determined whether Husserl's theory is a genuine, viable alternative to other, better known theories.

I. Introduction

It is well known that efforts to provide what Gottlob Frege once called "a more detailed analysis of the concepts of arithmetic and a deeper foundation for its theorems" (Frege [1879] 1967, 8) played a preeminent role in shaping the course of twentieth century philosophy. Frege's and Bertrand Russell's efforts to do so generated Analytic philosophy; Edmund Husserl's dissatisfaction with his own efforts to do so by applying Franz Brentano's techniques in the *Philosophy of Arithmetic* set him on the path to writing his groundbreaking *Logical Investigations* and eventually to phenomenology.

It is also well known that for a time many philosophers were inclined to interpret the evolution of Husserl's ideas about the foundations of mathematics from a Fregean perspective. This view was especially associated with Dagfinn Føllesdal's Master's thesis *Husserl and Frege: A Contribution to Elucidating the Origins of Phenomenological Philosophy* (Føllesdal [1958] 1994) and his article "Husserl's Notion of Noema" (Føllesdal 1969). Since the 1970s, it has been combated by Guillermo Rosado Haddock, J. N. Mohanty, myself, and others (Rosado Haddock

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1973, 1982; Mohanty 1974, 1982; Hill 1979; Hill and Rosado Haddock 2000).

Then, once the belief that Husserl was a Fregean subsided, the temptation arose to see links between Husserl's ideas on the foundations of mathematics and Brouwerian Intuitionism. This interpretation is associated with the work of Richard Tieszen (Tieszen 1989) and Mark van Atten (van Atten 2007). It is opposed by Jairo da Silva, Rosado Haddock, and myself. We consider that even a cursory examination of Husserl's ideas about axiomatization and numbers shows that Husserl's ideas could not in fact be more different from those of Brouwer. However, to my knowledge, the extent to which Husserl rejected the major tenets of Brouwerian Intuitionism has never been systematically demonstrated.

Now, the publication of Husserl's logic courses from 1896 and 1902/03 by the Husserl Archives in 2001 has made available the new material necessary to piece together a satisfactory picture of the development of Husserl's theories about axiomatization and the foundations of mathematics, a subject rich in interesting ramifications and implications for the philosophy of logic and mathematics. So, here I propose to take advantage of the raw material now available to piece together a picture of Husserl's ideas about axiomatization and arithmetic that can be used to lay the basic groundwork needed for exploration of those ramifications and implications.

In the course of my exposition of Husserl's theories about axiomatization and arithmetic, I draw attention to specific areas in which they are at odds with Brouwer's main theses. The considerations thus brought to the fore, I argue in the concluding sections, indicate that Husserl's theories were closer to those of Brouwer's opponent, David Hilbert (cf. Hintikka 1997, 2008), and belie claims of kinship with Brouwerian Intuitionism understood, in general, as the view that not only the collection of natural numbers, but all of pure mathematics, develops out of the self-unfolding of "the fundamental intellectual phenomenon of the falling apart of a moment of life into qualitatively different things, of which one is experienced as giving way to the other and yet is retained by an act of memory" that Brouwer called the Primordial Intuition of two-ity and considered to be the basis of the whole of Intuitionism (Brouwer [1912] 1983, 80; Brouwer [1929] 1998, 45–46; Brouwer [1930] 1998, 57). I further suggest that now that we are in possession of Husserl's theory, we need to give it a try in order to determine whether it is just an ingeniously worked out take on many of the issues in the philosophy of mathematics of his time by the father of phenomenology, or a genuine, viable alternative to theories more familiar to philosophers of logic and mathematics.

II. Husserl's Initial Opposition to the Axiomatization of Arithmetic

Husserl's position in his 1891 *Philosophy of Arithmetic* was resolutely anti-axiomatic. He attacked those who fall into remote, artificial constructions which, with the intent of building the elementary arithmetic concepts out of their ultimate definitional properties, interpret and change their meaning so much that totally strange, practically and scientifically useless conceptual formations finally result. Especially targeted was Gottlob Frege's ideal of the "founding of arithmetic on a sequence of formal definitions, out of which all the theorems of that science could be deduced purely syllogistically" (PA, 123–126).

As soon as one comes to the ultimate, elemental concepts, Husserl reasoned, all defining has to come to an end. All one can then do is to point to the concrete phenomena from or through which the concepts are abstracted and show the nature of the abstraction process. A verbal explanation should place us in the proper state of mind for picking out, in inner or outer intuition, the abstract moments intended and for reproducing in ourselves the mental processes required for the formation of the concept. He said that his analyses had shown with incontestable clarity that the concepts of multiplicity and unity rest directly upon ultimate, elemental psychical data, and so belong among the indefinable concepts. Since the concept of number was so closely joined to them, one could scarcely speak of defining it either (PA, 123–126). All these points are made on the only pages of *Philosophy of Arithmetic* that Husserl ever explicitly retracted (LI, 179n.).

Four years earlier, in *On the Concept of Number*, Husserl had set out to anchor arithmetical concepts in direct experience by analyzing the actual psychological processes to which he thought the concept of number owed its genesis. To obtain the concept of number of a concrete set of objects, say *A*, *A*, and *A*, he explained, one abstracts from the particular characteristics of the individual contents collected, only considering and retaining each one insofar as it is a something or a one. Regarding their collective combination, one thus obtains the general form of the set belonging to the set in question: one and one, etc. and. . . and one, to which a number name is assigned (Husserl 1887, 310, 352–356; PA, 85–86).

The enthusiastic espousal of psychologism of *On the Concept of Number* is not found in *Philosophy of Arithmetic*. Husserl later confessed that doubts about basic differences between the concept of number and the concept of collecting, which was all that could be obtained from reflection on acts, had troubled and tormented him from the very beginning and had eventually extended to all categorial concepts and to concepts of objectivities of any sort whatsoever, ultimately to include modern analysis and the theory of manifolds, and simultaneously to mathematical logic and the entire field of logic in general. He did not see how one could reconcile the objectivity of mathematics with psychological foundations for logic (Husserl 1975, 34–35; Husserl [1902/03a] 2001, 3–59).

III. Husserl's *Volte-Face*

In sharp contrast to Brouwer who denounced logic as a source of truth (Brouwer [1948] 1983, 90–96), from the mid-1890s on, Husserl defended the view, which he attributed to Frege's teacher Hermann Lotze, that pure arithmetic was basically no more than a branch of logic that had undergone independent development. He bid students not to be "scared" by that thought and to grow used to Lotze's initially strange idea that arithmetic was only a particularly highly developed piece of logic (Husserl [1896] 2001, 241, 271; Husserl 1902/03b, 19, 34; Hua 24, §15).

Many years later, Husserl would explain in *Formal and Transcendental Logic* that his "war against logical psychologism was meant to serve no other end than the supremely important one of making the specific *province* of analytic logic visible in its purity and ideal particularity, freeing it from the psychologizing confusions and misinterpretations in which it had remained enmeshed from the beginning" (FTL, §67). He had come to see arithmetic truths as being analytic, as grounded in meanings independently of matters of fact. He had come to believe that the entire overthrowing of psychologism through phenomenology showed that his analyses in *On the Concept of Number* and *Philosophy of Arithmetic* had to be considered a pure a priori analysis of essence (Husserl 1975, 42–43). For him, pure arithmetic, pure mathematics, and pure logic were a priori disciplines entirely grounded in conceptual essentialities, where truth was nothing other than the analysis of essences or concepts. Pure mathematics as pure arithmetic investigated what is grounded in the essence of number. Pure mathematical laws were laws of essence (Husserl [1905] 1994, 37; Hua 24, §13c).

He told students that it was to be stressed repeatedly and emphatically that the ideal entities so unpleasant for empiricistic logic, and so consistently disregarded by it, had not been artificially devised either by himself, or by Bolzano, but were given beforehand by the meaning of the universal talk of propositions and truths indispensable in all the sciences. This, he said, was an indubitable fact that had to be the starting point of all logic (Husserl [1908/09] 2003, 45).

All purely mathematical propositions, he taught, express something about the essence of what is mathematical. Their denial is consequently an absurdity. Denying a proposition of the natural sciences, a proposition about real matters of fact, never means an absurdity, a contradiction in terms. In denying the law of gravity, I cast experience to the wind. I violate the evident, extremely valuable probability that experience has established for the laws. But, I do not say anything "unthinkable," absurd, something that nullifies the meaning of the word as I do when I say that 2×2 is not 4, but 5 (Hua 24, §13c).

Husserl taught that every judgment either is a truth or cannot be a truth, that every presentation either accorded with a possible experience adequately redeeming it, or was in conflict with the experience, and that grounded in the essence of agreement was the fact that it was incompatible with the conflict, and grounded in the essence of conflict that it was incompatible with agreement. For him, that meant that truth ruled out falsehood and falsehood ruled out truth. And, likewise, existence and non-existence, correctness and incorrectness cancelled one another out in every sense. He believed that that became immediately apparent as soon as one had clarified the essence of existence and truth, of correctness and incorrectness, of *Evidenz* as consciousness of givenness, of being and not-being in fully redeeming intuition.

At the same time, Husserl contended, one grasps the "ultimate meaning" of the basic logical law of contradiction and of the excluded middle.

When we state the law of validity that of any two contradictory propositions one holds and the other does not hold, when we say that for every proposition there is a contradictory one, Husserl explained, then we are continually speaking of the proposition in its ideal unity and not at all about mental experiences of individuals, not even in the most general way. With talk of truth it is always a matter of propositions in their ideal unity, of the meaning of statements, a matter of something identical and atemporal. What lies in the identically-ideal meaning of one's words, what one cannot deny without invalidating the fixed meaning of one's words has nothing at all to do with experience and induction. It has only to do with concepts (Husserl 1902/03b, 33; Hua 24, §§13a, 50a; Husserl [1908/09] 2003, 45).

In sharp contrast to this, Brouwer saw intuitionistic mathematics as deviating from classical mathematics because the latter uses logic to generate theorems and in particular applies the principle of the excluded middle. He believed that Intuitionism had proven that no mathematical reality corresponds to the affirmation of the principle of the excluded middle and to conclusions derived by means of it. He reasoned that "since logic is based on mathematics—and not *vice versa*—the use of the Principle of the Excluded Middle is not permissible as part of a mathematical proof" (Brouwer [1921] 1998, 23; Brouwer [1929] 1998, 51–53; Brouwer [1948] 1983, 90; Brouwer [1928a] 1967; Brouwer [1928b] 1998).

IV. Analysis of the Concept of Number

According to Husserl, only concepts are purely logical that are not limited to a special field of objects, that not only actually figure and can figure in all the sciences, but are common and necessary to all sciences because they belong to what belongs to the ideal essence of science in general. So, all concepts relating to objects in general in the most universal ways, or to thought forms in general in which objects are brought to theoretically objective unity are purely logical.

In contrast to Brouwer's idea that it is the fundamental phenomenon of mathematical thinking, the intuition of two-oneness that in its self-unfolding creates not only the numbers one and two, but produces the collection of natural numbers and finally all of pure mathematics

(Brouwer [1912] 1983, 80; Brouwer [1929] 1998, 45–46), the concept of number stood as a paradigm of a purely logical concept in Husserl's sense. Each and every thing, he reasoned, can be counted as one. No science is conceivable in which the number concepts cannot find an application. All purely mathematical concepts like unit, multiplicity, cardinal number, order, ordinal number, and manifold are purely logical because they clearly relate in the most universal way to numbers in general and are only made possible out of the most universal concept of object. However, geometry, mathematical mechanics and all mathematico-natural scientific disciplines do not belong in pure logic since their concepts have real content (Husserl 1902/03b, 31–43, 49).

In his logic courses, Husserl taught that pure number theory is a science that unfolds the meaning of the idea number and arithmetic in a systematic theory of the laws unfolding the meaning of cardinal number, itself the answer to the question: "How many?" He illustrated what he meant by this "unfolding the meaning of the question 'How many?'" Since each and every thing can be counted as one, to conceive (*konzipieren*) the concept of number, or any arbitrarily defined number, we only need the concept of something in general. One is something in general. Anything can be counted as one and out of the units all cardinal numbers built: one and one or two, two plus one, etc. (Husserl 1902/03b, 31–43, 49). One pear and one man, one apple and one pear, one apple and one apple all have the form "one and one. . ." This form is the concept "one and one" or "two." Anything and anything, remains unchanged. It is different from "one and one and one, etc." (Husserl [1896] 2001, 102).

Eminent thinkers like Lotze, Husserl explained on another occasion, correctly recognized cardinal number as a specific differentiation of the concept multiplicity (*Vielheit*) and multiplicity as the most universal logical concept combining objects in general. This most universal concept of multiplicity splits into a series of different special forms and these are the cardinal numbers. Since an apple is not a multiplicity of apples, an A not a multiplicity of As, then an apple or an A cannot be designated by a cardinal number. The first number in the number series is 2 As. If from 2 As, we use definitions to form the new number 2 As and 1 A and designate them as 3 As, likewise 3 As and 1 A as 4 As, etc., then we obtain a series of the so-called natural numbers, infinite in one direction. The totality of numbers is not exhausted in so doing. For, we can also form the concept of the number of numbers of the natural number series, which can easily be shown not to be identical with any number of that series itself (Husserl [1896] 2001, 102, 241–242).

To questions as to how arithmetic came about and how the foundations of arithmetic were provided, Husserl answered that people analyzed the arithmetic propositions at first given as they were first entertained by people. They found that certain relations were grounded in the concept of number. For instance, any two numbers are either equal or one is larger or smaller than the other. They further noticed that certain combinations were grounded in the concept of number, first of all addition, then multiplication, raising to a higher power and the inverse of these operations, subtraction, division, extracting roots, logarithms. Given with the elementary combinations were certain simple, directly intelligible laws that careful analysis traced back to a certain minimal number of laws no longer reducible to one another. Since these laws lie in the simple meaning of the concepts founding them, they are a priori. They are not propositions about matters of fact drawn from experience, but propositions about relations of ideas obtained by analysis of the universal concepts by merely digging more deeply into their meaning (Husserl 1902/03b, 33).

The first law of arithmetic, Husserl taught, is a + b = b + a, or "For any two numbers there is a sum a + b." Denying its truth would be a contradiction. Anyone who does so uses "cardinal number" in some other way, does not know what the words mean or is abandoning its concept. It is a matter of a truth that could not possibly be false, of an analytic statement whose denial is self-contradictory (Husserl 1902/03b, 33, 35; Hua 24, §13c).

Mathematicians can set down $a + 1 = 1 + a$ in a single blow as something unconditionally valid and certain because it is part of the meaning of number (of cardinal number in the original sense) for that to be the case, and it would be tantamount to flying in the face of the meaning of the words "how many" if one wanted to deny it. Likewise, it is part of the meaning of talk of "cardinal numbers" that each number can be increased by one. To say that a cardinal number, a how many, cannot be increased is tantamount to not knowing what one is talking about. It is tantamount to contravening the meaning, the identical meaning, of talk of cardinal numbers. An elementary formula of this kind already contains infinitely many things in it. It gives not one basic law of arithmetic, but a whole series. Infinitely many laws are simply produced from a primitive number proposition like a + b = b + a by the fact that because of their universality a + b are substitutable (Husserl [1896] 2001, 250; Husserl 1902/03b, 33, 35; Hua 24, §13c).

Each genuine axiom is a proposition that unfolds the idea of cardinal number from some side or unfolds some of the ideas inseparably connected with the idea of cardinal number. These direct arithmetical laws develop directly in the evidence of certainty and this certitude and evidence carries over to all theses in deductive substantiation. And so these basic laws go on to serve as a basis for systematic deductions in which ever new laws are grounded (Husserl [1896] 2001, 39, 243; Husserl 1902/03b, 33, 35, 39; Hua 24, §13c).

On the basis of its axioms, the theorems of pure arithmetic are derived by pure deduction following systematic, simple procedures. The field branches out into more and more theories and partial disciplines, ever new problems surface and are finally solved by the expending the greatest mathematical acumen and following the most rigorous methods. So it is that all of arithmetic is grounded in the arithmetical axioms. The unending profusion of wonderful theories it develops is already fixed, enfolded in the axioms, and theoretical-systematic deduction effects the unfolding of them (Husserl [1896] 2001, 39, 243; Husserl 1902/03b, 33, 35, 39; Hua 24, §13c).

Such talk of a priori concepts and ideal entities stands in sharp contrast to Brouwer's mockery of what he called the "foolish superstition" to treat words as labels for "fetish-like" concepts which, along with the relations between them, are assumed to exist independently of the causal attitude of human beings. This was how, he thought, people came to believe that certain relations between concepts derived from axioms with the help of logical principles might be treated as ideal truths (Brouwer [1929] 1998, 49–50). Brouwer admitted that from certain relations among mathematical entities assumed as axioms, mathematicians deduce other relations in accordance with fixed laws in the conviction that they are deriving truths from truths by logical reasoning, but he maintained that this "non-mathematical conviction of truth or legitimacy has no exactness whatever, and is nothing but a vague sensation of delight arising from the knowledge of the efficacy of the projection into nature of the relations and laws of reasoning" (Brouwer [1912] 1983, 78).

V. Calculating with Concepts and Propositions

Husserl's search for answers raised by his earliest analyses of the concept of number in "On the Concept of Number" and *Philosophy of Arithmetic* led him beyond the confines of the mathematical realm to a universal theory of formal deductive systems in general. He saw that developments in formalization had unmasked close relationships between number statements and the propositions of logic and that this made it possible to develop a genuine logical calculus for calculating with propositions in the way mathematicians do with numbers, quantities, and the like (LI, 41–42; Husserl 1975, 16–17, 35; Husserl 1994, 490–491; FTL, §§23–27).

By 1896, he was teaching that the formal discipline of propositions in general and of concepts in general was a mathematical discipline that was of precisely the same nature and used the same methods as familiar mathematical disciplines like arithmetic and that there was nothing at all extraordinary about the idea of calculating with concepts and propositions. Practically speaking, he enthused, arithmetic actually represents the most marvelous tool devised by the human mind for purposes of deduction. It is the science in which the deductive relations are analyzed most carefully (Husserl [1896] 2001, 250, 271–272).

According to Husserl, only the completely unfounded prejudice that the essence of the mathematical lies in number and quantity could explain rejection of the new mathematical theory of conceptual and propositional inferences. But, what is mathematical in the procedure of arithmetic, he protested, does not hinge upon our having to do with numbers in them. The essence of the mathematical does not lie in being quantitatively determinable, but in establishing a purely apodictic foundation of the truths of a field from apodictic principles. It is a matter of a rigorously scientific, a priori theory that builds from the bottom up and derives the manifold of possible inferences from the axiomatic foundations a priori in a rigorously deductive way (Husserl [1896] 2001, 272–273; Husserl 1902/03b, 231–232, 239–249; Hua 24, 434).

To the question as to what it is that characterizes calculating in the field of numbers, Husserl answered that the calculating obviously involves operating with the signs, not with the concepts themselves. To solve a problem, to derive a proposition, we must not think at all about the concepts themselves, but by using procedures defined by set rules, we can be content to link signs to signs, replace combinations of signs by other combinations of signs, etc. At first, the result of the calculation is again purely a combination of signs on paper, but, the interpretation of the results of the inference yields precisely the proposition sought (Husserl [1896] 2001, 247).

In a similar fashion, he taught that every purely formal procedure that proceeds strictly deductively can be presented in algebraic forms and when this occurs scientific thinking first wins a free overview of all possibilities of deductive reasoning and that sovereign mastery of all possible problems and ways of solving them that is the prerequisite for the most exact and most universal solution of problems of the field concerned (Husserl [1896] 2001, 272–273; Husserl 1902/03b, 37, 231, 239–249).

In his courses, Husserl gave the details of his theory of inference in terms which, apart from some differences in notation, are familiar and intelligible to us nowadays (Husserl [1896] 2001, 250, 254; Husserl 1902/03b, 239–240). Among his laws and principles figured the identity principle, which Husserl considered to be just another way of expressing the principle of contradiction that was preferable for certain goals of inference and the law of the excluded middle, A or not A, it is not true that not not A and not A = it is not true that A and not A implies A or not A. He considered his Principle 6, which reads:

> If for every M and for every N, it is always true that M and N implies P, then it is always true that P,

to be especially important because it grounded the mathematical procedure according to which one could manipulate arbitrary number formulas in the calculation as if they were propositions with specifically given numbers. Every inference yields another formula and not just an individual proposition (Husserl [1896] 2001, 265).

VI. Three Levels of Logic

Developments in mathematics also led Husserl to detect a certain natural order in formal logic and to broaden its domain to include two layers above traditional Aristotelian logic. He considered the detection of these three layers of formal logic to be of the greatest importance for the understanding of logic and philosophy.

According to his theory, the lowest layer, traditional Aristotelian logic, makes up but a small area of pure logic. A logic of subject and predicate propositions and states of affairs, it deals with what is stated about objects in general from a possible perspective. The purely logical disciplines rising above that logic of subjects and predicates still deal with individual things, but these objects are no longer empirical or material entities. They are removed from acts, subjects, or empirical persons of actual reality. It is no longer a question of objects as such about which one might predicate something, but of investigating what is valid for higher order objects that are determined in purely formal terms and deal with objects in an indeterminate, general way.

The second layer is an expanded, completely developed analytics where one reasons deductively with concepts and propositions in a purely formal manner because each concept is analytic and each procedure purely logical. Husserl located the basic concepts of mathematics, the theory of cardinal numbers, the theory of ordinals, set theory here. Numbers no longer function as independent entities, but are dependent structures. One manipulates signs for which rules having such and such a form are valid, signs which like chess pieces acquire their meaning in the game through the rules of the game. One may proceed mechanically and the result will prove accurate and justified.

According to Husserl, the third layer of formal logic is that of the science of deductive systems in general, the theory of manifolds. Manifolds are pure forms, which, like molds, remain totally undetermined as to their content and not bound to any possible concrete interpretation, but to which thought must necessarily conform in order to be thought and known in a theoretical manner. Axiom forms define a manifold of anything whatsoever in an indeterminate, general way. A set of axioms of such and such a form that are consistent, independent, and purely logical in that they obey the principle of non-contradiction, yields the set of propositions belonging to the theory of such and such a form. After formalization, words are completely empty signs only having the purely formal meaning laid down for them by the axiom forms. A certain something must by definition stand in a certain relationship to something else in the defining manifold.

On the basis of the definition of the manifold, we can derive conclusions, construct proofs, and it is then certain a priori that anything obtained in this way will correspond to something in our theory. Only a form is defined. It exists insofar as it is correctly defined, insofar as the axiom forms are ordered in such a way as to contain no formal contradictions, no violation of analytic principles. But whether axioms as truths have existence in any objective real or ideal spheres corresponding to the prescribed form is left open. The theory of manifolds, or science of theory forms, is a field of free, creative investigation made possible once form is emancipated from content. Once it is realized that deductions and sequences of deductions continue to be meaningful and remain valid when another meaning is assigned to the symbols, we are free to reason completely on the level of pure forms where we can vary the systems in different ways (Hua 24, §§18–19; Hua 30, Ch. 11).

VII. Manifolds and Imaginary Numbers

In *Logical Investigations*, Husserl called his theory of complete manifolds the key to the only possible solution to how in the realm of numbers impossible, non-existent, meaningless concepts might be dealt with as real ones (LI, *Prolegomena* § 70). In *Ideas*, he wrote that his chief purpose in developing his theory of manifolds had been to find a theoretical solution to the problem of imaginary quantities (*Ideas*, §72 and note).

Husserl saw how questions regarding imaginary numbers come up in mathematical contexts in which formalization yields constructions which arithmetically speaking are nonsense, but can be used in calculations. When formal reasoning is carried out mechanically as if these symbols have meaning, if the ordinary rules are observed, and the results do not contain any imaginary components, these symbols might be legitimately used. And this could be empirically verified (PA, 411–413; FTL, §31; Schuhmann and Schuhmann 2001).

In a letter to Carl Stumpf in the early 1890s, Husserl explained how, in trying to understand how operating with contradictory concepts could lead to correct theorems, he had found that for imaginary numbers like $\sqrt{2}$ and $\sqrt{-1}$, it was not a matter of the possibility or impossibility of concepts. Through the calculation itself and its rules, as defined for those fictive numbers, the impossible fell away, and a genuine equation remained. One could calculate again with the same signs, but referring to valid concepts, and the result was again correct. Even if one mistakenly imagined that what was contradictory existed, or held the most absurd theories about the content of the corresponding concepts of number, the calculation remained correct if it followed the rules. He concluded that this must be a result of the signs and their rules (Husserl 1994, 13, 15–16). The fact that one can generalize, produce variations of formal arithmetic that lead outside the quantitative domain without essentially altering formal arithmetic's theoretical nature and calculational methods brought Husserl to realize that there was more to the mathematical or formal sciences, or the mathematical method of calculation than could be captured in purely quantitative analyses (LI, 41–43; Husserl 1975, 35).

Understanding the nature of theory forms, he explained in several texts, shows how reference to impossible objects can be justified. According to his theory of manifolds, one could operate freely within a manifold with imaginary concepts and be sure that what one deduced was correct when the axiomatic system completely and unequivocally determined the body of all the configurations possible in a domain by a purely analytical procedure (Hill and Rosado Haddock, Chapter 9).

It was the completeness of the axiomatic system that gave one the right to operate in that free way. A domain was complete when each grammatically constructed proposition exclusively using the language of the domain was determined from the outset to be true or false in virtue of the axioms, i.e., necessarily followed from the axioms or did not. In that case, calculating with expressions without reference could never lead to contradictions. Complete manifolds have the "distinctive feature that a finite number of concepts and propositions—to be drawn as occasion requires from the essential nature of the domain under consideration—determines completely and unambiguously on the lines of pure logical necessity the totality of all possible formations in the domain, so that in principle, therefore, nothing further remains open within it." In such complete manifolds, he stressed, "the concepts true and formal implication of the axioms are equivalent" (*Ideas*, §§71–72; Prolegomena, §70; Hua 24, §§19; Hua 30, §56; FTL, §31; PA, 439).

Husserl pointed out that there may be two valid discipline forms that stand in relation to one another in such a way that the axiom system of one may be a formal limitation of that of the other. It is then clear that everything deducible in the narrower axiom system is included in what is deducible in the expanded system, he explained. In the arithmetic of cardinal numbers, Husserl explained, there are no negative numbers, for the meaning of the axioms is so restrictive as to make subtracting 4 from 3 nonsense. Fractions are meaningless there. So are irrational numbers, $\sqrt{-1}$, and so on. Yet in practice, all the calculations of the arithmetic of cardinal numbers can be carried out as if the rules governing the operations are unrestrictedly valid and meaningful. One can disregard the limitations imposed in a narrower domain of deduction and act as if the axiom system were a more extended one (Hua 30, §56). We cannot arbitrarily expand the concept of cardinal number, Husserl reasoned. But we can abandon it and define a new, pure formal concept of positive whole number with the formal system of definitions and operations valid for cardinal numbers. And, as set out in our definition, this formal concept of positive numbers can be expanded by new definitions while remaining free of contradiction. Fractions do not acquire any genuine meaning through our holding onto the concept of cardinal number and assuming that units are divisible, he theorized, but rather through our abandonment of the concept of cardinal number and our reliance on a new concept, that of divisible quantities. That leads to a system that partially coincides with that of cardinal numbers, but part of which is larger, —meaning that it includes additional basic elements and axioms. And so in this way, with each new quantity, one also changes arithmetics. The different arithmetics do not have parts in common. They have totally different domains, but an analogous structure. They have forms of operation that are in part alike, but different concepts of operation (PA, 435–436).

Husserl concluded that formal constraints banning meaningless expressions, meaningless imaginary concepts, reference to non-existent and impossible objects restrict us in our theoretical, deductive work, but that resorting to the infinity of pure forms and transformations of forms frees us from such conditions and explains why having used imaginaries, what is meaningless, must lead, not to meaningless, but to true results (Hua 30, §57; Hill 2002b).

VIII. Mathematics and Phenomenology

Husserl wanted to hammer into people's minds a sense of the proper relationship between phenomenology and mathematics. He stressed that all fields of theoretical knowledge are particular instances of manifolds, but not all sciences are theoretical disciplines like mathematical physics, pure geometry, or pure arithmetic whose systemic principles are purely analytical. Theoretical disciplines have a systemic form that belongs to formal logic itself, that must be constructed a priori within formal logic itself and within its supreme discipline the theory of manifolds as part of the overall system of forms of deductive systems that are possible a priori. However, sciences like psychology, history, the critique of reason and, notably, phenomenology require one go beyond the analytico-logical model. When they are formalized and one asks what it is that binds the propositional forms into a single system form, one finds oneself facing nothing more than the empty general truth that there is an infinite number of propositions connected in objective ways that are compatible with one another in that they do not contradict each other analytically (FTL, §35a; Hua 30, §54).

We have the natural sciences of physical and mental nature, the mathematical sciences, logic, including formal logic, the sciences of value, ethics. None of that is phenomenology, Husserl underscored. Transcendental phenomenology has no dealings with a priori ontology, none with formal logic and formal mathematics, none with geometry as a priori theory of space, none with a priori real ontology of any kind (thing, change etc.). Transcendental phenomenology is phenomenology of the constituting consciousness, and consequently not a single objective axiom, meaning one relating to objects that are not consciousness, belongs in it, no a priori proposition as truth for objects, as something belonging in the objective science of these objects, or of objects in general in formal universality. The axioms of geometry do not belong in phenomenology, because phenomenology is not a theory of the essences of shapes, of spatial objects. Essence-propositions about objects do not belong in the phenomenology of knowledge, insofar are they are objective truths and as truths have their place in a truth-system in general (Hua 24, 411, 422–423).

The special interest of transcendental phenomenology does not lie in the theoretical concepts and laws to which the sciences are subject. Epistemological interest, transcendental interest, does not aim at objective being and laying down truths for objective being, consequently, not at objective science. What is objective belongs precisely to objective science, and what objective science still lacks for completion is its affair to obtain and its alone. The interest of transcendental phenomenology aims rather at consciousness as consciousness of objects (Hua 24, 425).

IX. What Numbers Could Not Be For Husserl

There is no longer any need to prove that Husserl was not a Fregean. Husserl's theory of arithmetic is not grounded in the unworkable theory of identity and reference that forced Frege to introduce the extensions of concepts and axiom of extensionality that he concluded led to Russell's paradox (Hill 1997). Husserl already spurned extensions in *Philosophy of Arithmetic*. In *Formal and Transcendental Logic*, he qualified extensional logic as naive, risky, and doubtful, and complained that it had been the source of many a contradiction requiring every kind of artful device to make it safe for use in mathematical reasoning. He condemned the work of extensionalist logicians, as fundamentally misguided and unclear (FTL, §§23b, 24, 26c; Hill and Rosado Haddock 2000). If, as Quine told us, the notion of essence is the forerunner of the modern notion of intension or meaning, and meaning is what essence becomes when it is divorced from the object of reference and wedded to the word (Quine [1953] 1961, 22), then it is clear from the above that Husserl's logic was resolutely intensional.

So Husserl was not a Fregean, but the theory that he was a Brouwerian still appeals to some and remains to be countered. In the course of this exposition of Husserl's ideas about axiomatization and arithmetic, I have pointed to some specific areas in which Husserl's and Brouwer's theories on the foundations of mathematics diverge. I wish now to reinforce what I have said by adding the following reflections.

Husserl's theory of the derivation of arithmetic from the unfolding of the concept "How Many?" could not in fact be more different from Brouwer's theory of self-unfolding of mathematics from the mathematical primordial intuition of two-ity. According to Brouwer, mathematics, science and language are the main functions of human activity by which human beings dominate nature and maintain order within it. These three functions originate in three forms of action of the individual human being's will to live: mathematical attention; mathematical abstraction; the use of sounds to impose his or her will on others. Mathematical attention comes into being in two phases. Time awareness, the first, is the fundamental intellectual phenomenon of the separating of a life moment into two qualitatively different parts that unfolds itself to create a time sequence of arbitrary multiplicity by giving birth to temporal two-ity, which can in turn be taken as an element of a new two-ity to create temporal three-ity, and so on. Mathematical attention receives its justification only by the "mathematical act," when "causal attention," the second phase of mathematical attention, enables people to force into being, "indirectly and by cool calculation," a particular event known as the aim that appears later in the sequence of phenomena. For Brouwer, the "causal coherence of the world is the outward-acting force of human thought, serving a dark function of will, making the world more or less defenseless like the snake that renders its prey powerless through its hypnotic stare or the inkfish through its darkening spray" (Brouwer [1929] 1998, 45–46).

In higher levels of civilization, Brouwer believed, mathematical abstraction enters in to divest two-ity of its material content, whereupon it becomes the empty form that is the common substrate of all two-ities that forms the Primordial Intuition of Mathematics that in its self-unfolding produces, "not only the numbers one and two, but also all finite ordinal numbers, inasmuch as one of the elements of the two-oneness may be thought of as a new two oneness, which process may be repeated indefinitely; this gives rise still further to the smallest infinite ordinal number ω.. . . gives rise immediately to the intuition of the linear continuum. . . ." and finally all of pure mathematics (Brouwer [1929] 1998, 45–46; Brouwer [1912] 1983, 80; Brouwer [1952] 1996, 1200).

In maintaining that temporal two-ity born from time awareness is the basal intuition of all of mathematics Brouwer saw himself as one holding resolutely to the Kant's apriority of time. He described intuitionistic mathematics as "an essentially languageless activity of the mind having its origin in the perception of a *move of time*" (Brouwer [1912] 1983, 80; Brouwer [1929] 1998, 45–46; Brouwer [1952] 1996, 1200). In contrast, Husserl taught that numbers could not concern what happens in or to real temporal matters of fact that we call mental experiences of experiencing individuals. He stated unequivocally that Kant had brought pure arithmetic into an entirely inadmissible relationship to time (Hua 24, §§11, 13b, 23). Theories of number based on intuitions of time were already spurned in *On the Concept of Number* and the *Philosophy of Arithmetic* (Husserl 1887, 320–329; PA, 22–35).

For Husserl, mathematics could not originate in the consciousness or possibly be developed from any intuition whatsoever. He taught that the laws of arithmetic just unfold what is found in the concept of number. They make no pronouncements about acts of counting, causal relationships, experiences of number, but are just about numbers (Husserl 1902/03b, 32). He insisted that the presenting or thinking of a number proposition must be distinguished from the number proposition itself. Thinking $2 \times 2 = 4$ is a phenomenon of my consciousness, but it is there when one turns to other objects. If one thinks again that 2×2 is 4, then that mental act is new. It is not the same, but what is thought is the same. Countless acts can objectively underlie the same thing, and in this case this identical thing is $2 \times 2 = 4$ (Husserl [1896] 2001, 19).

Brouwer contended that for Intuitionism, mathematical exactness exists in the intellect (Brouwer [1912] 1983, 78). In contrast, Husserl insisted that mathematical truth holds whether anyone has reason or not to believe it, or does not believe it, whether anyone sees it or does not see it (Hua 24, §11). In *Formal and Transcendental Logic*, he said that the problem guiding him originally was in isolating and determining the meaning of a pure analytic logic of non-contradiction was that the evidence of the truths of formal mathematics and formal logic is of an entirely different order than that of other a priori truths in that the former do not involve any intuition of objects or states of affairs whatsoever (FTL, Introduction and §§7–8).

Brouwer hoped to make it clear that "intuitionistic mathematics is inner architecture, and that research in foundations of mathematics is inner inquiry" (Brouwer [1948] 1983, 96). In contrast, Husserl's formal logic is a blueprint for limning the true and ultimate structure of reality by engaging in pure a priori analyses of essence that know no acts, subjects, or empirical persons, or objects belonging to actual reality. He taught that there was not to be any radical analysis of the psychological origins of the fundamental concepts of mathematics per se. He stressed that pure mathematics as pure arithmetic is not concerned with souls (*Seele*) (Hua 24, §§ 13c, 18; Hua 30, ch. 11).

