Chapter 6
Circular Membrane Plates

In this chapter circular plates in a membrane state require our attention. In Section 6.1 we study plates for axisymmetric load. Section 6.2 is devoted to non-axisymmetric load. Plate bending will be the subject of Chapter 7.

6.1 Axisymmetric Circular Membrane Problems

Figure 6.1 shows a homogeneous circular plate of constant thickness t and axisymmetric loading. For this type of problems it is convenient to change to polar coordinates. The position in the plate is specified by means of the radius r and the angle θ. The state of stress and strain is independent of θ, and there is just one displacement u in radial direction. Ordinary derivatives can be used since u depends only on the coordinate r, as does the load p per unit area. Only two membrane forces are present, n_{rr} and $n_{\theta\theta}$. The shear stress $n_{r\theta}$ cannot occur. Therefore, only two strains exist, ε_{rr} and $\varepsilon_{\theta\theta}$. The scheme for the essential quantities is displayed in Figure 6.2.

The strain ε_{rr} and the displacement u both act in the r-direction; they are related by $\varepsilon_{rr} = du/dr$. For the derivation of the tangential strain, $\varepsilon_{\theta\theta}$, a circle is considered with radius r. The circumference of this circle is $2\pi r$. After application of the axisymmetric load, each point of the circle displaces over a radial distance u. The new radius of the circle is $r + u$ and the circumference $2\pi(r + u)$. The increase of the circumference is $2\pi u$. Division of this increase by the original length $2\pi r$ provides the required strain $\varepsilon_{\theta\theta} = u/r$. So, the constitutive relations for plane stress are

$$\varepsilon_{rr} = \frac{du}{dr}, \quad \varepsilon_{\theta\theta} = \frac{u}{r} \quad \textit{Kinematic} \tag{6.1}$$

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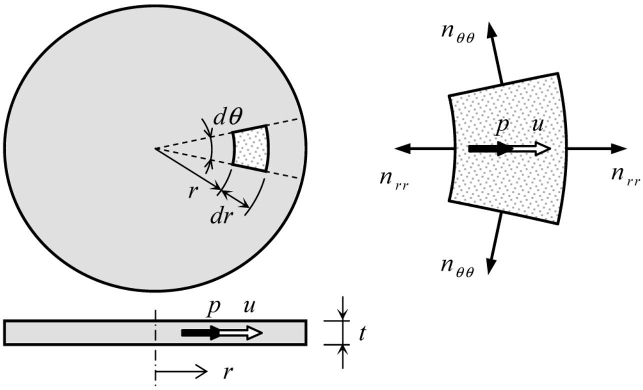

Figure 6.1 Displacement, load and membrane forces in axisymmetric plate.

The constitutive relations of Eq. (1.13) reduce to

$$\left.\begin{aligned} n_{rr} &= \frac{Et}{1-\nu^2}(\varepsilon_{rr} + \nu\varepsilon_{\theta\theta}) \\ n_{\theta\theta} &= \frac{Et}{1-\nu^2}(\varepsilon_{\theta\theta} + \nu\varepsilon_{rr}) \end{aligned}\right\} \quad \textit{Constitutive} \tag{6.2}$$

For the equilibrium equations we consider an elementary plate part of length dr and aperture angle $d\theta$ as shown in Figure 6.1. The length of the edge at the inside of the element is $r\,d\theta$. The total force on this edge is $n_{rr}r\,d\theta$ and points to the left. At the outside of the element, at a distance dr further, the force has increased $d(n_{rr}r\,d\theta)\,dr$, pointing to the right. The angle $d\theta$ is independent of r, which means that the increment can be written as $d(n_{rr}r)\,dr\,d\theta$. A force $n_{\theta\theta}\,dr$ is acting perpendicular to each straight edge of the element. Since the angle between the two forces is $d\theta$, there is a force $-n_{\theta\theta}\,dr\,d\theta$, where the minus sign indicates the direction of the force (negative r-direction).The distributed load p provides an outward-pointing force. For that purpose, p

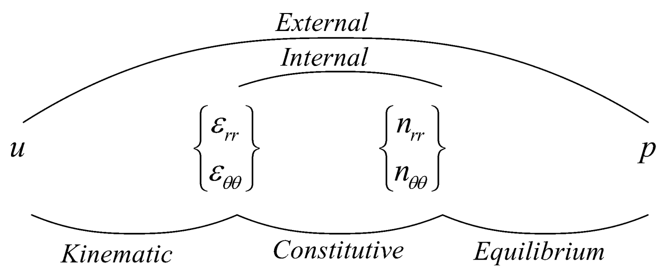

Figure 6.2 Scheme of relationships.