X. Conclusion

So, if Husserl was neither a Fregean nor a Brouwerian, nor really even a phenomenologist when it came to mathematics, what was he? When we find Husserl teaching that calculating in the field of numbers obviously involves operating with signs and not with the concepts themselves, that to solve a problem, to derive a proposition, one must not think at all about the concepts themselves, but by using procedures defined by set rules, link signs to signs, replace combinations of signs by other combinations of signs, etc. (Husserl [1896] 2001, 247), this automatically suggests kinship with the ideas of Husserl's colleague at the University of Göttingen, David Hilbert.

Husserl's teachings about axiomatization, arithmetic, completeness and consistency, the foundations of mathematics also display kinship with Hilbert's ideas, kinship that Husserl himself acknowledged in *Ideas I* §72, *Formal and Transcendental Logic* §§28–36 and *Crisis* §9f and note, where he also made it clear that he considered the fact of this kinship to be significant. In §31 of *Formal and Transcendental Logic,* he even went so far as to say that the close study of his analyses would reveal that the underlying, though inexplicit, reasons which had led Hilbert to attempt to complete a system of axioms by adding a separate axiom of completeness were much the same as those which had played a determinant role in Husserl's own independent formulation of his concept of completeness. In those texts, Husserl explicitly refers back to his theory of complete manifolds in the *Prolegomena* §§ 69–70 and to the then unpublished material from his Göttingen period now available in appendices to the Husserliana edition of his *Philosophy of Arithmetic* (Hill 1995).

However, caution also needs to be exercised in uncovering parallels in the ideas of original thinkers. Kinship can be superficial and it is not influence. In this case, it is important to remember that Husserl developed his ideas independently of Hilbert. Husserl's interest in axiomatization, completeness and formalist foundations for mathematics is traceable back to his early years in Halle, before he and Hilbert were together in Göttingen. They originally derived from problems regarding imaginary numbers which first came up while he was trying to complete *Philosophy of Arithmetic*. His 1896 teachings about the axiomatization of arithmetic antedated Hilbert's call to axiomatize arithmetic, which first went out in 1899 in "On the Concept of Number," Hilbert's first essay on the foundations of arithmetic (Hilbert [1900] 1996; Hill 1995).

Although Husserl acknowledged kinship as concerns completeness, he said that he developed his concept of completeness independently of Hilbert's axiom of completeness. His earliest ideas on completeness were tied in, not only with his inquiries into the logical foundations of the real number system, but also with a more specifically philosophical quest to clarify the sense of the analytic a priori and develop a pure analytic logic free of any taint of psychologism. Moreover, Husserl criticized Hilbert's appeal to the axiom of completeness in "On the Concept of Number." Husserl said that that kind of completeness can be of no use whatsoever, because it is not legitimate completeness, not something specifically characteristic of axiom systems. . . because any axiom system can be made quasi complete by appealing to an axiom of that kind (Hill 1995). As Jairo da Silva has pointed out, there are many senses of completeness (da Silva 2000a; 2000b; 2005).

When Husserl was appointed to the University of Göttingen in 1900, he was warmly welcomed into Hilbert's circle (Hua 21, XII). As documents in the Niedersächsische Staats- und Universitätsbibliothek Göttingen, Abteilung Handschriften und Seltene Drücke and in the Geheimes Staatsarchiv Preussicher Kulturbesitz in Berlin show, Husserl's colleagues in the philosophy department at the University of Göttingen did not consider him to be "a desirable addition to the faculty." Husserl did, though, have an ardent supporter in the person of Hilbert, who complained that people had not seen, or had not wanted to see, how important it was to support Husserl's efforts. In Hilbert's opinion, Husserl was viewed in professional circles as one of the most prominent and creatively most active scholars in the field of systematic, purely theoretical philosophy. Hilbert portrayed Husserl as an exception, as someone who was not tainted by relativism, someone who believed in the possibility of philosophical science and labored to make it a reality. He called the *Logical Investigations* epoch-making. Hilbert considered it no accident that Husserl had come to the mathematical environment cultivated there.

However, nowhere in his published writings on philosophy does Hilbert ever acknowledge being influenced by Husserl or having exercised influence upon him. Rather, in them, Hilbert affirms the abiding significance of what he called the most general fundamental idea of the Kantian theory of knowledge, namely the philosophical problem of establishing the intuitive a priori attitude and, with that, of investigating the prerequisites for the possibility of any conceptual knowledge and at the same time of any experience. Hilbert says that this is essentially what had happened in his own investigations into the principles of mathematics (Hilbert [1930] 1996, 383; Hilbert [1931] 1998, 266–267).

Be that as it may, once Husserl's theory of the axiomatization of arithmetic has been pieced together and the relationship of his ideas to Frege's, Brouwer's, and Hilbert's theories on the foundations of mathematics has been clarified, the really important question to be answered is whether his theory is really tenable and viable, whether it works, or whether it is not ultimately just an ingeniously worked out take on many of the issues in the philosophy of mathematics of his time by the father of phenomenology. Now that we have the material we need to piece together Husserl's theory, we need to give it a try. It needs to be tested to see whether it is tenable. That is the next step that needs to be taken.

I am personally of the conviction that such testing will unearth additional arguments to prove that Husserl had a deeper understanding of the issues that went into the investigations foundations of mathematics that generated analytic philosophy than analytic philosophers themselves have ever had. As the student and assistant of Karl Weierstrass, the long-time friend and colleague of both Georg Cantor and David Hilbert, Husserl was on the ground floor when it came to the grounding of mathematics. When he was teaching his logic courses, he was already in a position to take into account the shortcomings of both Cantor's and Frege's efforts. He was already perfectly lucid about those of the latter at the time he published *Philosophy of Arithmetic* (Hill 1991; Hill 2000a). Husserl's theory is grounded in an analytic derivation of number from the concept of "How Many" and not in the deeply flawed theories about identity and reference with which the mainstream philosophical theories of the foundations of mathematics which have struggled with for over a hundred years. Husserl's theory makes no appeal to the axiom of extensionality that still blights the axiomatizations of set theory to which, following Cantor, Frege, Russell and Whitehead, Zermelo-Frankel, Gödel, Quine, philosophers still appeal to ground mathematics (Hill 1997).

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CHAPTER IV

INTUITION IN MATHEMATICS: ON THE FUNCTION OF EIDETIC VARIATION IN MATHEMATICAL PROOFS

Dieter Lohmar

Abstract. Intuition is of central importance for the phenomenological access as a whole and it also turns out to be decisive in the analysis of cognition in mathematics. But sensibility can not be the source of intuition in abstract formal sciences. Therefore I first delineate some basic traits of cognition in terms of Husserl's categorial intuition. Then I investigate in a special case of cognition, i.e., the eidetic method of Husserl, the so-called *Wesensschau*. This intuition of essences is a special case of categorial intuition and it is characterized by apodictic evidence. Thus it may solve the question of intuitivity together with the problem of necessary validity. My way to argue for this thesis is to analyze some intuitive steps in simple proofs and point out some rules that enable an *implicit variation*. Following the traces of the eidetic method in mathematical intuition it turns out that not only in material mathematics like in geometry, but also in formal mathematics, there is a special kind of implicit eidetic variation that serves as the source of intuitivity in proofs.

The central line of investigation in phenomenology is dedicated to intuition, and from the very beginning it has been not only the search for sensible intuition, but also for intuition in knowledge, or as Husserl calls it, of categorial intuition. Sensible perception fulfills our intentions of simple real mundane objects, but categorial intuition is the way in which we intuit higher order objects, such as states of affairs. Most of the ordinary forms of knowledge can be traced back to sensibility, yet even in the simplest forms there are also elements in acquiring intuition that have only a peripheral connection to sensibility.

In higher forms of knowledge, especially in formal sciences, sensibility plays only an insignificant role. Nevertheless, we can gain intuition and evidence in formal contexts. We can even gain apodictic evidence, which is the highest form of evidence. It entails that the state of affairs given in this evidence are true and that they can not be otherwise. In my contribution I would like to make some detailed investigations into the

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special character of the process in which mathematics allows for apodictic evidences.

If we try to access apodictic evidence from the point of view of Husserlian thinking, this kind of evidence is closely related to the eidetic method, the so-called *Wesensschau.* The intuition of essences is a special case of categorial intuition and it is characterized by apodictic evidence. This position does not change throughout the whole development of Husserlian phenomenology. But it is not easy to understand how the eidetic method is used in the realm of mathematical proofs and how this evidence is reached in formal contexts.

The main interest of my investigation is, first, to delineate the abstract framework of the phenomenological theory of knowledge and, secondly, to provide a concrete description of the kind of intuition in mathematics and of its special evidence.[1] Therefore I am going to analyze phenomenologically some intuitive steps in simple proofs. My intention in doing this is not to remain within the higher level of the general theoretical framework but to provide a detailed description of what we are really doing in proofs. This simple descriptive work is the true strong point of Husserlian phenomenology. Husserl once characterized it with the words: We should not start with the big bills but with the small coins!

My presentation starts with some basic reflections about the source and the kind of intuition in formal sciences and in mathematics. I will outline some essential aspects of the phenomenological theory of knowledge in general and of mathematics in particular. In the second part I will address the question of how in phenomenological analyses of consciousness as well as in mathematics we gain apodictic evidence, which consists of necessary or apriori insights. The opinion I will try to establish is well known: Apodictic evidence in mathematics is due to the use of the eidetic method (*Wesensschau*) in mathematical proofs. The second part has three sections dedicated to the use of eidetics in real objects, in material mathematics and in formal axiomatic mathematics.

[1] Husserl's theory of categorial intuition is developed in (Lohmar 2002, 125–145, 2006, 109–126).

I. Some Basic Features of Husserl's Theory of Knowledge

Let me first shortly outline the basic features of Husserl's theory of knowledge. I thereby want to clarify the principal reasons why we can have knowledge not only of real things but also of objectivities in an axiomatic context.

Phenomenology starts the analysis of knowledge by examining the sensible givenness of objects and the activity of thinking. In perception what is sensibly given is apperceived, i.e., through a synthetic act interpreted as an object. In this synthetic act, the intended object is intuited as "something." And I am able to identify this "something" as the same in further acts of perception and knowledge.

Due to the character of apperception as a kind of interpretation, already the sensual perception of an object exceeds the actually given material by our senses. Sensibility gives us only a kind of material basis for our intentional apperception (*Auffassung*) but it remains a kind of "raw material" that has to be treated in the right way. An object is given in evidence when we do not just emptily intend it without the help of sensibility, but only when sensibility fulfills (makes true) what we intend. But this fulfillment is an active, synthetic process of collection and combination of the sensual material, which results in a sensual representation of the object.

Let us take a closer look: What fulfills my intentions in everyday insights such as "This book is green"? Admittedly, sensuality plays a decisive role in the fulfillment of such insights but, as we will see, it cannot fulfill this intention completely. In his analyses of cognitive acts alongside sensibility, Husserl finds another source of intuition in what he calls "synthesis of coincidence." To understand this better, let us consider the example of a green book: When I see a green book, there are different phases involved in this process. In the first phase of perception, i.e., the *general view* (*Gesamtwahrnehmung*), the book as a whole is given. In such a general view, all elements of sense entailed in this object are intended implicitly—but they are not explicitly noticed. In the second phase, I turn my attention to the particular color of the book, while still perceiving the book as a whole. Now I perceive the book *with the help of* its color. And: In the transition of these two acts, i.e., moving from the general view to the pointed special intending of the book's color, there arises a *synthesis of coincidence* of the intended senses of the two acts.

This means: The partial intention (*Partialintention*) toward the moment of greenness, which was only implicitly intended in the general view, *coincides* with the special intention to the green color that is now explicitly attended to. We realize that both intentions intend *the same*, but they differ in their character. One of these intentions is implicit, unnoticed, and performed casually; the other is active, deliberate, and explicitly directed toward a single sensuous element of the whole. Yet we are still able to notice that they aim at *the same*. This ability of the mind to notice similarity of sense in different modes of performance is called by Husserl *synthesis of coincidence*.

Syntheses of coincidence are passively given within the activity of consciousness, i.e., they either do or do not occur, and we cannot influence their occurrence. As in the case of the green book, the coincidence depends more on the object sensuously given than on our activity. Sensuality together with this synthesis of coincidence occurring in the framework of categorial activity forms the intuitive basis of our everyday insights. Thus even here the ground of intuition is not only sensibility, but also something more that relates back to our thinking activity.

One might object that in the initial general view we somehow already *knew* that the book was green because we intended it *as green*. This implicit and hidden knowledge could only become explicit and intuitively present by being highlighted in the second phase of categorial intuition. Afterwards, with the mediation of the syntheses of coincidence, it is interpreted as an existing state of affairs (*Sachverhalt*). Exactly this performance makes the difference between characteristics perceptually noticed only in passing (*beiläufig*) and those explicitly recognized.

Thus far I have stressed the function of the synthesis of coincidence that goes beyond sensuality in everyday insights, keeping in mind that my aim was to investigate the source of intuition in mathematics. Mathematical knowledge is genuine knowledge with its own intuitive source and in most cases it is independent of sensuality. But now we see that this is not so exceptional because we know that in everyday knowledge we also possess forms of knowledge that are to a great extent independent of sensuous intuition. In my view, this is true already of conclusions that we draw from previous cognitions in everyday cognition.

But what about formal axiomatic mathematics? Here it is possible to start in a context where there are only propositions *presupposed as valid*. In such contexts, we never move beyond judgments that are drawn from such presupposed propositions. But this concerns only judgments about properties of objects. We can, however, go beyond this. For example, consider the relations of *deducibility* between presupposed propositions: Concerning the question whether a proposition is deducible from a set of axioms, we can arrive at valid cognition, and this cognition has the highest degree of evidence, i.e., apodictic evidence.

But before going into a more detailed investigation concerning this special kind of evidence, let me stress the similarity of everyday and mathematical knowledge: Both are cases of knowledge and both kinds of knowledge are not only based on sensuality alone, because also in everyday cognition the syntheses of coincidence between partial intentions have a fulfilling function. Yet there is also an important difference between them: In everyday knowledge, the intentions are much more complex than in mathematics. Real things always have several different aspects like color, form, material properties, they are framed in causal relations, they have aesthetic aspects and are framed by an objective history, etc. In mathematics we are directed to numbers in general, operations in general, sets in general, etc.[2] They are all objects that rest on simple operations of the mind creating new objects and new kinds of evidences for these objects. And all these objects entail only a few partial intentions (no color, no causal relations, no objective history...). Therefore, the syntheses of coincidence appearing in this formally defined context are also much more easily structured and more distinct. This is due to the fact that the clouds of horizonal intentions, which usually accompany the intending of everyday objects, do not obscure such syntheses. As a result, evidence in formal sciences is much clearer. Yet this difference does not assure us that the truth in formal sciences is necessary and that its evidence is apodictic.

[2]For a more detailed investigation in the different kinds of mathematical objects, cf. (Lohmar 1989, 2006).

II. The Method of Seeing Essences in Mathematical Proofs

1. The Eidetic Method (Wesensschau) Used for Real Objects

Husserl's eidetic method of seeing essences claims that its results are not restricted to factual empirical matters-of-facts but also pertain to universal structures and apriori, necessary laws which are valid for all factual and all possible future cases of acts of consciousness.

Husserl's theory of seeing essences is worked out in the framework of his theory of knowledge, i.e., in the Sixth Logical Investigation. Seeing essences is analyzed as a special case of gaining knowledge. In the beginning it is a natural human ability to realize what is similar in different objects like trees, humans and lemons. But this simple mental ability can be methodically refined and in the end it becomes a basic method of the phenomenological analysis of the human mind.

In the first line, phenomenology is striving at apriori necessary insights into the universal structures of consciousness, i.e., structures that are independent from matters-of-fact. With the help of the eidetic method we can also have apriori insight into other regions of being, which Husserl names regional ontologies, such as sounds, colors, etc., as well as geometry, arithmetic and other parts of mathematics. The act of seeing essences is called *Wesensschau*, "*ideierende Abstraktion*" or "*Anschauung des Allgemeinen*." It is founded on and starts with simple perception of singular objects in a similar way like all other elementary forms of knowledge. But seeing essences demands also active performances of the mind, first of all the variation of the starting example.

The word "*Wesensschau*" (seeing essences) is, in my view, a very irritating choice in terminology. It suggests that phenomenology is a variant of the Platonic theory of ideas. This is surely not the case because Husserl never hypostatizes or substantializes the objects of the eidetic method: they are definitely not objects residing in another higher "reality," like they are in Plato. Husserl does not regard the essences as "more real" than objects of sensual perception.[3] For Husserl, the everyday world is the only reality. Nevertheless, mathematical objects and all other ideal objects are objects of thought and we can gain intuition of them as well. In order

[3]Cf. for Husserl's further demarcation to Platonism (Lohmar 1993, 73–87, 2000, 187, 215 f.).

for us to gain knowledge of these ideal objects, they must "participate" in some way in the only reality, at least when we refer to them with the help of (sensual) signs. Thus Husserl posits the incline in the grade of reality in a completely opposite way from Plato: Essences are dependent on the real world and they have no independent reality.

In the process of seeing essences we find exactly the same elements as in the simplest cases of cognition. The intuition of the identical traits in different objects (essences, universals) is expressed by the use of a universal name. For example to have an intuition of "green" is only possible if we run through a series of green objects in perception or imagination (Hua 19, 111–115, 176 ff., 225 f., 690–693). The intuitivity of my intending of what is common in all cases, i.e., of the universal, is due to the synthesis of coincidence that occurs in all cases.

All acts of categorial intuition have the characteristic threefold: simple perception of the object as a whole, pointed concentration on an aspect, and categorial synthesis. Seeing essences has exactly this threefold structure: it begins with a general view on the starting example and the second phase of pointed concentration on an aspect has a manifold of different objects, all of which are variations or modifications of the starting example. In running through these acts that result from variation of the starting example, we are directed to the identical aspects and thus there occurs a synthesis of coincidence. So the second phase consists in all possible variants of the starting example and the third phase is interpreting (or apperceiving) the special synthesis of coincidence as the intuitive representative of what is invariant in all possible cases.

If we realize that in all possible variations of the object (in perception and imagination) the same element is to be found—then we can be sure that it will also be there in all following cases, thus this insight is necessary and apriori in the special sense of the eidetic method.[4] For example: All extended objects necessarily have a certain color, all tones have a certain intensity and duration, etc.

In the act of eidetic abstraction we interpret (*auffassen*) the special synthesis of coincidence, which occurs in running through all possible

[4]Husserl points to the crucial difference of the sense of *apriori* in Kant and in his own understanding in a footnote in FTL, (cf. Hua 17, 255, Anm. 1).

variants as the intuitive givenness of the universal. By this method of making apparent the intuitive way of givenness, the universal becomes an object in the full sense, which is characterized by the following possibilities: empty intention, identification and fulfilled intuition.

We will now turn our attention to the acts of variation of the starting example. Husserl realizes the central role of variation only after the publication of the *Logical Investigations* (1900).[5] In the acts of variation there may and there must occur imaginative acts which in some way vary the starting example. This necessary phase of variation enables us in principle to reach the essence of an object on the basis of one single intuitively given object by varying this starting example. But somehow we have to reach all possible variants—and just this seems to be difficult.

Is it really necessary to perform *infinitely many* acts of variation to reach the specific generality of universals? Husserl poses and also answers this specific question only in his late genetic theory of eidetic variation, which can be found in the lecture on *Phenomenological Psychology* (1925) and in *Experience and Judgment*.

Nevertheless, the function of imagination is already discussed on many occasions in earlier writings. In the *Ideas I* (1913) and in *Phenomenological Psychology* (1925) Husserl points out explicitly the necessity and the priority of imaginative and "free" variation. This is in opposition to the presentation of his theory of seeing essences in the *Logical Investigations*, in which the matter is left undecided whether the acts of variation are intuitive perceptions or imaginations.[6] In *Ideas I* he clearly states that

[5] It is very difficult to find out about the historical sources of Husserl's idea of variation in the philosophical tradition. Not only Brentano and Bolzano, but also Berkeley and Hume might be possible sources, yet this is not my theme. We might also regard some mathematical theories as possible sources, like Klein's approach to characterize types of geometries by groups of transformation that leave essential properties of the objects of this geometry unchanged, cf. (Picker 1961, 266–355, Tieszen 2005, 153–173).

[6] On the function of free phantasy in eidetic variation cf. Hua 3/1, 146 ff., Hua 17, 206, 254 f., EU, 410 ff., 422 f. and (Ströker 1978, 21 ff.), Th. Seebohm proposes that there is a kind of variation within phantasy already in the *Logical Investigations,* cf. (Seebohm 1990, 14 f.). On the role of variation in phantasy cf. also (Lohmar 2003, IX–XLI). We begin variation in imagination with a vague idea of the starting object which we are going to make all possible variations of. In the beginning our idea of the colour green is

seeing essences is also possible on the basis of memories and imagination.[7] But even in *Ideas I* he makes clear that free imagination is privileged in the context of seeing essences in comparison to perception.[8] Using imaginative variation is necessary in this context (Hua 3/1, 148). The idea of a privileged status of imagination in this context leads Husserl to the proposition that fiction is the most important medium of intuition in phenomenology.[9]

In the lecture *Phenomenological Psychology* he works out the final form of his eidetic method as eidetic variation. Following this latest version we start the process of variation with some intuitively perceived or imagined object as a starting object, and then vary this object in the function of a guiding example in phantasy (Hua 9, 76). In this process there occurs a kind of overlapping in the sense of the different variations between the single variations.[10] In this overlapping between all possible variations, we experience a specific content that fulfills the intending of the invariant common features of all possible variants. We perform the act of categorial intuition, which in this case is an act of seeing essences, on the basis of this synthesis of coincidence. This special kind of content intuitively fulfills the intending of the invariant common feature.

But can we justify the claim for true universality by this performance? Husserl thinks that such a justification can be found due to the special character of the imaginative variation in the process of eidetic variation. In all variations we have to add an important element of sense: I can freely go on with this variation further and further.[11] This is the decisive new element in the specification of the eidetic method as an eidetic variation.

vague, but we can use it to create variations of green objects and in the further work of the eidetic method we are able to have an intuition of the general object.

[7] Husserl wrote that variation can be performed: "auf Grund bloßer Vergegenwärtigung von exemplarischen Einzelheiten" (Hua 3/1, 145 f.).

[8] Husserl wrote: "freie Phantasie [hat] eine Vorzugsstellung gegenüber der Wahrnehmung" (Hua 3/1, 147).

[9] He says: "daß die ‚Fiktion' das Lebenselement der Phänomenologie, wie aller eidetischen Wissenschaft, ausmacht, daß Fiktion die Quelle ist, aus der die Erkenntnis der, ewigen Wahrheiten' ihre Nahrung zieht" (Hua 3/1, 148).

[10] Husserl speaks of "überschiebender Deckung" (Hua 9, 77).

[11] Husserl wrote: "und so weiter nach Belieben" (Hua 9, 77).

We have to be aware that we are performing a free and in principle unlimited variation at every step of the factual variation. In an idealizing view we may interpret this as an "infinite variation" (Hua 9, 79).

Thus we find something like an idealizing step in every full act of eidetic intuition. And this is not only true for mathematical objects but also for everyday objects. We cannot claim the universality even in the variation of real objects without idealizing our possibility to act: "I can go on with this variation further and further." Thus we can be optimistic in our claim for universality and apriori insight in mathematics—but only if we are able to show that there is this element of varia and also of idealized anticipation of ever and ever new variation in mathematical proofs.

2. Eidetics in Material Mathematical Disciplines

At first glance, the method of seeing essences seems to be more appropriate for material regions of being, such as real objects, their relations in space, melodies, and colors. Therefore, the first line of argument in the analysis of intuition in mathematics leads to what Husserl calls the "material mathematical disciplines." More precisely, it leads to a certain way of understanding some basic disciplines of mathematics. These disciplines are characterized by properties of objects, such as natural numbers, lines, points and planes.

We know that Husserl was not only a philosopher of formal-axiomatic mathematics. His investigations also pertained to such "material mathematical disciplines" in which the basic objects and basic concepts are not completely replaced by algebraic variables. The disciplines he calls material mathematics are, for example, elementary arithmetic and Euclidean Geometry (Hua 17, 53, 84, 89, Hua 3/1, 150 ff.). Body, plane, line, point, angle, ordinal number, set, order, etc., are irreducible basic objects of geometry and elementary arithmetic.[12] Viewed as a material mathematical discipline, Euclidean Geometry is a science dealing with apriori structures of space.

[12]We have to keep in mind that in this context Husserl already presupposes the kind of idealization which differentiates the real drawing on the paper from the idealized object intended by this drawing.

The method of eidetic variation determines the specific phenomenological concept of apriori, and—most importantly—it cannot be regarded as equivalent to Kant's concept of apriori.[13] Kant regards knowledge as apriori if it can be reached independently from all experience and if it is valid before all experience.

In contrast to this, eidetic variation starts with an object of experience, which is then arbitrarily varied in imagination. For example, starting with an arbitrary human person we may vary his size, his volumeness, his color, his posture present his *Gestalt* in a certain way etc. But what we are able to identify in all these different variants is the general *Gestalt* of his body in a kind of upright and frontal, *normalized posture*. This means, we somehow use our ability to imagine the normalized *Gestalt* out of distorted perspectival presentations that appear to us.

During this process of variation, we attend to the properties that remain identical in every possible variation, for example, the normalized *Gestalt*. In the grasping of the identical aspects we are also oriented towards the synthesis of coincidence that occurs among all different variants of the same object. Thus eidetic variation is a case of cognition even if it to a great extent rests on the performance of imaginative acts. Eidetic experience is thus dependent on reality and its special configurations. We will only experience what is identical in all variants by means of the actual performance of this variation. Only afterward will we know which synthesis of coincidence occurs. Phenomenological apriori is valid for all possible experiences, but we will know about the concrete content of this apriori only after the actual performance of the variation and not *in advance*. And this remains true also for the use of the eidetic method in mathematics. But there is one important difference: The range of different properties that we find in real objects—and thus have to vary—is very rich. In contrast to this, the range of properties that are to be varied in mathematics is very limited. To argue for this I will discuss a very easy example of a geometrical proof.

How can we have the apriori insight "Two lines on a plane, not running parallel, intersect at one point"? In this case we are quite well off with the method of variation. We have to vary all kinds of imagined lines on planes

[13]Cf. the important note in (Hua 17, 255, Anm. 1).

not running parallel. And here we are able (by definition) to find in every imagined case one direction that shows a progressive approximation of the two lines imagined. Therefore we can be sure that there will be a point of intersection of the two lines in every possible case. And this insight has the apodictic evidence we are searching for. Thus it is valid apriori in the Husserlian sense.

Another example: Every pupil has to learn the well-known method of bisection of a line with the help of constructing two equal circles around the ends of the line and a connection of the two intersections of theses circles. What we are doing in this proof is a construction and in every step we are strictly oriented to general rules. These rules are:

1. Keep the identity of each value used for construction.
2. Do not limit the possibility [scope] of the construction to a special case or to a limited realm. Keep the full generality of the claim/proposition.
3. Choose the necessary elements of the construction in a way that it is possible to perform it.

The first rule demands no special comment, so I will concentrate on the last two.

Concerning 3: Even if we start the construction by adjusting the compass, we may ask ourselves: Is this construction possible in every case of a line? If we are limited to a sheet of paper of a certain size, then we will find arbitrary limits, but this does not limit the possibility of a construction in principle. But we might choose the value for the radius of the two circles too small in our first attempt and, as a consequence, the circles would not intersect. But we surely know that we may simply take the length of the line itself to adjust the compass. Then the construction of the two circles will result in two intersections that I can easily connect to a new line. And by this I can easily prove that the two parts of the starting line are equal in every possible case.

Concerning 2: In fact I did not try out every value for the radius used in constructing the two circles. I have to start with a radius that allows for intersections (i.e., longer than the half of the line). In fact I am using only one special value in my construction, but this does not limit my construction to a special case. The process of construction is directed to objects in general, i.e., in this case lines in general, circles on the basis of radii as values in general resulting in intersections and new lines in general, etc. In doing so, I have to make sure constantly and consciously that every step of my argument about the resulting triangles remains valid even if a different radius may have been chosen. So there is a kind of "implicit variation" at work that concerns only the radius of the two equal circles. It is not comparable with the broad variation of real things in all their aspects, but it is restricted only to the extension of the radius—for we are directed to objects that are very "poor" in regard to properties, since a circle is completely described by its middle point and the radius and, as we have already chosen the middle point, we can only and we have only to vary the radius. But we do not have to do this variation explicitly; it is enough to make sure in every step and consciously that no limitation of the following argumentation is implied.

In many mathematical proofs we find a formula like "without limitation of the generality" (*ohne Beschränkung der Allgemeinheit*) which indicates the conscious intention to make sure that all neccessary decisions to determine values chosen in the course of the proof will not limit the general validity of the conclusion reached. It is still true for all n ∈ N even if we have to make additional non trivial limitations (see also below).

Thus we have a construction on the basis of only one special radius resulting in a singular circle but this singular circle represents the full class of all possible variations regarding the value of the radius: a circle in general. And this allows us to gain insight into general states of affairs in apodictic evidence even if our concrete construction is only using a special value for the radius of the circle.

In Berkeley's and Hume's theory of abstract ideas we might find some elements of this solution that entails a certain variation, which, even though not explicitly performed, is nonetheless somehow "in the mind." Both are arguing against the idea of a *general triangle* in Locke. Locke tries to defend the concept that in regard to a geometrical object like a triangle, we are not directed to a certain object delineated on paper that has only one size, determined angles, etc. He proposes that the general triangle has none of the possible properties triangles can have and at the same time has all of them. This is a contradictory concept. Yet such a concept is attractive—to us, as well as to Berkeley and Hume—because it entails all the alternatives that have to be considered in a proof for reaching full generality.

The mathematician Berkeley's argumentation in § 16 of the introduction of his *Treatise Concerning the Principles of Human Knowledge* (cited by Husserl in the Second Logical Investigation) claims that the geometrical proof using the idea of a triangle works well because: In spite of the fact that we are drawing a determined triangle, we do not use its special properties in the proof.[14] By this a singular triangle serves as a representation on an unlimited series of possible triangles, and this does not imply contradictory properties of this triangle. This solution is exactly in the line of his nominalism. Berkeley thought of a singular word as being able to represent a whole class of objects that have something in common or are similar, by this avoiding the presupposition that there are abstract general ideas.

Hume tries to make use of Locke's intuitions in the same way as Berkeley. He adopts Berkeley's version of representation in his nominalism. Hume regards abstract ideas as themselves being individual, yet at the same time as general in regard to what they represent. Due to such a representative function, a single word calls alive a single idea, yet it also awakens a tendency to imagine other singular ideas that are alternatives of objects subsumed under a particular concept. For example, if we use the word "triangle" in an attempt to prove the proposition that "all angles of a triangle are equal," we may well start with an equilateral triangle (Hume 1888, Book I, Part 1, Sect. 7). But then other ideas of triangles arise, for example triangles that are neither equilateral nor rectangular, and this leads us to the insight that the proposed proposition is false. Here the element of implicit variation is present in a very clear version.

Berkeley and Hume mention this hidden tendency for variation only in the case of mathematical proofs. Disregarding Husserl's critique of the empiricists' theory of abstraction, there is in my view an important connection between the above mentioned element of empiricist theory of mathematical proof and the Husserlian eidetic method interpreted as a theory of mathematical proofs.

From our considerations of implicit variation in geometry—which may well be transferred to arithmetic—we see that there is a good sense

[14] "And that because neither the right angle, nor the equality, nor determinate length of the sides are at all concerned in the demonstration" (Berkeley 1901, Introduction, § 16).

in claiming apodictic evidence in material mathematics. But we can also immediately realize the difficulty for mathematical knowledge in a purely formal context. In formal mathematics we only refer to objects in general, which follow formally defined rules of operations. The question is: Are we really making use of eidetic variation in such formal contexts?

3. Eidetics in Formal-Axiomatic Contexts

In formal contexts we start with a set of elements and operations possible with them and presuppose that they are ruled by a set of axioms. If we consider natural numbers, then we start with Peano's Axioms about the natural numbers including the axiom of complete induction. In such a way we can easily start with the proof of simple propositions.

The heuristic way to find such propositions may use our knowledge about numbers, as we already know them from the material mathematical realm. We can create them and have them intuitively given by counting, and thus we know already that $2^2 \geq 2$, $3^2 \geq 3$, $4^2 \geq 4$, and so on. So we might suspect that $n^2 \geq n$ is true for all natural numbers, but if we want to prove that such is the case, we have to make use of the method of complete induction.

Proposition *For all $n \in N$, it is true that*: $n^2 \geq n$
Proof We start with complete induction:

(1) The proposition is true for $n = 1$, it is true that $1^2 = 1 \geq 1$
(2) Now we will make the step from n to n+1
(3) Let $n_o \in N$ be any element of N, then I assume that $n_o^2 \geq n_o$
(4) Now we investigate the value of $(n_o+1)^2$ The question is: Is it true that $(n_o+1)^2 \geq n_o + 1$?
(5) The multiplication of the first term results in: $(n_o + 1)^2 = n_o^2 + 2n_o + 1^2$
(6) Because of the assumption (3), it is true that $n_o^2 \geq n_o$, now (7) follows from (5) and (3), together with the more trivial lemma (6a):

Lemma[15] (6a): *For all* a, b, c, d $\in$ N *it is true that*
If $a = b$ and $c \geq d \Rightarrow a + c \geq b + d$

[15]This has to be proven on the basis of Peanos axioms, cf. (Landau 1930).

(7) $(n_o + 1)^2 = n_o^2 + 2n_o + 1^2 \geq n_o + 2\, n_o + 1 = 3\, n_o + 1$
(We realize this by insertion of (3) in (5) and the use of (6a), the last equation results by calculation)
(8) Now it remains to show that $3\, n_o + 1 \geq n_o + 1$ is true
(9) To show that (8) is true in every case of a natural number, using (6a) it is enough to show the validity of the following
Lemma[16] (9a): For all $n \in N$ it is true that $3\, n \geq n$
(10) From (7) and (8) it follows that $(n_o + 1)^2 \geq n_o + 1$ is true.

This is only a very simple example to characterize formal axiomatic proofs. Perhaps this is even not the easiest or most elegant way to prove the proposition, but we realize quite well how the single steps are connected. In this proof there are different types of argumentative steps: Some of the steps only express propositions already proven as true, or accepted as assumptions like (3), or they give an explanation about the next step, like (4). The conclusion (10) only declares that the proof by complete induction has been completed. In some of the steps there are only operations performed like in (5) or (7). Some of the steps inform us about the Lemmata we use and which we have to prove separately (6a) and (9a). Some steps incorporate logical conclusions following the *modus ponens*, like the conclusion that leads from (3), (5) and (6a) to (7). But not every proof is performed only by the use of logical rules. For example, we cannot interpret the procedure of complete induction as a logical rule.