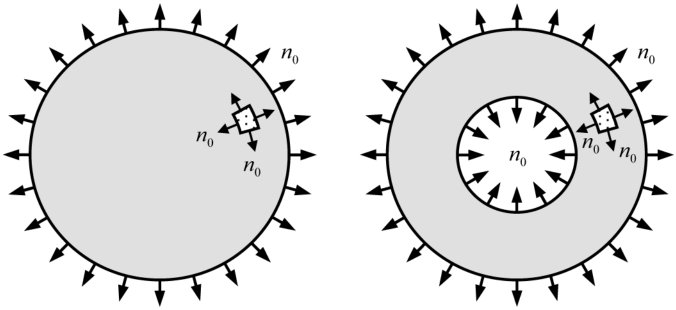

Figure 6.3 Only one constant stress-state exists with equal n_{rr} and $n_{\theta\theta}$.

has to be multiplied by the area $r\, rmd\theta\, dr$ of the plate element. The force equals $pr\, d\theta\, dr$. For equilibrium, the sum of the three forces has to be zero. After division by $d\theta\, dr$ the following equilibrium equation is obtained:

$$-\frac{d}{dr}(r\; n_{rr}) + n_{\theta\theta} = rp \quad \textit{Equilibrium} \tag{6.3}$$

In this stress problem, there are no rigid body displacements. For any displacement u there is a strain field. Further, only one combination of constant strains is possible. For the strain $\varepsilon_{\theta\theta}$ to have a constant value ε_0 a displacement is required of $u = \varepsilon_0 r$. The strain ε_{rr} then also equals ε_0. The only possible constant strains are identical strains $\epsilon_{\theta\theta}$ and ϵ_{rr}. Then, from the constitutive relations in Eq. (6.2) it follows that the membrane forces n_{rr} and $n_{\theta\theta}$ are equal and constant too. When the constant values $n_{rr} = n_0$ and $n_{\theta\theta} = n_0$ are substituted in the equilibrium equation (6.3) it appears that the distributed load p across the plate area has to be zero. The plate can be loaded only along the edge. Figure 6.3 shows two situations, a circular plate with and without a hole. In both plates, in each point, a membrane force n_0 is present and Mohr's circle is reduced to a point.

An alternative to deriving the three basic sets of equations is the consideration of work. Slightly different quantities are used, which we will show here. The equilibrium equation (6.3) comprises the terms rn_{rr} and rp. It makes sense to introduce new variables for these combinations. This will be done for $rn_{\theta\theta}$ too. We define

$$N_{rr} = r\, n_{rr}; \qquad N_{\theta\theta} = r\, n_{\theta\theta}; \qquad f = rp \tag{6.4}$$

The two quantities N_{rr} and $N_{\theta\theta}$ are normal forces with the dimension of force; f is a line load with the dimension of force per unit of length. Application of the transformations in Eq. (6.4) keeps the kinematic equations

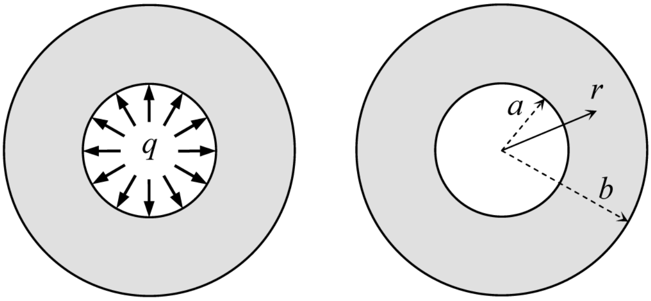

Figure 6.4 Thick-walled pipe with load at inner face.

(6.1), and changes the constitutive equations (6.2) and equilibrium equation (6.3) into

$$\varepsilon_{rr} = \frac{du}{dr}, \quad \varepsilon_{\theta\theta} = \frac{u}{r} \qquad \textit{Kinematic} \tag{6.5}$$

$$\left.\begin{aligned} N_{rr} &= \frac{Etr}{1-\nu^2}\left(\varepsilon_{rr} + \nu\varepsilon_{\theta\theta}\right) \\ N_{\theta\theta} &= \frac{Etr}{1-\nu^2}\left(\varepsilon_{\theta\theta} + \nu\varepsilon_{rr}\right) \end{aligned}\right\} \qquad \textit{Constitutive} \tag{6.6}$$

$$-\frac{dN_r}{dr} + \frac{N_{\theta\theta}}{r} = f \qquad \textit{Equilibrium} \tag{6.7}$$

We will continue with these three equations for the derivation of the differential equation. Substitution of the kinematic equations (6.5) into the constitutive equations (6.6) leads to

$$N_{rr} = \frac{Etr}{1-\nu^2}\left(\frac{du}{dr} + \nu\frac{u}{r}\right); \quad N_{\theta\theta} = \frac{Etr}{1-\nu^2}\left(\frac{u}{r} + \nu\frac{du}{dr}\right) \tag{6.8}$$

Substitution of this result into the equilibrium equation (6.7) leads to the differential equation

$$\frac{Et}{1-\nu^2}Lu = f \tag{6.9}$$

where the operator L is

$$L = r\frac{d}{dr}\frac{1}{r}\frac{d}{dr}r \tag{6.10}$$

This differential equation is of the second order.