The whole proof can be interpreted as an argumentation on the basis of the presupposed axioms, operations, and the use of logical rules. But: On the first view there seems to be no room left for variation. The only trace of this basic method is the generalized supposition about the $n \in N$ being arbitrarily chosen and therefore remaining object of an *implicit variation* with the following meaning: It can be taken really every n out of N. And to assure this, we have to watch carefully and consciously that this condition is not injured throughout the whole proof. This implicit variation is not carried out explicitly, as it was in the case of real objects, which are to be varied explicitly in various aspects of their appearance (for example,

[16]This can be also be proven by complete induction.

in a person concerning their size, weight, color, shape, etc). It remains an implicit variation like in material mathematics only indicated by the constant and conscious striving for not limiting the generality of the proof. In formal mathematics the objects have only a few properties that we may regard as changeable, for example, in natural numbers only its value (perhaps also further properties of the numbers, like to be prime or not, to have common prime factors or not, etc.). We might only vary the argumentation in the range of all possible n out of N and this is exactly the performance of the eidetic method. The only visible expression of this implicit variation is the rule to beware constantly and consciously of taking up additional suppositions that will limit the generality of the n.

It may appear as if there is a conflict between rule 2 and rule 3, for sometimes it looks like we have to introduce limitations for the sake of the argument. But in fact this is not the case, because if we take into concern additional suppositions in a certain proof it is of central importance to make sure that there is no limitation of the generality, i.e., that it remains possible to vary the values used in the full range for example $n \in N$.

Think for example of the classical proof for the incommensurability (irrationality) of $\sqrt{2}$. Here we start the indirect proof by supposing commensurability (rationality), i.e., that $\sqrt{2} = p/q$ where $p, q \in N$ and p and q have no common divisor (resp. no common factor in the prime factorization of the two numbers). This last presupposition—no common factor in p and q—seems to seriously limit the generality of the proof, nevertheless we will see that it is not trivial and it is essential for the proof.

Usually in mathematical literature you will find in a context like this a hint that we are allowed to introduce this additional presupposition "without limitation of the generality." And this hint points to our constant and conscious striving for full generality. But here we have to prove that the presupposition will not limit the generality of the whole proof (even if this lemma usually is not written down). Now the argument for the lemma: We know that in Euclidean rings we have the possibility of a full factorization of each element p (and q) in a product of prime numbers and we can always exclude in a next step the common prime factors of p and q (resulting in p' and $q' \in N$ with $p/q = p'/q'$). Thus the presupposition is not trivial but we are allowed to make it. And: it is essential for the argument.

We then start by posing $\sqrt{2} = p/q$ (where $p, q \in N$ and p and q have no common divisor) and conclude further $2 = p^2/q^2$ thus also $2\, q^2 = p^2$ and this implies that 2 is a divisor of p^2 and thus also of p. Now we play the trick in the other direction. For $p = 2\, n$ with $n \in N$, we can surrogate $2\, q^2 = (2\, n)^2 = 4\, n^2$ and this implies $q^2 = 2\, n^2$ and we see that 2 must also be a divisor of q. Thus we arrive at a contradiction.

So what we gain in using value of p, q ranging over the whole of all natural numbers is a proposition, which does not depend on the arbitrarily chosen value of p or q. Thus we may sum up: Even in formal contexts the evidence of the proof rests on an implicit variation and gains its special apodictic evidence from this source.

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CHAPTER V

HOW CAN A PHENOMENOLOGIST HAVE A PHILOSOPHY OF MATHEMATICS?

Jaakko Hintikka

Abstract. Husserl's philosophy of mathematics is interpreted as dealing with forms not unlike Aristotle's forms. They can be somehow immediately present in one's consciousness. Husserl's ideas are compared for similarities and dissimilarities with those of Aristotle, Mach, Russell and Wittgenstein. Husserl's main development is seen as making these forms more and more robust conceptually. It parallels the overall development of mathematics in the last 200 years from a study of numbers and space into a study of different structures. This development culminates in Husserl's unfinished project of a theory of all theories. This project has closer connections with Hilbert's axiomatic theorizing than with the ideas of the intuitionists.

In the nineteenth century, a French historian published a book purporting to prove that Napoleon never existed. So many different and even contradictory things were attributed to Napoleon, the main argument went, that they could not possibly be true of any one actual person. By a similar argument, a future historian of ideas might perhaps maintain that there cannot have been any single philosophical movement called phenomenology, for too many different and even contradictory things are said in the documents about it. There might seem to be as many phenomenologies as there are phenomenologists.

To begin with, what is meant by phenomenology in these days is not what this term meant for Husserl and his contemporaries in its main use. For them, "phenomenological" was an attribute of those physical theories that did not use unobservable variables. In our day and age philosophers seem to ignore this meaning. At one point I discovered that one of the most active *soi-disant* phenomenologists was unaware of this original meaning of the term. But what has taken the place of this meaning is far from clear. Not only are there different sects of phenomenologists, such as West Coast phenomenologists as distinguished from East Coast phenomenologists. Their substantial grasp of their subject is radically different. For some, phenomenology is very nearly descriptive psychology, presuppositionless examination of one's consciousness. Some

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philosophers even confuse phenomenology and phenomenalism. For others, the central idea in phenomenology is intentionality, the phenomenon of mind's intending or being directed to an object. Phenomenology then becomes a kind of generalized meaning theory, with emphasis on the meaning entities like noemata that are supposed to be the vehicles of such directness. In one recent dictionary of isms, phenomenology is defined as the philosophical study of the ways that the mind interacts with the world and itself. In view of such a definition, one cannot help asking what else might there be in philosophy besides phenomenology.

I thought that I had reached the ultimate absurdity in interpretations of phenomenology when long ago I found a respectable philosopher (Evert W. Beth) in effect accusing Husserl of plagiarizing the entire idea of phenomenology from the arch-positivist Ernst Mach. However, later I found that there was at least one other observer who proposed to look at phenomenology in the same way, albeit without any charge of plagiarism. In this perspective, phenomenology is nothing but a further development and radicalization of the ideas in the philosophy of science that were defended by the likes of Mach and Hering plus analogous developments in the philosophy of psychology represented by Brentano.

Now who on earth could have thought of phenomenology in such weird terms? The answer is: Edmund Husserl. The words I just used are in fact Husserl's characterization of his phenomenology in the beginning of his Amsterdam lectures (Hua 9, 302). Nor was his statement a casual *obiter dictum*. It occurs already in Husserl's definitive statement of his philosophy in the famous *Encyclopedia Britannica* article, although Husserl does not name any names there (Ibid, 277–278). Husserl even adds that that's where he got the term "phenomenology." Husserl's seriousness is measured by his resolute rejection of Heidegger's attempt to bowdlerize the *Encyclopedia* article by omitting this reference. (Ibid. p. 237 for Husserl's original statement and pp. 256–257 for Heidegger's attempt to put his words in Husserl's mouth.) In spite of these explicit acknowledgements, the Mach-Husserl connection has received scant attention in the literature. (The main exception is the work of Manfred Sommer; see Sommer 1985.) In the recent *Encyclopedia of Phenomenology*, we do not even find Mach listed in the index. This neglect makes it especially urgent to look at Husserlian phenomenology in terms of comparisons with Mach.

In the light of hindsight, it might be even more instructive to compare Husserl with Bertrand Russell than with Ernst Mach. (I have tried to do so in another essay; see Hintikka 1995.) This does not reflect on the interest of the Mach-Husserl relation, however.

But if Husserl was a follower of Mach, how could he have any philosophy of mathematics? Husserl professed agreement with Mach's criticisms of the excessive mathematization of the natural sciences. But for Mach all logical and mathematical reasoning is purely analytical in the sense of tautological. (See Mach 1905, 11: 299–313.) If a mathematician looks away from everything inessential, he or she can see the conclusion of a proof right there in the premises, according to Mach. A mathematical argument can therefore give new information only in a psychological sense.

This view of logical and perhaps also mathematical truths as tautologies is usually thought of as a novelty of Wittgenstein's *Tractatus*, a novelty that Russell claimed to have been shocked by. If Russell had been active on the continent, the tautologicity thesis would not have surprised him, for it was fairly common there. In Wittgenstein's immediate background it was represented among others by Mach and Schlick.

Not only does this kind of view on mathematics violate Husserl's canons of anti-psychologism. It can only yield a barren minimalist philosophy of mathematics. Hence, Husserl must have been advocating a philosophy of mathematics quite unlike that of his acknowledged predecessor. What is this philosophy?

Different readers have found an embarrassment of different mathematical riches in Husserl's writings. Or a more appropriate locution would be a confusion of riches. For it is far from obvious what the relevance of Husserl's ideas is to actual mathematics or to its foundations. This confusion is enhanced by the fact that Husserl kept on developing and changing his ideas. A complete account of Husserl's philosophy is therefore enormously laborious to give, especially if one wants to do justice to his development. In this paper, I will accordingly restrict my attention to a couple of main questions concerning Husserl and mathematics.

The most important such question is undoubtedly: How are Husserl's ideas related to and perhaps relevant to, the central development in mathematics and its foundations? Now Husserl's philosophy of mathematics is often thought to be akin to the ideas of the intuitionists. It is in fact the case that various intuitionists and related philosophers of mathematics—Hermann Weyl, Oskar Becker, etc.—found inspiration in phenomenological ideas. However, it seems to me that this engagement was rather superficial at both ends. It seems to me that this relationship is at best rather remote. And one way of illustrating this judgment is to take a clue from intuitionists' very label and examine briefly Husserl's notion of intuition. A comparative examination shows why it is tempting to think that Husserl's ideas must have pushed him toward a finitistic and in some sense intuitionistic position. Notwithstanding such a temptation, I will suggest that the mature Husserl did not fall to it.

Thus one way of looking for the wellsprings of Husserl's theory of mathematics is to examine his notion of intuition or *Anschauung* and related concepts. (Cf. here Hintikka 2003.) This notion is routinely used by philosophers without any reflection on its meaning. This is a dangerous procedure, for in reality intuition is among the most elusive notions in philosophers' vocabulary. It is used in two essentially different senses. Some philosophers and many laymen use the term to refer to a certain mental faculty, perhaps a kind of inner vision or like the Peircean faculty of "guessing well." However, as I have pointed out elsewhere, the postulation of such a capacity was in the course of history gradually deprived of all justification. By the eighteenth century the term "intuitive knowledge" could only be used in the minimal sense "immediate knowledge" or "non-discursive knowledge." This is by and large the basic *per definitionem* force of *Anschauung* in Kant. (Cf. Hintikka 1969.) This minimal sense (as I will call it) is in effect the force of the term among semantically careful subsequent philosophers including (I argue) Husserl. This is the sense in which Russell could list sense-perception as one kind of "intuitive knowledge." When it comes to direct knowledge of objects rather than facts, Russell does not speak of intuition but acquaintance.

Even several otherwise well-informed philosophers have overlooked this strong semantical tradition and (presumably following the vulgar usage) assumed that Husserlian intuition is some kind of mental faculty.

This minimal sense is by and large the way Husserl uses the notion of intuition. I have suggested that in Husserl it is a near counterpart to Russellian acquaintance (Hintikka 1995). This aligns Husserl's transcendental reduction with Russell's reduction to acquaintance.

Thus Husserl, Kant and Russell are all roughly in the same semantical tradition in that intuitiveness is characterized by immediacy, direct givenness. However, there are also major differences between these three thinkers. For Kant, only particulars can be intuitively given. Furthermore, we cannot anticipate what they are. The only nontrivial (synthetic) thing we can know about them is that they must conform to the structures of space and time that we humans impose on them.

According to Russell, we can be acquainted with universals and not only with particulars. But for him the basic objects of acquaintance are unanalyzable. They cannot be reduced any further. They are therefore simple, with no structure. The mere acquaintance with them does not enable us to anticipate them or have any a priori theory about them. They have to be given to me in experience. Even when Russell in 1912–13 construed logical forms as being among the objects of acquaintance, they too had to be given to me in my experience. Hence acquaintance with general concepts cannot serve as a basis of a genuine theory about them.

For Husserl, too, intuition can be about general concepts and even abstract ones. But in contrast to Russell, according to Husserl we can have direct access to complex general essences in what he calls *Wesensschau.* It was in Husserl a version of what he called "categorial intuition." We can freely theorize about the "essences" or "forms" that *Wesensschau* gives us, examine their *Wesenszusammenhänge,* etc. Indeed, *Wesensschau* provides us with an access to what Husserl calls "formal ontology" as well as various "regional ontologies," which enables us to foresee what the different general forms of future experience can be, in direct contrast to Kant's view according to which a priori intuition can only give the frameworks (space and time) in which we behold particular objects.

At this point, comparisons with still other philosophers (over and above Kant and Russell) are especially instructive. Where in the doctrines of earlier thinkers do we find theorizing about the complex forms that are given me directly and entirely in my experience? The obvious answer is: in Aristotle. (See here Hintikka 2004b.) In another paper (Hintikka 2004a) I have suggested at least as a heuristic idea for ambitious historians of philosophy to view phenomenologists as "raiders of the lost forms"—meaning, of course, the forms in the Aristotelian sense. Now we can see that in such a perspective some of the central Husserlian ideas find a natural niche. I do not think that Husserl's use of such Greek terms as *eidos* and *hyle* is mere stylistic show.

For one thing, in the light of this comparison, we can appreciate some of the uncertainties and controversies that surround the notion of the object of an act. Is it the object that we intend by means of a noema out there in the real "objective" world? Or must we take Brentano literally and say that the object "inexists" in the act? Aristotle would not have entertained such questions. For him in thinking (intending?) X, the form of X is fully actualized both in the external object and in the soul. If we express ourselves in the phenomenological jargon, this shows the sense in which the (formal) object of an act exists *both* in the reality *and* in the act.

Perhaps there is also a partial non-elective affinity between the respective philosophical techniques of Husserl and Aristotle. Aristotle thought we can form forms in our souls by starting from the bits and pieces of forms given to us in sense-perception and by deconstructing and reconstructing them build up ("form") the required forms. Husserl, too, envisages possible manipulations of forms in imagination, for instance in the method of free variation, and more generally in acts of constitution.

Another, temporally closer comparison is also of interest here (Hintikka 1990, Park 1998). Wittgenstein once admitted that "one can say of my philosophy that it is 'phenomenology'." The real meaning of this statement has not always been understood. Some commentators have searched for similarities between phenomenology and Wittgenstein's later philosophy while missing the massive, obvious fact that the simple objects of the *Tractatus* are phenomenological in the same sense as their older brothers, Russell's objects of acquaintance.

But at least in one direction the affinities between Husserl and Wittgenstein are more detailed than those between Husserl and Russell. There is an interesting similarity between the *schauen* of essences in Husserl and the way the logical forms that Wittgenstein postulates in the *Tractatus* are received by us. Wittgenstein's logical forms are not free-standing objects of acquaintance, as in Russell. Indeed, it is in Wittgenstein's early philosophy that we find the most instructive example of what a form-based phenomenology might look like in a modern context. They are forms of objects. They come to us as aspects of the experiences that yield the simple objects. This is not unlike the way Husserlian essences are extracted by means of *Wesensschau* from the stream of experiences.

Moreover, even though the Tractarian logical forms are forms of the most individual entities that there are, viz., simple objects, they have in a certain important sense structure, unlike Russellian unanalyzable objects of acquaintance. And these logical forms are in a sense general, viz., in the sense that they govern the way in which the owners of these forms can be related to and combined with each other. Wittgenstein's paradigmatic example is how my perception of a colored object *ipso facto* tells me that it is *conceptually* impossible for the same perceptual object to have another color (at the same moment). This illustrates Wittgenstein's puzzling idea that a direct experience of an object can *ipso facto* show me all the possible ways in which it can be combined with others into a fact.

An objection might perhaps be raised to this Husserl-Wittgenstein comparison. Wittgenstein in fact criticized Husserl in his discussion with Vienna Circle members. However, when we understand the precise point Wittgenstein was making, we can see that he is not contradicting my interpretation. What Wittgenstein criticizes is Husserl's doctrine of synthetic a priori truths. In order to employ the notion of synthetic a priori, you must have some theory to whose truths you could apply this label. Husserl clearly thought that he had such a theory, whether it is called ontology or not, in the form of a theory of those essences that *Wesensschau* can give us. But for Wittgenstein, logical forms were not expressible in language because they deal with the semantical relations that connect our language with reality and give it its meaning. Such relations are strictly inexpressible for Wittgenstein. Thus according to him we cannot put forward synthetic a priori propositions about logical forms because we cannot put forward any propositions about them. Hence what Wittgenstein is objecting to in Husserl is not the posssibility of a kind of *Wesensschau* (which he could not criticize without criticizing himself or at least his earlier self) but the expressibility of the testimony of such *Wesensschau*.

The inexpressibility of logical forms implies that there cannot be any literal discursive theory about them. Simple objects and their forms are the alpha and omega of the ontology of the *Tractatus*, but there cannot be any science or theory about them. Thus Wittgensteinean forms cannot be manipulated and experimented with in the same way as Aristotle's or Husserl's forms. This is another aspect of the impossibility of there existing a genuine theory of logical forms according to Wittgenstein.

Hence Wittgenstein's critical comments on Husserl do not reflect on the deeper kinship of their ideas. Perhaps the most striking facet of Wittgenstein's philosophy of logic and language is that the very basis of logic, the logical forms, are given to us phenomenologically. This is what he means by saying that "phenomenology is grammar"—the term "grammar" being the later Wittgenstein's euphemism for logic. This is why he gave up the idea of a sharp logic of language as soon as the phenomenologically given was no longer for him expressible in language. Usually, logic is thought of as the structure of our most general concepts and truths. For Wittgenstein, logic deals with the form of the most particular objects (of different logical types).

If Wittgenstein had thought it possible to express in language the forms of phenomenologically given objects and theorize about them, he would have been in a position to develop a truly phenomenological theory of logic in a wide sense of a system of conceptual necessities. In a sense, the *Tractatus* is calculated to be such a phenomenological theory of language and its logic. Alas, because of his highly restrictive commitment to a universalist view of language-world relations, he had to relegate in the penultimate paragraph of the *Tractatus* the logico-semantical theory he had already developed to the realm he called nonsense. (Once again, old as well as new Wittgensteineans have totally missed his point.) With this qualification, we can say that the *Tractatus* is the simplest example of what a phenomenological theory of logical (conceptual) necessity might look like, in particular, in this way we can see in what sense Wittgenstein could say, "phenomenology is logic."

This way of looking at Husserl as a raider of the lost Aristotelian forms also shows why it is tempting to overemphasize the affinity of his ideas with finitistic and intuitionistic ones. Now conceptual knowledge can be obtained according to Husserl by manipulating forms in intuition. However, the deep difference is that for Husserl these forms are not constructed by the human mind but are ultimately given in experience. Phenomenological reduction does not lead us to acts of construction but to what is given to me in intuition in the minimal sense.

However, it might still seem that the perspective in which I have considered possible phenomenological theories of mathematics is incoherent. Or are those phenomenological theories themselves incoherent? On the one hand in such aspects of his philosophy of mathematics as the theory of manifolds Husserl was concerned with arbitrary abstract structures of both the theories that could specify them, obviously including the infinite structures studied in mathematics. On the other hand, for such a theory to be phenomenological those forms presumably must be realizable in human consciousness, in Brentano's phrase, be capable of inexisting in consciousness. For how can one otherwise manipulate them in thought? But not all forms or structures seem to be capable of so doing. Can there be a *Wesensschau*, a kind of actual seeing of completely abstract structures? Furthermore, most of the interesting mathematical structures are infinite. How can such structures possibly be present in human mind?

This problem situation in Husserl is not new. It is connected with the question of the content of such ideas as form and structure. For Aristotle, thinking was operating with forms. (See here Hintikka 2004b.) And occasionally he seems to have assimilated these forms with structures. "All thinking takes place by means of images, as in seeing," he writes. The paradigm case is thus geometry, where Aristotle seems to say that all reasoning takes place by means of figures or diagrams. Hence something like the same structures as the figures have must be actually present in our minds. It is perhaps not unfair to suggest that his notion of form was not abstract enough for the purposes of mathematics. For one thing the figures needed in geometry must be arbitrarily large. How can such structures be actually instantiated in the soul? Aristotle is worried about this problem and proposes a solution, which I will not consider here. (Cf. *De Memoria* 452a11 ff.)

In his theory of mathematics, Husserl is faced with a similar problem. What are actually present in consciousness are finite structures. How can they give us access to the mathematical structures or forms that are in representative cases infinite? Cannot Husserl's own manifolds be infinite?

It is here that we can see why phenomenology has appealed to finitistic thinkers.

How did Husserl deal with this problem? I do not have an answer to offer at this time, but there are clues as to what he thought. Much of Husserl's early struggle with psychologism and with the role of formalism in transcending the limitations of what can be perceptually intuited can be seen as a part of this struggle. Husserl's own development led him to the view that we can entertain even completely abstract structures in our consciousness. This development is usually discussed in terms of a gradual liberation from psychologism. I suggest that the real gist of Husserl's development was not from one philosophical position to another, but a growing trust in the power of the human mind to grasp directly abstract structures. Thus we can perhaps say in Aristotelian terms that he was able to extend the notion of form from its simple geometrical sense to the abstract one needed in all mathematics. Those interpreters who assimilate Husserl's thought to intuitionism are therefore missing the most interesting aspect of his philosophy of mathematics.

The same systematic point has further historical manifestations. Husserl himself at one time emphasized the affinity of his ideas with Hilbert's axiomatic method. Now the typical models of Hilbertian axiom system are infinite, and so are clearly many Husserlian manifolds. But if he were a Hilbertian axiomatist, he could scarcely have been an intuitionist. Husserl's notions of essence and *Wesensschau* are part of this aspect. For Aristotle, form is a much more abstract notion than structure, even though in geometry they are hard to tell apart. But precisely what is there in Husserl's notion of *Wesensschau* that can acquaint us with forms in the sense of mathematical concepts and yet be part of our conscious experience? Wittgenstein believed that experience can give us logical forms of objects. Could Husserl's *Wesensschau* do the same thing? But then was Wittgenstein right? Wittgenstein changed his mind about many things, but I cannot help concluding that in an important sense he believed to the end that "phenomenology is logic."

Once we have overcome the temptation to find Husserl's soulmates among intuitionists and constructionists, we are free to see where the greatest philosophical and historical interest of his ideas lies. This interesting core idea is his project for a general theory of all structures. These structures are sometimes referred to as manifolds.

This interpretation has the advantage of relating Husserl's ideas to the actual development of mathematics. The most general feature of the history of mathematics in the last 200 years is its gradual transformation from a study of numbers (including functions of numbers to numbers) and of space to a study of different kinds of structures in general. Milestones in that development include Riemann's theory of manifolds, Cantor's set theory and Hilbert's axiomatics. What is common to all these is a more or less fully articulated idea of a general theory of different kinds of structures. They can for instance be different geometries, different sets or the different models of an axiomatic system.

The ultimate manifestation of this structuralist conception of mathematics would have been a vision of a universal "structure of all structures" or "model of all models." I believe that some such vision guided many mathematicians' thinking in the late nineteenth century even though it was never fully articulated by them. For instance, on Hilbert's early axiomatist view of mathematical theorizing, what an axiomatic theory does is to stake out a subspace of the space of all structures as the class of its models. The existence of such models was implied by the consistency of the axiom system. This explains not only Hilbert's misleadingly formulated claim that consistency implies existence. It also explains his preoccupation with such questions as consistency and completeness.

Both Riemann's theory of *Mannigfaltigkeiten* and Cantor's theory of sets (which at an early stage of the theory were also known as *Mannigfaltigkeiten*) can be thought of as partial realizations of a theory of all structures. Riemann's theory was only partial, for not all possible structures are manifolds in Riemann's sense. Cantor's theory was in a different way only partial, for in it we look away from the internal structure of a set. Yet it is hard to understand the history of set theory without appreciating the idea of a structure of all structures. For instance, why should there be a temptation to assume any strong form of the axiom of comprehension, which so easily has paradoxical consequences? If one accepts the idea of the structure of all structures, it is not unnatural to assume that any condition on the relevant structures (sets?) must pick out the structure (set?) that they form together.

Also set theory seems to have been thought of as supplying the models that an axiomatic theory needs. It was for this reason that Hilbert was so anxious to regain Cantor's paradise. Sometimes you can form models of a theory in another theory, as Klein could model certain non-Euclidean geometries in the Euclidean one. But when this fails, you have to find the relevant structure without the help of existing theories, other than the "model of all models" where they all come.

Without entering any details, it seems to me that the greatest interest of Husserl's philosophy of mathematics lies in his attempt to spell out systematically this idea, which apparently was tacitly on the mind of many of his contemporaries. (See here especially the texts at the end of

Hua 12, and cf. Hill 2000a; 2000b.) The manifestation of this attempt is undoubtedly Husserl's theory of what he called manifolds. Sometimes he thought of this vision in terms of a "system of all systems" not unlike Hilbert's axiomatic method. This directed his attention to some of the same concepts that Hilbert was developing, for instance to the notion of completeness. Indeed, for a while, Husserl professed a close kinship with Hilbert's axiomatic approach.

This project had two aspects. Husserl's theory dealt with certain abstract entities, viz. structures. But apparently Husserl thought that each such structure had to be capable of being specified by an axiom system. Thus this theory of "the model of all models" came to involve also a study of a "system of all systems." It was in this way that Husserl got involved in the intriguing but at his time confused problems of the completeness of logical and mathematical systems in different senses of completeness.

The main difficulty that an interpreter faces here and that in a different way haunted Husserl was a lack of distinction between what Beth later called formal derivability and logical consequence. This uncertainty affects all questions of completeness, for an axiom system can be complete with respect to logical (semantical) consequence relation and yet not admit of a complete formal proof procedure. Admittedly, this difficulty Husserl shared with all his contemporaries. Yet it makes his speculations about completeness, elementary equivalence, categoricity and *definitheit* to a considerable extent confused and confusing.

We can thus see one important direction that the philosophy of mathematics of a phenomenologist like Husserl can take—and in fact took.

Unfortunately, the actual development of logic and mathematics has not been very kind to Husserl's ideas, at least on the surface. In axiomatic set theory, all-comprehensive notions like the set of all sets have turned out to be difficult to implement. Likewise, Hilbert's use of the consistency of a system (structure) as a sufficient criterion of its existence likewise runs into trouble, in that such consistency turned out to be hard to prove in non-trivial cases. Further research will show whether these difficulties are insurmountable.

Another obstacle in the development of Husserl-type ideas was several leading logicians' commitment to what van Heijenoort (1967) called the idea of logic as language. (The general form of this idea has been called the idea of language as a universal medium.) As Martin Kusch (1989) has impressively shown, this idea was foreign to Husserl. This idea was understood as ruling out (among many other things) any general theory of all models. Early attempts to develop such a theory under the title "logical semantics" were shot down by mathematically sophisticated logicians, especially Tarski. Admittedly Tarski later came to build together with his associates what is now called model theory. However, Tarski's model theory was originally only a branch of the metatheory of algebra, not a universal theory with philosophical significance and general applicability. I suspect that it was Husserl's implicit rejection of the idea of language as a universal medium that made his theory *logica non grata* much more important than any particular technical result, such as Gödelian incompleteness.

On the philosophical level, Husserl nevertheless has not remained without followers—if that is the right word. No lesser a figure than Kurt Gödel has espoused a conception of universal theory of sets and structures, albeit without any explicit mathematical realization other than the usual set theory. However, for all his professed admiration of Husserl's ideas, Gödel departed significantly from them. For instance, in one important respect Gödel's envisaged theory of structures is only a flat shadow of Husserl's and Hilbert's ideas. Gödel was an actualist. For him, there was only this one reality of ours. Even the possible structures figuring as models of axioms must exist somewhere in this universe of ours, which therefore must encompass an abstract Platonic higher stratum. The truths of logic are therefore for Gödel among the truths about the actual world, not truths obtaining in all possible worlds, as for Leibniz.[1]

This is a much more timid idea than the Husserl-Hilbert idea of different mostly unactualized possibilities concerning the real world. For one thing, it means losing all connection with the idea of a variety of axiomatic theories each with its models and hence all connection with much of actual mathematical theorizing.

[1] Mark van Atten has challenged my attribution of actualism to Gödel. His evidence nevertheless supports my case. It is all from Gödel's paper on the general theory of relativity, where the relevant sense of possibility is a physical one, not conceptual (logical). Of course a philosophical actualist can speak of other physically possible worlds, meaning simply possible systems satisfying a given physical theory.

It also trivializes Gödel's notion of intuition. What Gödel called intuition is much more like Husserl's *Wesensschau* than Husserl's notion of intuition. Gödel's intuition could not be the discovery of new forms or structures in experience that might or might not be exemplified in the particular structure called the actual world. What it was calculated to do was to provide us with a peek of how things stand on the Platonic upper floor of our actual world. Gödel's intuition is therefore primarily intuition of truths, not objects. How such intuition operates as to give us any real insights remains a mystery.

Gödel compares his notion of intuition to sense-perception. It is supposed by him to be the basis of our knowledge in logic and mathematics in the same way as sense-perception is the basis of natural science. But this is a wrong conception of scientific method. Natural science, as distinguished from natural history, is not based on mere observation, but on experimentation. As was observed earlier, Husserl's notion of intuition allows manipulation and experimentation in thought. Gödel's does not, wherefore it is only a pale shadow of Husserl's ideas.

It might seem flattering to Husserl to have had admirers like Gödel. Unfortunately, it seems to me that Gödel's grasp of what Husserl's philosophy of mathematics amounts to was shaky and that as a philosopher Husserl had a better judgment than Gödel.[2]

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[2]I have profited from comments of several of the participants in our meeting, and I thank all of them. All my scholarly sins are original.

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CHAPTER VI

THE DEVELOPMENT OF MATHEMATICS AND THE BIRTH OF PHENOMENOLOGY

Mirja Hartimo

Abstract. The article examines Husserl's view of mathematics as a continuation of Weierstrass's project. While Husserl adopts the more modern axiomatic approach to mathematics as opposed to Weierstrass's genetic approach, he continues to be Weierstrassian in his preoccupation for foundational questions. The latter part of the article examines in what way the outcome is Platonistic in Husserl's own usage of the term. By Platonism Husserl means both a belief in immutable and unchanging ideal structures, as well as, a search for critical justification in reflection. In the latter sense of the term Husserl's "Platonism" can be seen as an outcome of Husserl's Weierstrassian *ethos*.

In the foreword to the *Prolegomena to the Logical Investigations* (1900) Husserl writes that the *Logical Investigations* (1900–1901) was born from his attempts to clarify the philosophical foundations of mathematics (Hua 18, 5).[1] Arguably, the most important discovery of Husserl's

[1]Husserl opens the *Logical Investigations* with the following claim about its origin: "Die logischen Untersuchungen, deren Veröffentlichung ich mit diesen Prolegomena beginne, sind aus unabweisbaren Problemen erwachsen, die den Fortgang meiner langjährigen Bemühungen um eine philosophische Klärung der reinen Mathematik immer wieder gehemmt und schließlich unterbrochen haben. Neben den Fragen nach dem Ursprung der mathematischen Grundbegriffe und Grundeinsichten betrafen jene Bemühungen zumal auch die schwierigen Fragen der mathematischen Theorie und Methode. ..." (Hua 18, 5). Husserl goes on to explain how he had realized that "das Quantitative gar nicht zum allgemeinsten Wesen des Mathematischen oder 'Formalen' und der in ihm gründenden kalkulatorischen Methode gehöre. Als ich dann in der 'mathematisierenden Logik' eine in der Tat quantitätslose Mathematik kennenlernte, und zwar als eine unanfechtbare Disziplin von mathematischer Form und Methode, welche teils die alten Syllogismen, teils neue, der Überlieferung fremd gebliebene Schlußformen behandelte, gestalteten sich mir die wichtigen Probleme nach dem allgemeinen Wesen des Mathematischen überhaupt, nach den natürlichen Zusammenhängen oder etwaigen Grenzen zwischen den Systemen der quantitativen und nichtquantitativen Mathematik, und speziell z. B. nach dem Verhältnis zwischen dem Formalen der Arithmetik und dem Formalen der Logik. Naturgemäß mußte ich

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Logical Investigations (1900–1901) is the notion of categorial intuition. In this chapter I will examine how Husserl's engagement in the problems about the foundations of mathematics led him to the discovery of categorial intuition. Roughly, the story goes as follows: At first, following his teacher Karl Weierstrass, Husserl held that mathematics should be erected on the concept of number. Accordingly Husserl's first philosophical works focused on the concept of number and the elementary arithmetic operations. During the latter half of the nineteenth century the mainstream approach to mathematics evolved from the Weierstrassian genetic approach into an axiomatic approach, to use terms introduced by Hilbert in 1900 (Ewald 1996, 1092–1093). Following the mainstream mathematicians Husserl adopted the modern axiomatic view of mathematics, according to which mathematics is about abstract structures and has little to do with numbers or counting. The development of mathematics thus produced new, independent, purely formal domains to be studied. The new ideal domains helped Husserl to develop his anti-psyhologistic point of view as well as led him to the discovery of categorial intuition in the *Logical Investigations*.

In the end I will briefly address the significance of categorial intuition to Husserl's philosophy. The notion takes Husserl's approach far beyond Kant, aligning Husserl's approach rather with Aristotle, as has been hinted at by Heidegger (Heidegger 1985, 2000; Taminiaux 1985) and emphasized by Richard Cobb-Stevens (1990, 2002, 2003, 2004), and Jaakko Hintikka in the present volume. I will discuss the nature of Husserl's Aristotelianism. I will argue that while Husserl's approach towards reality is certainly Aristotelian rather than Kantian or Cartesian, Aristotelian approach has difficulties in including Husserl's abstract view of mathematics into it. Indeed, Husserl's view of logic has a closer ancient counterpart in Euclid, and through Euclid, Plato. I will distinguish two senses in which Husserl's approach can be said to be Platonist: the first concerns his view about unchanging, identical mathematical objects, and the second relates to the demand for justification of the axiomatic

von hier aus weiter fortschreiten zu den fundamentaleren Fragen nach dem Wesen der Erkenntnis*form* im Unterschiede von der Erkenntnis*materie* und nach dem Sinn des Unterschiedes zwischen formalen (reinen) und materialen Bestimmungen, Wahrheiten, Gesetzen" (Ibid., 6).

theories. However, claiming that Husserl is a Platonist is not to say that Husserl was a metaphysical realist in a naïve sense (see Tieszen's paper in the present volume). Husserl's transcendental idealist interpretation of Plato is entirely in accordance with the non-metaphysical spirit of phenomenology.

I. Weierstrass and Mathematics as Rigorous Science

While the mathematical technique called "calculus" was invented by Newton, Leibniz, and others in the seventeenth century, only in 1821 Cauchy published the first systematic approach to analysis.[2] The logical structure of Euclidean geometry set the standard of rigor for Cauchy. The result was the first rigorous definitions of limit, convergence, continuity, and derivative. Many of Cauchy's discoveries were simultaneously arrived at by Bernhard Bolzano (1781—1848). But while Cauchy's *Cours* was extremely influential, Bolzano's work was relatively unknown until the latter half of the nineteenth century.

However, Cauchy still used expressions like "approach indefinitely," or "as little as one wishes" that left room for ambiguities and invoked geometrical intuition. Ultimately the needed rigor was established in the 1860s, when Karl Weierstrass presented several results that considerably clarified several fundamental notions of analysis. By means of ε-δ definitions he could define for example the notion of limit by using only the real numbers, addition and "smaller than" relation, instead of using vague natural language expressions (e.g., variable "approaching" something). Because of the overall rigor of his writings he is generally considered to be the founder of modern analysis.

In his lectures in the 1870s and 80s Weierstrass demanded stepwise demonstrations of the basic notions of analysis, beginning with the concept of number and operations on the numbers (Dugac 1973, 64–65, 73, 78). Later, his conception of rigor matured into a demand for clarity by means of a detailed mode of representation while attempting to manage a chapter of science as a whole. Weierstrass's student Gösta Mittag-Leffler

[2] I have discussed Husserl's relationship to Weierstrass as well as his *Philosophy of Arithmetic* (1891) in more detail in Hartimo 2006.

has characterized Weierstrass's *ethos* as a striving for an all-encompassing theory of the domain in question: "In one of his treatises Weierstrass expresses the conviction that the results he has attained 'will at least interest those mathematicians who find satisfaction, when it is possible to bring a chapter of science to a genuine conclusion.'" (Mittag-Leffler 1897, 79)[3]

With these plain words Weierstrass himself characterized his whole activity and articulated the goal which he strived for in all his works. The history of mathematics will also support this, that until now no mathematician was able to reach this goal to a higher degree and in greater extent than he, and the goal is to bring complete chapters of the science into actual conclusion. (Mittag-Leffler 1897, 79)

Initially Weierstrass's aim was to logically analyze the fundamental notions of analysis. While he wanted to articulate everything presupposed in the theory he also wanted to describe a branch of mathematics in its entirety. Thus Weierstrass's ethos is characterized by an aspiration to *descriptive* completeness, to capture everything there is to say about the domain of a theory. This *ethos* may well have had an enormous influence not only on Husserl and his view of *Definitheit* (See below Section III, also Hartimo 2007), but also many others in his generation of mathematicians, in particular, on Hilbert's Completeness Axiom.

II. Husserl in Weierstrass's Footsteps

Husserl went to Berlin to study mathematics in 1878. He attended the full cycle of lectures given by Weierstrass including the lectures on the theory of analytic functions (1878); the theory of elliptic functions (1878–1879), the lectures on the calculus of variations (1879); the lectures on the use of elliptic functions to solve selected geometrical and mechanical problems (1879); lectures on the theory on Abelian functions (1879–1880); again the lectures on the theory of analytic functions (1880–1881). Husserl's report about Weierstrass is the following:

[3]In einer seiner Abhandlungen spricht Weierstraß die Überzeugung aus, dass die von ihm erhaltenen Resultate "wenigstens diejenigen Mathematiker interessieren werden, welchen es Befriedigung gewährt, wenn es gelingt irgend ein Kapitel der Wissenschaft zu einem wirklichen Abschlusse zu bringen."

The great Weierstrass was the one who raised in me an interest for a radical grounding of mathematics during my student years in his lectures on the theory of functions. I learned to appreciate his efforts to transform analysis, which was so much a mixture of rational thought and irrational instinct and tact into a purely rational theory. He was after the original roots, attempting to postulate the elementary concepts and axioms to form a basis from which the entire system of analysis could be constructed and deduced with a completely rigorous, thoroughly insightful method. (Schuhmann 1977, 7)

Moreover, Malvine Husserl has reported that Husserl repeatedly said that he owes his scientific *ethos* to Weierstrass (Schuhmann 1977, 7; See also Becker 1970, 40).

To Husserl Weierstrass's importance is in his demand for rigorous foundations for mathematics. Despite of his changing view of mathematics from the more genetic approach to the more axiomatic approach, in the sense of demanding foundations for mathematics, Husserl remained Weierstrassian for the rest of his life.

Continuing Weierstrass's program Husserl wrote his *Habilitationsschrift Über den Begriff der Zahl. Psychologische Analysen* (On the Concept of Number. Psychological Analyses) under guidance of Carl Stumpf in 1887 (Hua 12, xxi-xxiii). In his *Habilitationsschrift* Husserl explains:

Today there is a general belief that a rigorous and thoroughgoing development of higher analysis . . ., excluding all auxiliary concepts borrowed from geometry, would have to emanate from elementary arithmetic alone, in which analysis is grounded. But this elementary arithmetic has, as a matter of fact, its sole foundation in the concept of number; or, more precisely put, it has it in that never-ending series of concepts which mathematicians call "positive whole numbers." . . . Therefore, it is with the analysis of the concept of number that any philosophy of mathematics must begin. (Husserl 2003, 310–311)·

Husserl explicitly takes on the task of continuing Weierstrass's program and providing foundations to Weierstrass's approach. In Husserl's *Habilitationsschrift* this meant providing an analysis for the concept of number. A similar view is also expressed in the introduction to Husserl's *Philosophie der Arithmetik, Psychologische und logische Untersuchungen* published in 1891. There Husserl states, that "[p]erhaps I have succeeded in preparing the way, at least on some basic points, for the true philosophy of the calculus, that *desideratum* of centuries" (Hua 12, 7, Husserl 2003, 7). The aim is thus to provide foundations to calculus by an analysis of the concept of number. Moreover, the analysis should render the number thoroughly intuitive.

III. Philosophy of Arithmetic as an Analysis of the Concept of Number

Husserl's first published work *Philosophy of Arithmetic* (PA) was originally supposed to appear in two volumes, but only the first one came out. The published work consists of two parts with the first four chapters of part one repeating the contents of Husserl's *Habilitationsschrift* "On the Concept of Number," according to Husserl, almost word for word (Hua 12, 8). The first part of the PA gives a psychological analysis of our everyday concept of number. Indeed, Husserl later claimed that also Weierstrass started his analyses from our everyday concept of number. Around 1930, in an unpublished manuscript, Husserl wrote that "Weierstrass admittedly started with *concepts that are already given in the natural thinking life and the tradition*. But they will not be accepted without hesitation, but only after deliberate proof, namely as intuitive in their meaning as clear and identifiable, like the individual 1, the operative construction of 1 + 1 etc. (addition), equality of the individuals, etc."(B II 23 8 a–b, my emphasis).[4]

A psychological analysis is to Husserl an analysis of an experience of the presentation of a number and in particular an elucidation of its "origin." Since our intellect and time are limited, we can have an intuitive understanding of only a small part of mathematics. In order to overcome the limitations of our intellect we make use of symbols to assist our thinking. This already takes place in such simple tasks as counting objects. Indeed, we know almost all of arithmetic only indirectly through the mediation of symbols. Accordingly the second part discusses the symbolic representation of numbers. As the subtitle *Psychologische und logische*

[4]Weierstraß beginnt zwar mit Begriffen, die von dem natürlichen Denkleben und der Tradition vorgegeben sind. Aber sie werden nicht unbesehen hingenommen, sondern nach bewusster Prüfung aufgenommen, nämlich als einsichtige ihrem Sinn nach klare und identifizierbare, wie das einzelne 1 die operative Bildung des 1 + 1 usw. (Addition), Gleichheit von Einzelnen etc. (B II 23 8a-b)

Untersuchungen suggests, it contains logical investigations of the concept of number.

In the first part on the psychological investigations Husserl appropriates Brentano's descriptive psychology to the problem of number. His aim is to describe what numbers are to us in an ordinary experience. In so doing Husserl develops several methods central to phenomenology. Indeed Husserl's later, properly phenomenological writings are easy to see as a continuation of the investigations Husserl started already in his *Habilitationsshcrift*. Indeed in the *Formal and Transcendental Logic* (1929) Husserl views it as a phenomenologico-constitutional investigation:

> I had already acquired the definite direction with regard to the formal and a first understanding of its sense by my *Philosophy of Arithmetic* (1891), which, in spite of its immaturity as a first book, presented an initial attempt to go back to the spontaneous activities of collecting and counting, in which collections ("sums," "sets") and cardinal numbers are given in the manner characteristic of something that is being generated originaliter, and thereby to gain clarity respecting the proper, the authentic, sense of the concepts fundamental to the theory of sets and the theory of cardinal numbers. It was therefore, in my later terminology, a phenomenologico-constitutional investigation; and at the same time it was the first investigation that sought to make "categorial objectivities" of the first level and of higher levels (sets and cardinal numbers of a higher ordinal level) understandable on the basis of the "constituting" intentional activities, as whose productions they make their appearance originaliter, accordingly with full originality of their sense. (FTL, §27a)

The second part offers an independent view of arithmetic, basing it on the use of signs. In the first part Husserl had explained that we are capable to have an authentic intuition only of numbers up to about 12. However, if we count the objects by enumerating them, we are already relying on symbolic methods (Hua 12, 104–105, Husserl 2003, 109–110). Thus it is clear that arithmetic in general cannot be authentically given in intuition (Hua 12, 192). The question that Husserl wants to answer in the second part of the *Philosophy of Arithmetic* is how the rest of arithmetic is given to us.

Husserl's answer to the problem lies in the complete parallelism between the system of concepts and the system of signs. The idea is to start from certain concepts, translate them into signs, and then to operate on the signs in accordance to given rules. The resulting sign is in the end interpreted as a concept. Thus Husserl's conception of the *arithmetica* *universalis* is based on the notion of computation, and the belief in, in modern terms, completeness and soundness of the two parallel systems. In the end, the basis for *arithmetica universalis* is in the sense-perceptible signs. Husserl also presupposes the existence of purely formal concepts that correspond to the results of the computations, thus maintaining a view that is rather independent of the first part of the *Philosophie der Arithmetik*.

The published volume of the *Philosophie der Arithmetik* was supposed to be followed by a second volume in which Husserl was supposed to discuss

> logical investigation of the arithmetical algorithm . . . and the justification of utilizing in calculations the quasi-numbers originating out of the inverse operations: the negative, imaginary, fractional and irrational numbers. Critical reflections on the algorithm repeatedly occasion a closer examination of the question whether it is the domain of cardinal numbers, or some other conceptual domain, that general arithmetic in the primary and original sense governs. To this fundamental question the Second Part of Volume Two is then devoted. (Hua 12, 7; Husserl 2003, 7)

In addition he also planned to develop a new philosophical theory of Euclidean geometry, possibly on the basis of Gauss's work *Anzeige der Theoria residuorum biquadraticorum, Commentatio secunda.* However, the second volume never appeared. Ironically, Husserl later claimed that the analyses of the *psychological* part of the PA already represent phenomenologico-constitutional investigation, while the second part on the *logical* investigations caused him problems and underwent several changes (Husserl 1994, 490–491, Hartimo 2007).

IV. Logical Investigations and the Axiomatic Approach

For the sake of the present argument I will not discuss Husserl's development between the *Philosophy of Arithmetic* and the *Logical Investigations* in detail here. But roughly what happened was that Husserl's investigations took him to the developing views of *Mannigfaltigkeitslehre*. He searched for a general framework in which one could examine individual theories and their relationships with each other. Wanting to find an approach to analysis free from geometrical intuition he was mostly interested in Hermann Grassmann's *Ausdehnungslehre*, coordinate-free geometrical calculus. However later he claimed interest also to Riemann's, Hamilton's, Lie's, and Cantor's approaches. He worked in Halle as Cantor's colleague. The two discussed each other's work, and for example, Schröder's attempt to deal with concepts of set theory by means of the algebra of logic (Schuhmann 1977, 52; Peckhaus 2004, 593–594).[5]

In his investigations Husserl followed the general trend in the mainstream mathematics. In the end of the nineteenth century mathematics developed from the Weierstrassian approach towards more abstract theories. By the 1880s Weierstrass, and with him, Berlin, lost the leading role in the world of mathematics. Göttingen with Klein and Hilbert in the lead assumed the role in the late 1890s. Around the turn of the century there were several competing "logics," i.e., different kinds of theories of manifolds, which were different kinds of suggestions for the general framework in which we could examine mathematical theories. The search for rigor in the foundations of mathematics reached its point of culmination in Hilbert's work. Incidentally, in 1901 Husserl moved from Halle to Hilbert's Göttingen.

Hilbert's work was a culmination in a trend to give analysis purely qualitative basis. To Hilbert, the ultimate rigor to analysis, as well as to the rest of mathematics, is given by axiomatization, not through arithmetization. Hilbert contrasted his axiomatic method to the genetic method, in which the number domain is generated from a number one. Instead, in the axiomatic method, the existence of the elements of the domain is presupposed, while the relationships between the elements are determined by means of the axioms. The axiomatic method then requires proving the completeness and coherence of the system. In 1900 Hilbert added the Completeness axiom to his axiomatization of the Euclidean geometry. With it he posited its categoricity, i.e., roughly that all of its interpretations are isomorphic to each other. A complete axiomatization has formally only one interpretation, and thus one could as well talk about tables, chairs, and beer mugs as points, lines, and planes. In such an axiomatization the elements of the theory are considered from a purely formal point of view and their material realization is entirely irrelevant.

Soon after having moved to Göttingen in 1901, Husserl gave a so-called *Doppelvortrag* in *Göttinger Mathematische Gesellschaft*. In it Husserl explained his view of a definite axiomatic system that justifies the usage

[5] About Husserl's development in more detail see (Hartimo 2007, 2008).

of the imaginary numbers. Definiteness to Husserl captures the formal domain in the manner of Hilbert's complete axiomatic system. I have discussed Husserl's view of definiteness elsewhere hence I will here only summarize the main point of Husserl's lectures. For Husserl a definite system is, like Hilbert's complete axiomatization, a categorical theory that has only one purely formal model (Hartimo 2007). The definite axiom system is Husserl's ideal of a pure logic. It is the basis for Husserl's arguments against psychologism as it shows an undeniable existence of an objective and purely formal theory. To Husserl, the problem with the psychological logicians is that they cannot account for something like that and hence to Husserl their views lead to relativism and skepticism.

Perhaps even more importantly, to Husserl the idea of pure logic is a source of a new philosophical problem: how is this newly found formal domain given to us? "So erwächst die große Aufgabe, die logischen Ideen, die Begriffe und Gesetze, zu erkenntnistheoretischer Klarheit und Deutlichkeit zu bringen. Und hier setzt die phänomenologische Analyse ein" (Hua19/1, 9). Instead of the givenness of numbers and sets, Husserl's problem is now the givenness of the theoretical structures. The idea of pure logic is purely formal, thus there can be no sensuous intuition of it. The task is now to give a descriptive analysis of the constitution of the ideality of the abstract domains rather than that of the givenness of different sizes of collections of objects.

It is worth emphasis that Husserl's problem is the *givenness* of the theoretical structures; Husserl does not postulate them, but he sees his task to be to describe what is handed to him by the mathematicians. His attitude towards the existence of mathematical objects and structures is largely neutral: he does not question them nor does he postulate them. He simply describes what the sciences, in the present case mathematics, have found there to be. In terms of ontological questions Husserl's approach is thus rather naturalist. But, unlike a typical naturalist, in order to understand the givenness of what there is, Husserl turns to a transcendental description of how the given is constituted.

V. Categorial Intuition

Categorial intuition is Husserl's initial answer to the givenness of the pure logic. Intuition of something according to Husserl means direct givenness of that something. Sensuous intuition means givenness of simple objects. Categorial intuition, on the contrary, means givenness of categorial formations, such as states of affairs, logical connectives, and essences. When we see that paper is white, we do not only see paper and whiteness but also that the paper *is* white. For Husserl this shows that what is directly given to us in our experiences is not restricted to sense data. The world appears to us as meaningful and structured. In addition to simple objects we intuit certain "surplus," which makes the world garbed with meaning.

As Heidegger points out, in formulating the notion of categorial intuition Husserl moves far beyond Kant. Heidegger only hints at Husserl's Aristotelianism, but some later phenomenologists have emphasized Husserl's close resemblance to Aristotle. (See in particular Cobb-Stevens 1990, 2002, 2003, 2004.) After the publication of the *Logical Investigations*, Husserl's interest in categorial intuition grows into all-encompassing analyses of the constitution of the given. In 1907, Husserl introduced the term *Wesensschau* for what in the *Logische Untersuchungen* he called idealizing abstraction. *Wesensschau* is a special case of categorial intuition, and it refers to our capability to "see" identical essences. Jaakko Hintikka bases his view of Husserl as an Aristotelian philosopher especially on Husserl's notion of *Wesensschau.*

VI. Aristotle or Plato (and Which Plato)?

In a very general sense Husserl's attitude towards the reality indeed is Aristotelian rather than Cartesian or Kantian. We intuit categories, which would be an oxymoron for Kant. Moreover, what we intuit is out there in the world. Intuition is not introspection.

But if we take into account the role of abstract structural mathematics to Husserl, some reservations are in order. The problem is that the role of Aristotle's syllogistic as well as his writings in mathematics, was to offer an *organon* for empirical sciences. Aristotle did not thematize axiomatizations of geometry or arithmetics in themselves. Consequently Aristotle's syllogistics is not abstract enough to be able to include Husserl's view of abstract structures. Aristotle's, or at least his students', approach to logic was guided by a practical interest, and it was not founded by pure logic. The main point of Husserl's *Prolegomena* is that logic as a practical discipline has to be justified by logic as a theory.

The Euclidean axiomatization rather than Aristotelian syllogistics has set the theoretical ideal for mathematicians ever since the ancient times. In this respect, Husserl follows the mathematicians: the Euclidean ideal sets the standard of pure logic to him, too. To Husserl Euclid systematized the axiomatic ideal presented in Plato's *Republic* (Hua 7, 34–35). Indeed, in 1918 in a letter to Julius Stenzel Husserl called himself a "phenomenological Platonist" (*Briefwechsel* 6, 427–428), and in the 1920s he repeatedly mentioned Plato as the most important philosopher in the history of philosophy. He also claims that instead of Aristotle, Plato is the one who establishes the ideal of rationality and logic (Hua 7, 34).

VII. Platonism of the Eternal, Self-Identical, Unchanging Objectivities

When Husserl's Platonism is discussed in the secondary literature, it usually refers to a view about abstract objects or mathematical objects. Indeed Husserl writes: "In fact, one cannot describe the given phenomena like the natural number series or the species of the tone series if one regards them as objectivities in any other words than with which Plato described his ideas: as eternal, self-identical, untemporal, unspatial, unchanging, immutable" (Hua 30, 34). Husserl's view about the abstract objects derives from Lotze and especially Lotze's reading of Plato's ideas (Hua 20/I, 297, *Briefwechsel* 6, 460, for a detailed discussion of Husserl's indebtedness to Lotze, see Hauser 2003). To Lotze Plato's ideas are self-same and eternal concepts that are objective. They are valid (within a web of logical theory) but they do not have the reality of *Sein* (Lotze 1928). Accordingly Husserl explains his plan to have been to take Lotze's view of the ideal domain and place all the mathematical and a good part of the traditional logic into it. Husserl's way of avoiding Lotze's occasional psychologism was to use the "new mathematical logic" of the late eighteenth century. In particular, in a categorical theory mathematical objects have an objective existence independently of our activities of judging. They exist *unter dem Blick der Theorie*, as Jocelyn Benoist (2003) has put it. This is a kind of Platonism, but Platonism without hypostatising, *ohne topos ouranios* (Hua 19/1, 106), and hence not naïve Platonism discussed by Tieszen nor the kind of Platonism Dieter Lohmar objects to in his contribution, but a kind that is consistent with Husserl's transcendental idealism. Moreover, Husserl's Platonism is metaphysically neutral in the sense that it only describes the way in which mathematicians relate to their subject matter. A *definite*, i.e., a categorical theory defines abstract objects uniquely, and the philosophers' task is to describe the givenness of these objects. Thus Husserl does not *postulate* the realm of abstract entities. Rather he *describes* what mathematicians show us there to be. And they show us structures.

VIII. Platonism as an Aspiration for Reflected Foundations

But there is more to Husserl's "Platonism" than the above described view of the objectivity of the concepts. Indeed, Husserl refers to Plato in a much deeper sense when he opens his *Formal and Transcendental Logic* (1929) saying that "Science in a new sense arises in the first instance from Plato's establishing of logic." Husserl goes on to explain that we inherit from Plato the attempt to gain genuine knowledge that is *fully justified by reflection*. This sense of Platonism however is also a part of Husserl's Weierstrassian inheritance and has motivated Husserl's work from the beginning as the search for intuitively evident foundations for mathematics. It is only later that Husserl found the justification for exact sciences to be thematized also in Plato's dialogues, in particular, the *Republic*. This sense of Platonism shows also as a striving to a complete theory about a subject matter in question. Early Husserl claimed that he owed his scientific *ethos* to his teacher Karl Weierstrass (Schuhmann 1977, 7). Above we mentioned how Weierstrass's close student Gösta Mittag-Leffler has characterized Weierstrass's *ethos* as a striving for an all-encompassing theory of the domain in question (Mittag-Leffler 1897, 79). In Husserl's writings Weierstrassian/Platonist *ethos* turns into aspiration for theoretical completion, but the completeness in question is not that of any kind of a fixed system, but rather that of a research project. Because of it the investigation of evidences has to be included into the domain of logic. Thus Husserl's quest for completeness takes him to a never-ending project of estimating the *telos* of logic and the transcendental examination of the way various layers of logic are given to us. The description of various evidences is to provide logic with Socratic self-examination. It is an investigation of sense, the outcome of which is increased understanding, and to quote Husserl who quotes Socrates in Plato's dialogues, "a life truly worth living, a life of 'happiness,' contentment, well-being, or the like…" (FTL, 5). Thus Husserl's call for a "foundation" or a "justification" for sciences should not be understood only in an epistemological sense but also in the sense of Socratic-Platonic examination. Thus the view of philosophy as a justifying reflection of the sciences ultimately relates to what Husserl sees the task of phenomenology to be in general. This is the sense in which Bernet, Kern, and Marbach relate Husserl's idea of philosophy to the Socratic-Platonic examination of the absolute knowledge in relationship to the self-knowledge (1989, 4). This aspiration to "know thyself" is, in Husserl's view, an infinite historical task.

IX. Conclusion

Plato emphasizes time and again the role of mathematics. To him the way into philosophy goes through mathematics. In this paper I have discussed how a similar road took Husserl from Weierstrass's lectures via the abstract structural mathematics in the end of the nineteenth century, to the knowledge theoretical problem of the givenness and justification of the abstract structures. All along Husserl remains Weierstrassian in his aspiration to ground mathematics by means of insight. However, the developing mathematics uncovered the purely formal structures, which gives Husserl the pure ideality. Husserl's phenomenology is an answer to the problem of combining these two tenets, and in both of these respects Husserl found echoes in Plato.

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CHAPTER VII

BEYOND LEIBNIZ: HUSSERL'S VINDICATION OF SYMBOLIC KNOWLEDGE

Jairo José da Silva

Abstract. For the entire span of his philosophical career, Husserl struggled with the epistemological problem posed by imaginary elements, that is, improper or objectless representations that are nonetheless treated as if denoting something. How can it be explained that we can obtain knowledge (*symbolic* knowledge), as is paradigmatically the case of mathematics, by operating with symbols according to rules – *even when these symbols do not represent anything*? This problem presented itself very early in Husserl's philosophical life and was a dominant factor in the development of his thought. From the first to the last work he published the task of clarifying the sense and delimitating the scope of symbolization and formalization in science and mathematics was one of Husserl's major concerns. In this paper I want to show how Husserl dealt with the problem of symbolic knowledge in mathematics, and the central role it played in his philosophical development.

I. Introduction

Since the writing of *Philosophy of Arithmetic* (*PA*, expanded version of the *Habilitationsschrift* of 1887, published in 1891), at latest, until the completion of the *Prolegomena to Pure Logic* (1896, first part of *Logical Investigations*—LI—published in 1900), or maybe until later, when he developed the ideas he presented in Göttingen in 1901, Husserl struggled with the problem of imaginary elements in mathematics. As he himself tells us, this problem forced him to broaden his philosophical horizons, opening up new perspectives on the role of symbolization in thinking and knowing processes and presenting new questions on the sense and scope of formal logic. In his own words:

Above all it was its [i.e., arithmetic's] purely symbolic procedural techniques, in which the genuine, original insightful sense seemed to be interrupted and made absurd under the label of the transition through the "imaginary," that directed my thoughts to the significance and to the purely linguistic aspects of the thinking—and knowing—processes

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and from that point on forced me to general "investigations" which concerned universal clarification of the sense, the proper delimitation, and the unique accomplishment of formal logic. (*Draft Introduction to Logical Investigations*, Hua 20/1: 272–329, 294, n.3. *Apud* Moran 2005, 90)

Imaginaries are improper representations, i.e., representations without object that nonetheless pass off as denoting something. Despite this apparent absurdity, imaginaries are in general harmless and often useful in mathematics. How can we explain that we can obtain knowledge by operating "blindly"[1] with symbols according to rules, even when these symbols do not represent anything? This epistemological problem presented itself very early in Husserl's philosophical career and was a dominant factor in the development of his thought. From the first to the last work he published the task of clarifying the sense and delimitating the scope of symbolization and formalization in science was one of Husserl's major concerns. Correlated problems, such as the interplay of representations without object and intuitions in the dynamics of knowledge, among others, were also prominent in his agenda. In this paper I want to show how Husserl dealt with the problem of imaginary elements and symbolic knowledge in mathematics and the central role it played in his philosophical development.[2]

[1] *Cognitio caeca* is one of the terms Leibniz—the man who brought this issue into philosophy—used for symbolic knowledge.

[2] We are talking, basically, of Husserl's philosophical development during, roughly, the last decade of the nineteenth century. The relevant textual sources are the *Philosophy of Arithmetic* (PA, 1891), the *Logical Investigations* (LI, 1900–01) and minor texts of that period published in *Husserliana* 12, 21 and 22. In this paper I will concentrate on those where Husserl's treatment of the many (basically three) versions of the problem of symbolic knowledge (concerning, respectively, symbolisms with and without interpretation, the role of imaginary elements in symbolisms with interpretation) comes out more clearly; namely, PA, LI, "Semiotic" (Hua 12, 340–373), the review of Schröder's book *Lectures on the Algebra of Logic* (Hua 22, 3–43), and the draft for the Göttingen lectures ("The Imaginary in Mathematics," Hua 12, 430–451). Other texts, such as "The Concept of General Arithmetic" (Hua 12, 375–379), "Arithmetic as an A Priori Science" (Hua 12, 380–384), a letter to Carl Stumpf (Hua 21, 244–251), "On Set Theory" (Hua 12, 385–407), and "Formal and Contentual Arithmetic" (Hua 21, 21–23), to mention a few, are either much shorter or not directly concerned with the problems discussed

II. Symbolic Knowledge

It is unquestionable that Husserl took for granted the epistemological relevance of symbolic representation. Consider the following quote of 1890:

Without the possibility of symbolic representations substituting more abstract proper representations, difficult to distinguish and handle, *or even representations that are not proper* [*my emphasis*], there wouldn't exist a higher spiritual life, and even less science. ("Semiotic," Hua12, 349)

It is that simple: no symbolic reasoning, no science, in particular, no mathematics. Husserl accorded vital importance to well-designed symbolic systems—of *calculation* as well as *derivation*, that is, *logical* systems—*even if they do not have a proper representational function*, as is paradigmatically the case of imaginary mathematical entities and purely symbolic logical systems (formal theories). However, he did not think that operating with the symbols of a system according to prescribed rules, by and in itself, constituted *knowledge*. A *calculus*, he thought, although a useful *technique*, does not necessarily produce science. According to Husserl's review of Schröder's *Lectures on the Algebra of Logic,*[3] where this distinction is introduced, a *language* is a vehicle for thought, whereas a *calculus* may not be, if it is not *logically* justified. A logical justification must *show* that the calculus in question leads to knowledge not only as a matter of *fact*, but as a matter of *right*. Husserl says that:

All the artificial operations on signs are in a way at the service of knowledge, but in fact they do not all lead to knowledge in the true and authentic sense of logical comprehension [*Einsicht*]. It is only if the process is itself a logical process, if we have logical comprehension that it must lead to truth, as it is, and because it is so, that its results are not only simply *de facto* true, but the knowledge of truth. ("Semiotic," Hua 12, 368–369)

here; they will be taken mostly as subsidiary sources of information, not as *loci classici* of Husserl's treatment of the problem of symbolic knowledge.

[3]Published originally in *Göttinger Gelehrte Anzeigen*, 1891, nº 7, 243–278; republished in Hua 22, 3–43.

Or still:

> A truly fecund formal logic is constituted first of all as a logic of signs, which, when sufficiently developed, will form one of the most important parts of logic in general (as the art of knowledge). The task of logic is here the same as anywhere: to become master of the natural procedures of the spirit that judges, to examine them, to understand the value they have for knowledge in order to assess with exactitude their limits, extent and range, and establish general rules concerning all this. ("Semiotic," Hua 12, 373)

So, this much is clear: as early as 1890 Husserl was already concerned with the task of justifying logically the purely symbolic aspects of mathematics. As a trained mathematician Husserl could not ignore the evidence that most of the *practice* and the *theory* of arithmetic rely on algorithms (for calculations) and formal systems (for theoretical development, particularly in the case of more general concepts of number, i.e., general arithmetic) and that maybe the most interesting new mathematical theories of the nineteenth century (Riemann's theory of manifolds, Hamilton's theory of quaternions, Lie's theory of transformation groups, Boole's logical calculus,[4] etc.) were purely formal. Since he was not willing

[4]Boole, whose logical calculus could be interpreted either as a calculus of classes or a calculus of propositions, was aware of the way winds were blowing in mathematics: "They, who are acquainted with the present state of the theory of Symbolical Algebra, are aware that the validity of the processes of analysis does not depend upon the interpretation of the symbols which are employed, but solely upon the laws of their combination. Every system of interpretation which does not affect the truth of the relations supposed is equally admissible, and it is thus that the same process may, under one scheme of interpretation, represent the solution of a question on the property of numbers, under another, that of a geometrical problem, and under a third, that of a problem of dynamics of optics" (Boole, in Bochenski 1970, §38.17). Although Brentano was an influence to be reckoned with, I believe Husserl's realization that purely symbolic reasoning has an *essential* role in knowledge is mainly due to his mathematical training and the awareness of the logical relevance of symbolization that came from reading Boole and Schröder, among other formal logicians of the time—it is clear, in Husserl's long review of Schröder's book, that he knew well Boole's calculus, maybe as "beautifully explained by Venn" (Schröder's review, p. 40). Of course, it was Leibniz (who was also well aware of the fact that some symbolic systems admit different interpretations, and that in this resides their main interest) the first to bring to philosophical attention the fact (and the problem) of symbolic knowledge; Boole, Schröder and Frege were Leibniz's natural heirs. I believe, however (hence the title of this paper) that Husserl took this problem a

to discredit these theories as mere playing with symbols or sterile formal exercises (as Kantians considered the non-Euclidian geometries)—quite the opposite—, Husserl, as the philosopher he had become, could not avoid taking the burden of giving mathematical symbolic knowledge, particularly if imaginaries were involved, a proper logical justification.

III. Meaningful Symbols in *PA*

Husserl's first extensive treatment of the logical problems posed by symbolic knowledge appeared in *PA*. There are in fact two variants of this problem there, one concerning the justification of the usual algorithms for carrying arithmetical computations, the other with treating the symbols 0 and 1 as numerical symbols proper alongside the numerals 2, 3, etc. One has to do with "blind" manipulations of meaningful symbols; the other with the use of meaningless symbols as if they had a meaning. Husserl treated these problems differently.

The algorithmic manipulation of numerals in the usual arithmetical operations is certainly not presided by accompanying intuitions; nonetheless, the numerical symbols involved (excluding 0 and 1) have a denotation and, moreover, the symbolic system constituted by numerals and symbolic operations is an *isomorphic* copy of the system of number concepts and conceptual operations (in Husserl's terminology they are *equiform*[5]). In fact, as Husserl showed in the second part of *PA*, the existence of such an isomorphism is the reason why purely *symbolic* manipulations can produce *true* statements, thus justifying these operations logically.

A closer inspection of the situation, however, reveals an important fact that Husserl did not emphasize, but that is central for a satisfactory explanation of the usefulness of imaginaries in mathematics: we can obtain information about a domain by handling an isomorphic copy of it only because all we want to be informed about the domain of interest has to do

step further by considering *purely formal*, non-*interpreted* systems; not only interpreted ones, as many of his antecessors, including Leibniz, did.

[5]It is worth noticing that the notion of isomorphism, as we have it today, had not yet by then come out clearly and distinctively in mathematics, although it was already operational, as the work of Dedekind, for instance, testifies.

with what this domain and its isomorphic copy have in common, which is *formal structure*. We can obtain arithmetical *knowledge* by playing with *proper* (i.e., denoting) arithmetical symbols algorithmically only because arithmetical truths are *formal*, i.e., they do not concern numbers strictly, but relations among and properties of whatever objects *behave like numbers*. Being formal, arithmetical truths can be obtained by investigating no matter which domain formally identical with the domain of numbers, the system of numerals and purely symbolic operations with numerals in particular.

Nonetheless, for Husserl, the system of arithmetical truths, despite their being *formal* is articulated internally by a unifying concept—the concept of number as a collection of units upon which we can operate (by inserting or removing units). Although formal, Husserl thought, arithmetical truths referred to *numbers*, not arbitrary number-like entities. According to Husserl, the fact that a system of formal truths—an articulated body of *formal knowledge*, as I call it—must refer ultimately to a *possible* system of objects unified under a concept whose formal properties the system of truths expresses, has to do with his persistent concern that symbolic systems must be safeguarded from degenerating into dead and dry formalism alien to knowledge, i.e., mere technicalities alienated from our *living experiences* (*Erlebnißen*) and the *Life-World* (*Lebenswelt*), as he would say later.[6] In *PA* and elsewhere, for him, arithmetic was the science of numbers *as particularizations of the concept of quantity*; a science whose *internal* unity is provided by the concept of number.[7] The unity of a formal domain (i.e., the form of an objectual domain ruled by a theory of a certain form) simply conceived by an act of formal imagination, as we might say, on the other hand, Husserl thinks, is only the *external* unity of a system whose elements hang together only in virtue of extrinsic formal relations.

[6]His critique of technization in *Crisis* spares symbolic mathematics for, as André de Muralt says: "[a]lthough mathematics is a symbolic knowledge, it can nevertheless be applied and is therefore not technized on its own account. Its original sense is therefore a logical sense" (1974, 183).

[7]Even though, let us keep this in mind, the insights we obtain by inquiring this concept cannot distinguish between numbers proper and anything that just look like numbers, that is, any domain equiform with (isomorphic to) the domain of numbers.

IV. Meaningless Symbols in *PA*

This leads us to 0 and 1, which, according to Husserl, belong to arithmetic only on the basis of their external or purely formal relations to numbers proper. According to Husserl, the symbols 0 and 1 do not denote numbers as he understood them, but merely intentional number-like entities, to use a later terminology. But the inclusion of these entities in the numerical domain was not adequately justified in *PA*, and this problem became dramatically serious when Husserl took upon him the task of providing logical foundation for more general systems of arithmetic (for the never published second volume of *PA*).

This is how Husserl handled the problem of justifying treating 0 and 1 as numbers. Firstly, he acknowledged the obvious fact: we *need* 0 and 1. He says: "We would be quickly led into embarrassing complications and even serious inconveniences in the theory of numbers if we wanted to keep 0 and 1 apart from numbers proper and renounce to give these two kinds a common denomination" (PA, Hua 12, 145).

Secondly, Husserl noticed that 0 and 1 are common products of the computation with numbers proper—we meet them often enough in our numerical calculations. But if we were to dismiss these results as nonsense, the algorithms would become worthless. He says: "If we consider that a uniform operational activity according to rules is not possible unless all imaginable results of an operation can be treated formally in the same way, it becomes clear why this enlargement of the domain of calculation [*via the introduction of 0 and 1 and all other number concepts, negative, rational, irrational and complex—my note*] was indeed an important progress towards the establishment of arithmetic" (ibid., 146–147).

In short, our usual numerical algorithms cannot do without 0 and 1; these imaginaries are, so to speak, engendered by the algorithms, which cannot work without them (a fact already appreciated by sixteenth century algebraists with respect to negative and complex numbers). The introduction of 0 and 1 in arithmetic on a par with the other numbers "makes it possible an arithmetical algorithm, i.e., a system of formal rules by means of which we can operate in a purely mechanical way in order to solve numerical problems, that is, to find unknown numbers from known numbers and relations among them" (ibid. p. 145).

The only reasons Husserl gives for accepting 0 and 1 as numbers (namely: (1) arithmetical operations among numbers proper produce them naturally—it is as if they were *required* as *necessary* completions of the arithmetical domain—and (2), the algorithms for solving numerical problems are worthless without them) are clearly not satisfactory answers from the perspective of a *logical* justification; obviously, Husserl could not accept this lame justification as the final word on the matter—after all, in strictly analogous way all number-like entities that Husserl called imaginaries could also be justified. So, what I call *the problem of symbolic knowledge*[8] had already stuck out his head in *PA*, but it remained unsolved therein, at least as far as imaginary elements were concerned. The problem only got worse when a logical justification for *general* arithmetic was required.

V. Logical Systems

Husserl's treatment of *interpreted* systems of derivation is similar to his approach to interpreted systems of calculation. The requirements for a logical justification of interpreted axiomatic theories, i.e., theories whose axioms are true by virtue of some sort of intuition into what gives the theory its internal unity are clearly stated in *Semiotic*. According to Husserl, logically sound symbolic reasoning within interpreted systems must fulfill two conditions: (1) "the systematic forms of junction of words [*i.e., the symbolic expressions*] must reflect exactly those of thinking [*i.e., the meaningful judgments*], otherwise the former could never become habitual substitutes for the latter" and (2) "the first part of the system, which contains the premises [. . .], must manifestly determine in a purely formal manner, univocally, the part that contains the conclusion [. . .] the set of premises determine univocally the conclusion." In short, logical *languages* (not simply *calculi*) must *represent* thinking proper, that is, the formal expressions must stand for meaningful judgments and the

[8]How can we *know* anything by simply "playing" with symbols according to prefixed rules?

formal machinery for drawing conclusions must produce logically sound inferences.[9]

The logical justification of computational algorithms involving *interpreted* symbols depends also on the existence of a representational relation based on strict formal identity, as is the case of arithmetical algorithms. Consider the following quote from another text of 1891 ("On the Concept of Operation," Hua 12, 408–429):

> Any algorithm first establishes a rigorous parallel correspondence between fundamental concepts, fundamental judgments and fundamental chains of reasoning and algorithmic elements. In fact, the objects of the domain, which are represented in an indeterminate manner, are replaced by simple signs; composites of objects, by composites of signs, established by means of signs of operation that correspond to the different concepts of operation; the relations, by signs of relation. Moreover, the fundamental propositions [are replaced] by symbolic conventions telling which are the permitted symbolic modifications (to the extent that they correspond to true judgments) and which are not. Concomitantly, conventional meanings are given to the symbols; hence, algorithmic concepts are one-to-one coordinated with the original concepts. (Hua 12, 418)

But a proper *logical* justification for *non-interpreted* symbolic axiomatic systems and *non-denoting* symbols cannot follow along similar lines. For in such cases, of course, we cannot speak of a parallelism between representations and represented, since non-denoting symbols do not represent. In other texts of the same and later periods, Husserl made it clear how he thought purely symbolic systems of calculation and derivation could be justified. In few words: as far as *determinate* objectual domains were concerned (for instance, the domain of numbers as answers to the question "how many?"), purely symbolic reasoning was acceptable, if useful, provided it were essentially unnecessary. This, for instance, was the justification for the use of imaginaries he presented in the Göttingen lectures of 1901. The problem with this approach, however (and in this resides

[9]It is already clearly discernible in this passage the tasks that Husserl will in later works impose upon formal logic, to elaborate a logical grammar so as to guarantee that the formulas of a logical language be meaningful—and consequently denote sates-of-affairs—and a theory of deduction so as to guarantee that formal derivations preserve truth. In fact, according to D. Willard (Husserl 1994, XIV) "Semiotic" was "apparently Husserl's first systematic effort towards a 'logic,' in his special sense, for symbolic calculation."

my criticism of Husserl's handling of the problem of imaginaries) is that it falls short of Husserl's own requirements for a *logical* justification of symbolic mathematics, since it does *not* explain *why* imaginaries are useful (*why* their manipulation leads to truth); it only tells us how they can be rendered harmless.

VI. Imaginary Elements: Earlier Treatment

In his 1891 review of Schröder's *Lectures on the Algebra of Logic* we can already see a hint of the mature treatment of imaginaries presented in Göttingen ten year later. Schröder, in the tradition of Boole, presented in this book a calculus that can be interpreted either as a logic of deduction, if the symbols are interpreted as denoting extensions of concepts, or a calculus of classes, if the symbols are interpreted as arbitrary collections of objects.[10] For Husserl, however, Schröder's algebra of logic is nothing but a *calculus*, not *logic* properly speaking; a "technique of the consequence," as he says, rather than the *science* of deduction. Interestingly, he compares Schröder's calculus with the *arithmetica universalis*, the theory of the most general concept of number whose logical justification was by then an important focus of interest for him and points out that neither is yet logically justified. As a logical calculus of *extensions*, Husserl thinks, Schröder's calculus cannot qualify as a pure theory of deduction, for deductions involve *concepts*, and extensions of concepts cannot determine their concepts, a task that only their *contents*, or contents of concepts that are materially equivalent to them, can accomplish. So, says Husserl, "[. . .] the ideal of a 'logic of extension', i.e., a logic that considers by principle *only* the extension of concepts is without value because it is without object" (Hua 22, 16).

Husserl thinks that in order to be logically justified, as we have already seen, a calculus needs to be adequately correlated with reasoning proper so as to be able to serve as a *substitute* for it. He says:

> [. . .] the proper task of a calculus is to be, for an entire domain of knowledge, a method of symbolic deduction of consequences; hence, an art for substituting, by means of an

[10] Husserl observes correctly (Hua 22, 42–43) that Schröder's calculus has little value interpreted as a logic of deduction, for the domain of deduction it formalizes is very restrict, whereas, as a calculus of classes, it can have many applications in mathematics.

appropriate designation of ideas, a calculus for effective deductions, that is, a conversion and a substitution according to rules of signs by signs and then, by virtue of the correspondence between signs and ideas, for obtaining from the final formulas the desired judgments. (ibid, 21)

The symbols and symbolic transformations of what Schröder presents as a logical calculus, Husserl thinks, cannot be adequately correlated with concepts and thinking proper; so, it cannot be a logically justified *logical* calculus (although it could be a logically justified calculus of classes). This is the same strategy of justification that Husserl employed for arithmetical algorithms in *PA*. But there are some novelties with respect to 0 and 1.

Schröder introduces 0 and 1 in a purely formal way: 0 as the class that can be subsumed under any class, 1 as the class that subsumes any other class. Husserl cannot, however, accept these purely formal definitions, for besides avoiding contradiction (*Widerspruch*), he says, a calculus must also avoid *conflict* (*Widerstreit*). That is, it must avoid opening the doors to imaginary entities, i.e., objects that do not exist but are treated, even in deductions, as if they did.[11] This is paradigmatically the case of 0. Husserl just cannot conceive of an empty extension; the idea of a class that is contained in any other class, he thinks, is absurd, for there are, after all, disjoint classes; 0 does not denote anything, which puts it on an equal footing with $\sqrt{-1}$ in general arithmetic.

There are, Husserl says, only two ways of accepting the symbol 0 as introduced by Schröder: (1) in the logical calculus, i.e., the calculus of classes as extensions of concepts, by giving it a meaning as the extension of the concept of non-existence, (2) in the calculus of identity, i.e., Schröder's calculus of classes in general, not only extensions of concepts, by treating it as a meaningless, perhaps useful, but essentially eliminable symbol (just like $\sqrt{-1}$). He says:

[. . .] this "creative" definition of 0 does not give it yet the right to exist in the system of the calculus [. . .]—however, is there anything that can give it such a right? *Of such a thing I cannot find the shadow of a proof. The 0 of the calculus of identity presents the same problem of* $\sqrt{-1}$ *in the arithmetical calculus* [*my emphasis*]. In one as in the other

[11] "A geometry is still geometry if after having defined square circles it uses them in deductions?" (Hua 22, 31) A *conflict* is an incongruity between a symbolic system and its *intended* objectual domain.

case, we can only give the correspondent proof by considering the corresponding algorithmic technique. Here, it would be necessary to show that any relation deduced with the help of 0, which involves moreover only symbols that are real [*i.e., meaningful, denotative symbol—my note*], must be a valid relation according exclusively to the meaning of these symbols and the laws that concern them. *Creative definitions do not contribute with anything, even if they preserve the internal consistency of the calculus. The question is not whether the calculus remains consistent, but whether it remains a calculus of classes* [*my emphasis*]. (Hua 22, 33)

Husserl insists on this, if a calculus is logically justified, it then "stands for" something (even if it can stand for different things), its basic principles and transformation rules are founded on the meaning of what it stands for, and the introduction of meaningless symbols in it can only be justified if these symbols—no matter how useful they are from a purely algorithmic perspective—are in the end unnecessary as far as the application of the calculus to its *intended* domain is concerned, despite the fact that their incorporation does not generate formal inconsistencies.

It is clear that by the time Husserl published *Philosophy of Arithmetic* and wrote the review of Schröder's book he did not see a calculus as a free creation; calculi had only a surrogatory function and could not stand by themselves. It is curious that essentially the same arguments appeared in Husserl's conferences of 1901 in Göttingen about imaginary entities in mathematics, when he was already in possession of much more sophisticated views concerning the epistemological value of purely formal symbolic systems per se. (According to these views, as we will see below, such systems provide us with formal knowledge, i.e., knowledge of formal manifolds or structures regardless of their eventual material fillings.)

In the review of Schröder as in the Göttingen talks, however, Husserl is *not* dealing with purely formal symbolic systems *for their own sake*, but with systems that have (in the case of symbolic arithmetic) or are presented as having (Schröder's calculus) *intended interpretations*. In such cases, Husserl thinks, the systems cannot conflict with their *intended* domains,[12] admitting by purely formal means the adjunction of entities that just cannot belong to these domains.

[12] I will take this opportunity to say that one of my main criticisms of Husserl's philosophy of mathematics is that it does not take into consideration the fact that a formal system, even when built on the intuitive apprehension of truths about a *determinate* objective domain, or the concept of which this domain is the extension, *is never only a theory of this domain*, even if it is categorical (for categoricity guarantees only the uniqueness of the *structure* of the domain, not its material content).

VII. Imaginary Elements: Later Treatment

I have already discussed in details Husserl's treatment of imaginaries in the Göttingen talks in other papers (and I do not have any reason to change my views);[13] so, I will be brief. Basically, he said to his audience in Göttingen in 1901 that the introduction of imaginary elements in a domain is allowed provided: (1) the (formal) theory extending the (formal) theory of the domain in question (obtained by formal abstraction from the contentual theory of this domain, i.e., the theory of which the domain is the intended model) by means of formal axioms introducing imaginaries in an extended language is consistent, and (2) the formal theory of the domain, written in the restrict language without imaginaries, is complete (Husserl's term is "*definite*").

If we compare this solution with the then ten-year old solution presented in the review of Schröder, one, and only one difference is noticeable. Whereas in the Schröder review Husserl says what amounts to saying that the extended theory must be *conservative* with respect to the narrower theory (a fact he confesses to be unable to prove), in the Göttingen talks he required the narrower theory to be *complete*—this, of course, implies the conservativeness of the extended theory, provided it is a consistent extension—(a fact that now he thinks to know how to prove, as far as arithmetic is concerned). Indeed, Husserl claimed in the talks that the arithmetic of the real numbers was complete and the sketches of what he took to be a proof of the completeness of different systems of arithmetic can be found in his notes for the talk (Hua 12, 442–443).[14]

In 1901 Husserl had already written the *Prolegomena*, so, we must compare the solution for the problem of imaginaries given in Göttingen with the views on the nature and role of formal mathematics presented in that work.

[13] See da Silva 2000a, 2000b and 2000c.

[14] He reasons thus: numerical equalities and inequalities are decidable on the basis of the axioms; algebraic equalities and inequalities and general assertions are decidable because their numerical instances are decidable (needless to remark that what Husserl understands by "decidable" has nothing to do with our notion of syntactic—or even semantic—decidability).

VIII. Formal Ontology

The scientific usefulness of symbolic reasoning is still fully appreciated in *LI*. According to Husserl:

> The solution of problems raised within a theoretical discipline, or one of its theories, can at times derive the most effective methodological help from recourse to the categorial type or (what is the same) to the form of the theory, and perhaps also by going over to a more comprehensive form or class of forms and to its laws. (Prolegomena, § 70)

That is, problems raised within a theoretical science—structured as an *interpreted* symbolic theory—can be solved by resorting to its formal *abstractum* (the formal theory obtained by divesting the original theory of any intended reference) or even to formal extensions of it. This obviously includes problems like, say, finding adequate formal procedures for solving arithmetical problems. As Husserl knew very well, the adequate solution of this problem required the extension of the formal manifold determined by the arithmetic of numbers seen as specifications of the notion of quantity to the manifold of complex numbers.[15] Again, formal mathematics can provide useful *techniques*. But can symbolic formal theories, *even if not given any interpretations*, lead to *knowledge*?

Clearly, in *LI* Husserl thought they could. Formal mathematics, he thought, is a province of formal logic, being pre-occupied with (logical) forms *independently of their eventual material fulfillments*. Formal mathematics studies formal manifolds, which are domains of objects determined only with respect to form, regardless of the particular nature of the objects they may contain, the properties these objects may have or the relations they may entertain among themselves (in short, mathematical structures in the sense of modern or abstract algebra; to use

[15] Another example may be this: the famous problems of ancient Greek geometry, the squaring of the circle, the trisection of the angle and the duplication of the cube could only be adequately dealt with—if not solved, at least shown to be unsolvable as stated—by going through an elaborate *algebraic*, i.e., formal analysis of geometrical constructions with straight edge and compass and what they can accomplish. This analysis interprets geometrical constructions formally in terms of algebraic field extensions.

a metaphor Berkeley made famous in another context, a formal manifold is the ghost of a departed objectual domain).[16]

It is an aspect of the pure theory of manifolds to study the interrelations of formal theories (or, equivalently, the manifolds they determine).[17] The key to understand *why*, say, complex numbers are useful for the theory of real numbers (for instance, in the theory of algebraic equations over the field of real numbers) lies in the *formal* relations between the *forms* of both real and complex number fields. Husserl seems to be suggesting this in the following quote:

> Not less important than [. . .] going back to pure form is the closely related ranging of each [. . .] form in more comprehensive forms or classes of forms. That we have here a central item in the wonderful, methodological art of mathematics, becomes plain if we look [. . .] at the first, simplest case of this sort, the extension of the field of real numbers (i.e., of the corresponding form of theory, the "formal theory of real numbers") into the formal, two dimensional field of ordinary complex numbers. *In this concept we indeed have the key to the only possible solution of the problem that has not yet been cleared up: how, e.g., in the field of numbers impossible (essenceless) concepts can be methodologically treated like real ones* [*my emphasis*]. This is not, however, the place to discuss this more closely. (Prolegomena, § 70)

Since the Göttingen talks of 1901 were given *after* Husserl had written and published the *Prolegomena*, it is safe to assume that the solution of the problem of imaginaries he presented there is the solution he

[16]Similar ideas were voiced in the sketches for the Göttingen talks: "[M]athematics in the highest and most general sense is the science of theoretical systems in general, abstracting the objects of theoretical interest of the given theories of different sciences; in no matter which given theory, in no matter which given deductive system, we abstract its subject matter, the particular types of objects it tried to theoretically master, and if we substitute the representations of objects materially determinate by simple formulas, that is, the representation of objects in general that is mastered by such a theory, by a theory of this form, we have then accomplished a generalization that considers the given theories as particular cases of a class of theories, or rather of a form of theory that we consider in a unifying manner and in virtue of which we can say that all these particular scientific domains have, as form is concerned, the same theory" (Hua 12, 430–431). "Mathematics is then, according to its highest ideal, a doctrine of theories, the most general science of deductive systems that are possible in general" (Hua 12, 432).

[17]Husserl was very consistent in his characterization of the doctrine of multiplicities: it is a science of forms of theories. See, for instance, *Einleitung in die Logik und Erkenntnistheorie* (Hua 24, §19, p.79), a work written 10 years after the *Prolegomena*.

alludes to in the quote above. According to the talks, imaginaries can be "methodologically treated" as real numbers because *methodologically* both can be treated as symbols subjected only to formal relations. But a good method is not yet a logically justified method.

In the talks Husserl is clearly making the logical justification of imaginaries rest on a *proof* of their dispensability by means of a *proof* of the completeness of the system to which they are added (since this narrower system has an intended model that must be completely mastered by the intuitive apprehension of its fundamental concept). As long as we consider the arithmetic of the real numbers as a theory in the pregnant sense, i.e., as founded on the concept of real number as, say, Cauchy sequences of rational numbers or Dedekind cuts, and its intuitive truths, then complex numbers had to be rendered dispensable by a *proof* of the conservativeness of the formal theory of complex numbers with respect to the formal theory of real numbers (in fact, Husserl says we have to prove the *completeness* of the narrower theory, but this would imply the conservativeness of any consistent extension of it, expressed in a language extending that of the narrower theory). I think this was the sort of task Husserl envisaged for the metatheory of formal systems that he located in the third level of apophantic formal logic.

The solution presented in the Göttingen talks for the problem of imaginaries tells that, from a formal perspective, imaginary entities can be treated like real ones, and establishes the *logical conditions* under which treating them so can be allowed, if useful, *considering conceptual knowledge exclusively*. Husserl insists that as long as we are interested in knowing the properties (even only the formal properties) of the *concept* that founds a theory (for instance, the properties of numbers as *numbers strictu sensu*), the use of imaginaries cannot be an essential one.

Although, as Husserl says in *LI*, purely symbolic theories are per se a form of knowledge, structural or formal knowledge precisely—they provide knowledge of formal manifolds independently of their interpretations, thus belonging to formal ontology—, they must be teleologically oriented towards objectual domains.[18] According to Husserl, formal

[18]A sign of this orientation is that Husserl sees even formal theories as referring to objects, *formal objects* precisely, indeterminate as to content, but determinate as to form by their theory.

theories are mere *forms of theories*, not theories in a pregnant sense (the chosen terminology is revealing, a "pregnant" theory is obviously a non-sterile theory); formal theories are like wandering spirits in search of a body to snatch to. For him, the creation and study of formal theories for their own sake, independently of intended applications, amounts to toying with what we can call "formalist alienation" (he will later attribute to its excessive technization—of which its mathematization is an aspect—the "crisis" of European science). As formal theories *of objectual domains*, on the other hand, formal theories can be enlarged to serve *methodological* purposes (whose logical justification depends on the completeness of the restrict theory).

If we admitted that a consistent formal theory always describes the formal aspects of an existing objectual manifold, as is the case of elementary, that is, first-order theories, Husserl's cautions would seem vacuous. But if we assumed that he did *not* take for granted that all *consistent* formal theories have models, his considerations would acquire some relevance, for what would be the point of investigating abstract formal manifolds that could not be given any objectual content, the only support that could subtract them from the world of fantasy in which they live?

But it may be the case that Husserl is saying something more prosaic, that formal theories are only interesting if they can be applied, or, in other words, that formal mathematics must keep its pragmatic sensibility alert so as to avoid indulging in sterile investigations. At first sight, this seems an honest and well-intended scruple. But the history of mathematics shows that hardly any formal mathematical device or theory has been created that did not prove its usefulness, the impossible and absurd numbers of the Italian algebraists of the Renaissance, the points at the infinite of Kepler, all the variants of the old Greek geometry, infinitesimals, quaternions, non-commutative algebras, etc. What makes Husserl's scruple undesirable is that we do not and can not know a priori when a mathematical purely "fictional" creation will prove its applicability, in mathematics itself, physics or any other field of knowledge. So, the best strategy is tolerance; to grant mathematicians their freedom and wait for the survival of the fittest (the probable reason we do not know many pure game-like formal theories is because the mathematical community do not take them seriously and they just vanish from sight—the community in the end takes care that mathematicians do not indulge in formalism for its own sake, alienated from our fundamental cognitive interests).

Being trained as a mathematician at the end of the nineteenth century made Husserl naturally suspicious of the possible excesses of the formalistic turn this science was undergoing (the proximity and influence of Kronecker may also have played a role). But he also saw the immense possibilities of purely formal mathematics; so, he endorsed it and gave it epistemological dignity (formal mathematics is a chapter of formal ontology), but with a note of caution (formal theories must be applicable).

Objectual domains of mathematics are in general infinite, so we cannot expect to have an intuitive access to its objects directly, or by inductive means, but only through a concept which unifies this domain as the extension of this concept.[19] In this case, we can intuitively access the infinite domain by intuitively accessing its concept. It may be the case that the formal theory of such a domain is already at our disposal (being previously or independently developed in the realms of formal ontology), in which case the formal theory finds its *raison-d'être* (for instance, the formal theory of complex numbers as the theory of the two-dimensional domain of displacements and their operations).

But consider that we have an objectual domain given from the start as the extension of a concept, such as, for instance, the domain of cardinal numbers as answers to "how many?" Do we have the right to use formal extensions of its formal theory, which may have completely unrelated models, in order to obtain knowledge, *even of a formal nature*, of such a domain, whose truths must be exclusively derived from the intuition of its ruling concept? Husserl thought that we have not, and that

[19]Husserl believed that any a priori contentual axiomatic theory is necessarily a *conceptual* theory, that is, the theory of a concept under which the objects of the relevant domain were assembled. In a text of 1891 ("Arithmetic as an A Priori Science," Hua 12, 380–384), inquiring on the nature of arithmetic as an a priori science, Husserl says that a science of such a nature "does not begin with single facts for then to obtain possibly true generalities by induction, but immediately by certain generalities that are apodictically certain and immediately evident, which it acquires by simply presenting to itself certain 'fundamental concepts' that give, by means of mediate evidence and certitude, all the sequence of theorems of this science."

the interference of formal manipulations could only be tolerated under the presupposition that conceptual intuition was in principle capable of providing the contentual theory of the domain with a *complete* set of basic principles and laws.

This was the solution Husserl presented in Göttingen for the problem of imaginary elements in mathematics. It is worth noticing that if indeed Husserl believed that even consistent formal theories may not have models (or, at least, *interesting* models), the attribution of a *pragmatic* role for formal theories *in general* would vindicate even those that cannot be given any (interesting) objectual domain, thus safeguarding mathematical methodology (of inventing vacuous symbols for solving problems) from falling into sheer nonsense (or formalist alienation). It is obvious that Husserl did not want to give up even the riskier formal procedures of mathematics, but it is also obvious that he was not willing to let the formalistic approach to mathematics be interpreted in a way that would alienate mathematics from a firm compromise with knowledge.

IX. Critical Considerations

I would like to conclude with some final comments on the *correctness* of Husserl's vindication of symbolic knowledge, in particular his treatment of imaginaries, and the role these questions played in the development of his philosophy. With respect to Husserl's cautious treatment of purely symbolic knowledge, I have already expressed my reserves above. We must let mathematicians do their work; no matter how inapplicable a formal theory may be, if it is consistent, it is the theory of a mathematical *structure* and time will decide if it is sufficiently interesting to survive. With respect to imaginaries, I have more serious concerns. I believe that Husserl was so worried about securing mathematics against a possible infection with imaginaries that he put more efforts into developing a protective vaccine than explaining *why* imaginaries are useful when they are. The vaccine, of course, was *completeness*, inoculated via *conceptual intuition*. Husserl believed that a priori mathematical contentual theories are in general *conceptual* theories and that imaginaries cannot substitute proper relevant intuition and be *essentially* involved in the business of proving theorems.

But the fact is that, contentual or purely formal, mathematical theories are invariably theories of *forms*, not *definite* and *singular* objectual domains, even when these are *categorical* theories.[20] No matter how clearly we intuit the concept of a non-negative integer, for instance, all we get in the process are formal or structural properties of ω-sequences. The intuitively apprehended fundamental facts about numbers, which constitute the axiomatic basis of arithmetic, are also true of no matter which domain of objects that is structured as an ω-sequence, regardless of the nature of these objects. So, there is not much difference between contentual or conceptual, on the one hand, and purely symbolic mathematical theories, on the other; theories of both types are in a sense formal, since their objects are only and invariably *forms* or *structures*.

The fact that even contentual mathematics is a formal science reduces Husserl's distinction between theories and mere forms of theories to, respectively, conceptual or eidetic formal theories (such as arithmetic and physical geometry) and hypothetical formal theories (such as Riemannian n-dimensional geometries). The reason Husserl insisted on keeping both types apart has to do with epistemological relevance: the former are already theories of something, the latter only describe possible hypothetical forms waiting for objectual domains to appear that can be in-formed by them. But this seems to me excessive caution, since hypothetical formal theories, far from mere symbolic games, also provide knowledge—formal knowledge, as Husserl himself acknowledged.

And it is precisely because of this that imaginaries can be useful. Despite the fact, for instance, that complex numbers do not measure quantity, introducing them in the domain of numbers and extending the arithmetic of non-negative integers to the arithmetic of complex numbers amounts to imbedding the original structure of arithmetic into a richer structural milieu in which problems originally posed for the restrict numerical domain can be adequately handled—for they may very well

[20]In a letter to Frege, summarized and commented by Husserl (Hua 12, 447–451) Hilbert says something relevant in this context: "Any theory can be applied to an infinite number of systems of fundamental elements. It suffices to apply a one to one reversible transformation and stipulate that the corresponding axioms for the thing thus transformed are the same (this is the case, for instance, with the principle of duality and my proofs of independence)" (Hua 12, 450).

be problems about non-negative integers *as complex numbers*, i.e., problems that demand the larger structure in order to be adequately stated and treated.

It is because arithmetic, even though founded on a conceptual intuition, is in a sense a formal theory—i.e., it does not tell us more about numbers proper than the formal theory abstracted from it tells us about number-like entities—that imaginaries can be added to its domain and be useful in solving numerical problems: imaginaries are after all number-like entities. The process of formal abstraction then amounts to no more than just forgetting we have an intended domain for a theory, since we cannot fix this domain as the *only* single model of the theory anyway. If we let the theory "talk" about structural aspects of any system of objects that satisfies its axioms, we can approach the phenomenon of imaginaries from the perspective of the interrelations between a given structure and structures extending it.[21]

Let us insist a bit more on this matter. If you look at a painting from the Romanic or Gothic periods the elements of the picture do not really fall together in the same space. The perspective invented by Brunelleschi solved this problem by introducing in the canvas an imaginary point that is not really there, but that organizes the whole into an articulated unity; a formal substitute of the eye in function of which all space relations are determined. The point at the infinite is not a point of the visual space, but helps to organize it. Imaginary elements work in analogous manner.

Pictorial perspective influenced some mathematical developments. The geometries of Kepler and Desargues were great improvements over Greek geometry—whose finitist sensibility could not conceive of points at the infinite—because they knew how to take advantage of the increased *formal possibilities* of an enlarged space which was not only potentially infinite, but actually had points at the infinite. In particular this made possible a treatment of conics much more elegant than Apollonius'.

The answer to the riddle posed by imaginary elements lies on this observation: like the points at the infinite of Kepler and Desargues, imaginary elements increase formal possibilities. Or still, they enrich structure.

[21] It is interesting to notice that the interplay of formal domains is, according to Husserl, a topic of study of formal ontology.

But if this is all they do, their utility can *only* be explained if the domains to which they are added interested us only insofar as their structure was concerned, which would not suffer from the substitution of the elements of the domain by others of a totally different nature. Otherwise the utility of imaginary elements would remain a mystery.

Imaginary elements such as $\sqrt{-1}$, introduced against good sense as *bona fide* numbers by the Italian algebraists of the sixteenth century were useful because they added necessary structure to the original domain of numbers to which they were added. Arbitrary algebraic operations on the system of proper numerical symbols quickly lead outside this domain; imaginary elements added necessary extra room to it so as to allow a bigger range of transformations. Analogously, if we tried to transform a right hand glove into a left hand glove using only rotations and translations we would necessarily fail, but if we were allowed to use inversions, we would easily succeeded. By so doing we increase the formal possibilities at our disposal. Since the original problem was essentially formal, it benefited from this enrichment. It is always thus with imaginary elements, they work because they enrich structure, and the problems they help to solve are structural problems.

The problem of finding algorithmic procedures for solving numerical problems is analogous to the problem of transforming a right-hand into a left-hand glove, as long as we allow only operations that are confined to the realm of positive integers (addition, multiplication and exponentiation) it is often difficult and in general impossible. But as long as we admit inversions of operations (subtraction, division and root extraction) we immediately escape that confined space and consequently allow imaginaries to come in. But this solves the problem, which was after all a purely formal one, at first unnecessarily restricted to a domain where it could not be adequately handled.

X. The Problem of Symbolic Knowledge in the Development of Husserl's Philosophy

As we have seen, the problem of imaginaries was fundamental in making Husserl consider questions such as the role of representations without object in the general scheme of knowledge; the interplay between conceptual theories (those that are based on intuition) and purely formal theories (whose objects are purely intentional) and, consequently, the interplay between empty intentions and intuitions in the dynamics of human knowledge; the need to enlarge well beyond Kantian limits the field of intuition in order to account for a priori mathematical contentual theories; the *logical* relevance of studying formal theories, their properties—such as consistency and completeness, in particular—and their mutual relations, and, consequently, the need for an enlargement of the field of formal logic vis-à-vis the tradition; the need for an adequate study of logical grammar and the theory of deduction so as to guarantee epistemological relevance for manipulation of signs within logical symbolic systems; and many other along the same lines. It is clear now why, in trying to come up with a philosophical account of general arithmetic he wrote the *Logical Investigations* instead of only the second volume of the *Philosophy of Arithmetic*.

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CHAPTER VIII

MATHEMATICAL TRUTH REGAINED

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Abstract. Benacerraf's Dilemma (BD), as formulated by Paul Benacerraf in "Mathematical Truth," is about the apparent impossibility of reconciling a "standard" (i.e., classical Platonic) semantics of mathematics with a "reasonable" (i.e., causal, spatiotemporal) epistemology of cognizing true statements. In this paper I spell out a new solution to BD. I call this new solution a *positive Kantian phenomenological solution* for three reasons: (1) It accepts Benacerraf's preliminary philosophical assumptions about the nature of semantics and knowledge, as well as all the basic steps of BD, and then shows how we can, consistently with those very assumptions and premises, still reject the skeptical conclusion of BD and also adequately explain mathematical knowledge. (2) The standard semantics of mathematically necessary truth that I offer is based on Kant's philosophy of arithmetic, as interpreted by Charles Parsons and by me. (3) The reasonable epistemology of mathematical knowledge that I offer is based on the phenomenology of logical and mathematical self-evidence developed by early Husserl in *Logical Investigations* and by early Wittgenstein in *Tractatus Logico-Philosophicus*.

I who erewhile the happy garden sung,
By one man's disobedience lost, now sing
Recovered Paradise to all mankind,
By one man's firm obedience fully tried
Through all temptation, and the Tempter foiled
In all his wiles, defeated and repulsed,
And Eden raised in the waste wilderness.

–J. Milton[1]

Pure intuition as Kant understood it was evidently supposed somehow to get us across the divide between the fuzzy *Lebenswelt* with its everyday objects and the sharp, precise realm of the mathematical, in terms of which mathematical conceptions of the physical world are developed.

–C. Parsons[2]

[1] (Milton, 1953b, 495, book I, lines 1–7)
[2] (Parsons 2008, 166)

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The epistemologically pregnant sense of self-evidence (*Evidenz*) . . . gives to an intention, e.g., the intention of judgment, the absolute fullness of content, the fullness of the object itself. The object is not merely meant, but in the strictest sense *given*, and given as it is meant, and made one with our meaning-reference. . .. It is said of every percept that it grasps its object directly, or grasps this object *itself*. But this direct grasping has a different sense and character according as we are concerned with a percept in the narrower or wider sense, or according as the directly grasped object is *sensible* or *categorial*. Or otherwise put, according as it is a *real* or *ideal* object.

–E. Husserl[3]

Self-evidence (*die Einleuchten*), of which Russell has said so much, can only be discarded in logic by language itself preventing every logical mistake. That logic is a priori consists in the fact that we *cannot* think illogically.

–L. Wittgenstein[4]

I. Introduction

Benacerraf's Dilemma, or BD, as originally formulated by Paul Benacerraf in 1973 (Benacerraf 1973, 672–673), is about the apparent impossibility of reconciling a *standard, uniform* semantics of natural language with a *reasonable* epistemology of cognizing true statements, when the relevant kind of true statement to be semantically explained is mathematical truth and the relevant kind of cognition to be epistemologically explained is mathematical knowledge. A "standard, uniform" semantics in Benacerraf's terminology is a Tarskian satisfaction-theoretic and model-theoretic semantics applying across natural language as a whole. This semantics, together with some natural assumptions about standard mathematical linguistic practices, very plausibly, smoothly, and jointly yield classical Platonism about mathematics. And a "reasonable" epistemology is an epistemology that ties a human linguistic knower causally to the known objects themselves. This epistemology very plausibly and smoothly yields the *denial* of classical Platonism about mathematics. Hence BD.

[3](LI, pp. 765 and 787, texts combined)
[4](Wittgenstein, 1981, prop. 5.4731, p. 129)

In this paper I will spell out a new solution to BD. I call this new solution a *positive Kantian phenomenological solution* for three reasons:

(1) It accepts Benacerraf's preliminary philosophical assumptions about the nature of semantics and knowledge, as well as all the basic steps of BD, and then shows how we can, consistently with those very assumptions and premises, *still* reject the skeptical conclusion of BD and *also* adequately explain mathematical knowledge.
(2) The standard semantics of mathematically necessary truth that I offer is based on Kant's philosophy of arithmetic, as interpreted by Charles Parsons and by me. (See Parsons 1983; Hanna 2002, 2006a, Chapter 6.)
(3) The reasonable epistemology of mathematical knowledge that I offer is based on the phenomenology of logical and mathematical self-evidence developed by early Husserl in *Logical Investigations* and by early Wittgenstein in *Tractatus Logico-Philosophicus*.

More precisely, however, what I will argue is that we can solve BD in three stages:

First, I accept Benacerraf's preliminary philosophical intuitions about the nature of semantics and knowledge, as well as all the basic premises of BD.

Second, I hold that mathematical truth is adequately explained by accepting the following three claims:

(1) that the natural numbers are essentially positions or roles in the mathematical natural number structure provided by Peano arithmetic,
(2) that the mathematical natural number structure provided by Peano arithmetic is abstract only in the sense that it is *transcendentally ideal*, which is to say that this structure is identical to the formal structure of time insofar as we consciously represent it in sense perception, together with all the formal concepts and other logical constructions, including specific logical inference patterns such as mathematical induction, needed for an adequate rational human understanding of Peano arithmetic, and
(3) that in our actual world, the unique, intended model of the natural number structure provided by Peano arithmetic is just the set of

manifestly real directly perceivable spatiotemporal material objects—the natural inhabitants of Parsons's "fuzzy *Lebenswelt* with its everyday objects"—insofar as *they are the role players of the Peano-arithmetic-specified natural number roles* in the abstract formal structure of time as we consciously represent it in sense perception, together with all the formal concepts and other logical constructions, including specific logical inference patterns such as mathematical induction, needed for an adequate rational human understanding of Peano arithmetic.

Third, I hold that mathematical knowledge is grounded on

(1) a rational human agent's *mental-model-manipulating* abilities, which are innately specified in the agent's mind and also inherently present, as necessary ingredients, in human sense perception, and which entail her self-conscious cognition of phenomenologically self-evident formal non-conceptual structures of human sense perception, together with
(2) a rational human agent's *logic-and-language-constructing* abilities, which are innately specified in the agent's mind and also inherently present, as necessary ingredients, in human empirical conceptualizing and perceptual judgment, and which entail her self-conscious cognition of phenomenologically self-evident formal conceptual contents and specific patterns of logical inference in classical or non-classical logics.

The second and third stages of this argument respectively invoke what I call *Kantian Structuralism* about the nature of numbers and mathematical truth, and also what I call *the Husserl-Wittgenstein Theory of Logical and Mathematical Phenomenological Self-Evidence*, or the HW Theory, about the nature of logical and mathematical a priori knowledge. As the labels clearly indicate, this part of the argument has historical foundations in the work of Kant, early Husserl, and early Wittgenstein. At the same time, however, Kantian Structuralism and the HW Theory are intended to be fully rationally defensible on their own merits.

Now Milton's *Paradise Lost* and *Paradise Regained*, as I read them, are about the necessary transition from the impossibly super-human conception of moral virtue embodied in pre-lapsarian Adam and Eve, and our consequent tragic Fall and expulsion from the Garden of Eden, towards a fully realistic knowledge of our own moral limits and of our inescapably finite, mortal role on this desperately imperfect Earth. Correspondingly, the philosophical story I am telling in this paper is about the necessary philosophical transition from the impossibly superhuman conception of mathematical truth and knowledge offered by classical Platonism, and our consequent tragic Fall and collapse into BD, towards a fully realistic and also inescapably anthropocentric conception of mathematical truth and knowledge, but without either finitism or nominalism—*real mathematics for humans*. So if my argument is sound, then the result will be, in effect, a semantic and epistemic Paradise Regained—with Kantian, Husserlian, and Wittgensteinian bells on.

II. Benacerraf's Dilemma and Some Negative or Skeptical Solutions

Here is Benacerraf's own formulation of BD:

> As an account of our knowledge about medium-sized objects, in the present, this is along the right lines. [A reasonable epistemology] will involve, causally, some direct reference to the facts known, and, through that, reference to those objects themselves. . . . [C]ombining *this* view of knowledge with the "standard" view of mathematical truth makes it difficult to see how mathematical knowledge is possible. If, for example, numbers are the kinds of entities they are normally taken to be, then the connection between the truth conditions for the statements of number theory and any relevant events connected with the people who are supposed to have knowledge cannot be made out. (Benacerraf 1973, 672–673)

And here is my rational reconstruction of that argument:

(1) Natural language requires a standard, uniform semantics. (Preliminary assumption I.)
(2) A reasonable epistemology of cognizing true statements should be modeled on sense perception. (Preliminary assumption II.)
(3) Mathematical knowledge in a classical sense (i.e., as a priori knowledge) exists as a feature of standard mathematical linguistic practices, so mathematical truth in a classical sense (i.e., as necessary truth) also exists as a feature of those standard practices.

(4) Given (1) and (3), our standard, uniform semantics of natural language, as applied to mathematical truths, commits us to a truth-making ontology of abstract mathematical objects and also to the non-empirical knowability of true mathematical statements.
(5) On the one hand, given (2), that fact that a reasonable epistemology of cognizing true statements should be modeled on sense perception entails that knowledge involves causally efficacious, contact-involving or efficient, directly referential, non-inferential, and spatiotemporal relations between human linguistic knowers and the known objects themselves.
(6) But on the other hand, given (4), and since all abstract objects are causally isolated and inert, it then follows that all abstract mathematical objects are causally isolated and inert.
(7) So if we accept all of (1)–(6), then mathematical knowledge in the classical sense is both possible and impossible, which is absurd.

I will say that any proposed solution to BD is *negative* or *skeptical* if it rejects either of Benacerraf's preliminary philosophical assumptions about a standard uniform semantics and a reasonable epistemology or else rejects one or more of steps (3) to (6). Then there are at least six different categories of possible negative or skeptical solutions to BD. The first two categories I will call *pre-emptive* negative or skeptical solutions, since they consist in pre-emptively rejecting at least one of the two preliminary assumptions.

1. Pre-emptive Negative or Skeptical Solutions

(1) *Reject the preliminary assumption (I) that natural language requires a standard, uniform semantics.*

This in turn entails either

(1.1) rejecting Tarskian semantics or
(1.2) accepting a multiform semantics of natural language.

(2) *Reject the preliminary assumption (II) that a reasonable epistemology of cognizing true statements should be modeled on sense perception.* (See, e.g., Katz 1995)

This in turn entails either

(2.1) modelling the epistemology of cognizing true statements on conceptual reasoning or concept-possession (see, e.g., Hale and Wright 2002),
(2.2) modelling the epistemology of cognizing true statements on self-consciousness,[5] or
(2.3) modelling the epistemology of cognizing true statements on the imagination.[6]

The other four categories I will call *concessive* negative or skeptical solutions, since they involve conceding both of the preliminary assumptions I and II, and then rejecting at least one of the other steps leading to the unacceptable conclusion.

2. Concessive Negative or Skeptical Solutions

(3) *Reject the classical necessity or apriority of mathematical truth.*

This entails accepting either

(3.1) the contingency of mathematical truth, or
(3.2) the aposteriority of mathematical truth.

[5]In (Hanna, 2006a Chapters 6 and 7), I work out Kant's idea that mathematical knowledge is grounded on reflective self-consciousness together with the imagination.
[6]One way of doing this would be via "plenitudinous platonism": For every consistently imaginable mathematical statement, there is a corresponding mathematical object. (See, e.g., Balaguer, 1998.) This construes imaginability as conceivability. But there are other ways of thinking about the imagination, e.g., Kant's conception of the productive imagination as a "schematizing" (i.e., mental modeling) capacity (Kant 1997, A84–147/B116–187, and esp. A120 n.). In (Hanna, 2006b, Chapter 6), I extend BD to logical knowledge, and then develop a strategy for solving the extended BD that starts with the thesis that a reasonable epistemology should be modeled on the imagination, not on perception. So by the classification scheme described here, strictly speaking, that earlier solution counts as a pre-emptive negative or skeptical solution. But to the extent that the present solution postulates the innate presence of mental modeling abilities in sense perception, it also postulates the innate presence of the capacity for *imagination* within the capacity for sense perception. So in that sense, the present positive or anti-skeptical solution is really only an extension and refinement of the earlier solution.

(4) *Reject the truth-making ontology of abstract mathematical objects.* (See, e.g., Shapiro 2000, Chapters 6, 7, and 9.)

This in turn entails accepting either

(4.1) empirical or phenomenal idealism (whether communal or solipsist),
(4.2) intuitionism,
(4.3) formalism,
(4.4) conventionalism,
(4.5) fictionalism or some other form of nominalism, or
(4.6) non-cognitivist anti-realism.

(5) *Reject the thesis that sense perception involves causally efficacious, contact-involving or efficient, referentially direct, non-inferential, and spatiotemporal relations between human cognizers and the cognized objects.*

This in turn entails accepting either

(5.1) the replacement of causal efficacy by causal relevance,
(5.2) the counterfactual theory of causation,
(5.3) the probability-raising theory of causation,
(5.4) a non-causal theory of perception,
(5.5) an indirect causal theory of perception (whereby a perceptual subject *S* can sense perceive a universal *U* or type *T* just by standing in a direct causal sense perceptual relation to an instance of *U* or a token of *T*),
(5.6) referential descriptivism, or
(5.7) cognitive inferentialism.

(6) *Reject the thesis that abstract objects are causally isolated and inert.*

This in turn entails accepting either

(6.1) the causal relevance of abstract objects, or
(6.2) the causal efficacy of abstract objects.

Obviously, some of these negative or skeptical solutions logically entail or logically exclude others. But at the same time many of the negative or skeptical solutions are also consistent with others, which gives rise to a large number of distinct possible *combined* negative or skeptical solutions.

This in turn makes the strategy of proving the rational superiority of one or another of the given negative or skeptical solutions by attacking all the other possible negative or skeptical solutions somewhat strenuous, and possibly even unfeasible, given the usual limits on human time, energy, and patience.

In what follows in this paper, I will attempt to work out a *positive* or anti-skeptical solution to BD, but not explicitly to criticize or defeat the possible negative or skeptical solutions, which would require a separate book-length treatment on its own. As I said already, I call my solution to BD a "positive" or anti-skeptical one because it accepts Benacerraf's preliminary philosophical assumptions about the nature of semantics and knowledge, as well as all the basic premises of BD—captured in steps (1) to (6)—and then shows how we can, consistently with those very assumptions and premises, still reject the skeptical conclusion of BD—captured in step (7)—and also adequately explain mathematical knowledge. On the face of it, any positive or anti-skeptical solution should have a distinct rational edge over any negative or skeptical solution because only a positive or anti-skeptical solution will adequately preserve the rational force of all the original philosophical intuitions that generated the dilemma in the first place. If any of these intuitions did not have rational force, then BD would not be a *genuine* dilemma. So the fact that we *do* take BD seriously clearly entails that if there really *is* a positive or anti-skeptical solution, then prima facie *it* will trump any of the negative or skeptical solutions.

III. Benacerraf's Dilemma and Kantian Structuralism

> *Number* . . . is a representation that summarizes the successive addition of one homogenous unit to another. Number is therefore nothing other than the unity of the synthesis of the manifold of a homogeneous intuition in general, because I generate time itself in the apprehension of the intuition.
>
> –I. Kant (Kant 1997, A142–143/B182)

> Time provides a universal source of models for the numbers. . .. What would give time a special role in our concept of *number* which it does not have in general is not its necessity, since time is in some way necessary for all concepts, nor an explicit reference to time in numerical statements, which does not exist, but its sufficiency, because the temporal

order provides a representative of the number which is present to our consciousness if any is present at all.

–C. Parsons (Parsons 1983, 140)

The key to achieving a positive or anti-skeptical solution to BD, I think, is precisely how one interprets step (4) in my reconstruction, which says:

(4) Given (1) and (3), our standard, uniform semantics of natural language, as applied to true mathematical statements, commits us to a truth-making ontology of abstract mathematical objects and also to the non-empirical knowability of these statements.

It is *very* natural, and all-too-easy, to interpret the notion of "a truth-making ontology of abstract mathematical objects" in terms of classical Platonism. Classical Platonism about mathematics says that mathematical objects, which are the truth-makers of mathematical statements, have a mind-independent, substantial existence in a separate non-spatiotemporal realm, and that their nature is strictly determined by intrinsic non-relational properties of those objects. In short, classical Platonism interprets mathematical objects as what Kant would have called *things-in-themselves*. (See Hanna 2006a, esp. Chapters 1, 2, 3, 4, and 6.) This classical Platonist interpretation of the truth-making ontology of abstract mathematical objects postulated in step (4), I think, is precisely *the snake in the Garden of Eden*, by which I mean that I think that this interpretation is precisely the false and vitiating assumption which leads inevitably to Benacerraf's Dilemma and to skepticism, and I hereby reject it.

Granting that rejection as a starting point, my positive or anti-skeptical Kantian phenomenological solution to Benacerraf's Dilemma—as I previewed it in section I—then has two parts:

(1) *Kantian Structuralism*, which says

 (1.1) that the natural numbers are essentially positions or roles in the mathematical natural number structure provided by Peano arithmetic,

 (1.2) that the mathematical natural number structure provided by Peano arithmetic is abstract only in the sense that it is *transcendentally ideal*, which is to say that this structure is identical to the formal structure of time insofar as we consciously represent it in sense perception, together with all the formal concepts and other logical constructions, including specific logical inference patterns such as mathematical induction, needed for an adequate rational human understanding of Peano arithmetic, and

(1.3) that in our actual world, the unique, intended model of the natural number structure provided by Peano arithmetic is just the set of manifestly real directly perceivable spatiotemporal material objects—the natural inhabitants of Parsons's "fuzzy *Lebenswelt* with its everyday objects"—insofar as *they are the role players of the Peano-arithmetic-specified natural number roles* in the abstract formal structure of time as we consciously represent it in sense perception, together with all the formal concepts and other logical constructions, including specific logical inference patterns such as mathematical induction, needed for an adequate rational human understanding of Peano arithmetic.

(2) *The Husserl-Wittgenstein Theory of Logical and Mathematical Self-Evidence* (the HW Theory), which holds that a priori knowledge in logic and mathematics is the joint product of two rational human abilities operating in tandem:

(2.1) a rational human agent's *mental-model-manipulating* abilities, which are innately specified in the agent's mind and also inherently present, as necessary ingredients, in ordinary sense perception, and which entail her conscious cognition of phenomenologically self-evident formal non-conceptual structures of human sense perception, together with

(2.2) that rational human agent's *logic-and-language-constructing* abilities, which are innately specified in the agent's mind and also inherently present, as necessary ingredients, in ordinary empirical conceptualizing and perceptual judgment, and which entail her conscious cognition of phenomenologically self-evident formal conceptual contents and specific patterns of logical inference in classical or non-classical logics.

In the rest of this section I want to unpack and rationally motivate the basic features of Kantian Structuralism. Then I will come back to the HW Theory in section IV.

Mathematical Structuralism, as an explanatory metaphysical thesis in the philosophy of mathematics—defended for example by Benacerraf himself, and in a different way by Stewart Shapiro (see, e.g., Benacerraf 1965; Shapiro 1997, 2000, Chapter 10), and most recently in another different way by Charles Parsons (Parsons 2008, esp. Chapters 3, 5, 6 and 9)—says that mathematical entities (e.g., numbers or sets) are not ontologically autonomous or substantially independent objects, but instead are, essentially, *positions* or *roles* in a mathematical structure, where a mathematical structure is a complete set of formal relations and operations that defines a mathematical system. What counts as an individual object of the system is thereby uniquely determined by the system as a whole—that is, any such individual object is identical to whatever possesses a specific set of intrinsic structural system-dependent properties. So every individual object of the system is essentially a role in the relevant mathematical system, and thus strongly metaphysically dependent on the whole system.

The significant philosophical payoffs of mathematical Structuralism are twofold. First, Structuralism gets between Platonism and Nominalism, because according to Structuralism mathematical objects are metaphysically absorbed into mathematical structures, hence they lack independent existence (contra Platonism), and yet it is also not true that there are no mathematical objects (contra Nominalism) since the objects continue to exist in a theoretically transformed way *as* roles in the structure. Second, because according to Structuralism the mathematical objects, as embedded in the relevant mathematical structure, continue to have whatever metaphysical status the relevant embedding structure has, then there is no longer any serious metaphysical "identity problem" of precisely *which* objects should be identified with the natural numbers, since we look to the embedding structures and not to the objects for any relevant metaphysical identity conditions.

In a way that is highly analogous to Functionalism in the philosophy of mind (see Block 1980b; Kim 2006, Chapters 5 and 6), there are at least two distinct ways we can interpret mathematical Structuralism. On the one hand, we can identify mathematical objects with *the roles* determined by the mathematical system as a whole. Or on the other hand, we can identify mathematical objects with *the role players* of the mathematical roles determined by the system as a whole. Which interpretation of mathematical Structuralism should we accept?

In the analogous case of Functionalism in the philosophy of mind, I think that there is good reason to take the Role-Player interpretation seriously because we think that it is intuitively plausible to identify a mind with whatever it is that actually does all the things that cognitive systems are supposed to do, and not merely to identify it with the set of causally relevant abstract patterns or rules that actual cognitive systems follow. If a mind were merely identical with a set of causal-functional roles, then it would be open to the classical inverted qualia argument, Searle's Chinese Room argument, and Block's Chinese Nation argument (a.k.a. "the absent qualia argument") (see ibid., and also Block 1980a; Searle 1984), not to mention the deeper worry that causal relevance does not entail causal efficacy (see, e.g., Jackson 1996), which yields the unhappy result that even *the representational mind* would be epiphenomenal if the Roles interpretation were true. Correspondingly, and to use an everyday non-philosophical analogy now, it seems intuitively right to say that a hockey player is a person who actually and in a causally efficacious way does all the things that hockey players are supposed to do, according to the rules of hockey—and obviously, a real hockey player is *not merely* the same as a set of causally relevant abstract rules that hockey players follow. So if we want minds to be *real causal players*, as it were, in physical nature, not to mention being *really capable of qualitative conscious experience* in addition to mental representation, then I think that we should defend a *dual* Roles interpretation *and* Role-Player interpretation of Functionalism, as opposed to a Roles interpretation alone or a Role-Player interpretation alone. We should say that for *some* rational purposes, the mind should be identified with functional roles, and also that for *other* rational purposes, the mind should be identified with the role-players of the roles.

By analogy, then, and for essentially the same basic reasons, I will adopt a *dual* Roles interpretation *and* Role-Player interpretation of mathematical Structuralism, as opposed to a Roles interpretation alone or a Role-Player interpretation alone. We want the natural numbers to be identified for many rational purposes with their roles in the mathematical structure of Peano arithmetic. But for other rational purposes we also want the unique, intended model of Peano arithmetic *to be consciously knowable according to a reasonable epistemology*, which is the direct analogue of the problem of qualitative conscious experience for the Roles interpretation of Functionalism. And we also want natural numbers and true statements about natural numbers *to be applicable to the actual spacetime world*, which is the direct analogue of the problem of epiphenomenalism for the Roles interpretation of Functionalism

So as I see it, mathematical Structuralism should hold that mathematical objects are essentially the same, for some rational purposes, as *the roles* in a given mathematical structure, and also essentially the same, for some other purposes, as *the role players* of the specific mathematical roles in a given mathematical structure, and *not* reducible either to those roles themselves or to the role-players themselves. The roles tell us precisely what will *count* as the unique intended model of that mathematical structure, but they neither *exhaust* the total nature of the mathematical objects nor do they *eliminate* the objects altogether. The mathematical objects are *strongly superveniently determined* by the structure as regards the precise roles they play, but they are also *something over and above* the structure as regards their role-player status. Different objects can play the same mathematical roles; the same objects can play different mathematical roles; and as a consequence, there is no intelligible worry whether the natural number 12 is the same as or different from the real number 12. This metaphysical dependency relation between mathematical structure and mathematical object in Structuralism thereby provides a precise analogue of *natural or nomological strong supervenience*, as opposed to *logical or reductive strong supervenience*, in the philosophy of mind.

Now BD clearly and distinctly shows us that we do not want the numbers to be the kinds of abstract entities that are also unknowable things in themselves and inapplicable to the actual spacetime world, lest we render mathematical truth and knowledge impossible. Or otherwise put, BD clearly shows us that the abstractness of the numbers must somehow correlate directly with what is consciously knowable according to a reasonable epistemology. This is possible, I think, if (and perhaps also only if) the abstractness of the numbers is *not* the abstractness of independent objects in a causally inert non-spatiotemporal realm, but instead just the abstractness of the roles *in a non-empirical or* a priori *consciously-accessible* *cognitive structure*. On this philosophical picture, the natural numbers are abstract because they are essentially roles in a *transcendentally ideal* structure.

In other words, I am proposing a broadly Kantian version of what Parsons calls "non-eliminative structuralism" (Parsons 2008, 100–116). More specifically, however, I think that the natural numbers are essentially the same, for *some* rational purposes, as roles in the abstract structure provided by Peano arithmetic, when this is interpreted as certain kind of non-empirical or a priori consciously-accessible cognitive structure, and also that the numbers are essentially the same, for *other* rational purposes, as the role players of the natural number roles in the real spacetime world, i.e., the natural numbers are just the set of manifestly real directly perceivable material objects intrinsically embedded in actual spacetime, insofar as they fall under the elementary or Peano arithmetic of the natural numbers. I will come back to this thesis again shortly.

Even if we have decided to adopt a dual Roles interpretation and Role-Players interpretation of structuralism, there are also several further basic distinctions between different kinds of Mathematical Structuralism that need to be made more explicit. The two main divisions are these:

(a) *Reductive* Structuralism vs. (b) *Non-Reductive* Structuralism,
(c) *In Rebus* Structuralism vs. (d) *Ante Rem* Structuralism.

Reductive Structuralism, as I am interpreting it, says that the objects of the mathematical system are either strictly identical with various elements and relations of the system or logically supervenient on the whole system and thus *nothing over and above* the whole system. By contrast, Non-Reductive Structuralism says that the objects of the system are strongly supervenient on the whole system but still *something over and above* the whole system, hence neither strictly identical to various elements and relations of the system nor logically supervenient on the whole system. In other words, the Reductive vs. Non-Reductive distinction applies to the *objects* of mathematical structural systems. Correspondingly, the Role-Players interpretation, on its own, entails Non-Reductive Structuralism, and the Roles interpretation, on its own, is consistent with both Non-Reductive Structuralism and Reductive Structuralism.

In Rebus Structuralism, as I am interpreting it, says that both the existence and specific character of the mathematical system are necessarily dependent on and determined by material things in the natural world, and that the systemic structures are not only literally proper parts of those material things but also ontologically *non-detachable* and epistemically *non-abstractible* from them. By contrast, *Ante Rem* Structuralism says that the existence and specific character of the system are neither necessarily dependent on nor determined by the existence of material things, and that the systematic structures are both ontologically *detachable* and also epistemically *abstractible* from those material things, even if they are also literally proper parts of them. In other words, the *In Rebus* vs. *Ante Rem* distinction applies not to the objects of mathematical structural systems, but instead to the *structural systems* themselves. For example, *In Rebus* Structuralism would be defended by a mathematical structuralist who is both a *reductive or scientific naturalist* and also an empiricist/nominalist, like Hartry Field (see e.g., Field 1980, 1989), whereas *Ante Rem* Structuralism would be defended by a mathematical structuralist who is both a *platonist* and also a rationalist/realist, like Shapiro.

Significantly, and perhaps because of the example set by Field, Shapiro identifies Reductive Structuralism with *In Rebus* Structuralism, and Parsons identifies both Reductive Structuralism and *In Rebus* Structuralism alike with what he calls "eliminative structuralism" (Parsons 2008, 80–100). But strictly speaking, at least in principle, one could consistently defend both *In Rebus* Structuralism and Non-Reductive (a.k.a. "non-eliminative") Structuralism. Consider, e.g., a specifically *Wittgensteinian* mathematical Structuralism (Wittgenstein 1983), in which numbers are identified with the entities that play the roles specified by mathematical linguistic practices, and not identified with the practice-specified roles, and in which those living mathematical linguistic practices *themselves*, conceived as rule-systems, are the enframing mathematical structural systems in which mathematical objects are embedded as the role-players of the roles in the structures. This Wittgensteinian Structuralism would be both *in rebus* and non-reductive. I myself am not going to defend such a Structuralism. But the very possibility of it does have a relevant bearing on the HW theory of mathematical apriori knowledge that I will defend in section IV, because I do think that mathematical *knowledge* is partially determined by living mathematical linguistic practices, even if mathematical *truth* is not so determined.

The brand of Structuralism I favor, Kantian Structuralism, is a non-reductive and *ante rem* version of mathematical Structuralism, doubly based on the abstract formal structures of space and time insofar as we consciously represent them in sense perception, together with formal concepts and the ramified abstract formal structures of classical logic and conservative extensions of it, insofar as rational human agents are capable of understanding those, that intends to take the necessity and apriority of mathematical truths at face value and then metaphysically explain those semantic features in terms of transcendentally ideal spatiotemporal structures, conceptual structures, and logical structures. By sharp contrast to Kantian Structuralism, however, Field's Structuralism is both reductive and *in rebus* because it says that numbers are nothing over and above their being positions in modal structures and that mathematical truth is reducible to fundamental physical facts about the physical world. And by another sharp contrast to Kantian Structuralism, Shapiro's Structuralism is both reductive and *ante rem* because it says that numbers are nothing over and above their being positions in non-modal structures and that mathematical truth is reducible to non-physical facts about non-spatiotemporal classically platonic structures.

But more precisely, and with respect to the elementary arithmetic of the natural numbers, i.e., Peano arithmetic, in particular, Kantian Structuralism says the following:

(1) that the natural numbers are essentially roles in the mathematical natural number structure provided by Peano arithmetic,
(2) that the mathematical natural number structure provided by Peano arithmetic is abstract only in the sense that it is *transcendentally ideal*, which is to say that this structure is identical to the formal structure of time insofar as we consciously represent it in sense perception, together with all the formal concepts and other logical constructions, including specific logical inference patterns such as mathematical induction, needed for an adequate rational human understanding of Peano arithmetic, and
(3) that in our actual world, the unique, intended model of the natural number structure provided by Peano arithmetic is just the set of

> manifestly real directly perceivable spatiotemporal material objects—the natural inhabitants of Parsons's "fuzzy *Lebenswelt* with its everyday objects"—insofar as *they are the role players of the Peano-arithmetic-specified natural number roles* in the abstract formal structure of time as we consciously represent it in sense perception, together with all the formal concepts and other logical constructions, including specific logical inference patterns such as mathematical induction, needed for an adequate rational human understanding of Peano arithmetic.

In this way, Kantian Structuralism adequately explains why something that is *abstract, ideal,* and *necessary* like the elementary arithmetic of the natural numbers, i.e., Peano arithmetic, can really and truly apply to the hurly-burly *concrete, manifestly real,* and *contingent* world of rational human animals and other natural things and processes, and thereby really and truly apply to all the manifestly real directly perceivable material spatiotemporal objects in our actual world. According to Kantian Structuralism, since the formal structure of time as we consciously represent it in sense perception is intrinsic to all manifestly real directly perceivable material spatiotemporal objects, and since the formal structure of time as we consciously represent it in sense perception together with anything isomorphic to the formal structure of time as we consciously represent it in sense perception, is the unique, intended model of Peano arithmetic, it follows as a matter of synthetic a priori necessity that Peano arithmetic applies to all manifestly real, directly perceivable, material spatiotemporal objects. The abstractness, ideality, and necessity of Peano arithmetic is captured by the *number roles* in the composite structure of time and Peano arithmetic and its conservative extensions, insofar as it can be understood by rational human agents. Correspondingly, the concreteness, reality, and contingency of the things and people to which arithmetic applies is captured by the *number role players* in the composite structure of humanly cognizable time and humanly cognizable Peano arithmetic and its conservative extensions. Thus consciously-representable time-structure is the metaphysical glue that ineluctably binds Peano arithmetic to our manifestly real natural world; or to re-use Parsons's apt phrase, consciously-represented time-structure is precisely what get[s] us across the divide between the fuzzy *Lebenswelt* with its everyday objects and the sharp, precise realm of the mathematical, in terms of which mathematical conceptions of the physical world are developed.

Otherwise put, Kantian Structuralism clearly solves the classical *application problem* for the philosophy of arithmetic. (See Potter 2000.)

So I am now in a position to solve BD by using Kantian Structuralism. I will begin by supposing that the two preliminary assumptions of BD are true. That obviously satisfies steps (1) and (2) of BD. Then I will further suppose that Kantian Structuralism is true, and that it adequately explains the apriority and necessity of mathematical truth. This satisfies step (3) of BD. This in turn allows me to re-interpret the truth-making ontology of abstract objects described in step (4) of BD as the transcendentally ideal abstract formal structure of time, and of anything isomorphic to time, insofar as we consciously represent it in sense perception, together with the transcendentally ideal abstract formal structure of any classical logical system rich enough to capture Peano arithmetic and conservative extensions of it, insofar as it can be understood by rational human agents. This dual abstract structure is itself of course causally isolated and inert, which satisfies step (6) of BD. But this dual abstract structure is also *intrinsically temporal*, and in our actual world it uniquely determines the unique intended model of the natural number structure, which then *just is* the directly perceivable manifestly real material world of spatiotemporal objects *insofar as* they are the role players of the Peano-arithmetic-specified natural number roles in the abstract structure of time. So the dual abstract structure consisting of the consciously-representable abstract formal structure of time together with Peano arithmetic and its rationally understandable conservative extensions is *causally relevant*, even though it is not *causally efficacious*. Therefore in our actual world the unique intended model of the natural number structure is identical to the world of causally efficacious manifestly real, directly perceivable material spatiotemporal objects, which obviously solves the application problem for Peano arithmetic, and mathematical knowledge is thereby possible on the assumption that a reasonable epistemology of cognizing true statements is modeled on a theory of sense perception which includes

causally efficacious, contact-involving or efficient, directly referential, non-inferential, and spatiotemporal relations between human linguistic knowers and the known objects themselves, understood by me to be some or another version of *direct or naïve perceptual realism* (see, e.g., Martin 2006; Haddock and F. McPherson 2008; Byrne and Logue 2009), which satisfies premise (5) of BD. Hence if Kantian Structuralism is true, then all of (1)–(6) are true, but the unacceptably skeptical conclusion of BD—step (7)—is avoided, and mathematical knowledge is still possible.

Considered for a moment apart from its ability to help us achieve a positive solution to BD, and also to solve the classical application problem for arithmetic, what other reasons could we have for defending Kantian Structuralism? I think that there are at least four other very good reasons.

First, Kantian Structuralism offers a clean-and-simple solution to another important problem pointed up by Benacerraf, which is that many different models satisfy the abstract structure of any logical system rich enough to express Peano arithmetic, so the second-order logic of Peano arithmetic underdetermines the natural numbers.[7] Otherwise put, Benacerraf's *other* problem is that there seems to be in principle no way of determining or identifying just *which* of the many distinct models that satisfy the logic of Peano arithmetic is *really* the natural numbers. This is what Parsons calls the "multiple reduction" problem (Parsons 2008, 48), and what others, following Frege, have called the "Caesar" problem or the "identification" problem. According to Kantian Structuralism, however, the abstract formal structure of the asymmetric successively synthesized series of moments (or simple events) in time insofar as we consciously represent it in sense perception is *the unique, intended model* of Peano arithmetic. On this picture, a "standard" model of Peano arithmetic is any possible world in which either time as we consciously represent it in sense perception exists, or else something isomorphic to the time-structure exists. (See, e.g., Parsons 2008, 272–293.)

But then the part of the model that satisfies a particular natural number-role in the abstract system of Peano arithmetic *just is* anything in our actual world that occurs in time as we consciously represent it in sense perception *insofar as* it intrinsically instantiates the thermodynamically asymmetric successive serial structure of time insofar as we consciously

[7] (See Benacerraf, 1965). This problem, in turn, is closely connected to Frege's "Caesar" problem. (See Frege, 1953, p. 68.)

represent it in sense perception, and thereby plays at least some of the Peano-arithmetic-specified natural number roles. The natural numbers themselves exist in non-actual possible worlds as *the Peano-arithmetic-specified and temporally-specified natural number roles*, and in our actual world as the unique intended model of Peano arithmetic, namely *the totality of real-world Peano-arithmetic-specified and temporally-specified natural number role-players.* Now the actual inhabitants of time insofar as we consciously represent it in sense perception are directly perceivable, manifestly real material spatiotemporal objects that contain spatiotemporal intrinsic structural properties. So in our actual world, the unique intended model of the natural number structure is identical to the totality of directly perceivable, manifestly real material spatiotemporal objects insofar as they are the role players of the Peano-arithmetic-specified natural number roles in the abstract formal structure of time insofar as we consciously represent it in sense perception.

Second, if Kantian Structuralism can offer a unified solution to BD *and* Benacerraf's other problem, then that seems to be another important point in its favor. For as Benacerraf himself has argued, BD and Benacerraf's other problem are essentially interdependent. So an adequate solution to BD must *also* solve Benacerraf's other problem (Benacerraf 1996).

Third, Kantian Structuralism crisply explains why classical Logicism failed, and why it seems that the arithmetic of the natural numbers is not reducible to second-order logic plus the Peano axioms alone. According to Kantian Structuralism, the elementary or Peano arithmetic of the natural numbers can be determined only by the ramified logical formal structure of Peano arithmetic and its conservative extensions insofar as it can be understood by rational human agents, together with any formal structure that is isomorphic to the structure of time insofar as we consciously represent it in sense perception. To be sure, contemporary neo-Logicists have shown that adding Hume's Principle (which says that the number of Fs = the number of Gs if and only if there are as many Fs as Gs) to second-order logic plus the Peano axioms logically entails the elementary arithmetic of the natural numbers. (See Wright, 1983; Hale 1987; Hale and Wright 2001.) But it seems to be intelligibly arguable that Hume's Principle is *not* an analytic truth precisely because it *presupposes* the formal structure of time insofar as we consciously represent it in sense perception, and also whatever is isomorphic to the formal structure of time insofar as we consciously represent it in sense perception. If so, then ironically enough the actual success of neo-Logicism is metaphysically best explained by *Kantian* Structuralism, and not by postulating the analyticity of *Hume's* Principle, as the neo-Logicists have done.

Fourth, if that is true, then Kantian Structuralism would also crisply explain why, contrary to both classical Logicism and neo-Logicism, mathematical truths seem *not* to be analytically necessary truths, but instead *synthetic* a priori *truths*. One good reason for thinking that mathematical truths are not true in every logically possible world, hence not analytic, is the clear and distinct conceivability and hence logical possibility, of either

(1) worlds with *nothing whatsoever* in them—which would of course entail the non-existence of numbers in those worlds, and thus the non-truth of many sentences of Peano arithmetic in those worlds (Parsons 1983, 131; Shapiro 1998, 604), or
(2) worlds with *non-standard arithmetics* of the natural numbers in them, e.g., a world in which "plus" is replaced by Kripke's "quus"—which would of course directly entail the non-truth of many sentences of Peano arithmetic in those worlds. (See Kripke 1982)

If mathematical truths are necessarily true but not analytically necessary, then according to Kantian Structuralism the explanation for this striking fact is that the truth and meaningfulness of mathematical propositions presuppose the abstract formal structure of time insofar as we consciously represent it in sense perception, which is not itself a purely logical or conceptual fact that attaches to every logically possible world. On the contrary, the presence either of the abstract formal structure of time insofar as we consciously represent it in sense perception, or of some other abstract structure isomorphic to the abstract formal structure of time insofar as we consciously represent it in sense perception, in a given possible world, is *a special metaphysical fact* that attaches to only a *restricted class* of logically possible worlds, i.e., all and only the logically possible worlds in which the very same spacetime structure, causal-dynamic structure, and mathematical structure as that of our actual world, also exist. This is also the special class of possible worlds in which *consciousness like ours* is really possible (see Hanna and Maiese 2009, esp. Chapters 1, 2 and 6, 7 and 8).

On this view, possible worlds without denumerable objects in them are all time-structureless worlds, and all time-structureless worlds are possible worlds without denumerable objects in them. So if Kantian Structuralism is true, then the metaphysical explanation for *Modal Dualism*—which is the classical Kantian thesis that there are two essentially different kinds of necessary truth, namely

(1) analytic necessary truth, i.e., truth about the kind of necessity which flows from the nature of logic and concepts, which thereby includes logical truth and conceptual truth, and
(2) synthetic necessary truth, i.e., truth about the kind of necessity which flows from the nature of things in the world, which thereby includes mathematical truth (Hanna 2001, Chapters 3, 4 and 5)

—comes along for free.

Now if Kantian Structuralism is true, then it fully explains how the elementary arithmetic of the natural numbers, i.e., Peano arithmetic, is true. What about the rest of mathematics? The clean-and-simple answer provided by Kantian Structuralism is that all of the rest of mathematics, *including* its most abstruse and ontologically rich parts—e.g., iterative set theory—can be built up from Peano arithmetic and the abstract formal structure of time insofar as we consciously represent it in sense perception, together with all the formal concepts, classical logical constructions, and specific patterns of logical inference required by those other parts of mathematics, encoded in standard mathematical linguistic practices, insofar as mathematical language can be understood by rational human agents. Leopold Kronecker famously said that God made the integers and everything else was done by humans. (See, e.g., Struik 1967, 160.) Kantian Structuralism is even more radically anthropocentric. According to Kantian Structuralism, *the formal constitution of rational human nature* made the natural numbers, and *logico-conceptual construction by rational human agents, together with their innate capacity for linguistic understanding* did all the rest.

Now of course the Kantian structuralist still needs to explain more precisely *how* mathematical a priori knowledge is possible. And that is where the HW Theory comes in.

IV. The HW Theory

As we have seen, the Husserl-Wittgenstein Theory of Logical and Mathematical Phenomenological Self-Evidence holds that a priori knowledge in logic and mathematics is the joint product of two rational human abilities operating in tandem:

(1) a rational human agent's *mental-model-manipulating* abilities, which are innately specified in the agent's mind and also inherently present, as necessary ingredients, in ordinary sense perception, and which entail her conscious cognition of phenomenologically self-evident formal non-conceptual structures of human sense perception, together with
(2) that rational human agent's *logic-and-language-constructing* abilities, which are innately specified in the agent's mind and also inherently present, as necessary ingredients, in ordinary empirical conceptualizing and perceptual judgment, and which entail her conscious cognition of phenomenologically self-evident formal conceptual contents and specific patterns of logical inference in classical or non-classical logics.

And as its name clearly indicates, there are two historical provenances for the HW Theory: Husserl's specifically *phenomenological* approach to the epistemology of necessary truth in *Logical Investigations*, and Wittgenstein's specifically *linguistic* approach to the epistemology of necessary truth in the *Tractatus*. The historico-philosophical task of correctly interpreting each of these books is both highly strenuous and highly tricky, and, especially in the case of the *Tractatus*, currently quite controversial. In this context, I want to bracket those hard interpretive questions, and just state what I take to be the deep epistemological ideas lying behind Husserl's doctrine of "categorial intuition" and also behind Wittgenstein's doctrine that "language itself prevent[s] every logical mistake" by virtue of the fact that "we *cannot* think illogically."

For our purposes here, Husserl's deep epistemological idea is that the abstract formal structures characteristic of logic or mathematics are immediately represented in our inherently non-conceptual, pre-reflectively conscious awareness of the logico- syntactic and sortal-semantic structures of the meaningful sentences we use to frame true logical or mathematical judgments, and that the truth of those judgments is directly verified in direct perceptual experience of the manifestly real and intrinsically spatiotemporal natural world. This direct verification, in turn, is *phenomenological self-evidence*.

To understand the notion of phenomenological self-evidence properly, we need to sketch the basic concepts of Husserl's early phenomenology. Phenomenology, as Husserl understood it in 1900 in the first edition of the *Logical Investigations,* is an elaboration of "descriptive psychology" in Brentano's sense. More precisely, phenomenology is the first-person, introspective, non-reductive philosophical psychology of consciousness and intentionality, as opposed to the natural science of empirical psychology (LI 5, §7). As a specifically *philosophical* psychology, its basic claims, if true, are non-logically or synthetically necessarily true and a priori.

As Husserl points out in Investigation 5, consciousness (*Bewusstsein*) is a subject's capacity for "lived experience" or *Erlebnis*, i.e., phenomenal awareness, together with her capacity for *intentionality*. Intentionality, in turn, is essentially the same as what Kant would have called "directed experience" or *Erfahrung*. So more comprehensively, as I will put it, consciousness is *subjective experience*.

Now all subjective experience, insofar as it is "directed experience," or intentionality, is either dispositionally or occurrently directed towards *targets* of various kinds—objects (of all sorts), events (including intentional actions), and subjects (including oneself or others). Conversely, all "directed experience" or intentionality is either dispositionally or occurrently conscious in the sense of phenomenal awareness or "lived experience." In turn, every conscious intentional mental state *M* has four individually necessary and jointly individuating features:

(1) *M* is a mental *act* (*psychischer Akt*) with its own "immanent content" or "act-matter" and its own specific character (i.e., phenomenal character) (LI 5, §§11, 14, 20),
(2) *M*'s mental act falls under a specific intentional *act-type* or "act-quality," e.g., perceiving, imagining, remembering, asserting, doubting, etc. (LI 5, §20),
(3) *M*'s mental has an intentional *target*, which at the very least has ontic status or "being" (*Sein*) and perhaps also actual existence or "reality"

(*Wirklichkeit)*, although this target need not necessarily have reality—hence intentional targets can include fictional objects, impossible objects, abstract objects, ideal objects, etc. (LI 5, §§11, 17, 20), and

(4) *M* has an intentional *meaning content* or "semantic essence" (*bedeutungsmässige Wesen*), which presents its target in a certain specific way, where this meaning content is either *propositional* or *referential* (LI 5, §§21, 31–36).

It is crucial to note that this general phenomenological analysis holds *both* for the intentionality of judgment and belief, which presupposes pure formal logic and necessarily requires the existence of natural language and the intentional subject's linguistic competence, *and also* for the intentionality of perception and other modes of sensory cognition such as imagination and memory, which do not presuppose pure formal logic or necessarily require the existence of natural language or linguistic competence.

In Investigation 6, Husserl argues that truth (*Wahrheit*) is the structural and semantic conformity of a judgment to the very fact that satisfies its propositional content, and that authentic knowing (*Erkennen*) or "self-evidence" (*Evidenz*)—whether authentic a priori knowledge or authentic a posteriori knowledge—is the sufficiently justified conscious intentional recognition of truth (LI 6, §§6–12, 20, 28, 36–39). Moreover, self-evidence has its own characteristic phenomenology. The essential structure of the phenomenology of self-evidence is the advance from "empty" intentions to "filled" intentions, where

(1) empty intentions are logico-linguistically structured propositional contents insofar as they are *conceptually understood* by an intentional subject to specify the very facts that *could or would* satisfy those contents and thereby *make* those propositions true, and
(2) filled intentions are logico-linguistically structured propositional contents insofar as the very facts that could or would satisfy them are also *non-conceptually intuited* by an intentional subject as *actually satisfying* those contents and thereby *making* those propositions true.

In other words, and now formulated in an explicitly Kantian way, for early Husserl the phenomenological profile of authentic knowledge or self-evidence is a systematic advance from conceptual "understanding" (*Verstand*) to non-conceptual "intuition" (*Anschauung*), and this holds whether the authentic knowledge is a priori or a posteriori, and whether the truth-making fact that is intuitively experienced in intentional fulfillment as satisfying the relevant propositional content is a non-empirical or ideal (necessary or possible) abstract fact, or an empirical or real (contingent) concrete fact.

In the case of non-empirical or ideal facts, then the non-conceptual intuition by which the fact is self-evidently known is a *categorial* intuition (LI 6, §§40–58). Categorial intuitions are intentional states containing phenomenal characters that specifically pick out the formal and structural elements of the very facts that are known via intentional fulfillment, either by means of formal elements of perceptual consciousness, or by means of formal elements of logico-linguistic consciousness. The two paradigmatic examples of this special sort of a priori intuition would be the way in which aggregates of directly perceived objects (say, beer bottles) are non-conceptually and pre-reflectively "subitized" into finite groups (say, groups of 5 or 7), and the way in which a state-of-affairs as described by a statement or judgment (say, "The twelve beer bottles are all lined up in the shelf on the wall") appears to have the very same grammatical form as the sentence used to describe it.

What this all means, again for our purposes here, is that when we use very simple arithmetic sentences like "7 + 5 = 12" in making statements like "7 + 5 = 12," we are non-conceptually and pre-reflectively consciously aware of a temporal flow of mental images associated with our visual or auditory cognition of those inscriptions or utterances. Indeed, recent empirical research on memory strongly indicates that the non-conceptual, pre-reflectively conscious phenomenal look and sound of language is processed separately from the propositional cognition of linguistic meaning (see Schacter 1990). For example, I can vividly recognize and remember the look or sound of German sentences and words—*Die Welt is alles, was der Fall ist* or *Wovon man nicht sprechen kann, darüber muss man schweigen* (as, perhaps, screeched by the brilliant Finnish absurdist composer and singer M.A. Numminen[8])—without recognizing or remembering what they mean. Thus the mathematical propositions that

[8] See (and hear) (Numminen, 2009).

we express by means of the *self*-conscious intentional *conceptual* acts of cognizing the linguistic meanings of arithmetic sentences is directly combined with a *non*-conceptual, *pre-reflectively* conscious grasp of the formal structure of experiential or lived time.

And in turn, whenever we directly perceive a configuration of manifestly real material objects in the natural world that partially confirms the arithmetic propositions we express—say, we see seven bottles of beer on the wall sitting alongside five more bottles of beer on the wall, yielding the look of twelve bottles of beer on the wall—then the non-conceptual, pre-reflectively conscious direct sense perceptions of those manifestly real material objects supplemented by the self-conscious epistemic perceptions based on those direct perceptions, taken together with their perceptual, imaginational, and memory-based synthesis in time as we explicitly or implicitly count them up, collectively immediately deliver to us a phenomenological formal structure that is also isomorphic to the addition operation over the natural numbers 7 and 5 in the system of Peano arithmetic. That non-conceptual, pre-reflectively conscious visual experience is a categorial intuition in Husserl's sense that necessarily impresses itself upon us as *mathematically self-evident*, where "self-evident" also means "inherently compelling," and as thereby conferring a *defeasible* epistemic certainty (Giaquinto 2007): As a rational human conscious intentional subject, you cannot help believing the propositional content associated with precisely that non-conceptual, pre-reflectively conscious subjective visual experience, or categorial intuition, precisely because it is inherently compelling. But correspondingly, the statement "7 + 5 = 12" is true if and only if there really is an appropriate mathematical truth-maker in the actual world that makes it true. This Husserlian doctrine, I think, provides a robustly realistic phenomenological interpretation of the classical Cartesian idea of "clear and distinct intuition."

Correspondingly, as I see it, the Tractarian Wittgenstein's equally deep epistemological idea is that to have logical or mathematical a priori knowledge is just

(a) to be a conscious rational human agent who possesses an innate conceptual cognitive capacity for non-conceptually and pre-reflectively consciously constructing, understanding, and using natural languages:

> Human beings possess the capacity of constructing languages, in which every sense can be expressed, without having an idea of how and what each word means—just as one speaks without knowing how the single sounds are produced. Ordinary language is a part of the human organism and is not less complicated than it. (Wittgenstein 1981, prop 4.002, 61–63. Translation slightly modified)

and

(b) then actually applying the meaningful logical and mathematical sentences of those natural languages—e.g., "7 + 5 = 12"—according to the implicit normative rules of logic and natural languages, to a world of directly perceivable manifestly real material objects whose configurations inherently satisfy those sentences.

So if, plausibly, we take early Wittgenstein's remarks about cognizing language to be anticipations of a broadly *Chomskyan* theory of language (e.g., Chomsky 1986), then non-conceptually, non-self-consciously, and thus "tacitly" consciously knowing the logical and mathematical parts of natural languages is just a sub-species of non-conceptually, non-self-consciously, and thus "tacitly" consciously knowing a natural language more generally. This is a priori knowledge in the mode of *knowing exactly but also only non-conceptually and pre-reflectively consciously how to construct and use the language according to categorically normative rules of human rationality* (Hanna 2006c, esp. Chapters 4, 5, 6 and 7), and not a priori knowledge in the mode of self-consciously knowing exactly *what* one is doing or *that* one is doing it, whenever one actually does it. Or in other words, Wittgenstein is adumbrating the notion of a *conceptually-driven but also non-conceptually and pre-reflectively conscious* a priori *logical and mathematical linguistic competence*.

According to the HW Theory then, our knowing mathematical truths by means of mathematical judgments involves the very same sorts of non-conceptual, pre-reflectively conscious but also conceptually-driven cognitive activities as knowing factual truths by means of ordinary linguistic perceptual judgments, in accordance with direct or naïve perceptual realism. In this way, our innate conceptual capacity for constructing, understanding, and using the logical and mathematical parts of natural language, together with our innate non-conceptual capacity for direct sense perception and pre-reflective consciousness, when conjointly triggered appropriately by the world of directly perceivable manifestly real material spatiotemporal objects, and when correctly conjointly implemented by us, just *is* mathematical a priori knowledge in the classical sense. That is, and more briefly: You know some mathematical truths a priori when you are both non-conceptually and pre-reflectively consciously and *also* conceptually and self-consciously thinking or talking about mathematics correctly, and furthermore the manifestly real natural world also uniquely satisfies the mathematical statements generated in your language of thought or in your outer speech.

It is plausible to think, for reasons supplied by classical Constructivist theories of arithmetic, that the precise class of arithmetic statements that would be satisfied under phenomenologically self-evident mathematical a priori knowledge is primitive recursive arithmetic, or PRA, which is a fundamental fragment of elementary or Peano arithmetic containing the quantifier-free theory of the natural numbers and the primitive recursive functions. (See Skolem 1967; Troelstra and Dalen 1998, 120–126; Hanna 2006a, Sec 6.2.) More precisely, it is plausible to think that our directly perceivable and linguistic access to the unique intended model of Peano arithmetic will not permit us to verify all of Peano arithmetic with phenomenological self-evidence. Peano arithmetic is of course defined by the following five axioms:

(1) 0 is a number.
(2) The successor of any number is a number.
(3) No two numbers have the same successor.
(4) 0 is not the successor of any number.
(5) Any property which belongs to 0, and also to the successor of every number which has the property, belongs to all numbers,

together with the primitive recursive functions (basic calculations) over the natural numbers—the successor function, addition, multiplication, exponentiation, etc. But axiom (5) is not verifiable in an inherently *non-conceptual* way, and on the contrary requires the inherently *conceptual* ability to grasp quantifications over all the numbers. Nevertheless, given our grasp of all the arithmetic statements covered by the first *four* axioms, together with a grasp of the primitive recursive functions, and thus for PRA, there is no need whatsoever for a further theory of sufficient justification by epistemic reasons, nor for any sort of reply to skepticism. Therefore PRA is phenomenologically self-evident in the Husserlian and Wittgensteinian sense, precisely because the cognitive abilities required to grasp it are inherently non-conceptual and pre-reflectively conscious, and fall within the scope of categorial intuition.

Presumably there are also humanly-graspable, categorially-intuitable structural analogues of PRA in elementary geometry, elementary set theory, and elementary logic—e.g., *Euclidean* geometry, *basic* set theory (see, e.g., Potter 1990, Chapter 3), and *monadic* logic. If so, then Euclidean geometry, basic set theory, and monadic logic are all phenomenologically self-evident *too,* along with PRA.

It is crucial to note that a priori knowledge in mathematics and logic far exceeds the scope of phenomenological self-evidence and categorial intuition. Non-self-evident a priori mathematical and logical knowledge—e.g., a priori knowledge in non-Euclidean geometry and topology, Zermelo-Fraenkel set theory, and classical first-order polyadic logic—is inferential, conceptual, and of course also defeasible. But non-self-evident mathematical and logical a priori knowledge presupposes the phenomenologically self-evident and categorially intuitable parts of mathematics and logic, and constantly draws upon them as it carefully advances from the less defeasible, virtually uncontested, and more epistemically secure domains, towards the more defeasible, more contested, and less epistemically secure domains. This epistemic advance from the self-evident a priori to the non-self-evident a priori is beautifully symbolically mirrored in the situation of Adam and Eve as they leave Paradise at the end of *Paradise Lost*, with a hard-won awareness of what is and what is not really possible for creatures like us, in our rational *human* condition:

> They looking back, all the eastern side beheld
> Of Paradise, so late their happy seat,
> Waved over by that flaming brand, the gate
> With dreadful faces thronged a fiery arms.
> Some natural tears they dropped, but wiped them soon;
> The world was all before them, where to choose

Their place of rest, and Providence their guide.
They hand in hand with wandering steps and slow,
Through Eden took their solitary way.[9]

We can now see that the HW Theory is breathtakingly elegant. It also coheres perfectly with Kantian Structuralism and direct perceptual realism. For if Kantian Structuralism and direct perceptual realism are both true, then the HW Theory makes *perfect sense,* precisely because our actual world of directly perceivable manifestly real material spatiotemporal objects intrinsically carries with it the abstract formal structures of the system of Peano arithmetic and its conservative extensions, and thus directly perceptually presents the system of natural numbers, i.e., the intended model of Peano arithmetic, via the self-evidence of primitive recursive arithmetic or PRA, to any rational human conscious intentional subject who is also competent in the mathematical parts of her own natural language.

V. Conclusion: Benacerraf's Dilemma Again and "Recovered Paradise"

If Kantian Structuralism, direct perceptual realism, and the Husserl-Wittgenstein Theory of logical and mathematical self-evidence are all true, then both of Benacerraf's preliminary philosophical assumptions about a "standard, uniform" semantics of natural language and a "reasonable" epistemology of cognizing true statements are true, and the other four steps of Benacerraf's Dilemma are also true, but the unacceptably skeptical conclusion does *not* follow. Mathematical a priori knowledge in at least the classical, Kantian sense still *is* possible. Kantian Structuralism together with direct perceptual realism *also* together solve the classical application problem for mathematics; they solve Benacerraf's other problem about what the numbers could not be; they explain why classical Logicism failed; and they account for the synthetic necessity of mathematical truth. All of these very important individual theoretical virtues then seem to me to add up very naturally to one great big sufficient

[9] (Milton, 1953a, p. 487, book XII, lines 641–649).

reason for accepting Kantian Structuralism, direct perceptual realism, and the HW Theory as a single package.

Moreover, the conjunction of Kantian Structuralism, direct perceptual realism, and the HW Theory yields a phenomenologically-enriched *Kantian logico-linguistic constructivism* as a serious alternative to classical Platonism about mathematics on the one hand, and also to all the more or less skeptical recent and contemporary theories of mathematics—i.e., the full range of pre-emptive or concessive negative solutions to Benacerraf's Dilemma—on the other. This, in turn, suggests a fundamental Kantian, Husserlian, and Wittgensteinian insight into the nature of a priori knowledge. Given this phenomenologically-enriched Kantian logico-linguistic constructivism, what is required for mathematical knowledge is just a linguistically competent, healthy, developmentally normal, and relatively mature rational human conscious intentional subject, who can grasp both the non-conceptual content of perception and also the conceptual and propositional content of statements or judgment, who has also learned the basics of PRA, and who is thus primed and ready for speaking her own natural language, and for non-conceptually and pre-reflectively consciously but also conceptually and self-consciously intaking her manifestly real world through direct sense perception. And that is *all* that is required. Mathematics, just like *logic*—as I have argued elsewhere (see Hanna 2006b), is an *exact science* and yet also inherently a *human* or *moral* science. In this way, by equally rejecting *both* classical Platonism *and* post-Benacerrafian skepticism about mathematical truth and knowledge, we find

> Eden raised in the waste wilderness.

So let us go forth and multiply. And of course also add, subtract, divide, and correctly perform the other primitive recursive functions over the natural numbers too.[10]

[10] I am very grateful to the organizers (especially Mirja Hartimo, Leila Haaparanta, Juliette Kennedy, and Sara Heinämaa) of and also the participants in (especially William Tait), the Phenomenology and Mathematics conference at the University of Tampere, Finland in May 07, where I presented an earlier version of this paper, for all their help—critical, editorial, philosophical, and otherwise.

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CHAPTER IX

ON REFERRING TO GESTALTS

Olav K. Wiegand

The time has come to enrich formal logic by adding to it some other fundamental notions.

Stan Ulam[1]

Abstract. This paper discusses a fresh approach to formal semantics based on mereology and Gestalt Theory. While Wiegand (2007, *Spacial Cognition & Computation, Mahwah, NJ: Erlbaum*) unfolds the technical details of this new approach, the following paper aims to discuss the philosophical motivation an implications of what I have called *mereological semantics*. Particular attention will be given to an ongoing debate on the nature of relations.

I. Introduction

In what follows I will be confronting two tasks: (a) I will sketch the philosophical motivation for what I have called *mereological semantics* (MS).[2] This motivation consists of ingredients from Gestalt psychology and phenomenology, and will explicate a certain conception of structured objects of reference that underlies MS. (b) I will then—from the point of view of this background philosophy—comment on some philosophical aspects of an ongoing debate on the nature of relations.

MS interprets a more or less complex formal *language of mereology* (into which portions of natural language may be translated) with respect to a modularly structured domain of quantification. Various sorts of objects may be axiomatically defined and then included in the domain, so that it may encompass various sorts of *aggregate wholes* and sorts of *structured wholes* at the same time. The concept of structured whole is a generalization of the concept of Gestalt which was originally developed as a means

[1] This motto is a remark made by Stan Ulam in a conversation with Gian-Carlo Rota. The conversation is summarized in Rota (1985), see also Barwise's epilogue to his *Situation in Logic*, entitled "Toward a Mathematical Theory of Meaning."
[2] See Wiegand (2007). Mereology is the theory of Parts and Wholes.

M. Hartimo (ed.), *Phenomenology and Mathematics*,
Phaenomenologica 195, DOI 10.1007/978-90-481-3729-9_9,

for interpreting empirical findings within the psychology of perception. However, gestaltists themselves soon transgressed these limits and applied the concept to all sorts of cognitive activity (perceptive or predicative), and have even argued for analogies between the physical concept of field and the concept of Gestalt. In what follows we will employ the concept of structured whole as opposed to the notion of aggregate whole, whereby the latter does not imply the interdependence of all parts of the common whole, a feature that is characteristic of the concept of Gestalt when we take the following definition as a starting point:

> A Gestalt is an ensemble of items which mutually support and determine one another. Thus they realize a total structure which governs them and assigns to each of them (as a part of the whole) a function or a role to be performed as well as a determinate place in that whole. Each detail exists only at the place at which it plays the role assigned to it by the whole of which it is a part. (Gurwitsch 1936, 25)

Various attempts to formalize the notion of Gestalt were put forward (see e.g., Rescher 1955, Rescher and Oppenheim 1955; Simons 1987). The following constructions will be based on *mereological concepts*, since the idiom of "wholes" and "parts" has been characteristic for gestaltists' analyses since the very beginning. The modelling offered within the framework of MS wants to satisfy two main intuitions concerning Gestalts and their modifications:

(i) We need to reflect the interconnectedness of all parts of a Gestalt: no part of a Gestalt can undergo a change without affecting all the other parts of the common Gestalt contexture.[3]

(ii) From a gestaltist point of view *courses of reasoning* (formally) show themselves as sequences of gestalt-modifications like breaking up, inverting, differentiating, rearranging, widening or narrowing the Gestalt contexture. Whether these modifications are brought about as an effect of a change in the subject's focus of *attention* or whether they "just emerge," we need the means to model courses of reasoning as sequences of Gestalt modifications (which has, of course, nothing to do with *explaining* reasoning itself).

[3]See Rock and Palmer (1990), Rock (1985), Kellman (2000) for a more detailed description of the notion of Gestalt.

In her book *Parts and Wholes in Semantics* Moltmann defined the concept of *R*-integrated Whole[4] that may be viewed as another formalization of the notion of Gestalt, particularly attempting to capture intuition (a) for use within linguistics. Although we will build upon some definitions that she used for her construction of *R*-integrated Wholes, the concept of an *R-structured whole* will deviate from her concept of an integrated whole. As regards intuition (b) Wiegand (2007) provides a definition of "courses of reasoning" as sequences of Gestalt-modifications. We will, however, not deal with that topic within the framework of this essay.

The primary intention for developing MS was to provide a formalization of the notion of structured whole for use within formal semantics (and as such to supplement the informal analyses of *cognitive semantics*).[5] In what follows the motivation for MS—more precisely: the motivation for viewing reference as reference to structured wholes—will be unfolded from a phenomenological point of view that leans heavily on the work of Aron Gurwitsch, and his conception of intentionality as the objectivating function of consciousness. Gurwitsch combined the tenets of Gestalt theory with that of phenomenology, and conceived of objects as structured wholes. He also interpreted basic cognitive operations (that he referred to as attentional modifications) in terms of Gestalt operations. Gurwitsch has also criticised Husserl's conception of mereology from a gestalt theoretical point of view.[6]

II. R-Structured Wholes

Let us first go step by step through the construction of what we will call an *R-structured whole*. To that end we will at the outset recall a couple of basic notions from the theory of relations, and partial orders in particular.

1. Preliminaries

Let A and B be arbitrary sets, not necessarily distinct. Then any subset R of $A \times B$ is called a *binary relation from A to B* (or simply a *relation from A*

[4] See also Simons (1987) on that concept.
[5] See Lakoff (1977, 1986), Talmy (2000), Langacker (1987, 1991) et al.
[6] See Gurwitsch (1929), 364 ff, see Wiegand (2001).

to B). If $A = B = M$, i.e., if $R \subseteq M \times M$, R is called a *relation on M.* For the pair $(a,b) \in R$, we write $R(a,b)$ or aRb. The set $\{x: \exists y\ (x,y) \in R\}$ is usually referred to as the *domain* $Dm(R)$, the set $\{y: \exists x\ (x,y) \in R\}$ is called the *range* $Rn(R)$ and we will refer to $Dm(R) \cup Rn(R)$ as the *field* of R, written $Fd(R)$.

Now suppose P be a set. An *order* (or *partial order*) on P is a binary relation $\sqsubseteq$ on P such that, for all $x, y, z \in P$,

(Reflexivity)	$x \sqsubseteq x$,
(Anti-symmetry)	$x \sqsubseteq y$ and $y \sqsubseteq x$ imply $x = y$,
(Transitivity)	$x \sqsubseteq y$ and $y \sqsubseteq z$ imply. $x \sqsubseteq z$.

A set P equipped with an order relation $\sqsubseteq$ is said to be an *ordered set* (or *partially ordered set* or simply *po-set*), written $\langle P; \sqsubseteq \rangle$. We usually just say "$P$ is a po-set." Instead of $x \sqsubseteq y$ we may also write $y \sqsupseteq x$. We write $x \parallel y$ if $x \not\sqsubseteq y$ and $y \not\sqsubseteq x$.

Let P be a po-set:

(1) P is a *chain* if, for all $x,y \in P$, either $x \sqsubseteq y$ or $y \sqsubseteq x$. P is an *anti-chain* if $x \sqsubseteq y$ in P only if $x = y$. A chain P is finite if its cardinality $|P|$ is a natural number.

(2) Let Q be an arbitrary subset of P, and $x \in P$. Q is a *down-set* if, whenever $x \in Q$, $y \in P$ and $y \sqsubseteq x$, we have $y \in Q$. We define:

 (i) $\downarrow Q = \{y \in P: (\exists x \in Q) y \sqsubseteq x\}$;
 (ii) $\downarrow x = \{y \in P: y \sqsubseteq x\}$;

 to be read *"down Q"* and *"down x."* Q is a down set iff $Q = \downarrow Q$, and $\downarrow\{x\} = \downarrow x$.

2. The Part-of Relation

We will now consider a special partial order written $\preceq$ which is the *part-of relation*. For what follows let G be a non-empty, at most countable set, and $\preceq$ a partial order on G.

(3) Suppose $x,y \in G$. We will call x a *part of y* if $x \preceq y$. x is a *proper part of y*, written $x \prec y$, if $x \preceq y$ and $x \neq y$. In case that $x = y$ one says that x is an *improper part of y*.

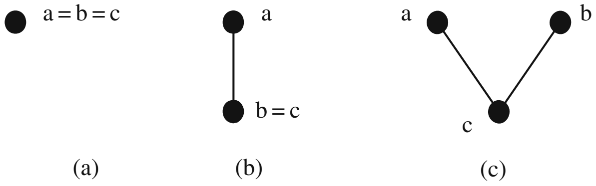


Fig. IX.1

(4) a,b not being necessarily distinct elements of G, we will say that a and b *overlap*, written $aO_{\preceq}b$, if $\exists z(z \preceq a \wedge z \preceq b)$.
(5) $x,y \in G$, we will say that x is an *immediate part of y*, and write $x \lhd y$ or $y \rhd x$ if $x \prec y$ and $x \preceq z \prec y$ implies $z = x$.

We will use *Hasse-Diagrams*[7] to depict configurations of parts and wholes. In all of the following examples a and b overlap because they have a part c in common Fig. IX.1.

Having the usual sentential connectives in mind, we can verify that a statement like "$c \preceq a \wedge c \preceq b$" is true with respect to figure (c).

In his 3rd *Logical Investigation* the phenomenologist Edmund Husserl divides objects into simple and complex. The latter are defined such that they contain at least two "disjoined parts," that is parts that have no common parts. This intuition leads to the formulation of the *principle of complex objects* (PCO):

(6) $\forall x \forall y(x \prec y \rightarrow (\exists z\, z \prec y \wedge \neg z O_{\preceq} x))$.[8]

In Fig. IX.2 the po-sets[9] (d), (e) and (f) do not meet condition (PCO), the po-set depicted in diagram (g), however, does.

Suppose $\langle G; \preceq \rangle$ be a po-set that meets condition (PCO), and $a \in G$. A set $P_a \subseteq G$ will be called the *part-expansion of a in G* if:

[7] See Davey and Priestley (2002).
[8] PCO is a modified version of SA3 in Simons (1987), 28. We subscribe to his view that the chain-models (d) and (e) are counter-intuitive in that they imply the idea of a whole containing merely a single part. (f) would have to be understood as a whole, all of whose parts overlap each other. It seems, however, counter-intuitive to assume such objects. See the discussion of SA3 in Simons (1987).
[9] As regards the notion of partially ordered set (po-set for short) see Section I.I above.

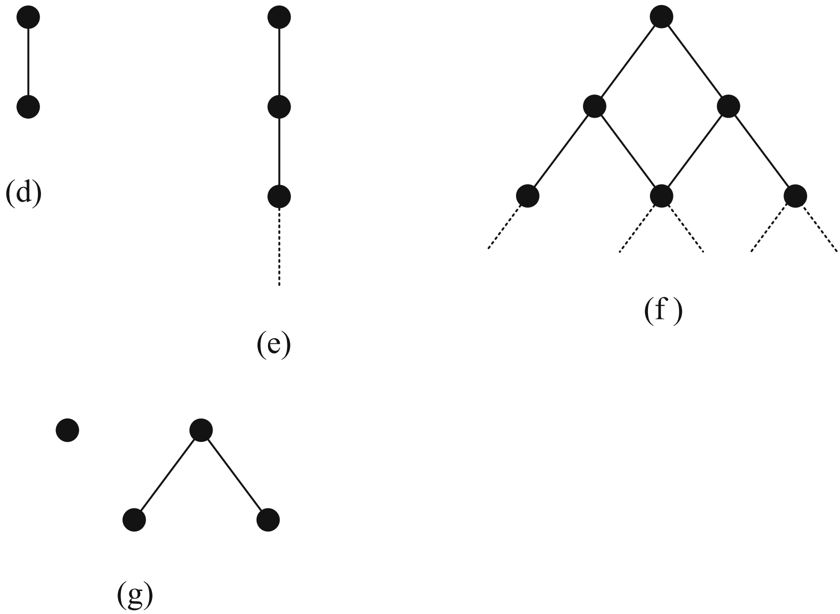


Fig. IX.2

(7) $P_a = \downarrow a$,
(8) $(\exists x \in G)\ x \prec a$,
(9) All chains in P_a are finite.[10]

Given an element a of G, and P_a its part-expansion in G, $P_a \setminus \{a\}$ will be referred to as the *proper part-expansion* $\mathbb{P}_a$ *of a in G*. The elements of the part-expansion of a will be called the *parts of a in G*, the elements of $\mathbb{P}_a$ the *proper parts of a in G*, and the minimal elements of P_a will be called the *atomic parts of a in G* (or simply the *atoms of a in G*).

Part-expansions are defined as finite sets. This is an immediate implication of the strictly epistemological orientation we have chosen to assume. Wiegand (2007) provides a means to differentiate part-expansions gradually and incrementally, which is important for any logic wishing to model courses of reasoning. Taking up a famous Husserlian example, if we observe a shop window from a distance, we may not be able to

[10]See definition (1) above.

decide whether there is a display dummy or a real person behind the glass. In his *Experience and Judgement* (see §21b) Husserl speaks of two perceptual situations, each possessing its respective power and graduality, that overlap each other and thus alternatively win the upper hand for a brief duration. If we walk closer to the shop window we may eventually see more and more details (parts of the whole) so that a new cognitive situation emerges in which it eventually becomes clear whether the object is a person or a dummy. It will, however, never be the case that a subject apprehends an object qua an *infinite* totality of parts.

3. One Sort of Structured Wholes: R-Structured Wholes

We will now comment on the definition of *R*-structured wholes. For that end we will start with an intuitive understanding of the Hasse-diagram of Fig. IX.3, The Diagram depicts a complex *R*-structured whole *h*. The nodes *a, . . ., g* symbolize proper parts of *h*. These are grouped by dotted rectangles that indicate fields of relations R_x that, in turn, are meant to capture the notion of *immediate gestalt-contexture of object x*. The gestalt-contextures of the objects *h*, *f* and *g* taken together represent the internal structure of the given whole *h* (as it shows itself within a certain cognitive situation). Informally we may speak of wholes and subwholes in the sense of Wertheimer:

> Proceeding from above, from the structure of the whole and descending from there to the subwhole and to the parts, the parts are not mere pieces in additional relation together, but parts of the whole; these parts are in hierarchical relation together (Wertheimer 1922).

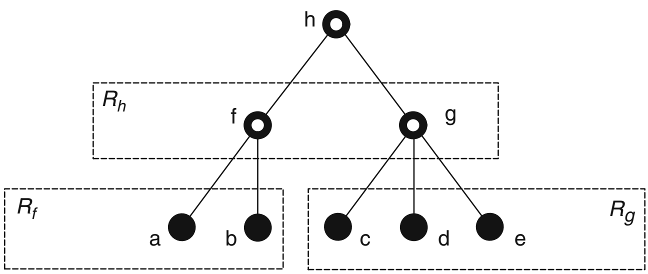


Fig. IX.3

We may now, for reasons of illustration, assume that *h* is a musical duet, and that *f* and *g* are the musicians: both are part of the *R*-structured whole *h*, they belong to the immediate gestalt-contexture of *h*, and therefore they make up the field of the relation R_h. That both musicians belong to the gestalt-contexture of *h* means that *g* only "makes sense" in connection with *f*, and vice versa. If *g* leaves the room, then *f* is no longer part of a musical duet. (If we are still inclined to call *g* or *f* "member of a duet", then this is in a more abstract sense.) We note that *f* and *g* belong to the gestalt-contexture of the duet, but certain parts of *h*, for example *c*, . . ., *e* do not belong to the immediate gestalt-contexture of *h*, these belong to the immediate gestalt-contexture of *g*.

In his 3rd Logical Investigation Husserl distinguishes pieces from dependent parts (i.e., moments) in various ways. For our present concern one definition advanced in the Third Investigation is important. There he defines the concept of piece (and, indirectly, of moment) by means of the absence of those functional dependencies to which dependent parts, along with other parts of a given perceived whole, are subject:

> The head of a horse can be presented "on its own" or "cut off," i.e., we can hold it in our imagination, while we allow the other parts of the horse, and its whole intuited setting, to alter and vanish at will. . . .the content of such a "phenomenon" does not in the least involve anything entailing a self-evident, necessary, functional dependence of its changes on those of coexistent phenomena. (LI, 234/439)[11]

This definition is, however, somewhat problematic (see Gurwitch 1929, Wiegand 2001) because the notion of Gestalt contexture implies that *no* part of the whole is (functionally) independent of the other parts of the gestalt contexture. We may, however, speak of parts that are capable of being singled out (namely pieces), and parts that are not capable of being singled out (namely, moments).

In our example of Fig. IX.3, *h*,*f*, and *g* are pieces (which is indicated by drawing those nodes as **o**): *g* can leave the room, she can be "singled out" from the contexture that is created by their playing together in the form of a duet within a certain time frame. Parts *a* to *e* on the other hand are "moments," i.e., they are parts that cannot be singled out. If one wishes to

[11] See Wiegand (2001) for an explication of the notion of being singled out.

interpret these parts then they could be materialized as e.g., the height of one of the musicians, her shape, certain "Gestalt qualities" of her movements or personality, etc. Husserl's example from the 3rd Investigation for mutually dependent parts is the coloring of a particular object, and the extension covered by that color. However if a part like *c* is interpreted, in the drawing of Fig. IX.3 it does not play a role for the first-level gestalt-contexture of the duet that is modelled by the field of R_h.

We will now further explicate the role played by the relation *R*. Definitions (10) to (12) are used in Moltmann (1998, Section 1.5.2) to prepare the definition of the concept of integrated whole.

(10) For a non-empty set *X* and a two-place relation *R*, *X* is *closed* under R $(Cl(R,X))$ iff $\forall x \forall y \left(x \in X \wedge \left(xRy \vee yRx\right)\right) \rightarrow y \in X$.
(11) For a non-empty set *X* and a two-place relation *R*, *X* is *connected* under R $(Con(R,X))$ iff $\forall x \forall y \left(x \in X \wedge y \in X\right) \rightarrow \left(xRy \vee yRx\right)$.
(12) R^{tr} the *transitive closure of* *R*, i.e., $xR^{tr}y$ iff there are objects $x_1, \ldots, x_n$ such that $xR^{tr}x_1 \wedge \ldots \wedge x_n R^{tr} y$.

Let us now consider a non-empty, at most countable po-set *G*, and let us, furthermore, distinguish a subset $W \subseteq G$ that encompasses all and only the *pieces* in *G* (i.e., the elements of *W* are those and only those objects in *G* that we depict as **o**). What we have in mind is, of course, that *G* is a universe of only structured wholes and their parts. We have called these auxiliary objects *mereological ground-structures:*

(13) A triple $\mathbf{G} = \langle G; W, \preceq \rangle$ will be called a *mereological ground-structure* if:

  (i) $\langle G; < \rangle$ meets condition (PCO),
  (ii) *W* is a non-empty subset of *G* such that for every object $x \in W$ there is a part-expansion P_x of *x* in *G*, and $G = \bigcup_{x \in W} P_x$.

The drawings of diagram (h) could be seen as depicting the finite mereological ground-structure $\mathbf{G} = \left\langle \left\{a,b,c,d,e,f,g,h,i,j\right\}; \left\{c,g,i,j\right\}, \preceq \right\rangle$.

Let now $\mathbf{S} = \langle M; S, \preceq \rangle$ be a mereological ground-structure, $a \in S$, and $\mathbb{P}_a$ be *a*'s proper part-expansion in *M*.

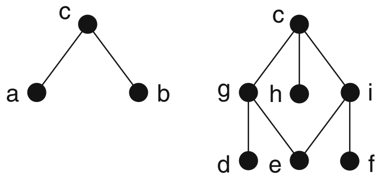


Fig. IX.4

(14) Let A_a be the set of all chains $B = \{x_1, \ldots, x_n\} \subset \mathbb{P}_a$ such that:

(i) x_1 is a maximal element of $\mathbb{P}_a$,
(ii) for $n \in \mathbb{N}$:$x_1 \succeq \ldots \succeq x_n$ is a sequence, and x_n is either an atom or n is the smallest number such that $x_n \in S$.
A set $C_a = \bigcup_{B \in A_a} B$ will be called an *immediate gestalt-contexture in* **S**.

The sense of definition (14) is to collect all the parts that belong to the immediate Gestalt-contexture of a given piece a. In order to do so we check all the chains beginning with an immediate part $x_1 \lhd a$ until we reach an element x_n that is either a minimal element of $\downarrow x_1$ or a piece in the set $\downarrow x_1$, such that there is no other independent part x_k for $1 < k < n$.

Applied to the drawing of Fig. IX.3 above we may, for example, collect those and only those parts that belong to the immediate Gestalt-contexture of h. When this is done we collect those that belong to the immediate Gestalt-contextures of f and g respectively. The proper part expansion $\mathbb{P}_h$ is the set $\langle \{a, \ldots, g\}, \preceq \rangle$. From that set we may start with elements f and g in order to construct chains that meet conditions (i) and (ii) of (14). The immediate gestalt-contexture C_h will therefore simply be $\{f\} \cup \{g\}$.

Let us now return to the role of R for the constitution of *R-structured wholes*.

(15) a is called an *R-structured whole in* **S** (written $R\text{-}Stc\text{-}Wh_{\mathbf{S}}(a)$) if there is a non-trivial[12] two-place relation R_a defined on M such that

[12]"The specification 'nontrivial' is required in order to exclude integrated wholes being defined on the basis of relations such as difference or identity" (Moltmann 1997).

$Cl\left(R_a^{tr},C_a\right)$ and $Con\left(R_a^{tr},C_a\right)$. The relation R_a will be called the *R-foundation relation of a in* **S**.

(16) Suppose there is given an *R*-structured whole *a* in **S**. For each $y \in C_a$, the field of the *R*-Foundation Relation R_a is called the *immediate gestalt-contexture for y in* **S**.

The notion of immediate gestalt-contexture helps to formalize complex objects ("configurations"), i.e., objects that encompass more then one structured whole. It tries to capture Wertheimer's terminology of "wholes and subwholes." The notion of immediate gestalt-contexture also adds to the philosophical discussion centered around the question of the transitivity of the part-whole relation. Within classical mereology this problem has been addressed as the question as to whether the part-of-relation is transitive or not (see, e.g., Johansson 2004, Varzi 2006, Talmy 2006): Jim's nose is part of Jim, and Jim is part of the Berlin Symphony Orchestra, but is Jim's nose part of the Berlin Symphony Orchestra? The intuitive answer is no. Moltmann (1997) has justly pointed out that transitivity is somehow blocked by the bounds of a Gestalt (the bounds of Jim *qua Gestalt* block transitivity). So should one relinquish the transitivity of the parthood-relation? Within the framework of the present approach we have chosen to tackle the problem as follows: clauses (i) and (ii) of (14) provide instructions of how to select those and just those parts that belong to the immediate gestalt contexture of a particular structured whole *x*. Other objects in the universe may then be part of elements of the field of R_x *but they may not belong to the Gestalt contexture of x*. This allows us to understand the part-of-relation as a partial order, which is a solid technical framework. Our approach also sheds light on the intuitive problems with transitivity enumerated, e.g., in Johansson 2004. To say that "Jim's nose is not part of' the Berlin Symphony Orchestra" should be reformulated as "Jim's nose does not belong to the gestalt-contexture of the BSO"—it may be part of one of its subwholes, but Jim's nose does not belong to the material context of a symphony orchestra. It may be an extraordinary snub nose, so that it came to my attention while I was listening to a performance of the orchestra. In this sense a node must be dedicated to Jim's nose when I were about to model what I perceived. As I have (inevitably) perceived the scene as structured in a certain way, and if I wanted to capture that structure, the modelling would have to place the snub nose as part of one subwhole. Had I executed other cognitive activity (like dividing the orchestra into blondes and non-blondes) the modelling would have to capture this.

We are now left with the question of how to address the *total* gestalt-contexture, the gestalt at large. This is the question of how to view a *configuration* (i.e., a whole that encompasses more than one structured whole) as an organic totality. If one were to provide a formal definition, then such a definition should conceive of the whole as a special arrangement of R-relations, pieces and moments. Such an arrangement obeys an important gestalt theoretical principle, namely the *primacy of the whole*, which says that it is not possible to remove or add parts without modifying a Gestalt contexture as such. Removing a part or supplementing the whole with additional parts results in a different cognitive situation. We will therefore have to include considerations as to how to define identity.

4. *Questions of Identify*

From the point of view of ordinary language it seems absurd to speak of two things being the same thing. But it is not nearly as absurd to claim that, in order to be distinct, they must be discernibly distinct in the sense of there being one property not common to both of them.[13] The latter seems to be the rationale behind defining identity in terms of

Leibniz-Identity: $x = y := \forall P(Px \equiv Py)$.

Let now A be an ordered set, and denote the family of all down-sets[14] of A by $O(A)$ (it being in itself an ordered set, under the inclusion order). An analog to classical Leibniz-Identity for R-structured wholes x, y could then be formulated as

abstract identity of wholes: $x\, Id_{\text{abstract}}\, y := O(x) = O(y)$

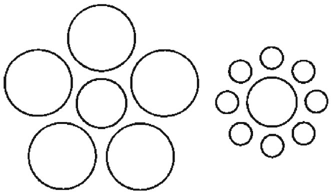
We are speaking of *abstract identity* because we abstract from the R-relations that may be present. Abstract identity in that sense may not be given because:

[13] See Barcan-Marcus (1962).
[14] See (2) of Section I.I.

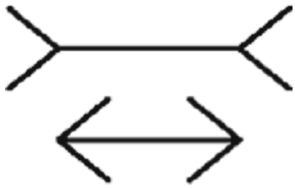
(i) there is at least one part that *x* and *y* do not have in common or
(ii) *x* and *y* may have all properties in common, but in a different arrangement.

For the previous and all of the following definitions we presume that objects *x* and *y* are given within the framework of one cognitive situation (like within the unity of one perception).[15]

However, the question of the identity of structured wholes is much more complicated. We might briefly examine just one other aspect of it, namely that of configuration-dependent changes as is the case in the following famous examples:



Ebbinghaus



Müller-Layer

When we examine e.g., the Ebbinghaus case, then the circle *c* in the middle looks larger in the right configuration (although from an abstract point of view they are equal in size and shape). This accords with the gestalt theoretical tenet of the *primacy of the whole*, namely the supremacy of the whole over its parts. Configurations like these were used by gestaltists to demonstrate the impact of contextuality: it is not possible to isolate a part, take it out of a given configuration (for example the left central circle of the Ebbinghaus case) and insert it into another configuration in such a manner that the part remains the same—like taking *c* out of the left configuration of the Ebbinghaus and insert it into the right one.

We could informally examine two options here:

(a) One possibility is to depict the central circle as an immediate part of the configuration, on a par with the other circles that surround

[15] See Wiegand 2007, and Moltmann's concept of reference-situation.

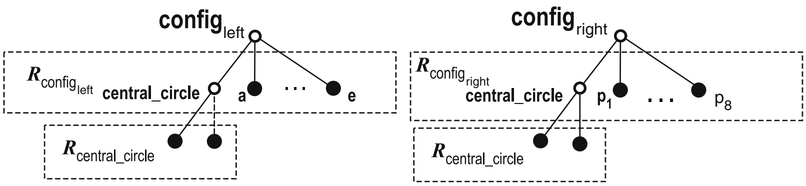


Fig. IX.5

it. They make up one organic whole. Parts *a*, . . ., *e* (the circles that surround the central one) are not further differentiated (we assume that the perceiver does not perceive any particular details of one of those parts)Fig. IX.5.

The visual difference between the central circle as part of the left and the right configuration of the Ebbinghaus respectively may correspond to the difference of the contexts (i.e.,$R_{\text{config}_{\text{left}}} \neq R_{\text{config}_{\text{right}}}$), in spite of the fact that the central circles have a part structure that makes them as equal as Leibniz' two drops of water (which are equal with the exception of their position in space).

(b) Another possibility is to say that the central circle in the middle is perceived as a figure, whereas the other circles around it are perceived as a background. In this case, the modelling needs to be altered as follows Fig. IX.6.

Here the configurational contexts of which the figure is part is equal in both configurations (i.e., $R_{\text{config}_{\text{left}}} = R_{\text{config}_{\text{right}}}$). The fact that *not* (background Id_{abstract} background´) accounts for the visual difference.

Therefore it seems as if contextual identity could be sketched as follows: $xId_{\text{contextual}}\ y$ iff there are wholes x', y', and foundation relations

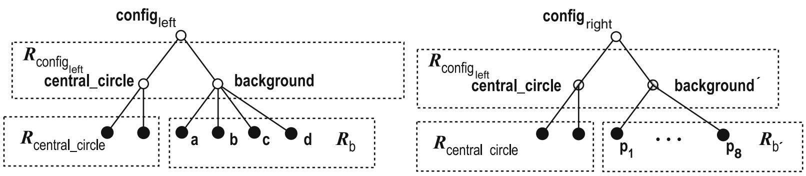


Fig. IX.6

$R_{x'}$, $R_{y'}$ such that $x \in Field(R_{x'})$, $y \in Field(R_{y'})$, and for all objects $v \in Field(R_{x'})$, there is an object $w \in Field(R_{x'})$ if and only if $O(v) = O(w)$. It is obvious that contextual identity is not given in our example, which in other words means that the (general) gestalt-contextures of $\text{config}_{\text{left}}$ and $\text{config}_{\text{right}}$ are different.

Again we point out that our modelling is based on empirical psychology. We are not attempting to provide smooth mathematical properties. It is also important to see that we have not attempted *to explain* why the circles in the middle of the Ebbinghaus look different. We have attempted to model the gestalt-structures of the respective configurations, and obviously the gestalt-structures show perceptual differences.

III. On Relations

Determining a relation is an important part of productive thought. From a phenomenological point of view we will now comment on a philosophical discussion of the ontology of relations that has persisted over a number of years. As a starting point I will take up a critique of von Wachter on Mulligan (1998).

There Mulligan distinguishes between what he calls "external relations" on the one hand and "internal relations" on the other. In order to explicate the use of the notion "external relation," let us assume that a given stone *a* has mass-trope α and a certain other stone *b* has mass-trope β. The view that is rejected by Mulligan is the view that the statement "This stone *a* is heavier than *b*" is made true by a relational trope $\rho(a,b)$ that exists in addition to mass-tropes α and β. This *external relation* between *a* and *b* is a relation between things. It is supposed to be an *entity* that is borne by *a* as well as by *b*. Mulligan and von Wachter both wish to reject those theories that hold that $\rho(a,b)$ is the truthmaker of the statement "This stone *a* is heavier than *b*."

Mulligan suggests reducing "thick relations" to "thin relations" as their truth-makers. Besides similarity thin relations are also identity, greater/smaller/less than, distance, dependence, implication/causality, justification and exemplification. Mulligan characterizes thin relations as *topic-neutral, formal* and *internal*. A relation that is *topic-neutral* says nothing materially about the relata it connects. It connects relata without having material content itself. In particular similarity (a thin relation) does not express any content; it is neutral, it receives its meaning through its relata. A relation is *internal* if it necessarily follows from the existence of its relata. Whether Adam and Eve love each other is contingent (i.e., love is an external relation, also expressing content by itself) whereas the redness of their cheeks is necessarily similar. "Resemblance, too, is an internal relation if it is a relation between tropes. If two things, *a* and *b*, resemble one another this is because there are tropes of the one which stand in an internal relation of similarity to tropes of the other" (Mulligan 1998, 345). Calling a relation *formal* comes close to emptying it from material content with regard to its relata. Similarity cannot be seen *qua* similarity between two objects—there are just two things that are similar. *Love* on the other hand has a meaning in itself and can—so the theory—in a certain sense be seen in isolation from its relata.

Mulligan suggests that a relation between things (i.e., an external relation) obtains because a certain *internal* relation between properties of these things obtains. So the relation of *being heavier* that obtains between *a* and *b* is made true by an internal relation between the mass tropes α and β. The needed internal relation, Mulligan suggests, may be the relation of *being greater than* which, in turn, may be constructed from the relation of *resemblance*. I do not wish to repeat details of the discussion of resemblance. It does, however, make use of the notion of the *order of masses* which—from the point of view of genetic phenomenology—is a highly abstract notion. Eventually views differ between Mulligan and von Wachter: While Mulligan holds that internal relations between tropes are "irreducibly relational entities," von Wachter holds that one can do without relations at all, they are—eventually—just a *façon de parler*. The point he makes seems reasonable (though merely as regards the result): all relations have to be seen as entities in addition to the individuals related.

Before we come to the point of how relations could be conceived of from the point of view of MS, the framework within which the argumentation makes sense should be addressed. This framework is the genetic phenomenology of Edmund Husserl as unfolded in *Experience and Judgement* (*EJ*) or *Analyses of Passive Synthesis* (*APS*). From that point of view the discussion about the ontological status of relations is flawed by its abstractness. For genetic phenomenology, whose orientation is *epistemological*, objectivities like "mass-tropes" are the result of objectivating acts at a very high layer within the stratified conception of consciousness.

In this sense the notion of "mass-trope" has certainly nothing to do with Husserl's notion of a "quality moment" *qua* ingredient of the "noematic sense." This means that a child would not be able to speak about masses or mass-tropes until it is capable of performing formalizing abstraction, idealization or generalizing abstraction.

The method of genetic phenomenology traces abstract concepts back to "proto-objects" on the prelinguistic level of perception. Wiegand (1998) aims to show that—from the point of view of genetic phenomenology—the concept of *individual* as used in formal languages of modal logic is usually too abstract to model the unsharp and fuzzy notion of individuality dominating natural language and perception (see also Wiegand 2000, 2001 and the literature mentioned there). When a phenomenologist reads statements like "I assume that relations between things always obtain *because* the things in question have certain (monadic) properties" (Von Wachter 1998, 356), then expressions like "monadic properties," "relations," or the use of the expression "because" are understood as being located on a relatively high level of abstraction, well above perceptual (proto-)objects.

Intentionality is the objectivating function of consciousness. The phenomenological theory of intentionality is the phenomenological theory of objectivity (and its structure and forms).[16] Intentionality is defined as a correlation of subjective (noetic) and extra-subjective (noematic) factors—two strata that, by themselves, make up a structured whole. Aron Gurwitsch associates this conception of intentionality with equation (ii), and points out the difference to a conception that could be associated with (i):[17]

(i) $P = f(x_e) + f(x_i)$
(ii) $P = f(x_e, x_i)$.

If one thinks of the variables x_e and x_i resp. as ranging over *external* and *internal* factors, then the difference between the two competing conceptions of perception (P) is analogue to the difference between equations (i) and (ii). As regards equation (ii) we might vary x_e as well as x_i and

[16] See Gurwitsch (1959)
[17] Cf. Gurwitsch (1936), Kap. III.

the value *P* will, in both cases, vary accordingly. However, whether one parameter is held constant or is *ignored* (as would be the case with x_e in equation (i) when x_i is varied), makes a significant difference.

It is quite clear that putting emphasis on cognitive activity does not have anything to do with idealism. A phenomenological point of view that is oriented towards gestalt psychology might be accused of *psychologism*,[18] but this challenge seems preferable to a position of "ultimate grounding" or speculative ontological thought. The epistemological orientation of genetic phenomenology and gestalt psychology (at least in the sense of the "Berliner Schule") concerns *meaning*. 'There are' external factors but there would simply not be a dream of *sense* in the world if there was no subjectivity *for which* flowers, theories or tropes made sense. This "making sense for a subjectivity" needs a theory of stratification of consciousness and (noematic) sense as is provided by genetic phenomenology, and "making sense" begins at a rather low level:

In Köhler (1971) we are informed about how anthropoid apes deal with "relations." The information stems from his well-known work on chimp cognition carried through on his Primate Research Centre in Tenerife. There he investigated in particular the chimp's use of tools to obtain food. In one of his experiments a chimp needed to stack boxes in order to reach the famous banana. While an obviously bright chimp called Sultan never hesitated to move boxes in an appropriate way, and stacked them to reach a banana that was suspended out of reach, one other chimp Rana came into serious trouble. Köhler vividly describes Rana's failed effort to imitate Sultan. In one attempt, for example, Rana moved a box (not below the banana, though) climbed onto it, and then—obviously highly concentrated—ran below the banana and jumped high, just as Sultan did. The difference, however, was that Sultan had moved the box right under the banana so that only a jump was needed to gain the food. Rana was obviously not able to merely imitate what the other ape had demonstrated. Köhler says:[19]

> A chimp, however, who is particularly unintelligent, may be fully unable to repeat something simply because certain relations that were essential to the other's demonstration

[18] See Seebohm (1991)
[19] Quotes from Köhler (1971), my italics. See also Köhler [1925], 1999.

eluded him (p. 117) ... Already when Sultan begins to move the box, he moves it in the direction of the bananas. For the simple-minded Rana, however, there is no compelling reason to set the beginning of the movement in relation to the location where the box may then serve to diminish the distance between the ground and the fruit. To Rana the beginning of the movement may perhaps appear as a simple form of game. In fact chimps often simply shove boxes around when they play. Or Rana may regard the beginning of the movement as a movement away from the original location of the box, which would be a further relation, but again not the one demanded. Furthermore the movement could be regarded as a movement parallel to one of the walls or the like (p. 119).

There are infinitely many possibilities for moving a box around. What about the infinitely may *internal relations* whose relata already "exist" in the cage? Following the analyses of *EJ* the concept of a *relatum*, even that of a relatum *qua individual object in the perceptual field* is located on a rather high level of cognition, at *proto-relations.* These emerge out of what Husserl calls the "outer horizon" of a (proto-)object (*EJ* Section III; see also Wiegand 1998, Chapter VI.3). This horizon changes when the nucleus of the horizon is inserted into another context, when it becomes part of another structured whole or when the part structure of the nucleus has changed (the parts may be rearranged, the whole may be ruptured, etc.). Depending on these changes the outer horizon changes and certain relations become possible while others simply disappear. Which relation is eventually picked out by the *attention of the subject* (see *EJ,* §§ 17 ff.) depends on the higher primate's (proto-) goals, drives, and past experiences. The nuclei and their outer horizons are seen respectively in different lights, are associated with different drives and goals, and therefore, just *make sense* in relation to particular relata, and *do not make sense* in relation to others.

The abstraction that characterizes ontological reasoning does not only guarantee the identifiability, and therewith the objectivity of the entities under discussion (see Wiegand 2000, Section 4; 2001, §4; see also Hume 1739, 199 ff). It seems as if this abstract reasoning also leads authors to a certain preference for a particular type of relations, namely those that are to be located within the quantifiable or measurable realm. Contrary to *love*, relations like greater / smaller / less than, distance, dependence (when suitably formalized), etc. are all *formal* in the sense of the phenomenology of mathematical objects (see Wiegand 1998, 2000).

Hochberg (1992) groups relational tropes (we recall that tropes are usually understood as coming close to Husserl's qualitative moments) into similarity classes. He tries to provide a solution to the problem of how "greater than" could represent a trope similiarity class. From the point of view of genetic phenomenology, it is a commonplace that the objects he discusses are *eo ipso* devoid of all content; they are the result of what Husserl called a "process of algebraization" by which he meant a process of abstraction that characterizes (modern) mathematics (see *FTL* Part I; Wiegand 2000). But already on the level of natural language (not to speak about the prelinguistic level) the well-known mathematical relational properties like "transitivity," etc. are not generally present.

There is good cause to dwell on this important point for a while. In order to clarify the issue, I will summarize the gist of Leonard Talmy's analyses on the general conceptualization of figure and ground in natural language. The examples are all taken from Talmy 2000, Vol. I, Chapter 5. There he defines *the figure* as a moving or conceptually movable entity whose path, site, or orientation is conceived as a variable, the particular value of which is the relevant issue. *The ground* on the other hand is a reference entity, one that has a stationary setting relative to a reference frame, with respect to which the figure's path, site, or orientation is characterized. What is figure and what is ground can be identified by applying the method of reversing the nominals in a sentence:

(1) a. The bike (F) is near the house (G).
    b. ? The house (F) is near the bike (G).
(2) a. John (F) is near Harry (G).
    b. Harry (F) is near John (G).

The method of reversing the nominals in a sentence to highlight the existence of figure and ground roles in a locative event can use the otherwise symmetric relation "near." It can also use an asymmetric relation if we consider the relation and its inverse, like "above / below":

(3) a. The TV antenna (F) was above the house (G).
    b. ? The house (F) was below the TV antenna (G).

The figure / ground functions extend to (some) nonphysical situations. The relation of resemblance, we recall, was classified as an internal relation according to Mulligan, and should therefore be conceived of as symmetric and devoid of content. The following examples from natural language show that resemblance has nevertheless some content, since otherwise symmetry could not occasionally fail on the level of natural language. Furthermore we are confronted with the question of deciding *when* resemblance is a relation between objects or between tropes. Example (4) may be viewed as a relation between tropes ("unique appearances"). As regards example (5) it is obviously a relation between things (it may, however, be reducible to a resemblance between tropes):

(4) a. She resembles him.
b. She is similar in appearance.
c. ? He resembles her.
(5) a. My sister (F) resembles Madonna (G).
b. ? Madonna (F) resembles my sister (G)

The reference of figure and ground to the relative location of objects in space can be generalized to the relative location of events in time:

(6) He exploded after he touched the button.

This statement seems to assign a ground interpretation to the button-touching event (setting it up as a fixed, known reference point), while the next example has a different meaning:

(7) He touched the button before he exploded.

Now, what in fact *are* relations from the point of view of MS? Let us at first point out that Wiegand (2007) offers a simple language of mereology that contains symbols for relational predicates that are to be interpreted in the vain of traditional logic (i.e., as sets of ordered pairs, triples etc.). There are no variables that range over relations. In this sense the above mentioned discussion of the ontological aspects of relations does not immediately apply to MS in its present shape.

If, however, one insists on asking how relations should be conceived of from the point of view of the *background-philosophy* of MS, then it needs to be repeated that this point of view is actually that of genetic phenomenology. As such it heavily leans on the phenomenological analyses of the genesis of relational structures in the prepredicative field (see *EJ* Section III; see also Wiegand 1998, Chapter VI.3). Modelling proto-relations on a level where they are still *gestalthaft* (the interesting case) seems hardly possible since proto-relations do not generally come along with certain relational properties. Relations on that level are marked by a certain plasticity that does not generally allow for distinctions like that between symmetric and non-symmetric relations etc.

However, the formal language of mereology (into which portions of natural language may be translated) that is provided in Wiegand (2007) is kept rather simple. More complex languages may be developed that allow for other sorts of structured wholes in the domain of quantification (here we agree with Moltmann 1997, when, with regard to the notion of *R-integrated whole*, she says that other sorts of integrated wholes are conceivable). It was stressed that MS is a technical framework in the first place. A modularly structured domain of quantification may encompass various sorts of suitably defined objects. Including structured wholes into the universe was suggested by the author under the premise that this notion may be found to be a handy formalization of principles of the notion of Gestalt.

In order to sketch how relational objects could be incorporated into the framework of MS let us contemplate the following types of objects (Figs. IX.7 and IX.8):[20]

Informally speaking (j) shows a simple structured whole. We have tried to unfold the *raison d'être* for that type of objects in section one above. If we understand tropes in the sense of Husserl's qualitative moments, an element like *d* (e.g., a color) could be called a trope, having, in turn, two moments (like extension and a certain hue) as parts. If, in addition, one wishes to speak about universals, one would have to define objects (possibly equipped with some suitably defined foundation-relation) that can be part of more than one structured whole at the same time. These objects

[20] See also Johansson (2006)

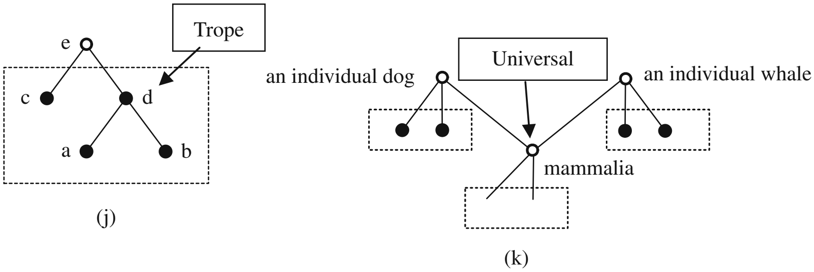
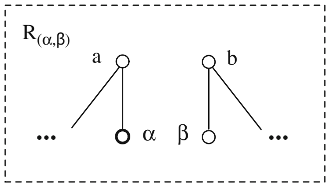


Fig. IX.7



(1)

Fig. IX.8

should then look like the schema (k) where an uncompleted Hasse-Diagram (together with dotted rectangulars for foundation-relations) is indicated. A third sort of structured objects would now be necessary to model relations as Gestalts. Objects of that sort should also be characterized by the functional interdependence of the parts (the "relata") of the whole and the principle of the *primacy of the whole*. Schema (l) indicates how—in the simple case of a relation between two objects—such a relational whole could look like. The drawing may be interpreted as indicating a formalization of "objects *a* and *b* bear a relation because of tropes α and β, resp." Changing the part-structure of *a* or *b* would have an effect on the field of the foundation relation $R_{(\alpha,\beta)}$. The definition of $R_{(\alpha,\beta)}$ may include relational properties.

Abstraction is the prize of formalisation. On the level of the *Lebenswelt* relations like *a* loves *b* are characterized by a certain *Gestalthaftigkeit*. If *b* dies, then *a* is no longer the lover she used to be. If *b* doesn't love *a* then the relation lacks the symmetry that—to a certain degree—characterizes lovers. Perceiving a couple in love has the form of perceiving a figure (the couple) before a background of hastily moving people or on the background of a certain landscape, etc. Love can—to use a Husserlian technical term—be *nominalized*, i.e., it can be objectified (*EJ*, §58). This means that we have a name that does in a rather fuzzy way apply to the various expressions, emotions and to the behavior of lovers (all that *depends*, of course, on the "relata" involved). As long as we remain on the life-world level love has certain (unsharp and fuzzy) properties that vanish when we speak about relations from an abstract (mathematical or formal ontological) point of view. In the latter case all relations (love or greater than) loose their plasticity and become defined formal objects. MS was designed to provide formal techniques for use in cognitive semantics, i.e., for use in the analyses of natural language reference in the life-world.

IV. Mereological Semantics: Logig As Philosophy?

In this paper we have tried to sketch the philosophical motivation for what I have called *mereological semantics* (MS) (see the introduction above). The formalism of mereological semantics has been unfolded in a more detailed fashion in Wiegand 2007. Within the present paper we have in the first place tried to explicate how details of the concept of Gestalt have led to particular details of the formalism of MS. A crucial part in the formalization of the concept of Gestalt is played by the formal foundation-relation *R*. The relational character of Gestalts has also been pointed out by Moltman 1997, Rota 1985 and 1989, as well as Rescher and Oppenheim 1955. As a first application of the theory we have thus critically commented on a still ongoing discussion within ontology on the nature of relations. From the point of view of the background philosophy of MS we have taken a strictly constructive stance. Let us now dwell for a while on the connection between MS and phenomenology in general. We may do that in two steps.

At first we have to recall what Edmund Husserl notes in his Third Logical Investigation with regard to his mereological analyses: "A true realisation of the pure theory, in the sense we are developing it, would have to define all of its concepts with mathematical exactitude and define its theorems by means of argumenta in forma, that is through mathematical deduction" (Hua 19/1, 294). In Chapter 2 of the Third LI Husserl makes several proposals for such precise definitions, however he calls them "mere suggestions" (Hua 19/1,294). But he never did develop these initial suggestions in his later works. Husserl also points out that his theory is "of greatest importance for all phenomenological investigations" (introduction to Third LI). However, the layers of consciousness (from the level of passive synthesis up to the level of formalizing abstraction), the poles of intentionality (defined as noetic-noematic correlation),[21] concepts like (identity and temporality),[22] and even *formal logic* and *mathematics* as parts of what Husserl calls *mathesis universalis*[23] can certainly not be viewed as parts of an aggregate but should rather be seen as parts (respectively, poles) of structured wholes. One important application of mereology, namely to the phenomenological study of natural language, has been suggested by Husserl in his Fourth Logical Investigation.[24] In Wiegand 2007 we have made a suggestion what a formal mereological semantics—into which portions of natural language may be translated—could look like. Our idea of how elementary linguistic predication could be interpreted with respect to a universe of parts was inspired by Husserl's dictum: "Every non-relative 'real' [*reale*] predicate therefore points to a part of the object which is the predicate's subject: 'red' and 'round,' e.g., do so, but not 'existent' or 'something'" (Hua 19/1,231).

Gestalt theory enters the scene when one adopts the convincing criticism of Aron Gurwitsch on Husserl's mereology (see Wiegand 2001) according to which both Stumpf and Husserl were holding positions close to associationist psychology,[25] and thus do not stick to a

[21] See Gurwitsch (1982) or Gurwitsch (1940).
[22] See Gurwitsch (1940).
[23] See Wiegand (2000).
[24] Cf. the introduction to the Fourth Logical Investigation. See also Appendix I of *FTL*.
[25] Associationist psychology is opposed to the Gestalt approach. Basically, the former kind of psychology understands conscious acts merely as the results of the composition and modification of sensual contents. The psychological laws in accordance with which those compositions and modifications work are called "laws of association". Associationism can be traced back to British empiricism, but Joseph Priestley, James Mill, and Johann F. Herbart are also eminent figures in that tradition. From the viewpoint of Gurwitschian phenomenology, the psychological atomism and the "psychophysics" of Gustav Theodor Fechner and Hermann von Helmholtz must also be regarded as a physicalistic branch of associationism. The main reason why Gurwitsch has severely criticized associationist psychology is that physical stimuli or psychological

consistently descriptive point of view. MS claims to provide a formalization that (a) meets Husserl's postulate of a formalization and (b) captures Gurwitsch's phenomenological tenets of a consistently descriptive theory of parts, wholes and foundation. But MS may even claim more.

In the second step MS claims to provide the means for an analysis of intentionality. In §§ 30–32 of *EJ* Husserl shows how his analyses of the constitution of objects may be reformulated in terms of parts and wholes. Taking our starting point from that possibility we formulated the basic tenet of MS, namely that *reference is reference to structured wholes*. Intentionality, defined as the *objectifying function of consciousness* (Gurwitsch 1940) may now be analyzed in a twofold way: (a) in the sense of a *subjective* logic of truth. This subjective logic of truth, has—by Seebohm and others—been developed as a *phenomenological semantics* (see Wiegand 2000). (b) Intentionality may also be analyzed in terms of formal logic. Focusing on especially this latter formal logical aspect Hintikka has said in his 1969 book that "... possible–worlds semantics is the logic of intentionality, and intentional is what calls for possible–worlds semantics" (p. 195). The formalism of MS (as unfolded in Wiegand 2007) is abstract in nature, but is motivated by phenomenological semantics, and a phenomenological mereology that has incorporated Gurwitsch's critique from the point of view of Gestalt-theory. MS claims to be *a logic of intentionality*—rough and abstract, but modelling the basic structures of our reference to extralinguistic structured wholes as unfolded in informal descriptive analysis.

Max Wertheimer has defined productive thought as a process that (ideally) leads from an initial state of approximate chaos to

laws of association—they need not be the Humean laws—assume the role of causes. In this sense causal explanation is the main methodological tool of associationist psychology, whereas Gestalt psychology—like phenomenological method—descriptive in nature. Gestalt psychology does not distinguish between the stimuli and the laws of association that cause a certain unity among an in-itself scattered and unstructured manifold of sense data. Gurwitsch formulates the main tenet of a strictly descriptive approach to the psychological as follows: "for intentional analysis the ultimate fact and datum is the sense or meaning itself as a structured whole" ("Phenomenology of Thematics and of the Pure Ego," 257). For a summary of Gurwitsch's critique of associationism (where he also mentions Hobbes, Locke, and Herbart) see "The Place of Psychology in the System of Sciences," in *Studies in Phenomenology and Psychology,* 56–68.

an organized totality of parts. The process is described in terms of Gestalt-transformations that are goal-directed.[26] Instead of Gestalt-transformations Aron Gurwitsch speaks of "attentional modifications."[27] In MS we speak about "intentional situations" which may—informally speaking—be understood as the cornerstones of productive thought. These situations are connected by modification relations that model "attentional modifications" (see (ii) in the introduction above). This class of relations is meant to model those laws of cognitive psychology (Gestalt theory) that rule over the modification of objects, and so it obtains between cognitive situations. The stream of (intelligent) cognitive activity is characterized by constantly inverting, synthesizing, extending, or otherwise modifying cognitive situations *qua* structured wholes. In this sense transformation relations are of *empirical origin* and are to be considered "general" to the extent that the basic laws of Gestalt theory are "general." Transformation relations may be seen as analogous to what is called an "accessibility relation" in the context of "possible worlds semantics." Since transformation relations lead from one cognitive situation to the next, they serve to model "courses of reasoning."

In the sense of what has been said within this chapter it may have become plausible that MS claims to be a logic of intentionality, and is therefore to be seen in the tradition of a vein of thought that has been initiated by Hintikka, Seebohm, Mohanty and many others who have tried to combine the descriptive study of intentionality with formal logical techniques.

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