6.1.1 Thick-Walled Tube

The differential equation can be used to determine the stress state in a thick-walled tube subjected to an internal gas pressure q. Figure 6.4 defines the tube. Flat sections remain flat after deformation, but the strain ε_{zz} in the axial direction will not be zero. On average, σ_{zz} will be equal to zero. Therefore, the problem will be treated as a plane stress state. A slice of unit thickness of the tube is considered, which is cut perpendicularly to the axial direction. This means that n_{rr} and $n_{\theta\theta}$ are equal to σ_{rr} and $\sigma_{\theta\theta}$, respectively. For this load case it holds that the distributed load f is zero and therefore

$$Lu = 0 \tag{6.11}$$

In the general solution of this second-order differential equation there are two coefficients which must be determined from the boundary conditions. Choosing the trial solution Cr^m, we obtain two roots $m = -1$ and $m = 1$. Therefore

$$u = C_1 \frac{1}{r} + C_2\, r \tag{6.12}$$

The coefficients C_1 and C_2 follow from the two boundary conditions

$$\begin{aligned} r = a \;&\rightarrow\; n_{rr} = -q; \quad N_{rr} = -qr \\ r = b \;&\rightarrow\; n_{rr} = 0; \quad N_{\theta\theta} = 0 \end{aligned} \tag{6.13}$$

The results for the displacement u and the stress quantities n_{rr} and $n_{\theta\theta}$ become

$$u(r) = \frac{a^2}{b^2 - a^2}\,\frac{q}{Et}\left\{(1+\nu)\,\frac{b^2}{r} + (1-\nu)\,r\right\} \tag{6.14}$$

$$n_{rr} = \frac{a^2}{b^2 - a^2}\left(-\frac{b^2}{r^2} + 1\right) q; \quad n_{\theta\theta} = \frac{a^2}{b^2 - a^2}\left(\frac{b^2}{r^2} + 1\right) q \tag{6.15}$$

The results are presented in Figure 6.5. Both stresses σ_{rr} and $\sigma_{\theta\theta}$ are nonlinear over the thickness of the tube.

6.1.2 Circular Hole in a Homogeneous Stress State

We want to compute the stress concentration factor on the edge of a circular hole in a large plate with a homogeneous stress state of equal principal membrane forces n. The radius of the hole is a. The homogeneous membrane forces without the hole are

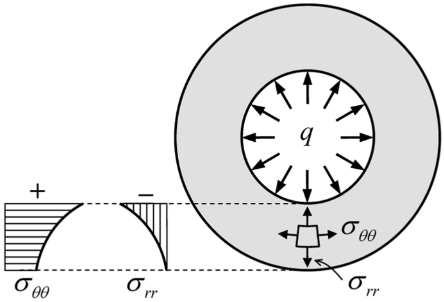

Figure 6.5 Stresses in a thick-walled pipe due to gas pressure.

$$n_{rr} = n; \quad n_{\theta\theta} = n \tag{6.16}$$

In order to make the edge of the proposed hole to be stress-free a loading case must be superimposed in which the edge is loaded with an opposite load on the boundary $r = a$. The boundary condition is $n_{rr} = -n$. Further it is known that the stresses will vanish for large radius r, which requires C_2 to be zero. The result is

$$n_{rr} = n\left(-\frac{a^2}{r^2}\right); \quad n_{\theta\theta} = n\left(\frac{a^2}{r^2}\right) \tag{6.17}$$

Still these membrane forces must be superimposed on the constant equal stresses for the case without a hole. The final result for the large plate with hole is

$$n_{rr} = n\left(1 - \frac{a^2}{r^2}\right); \quad n_{\theta\theta} = n\left(1 + \frac{a^2}{r^2}\right) \tag{6.18}$$

Due to the hole, the maximum value of the membrane force $n_{\theta\theta}$ is twice the value n of the homogeneous stress state. The stress concentration factor is 2, see Figure 6.6.

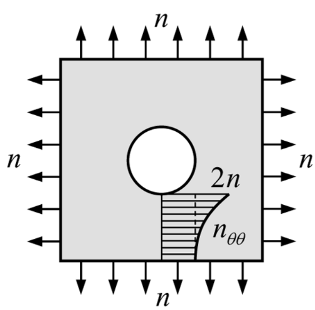

Figure 6.6 Concentration factor 2 for equal principal stresses.

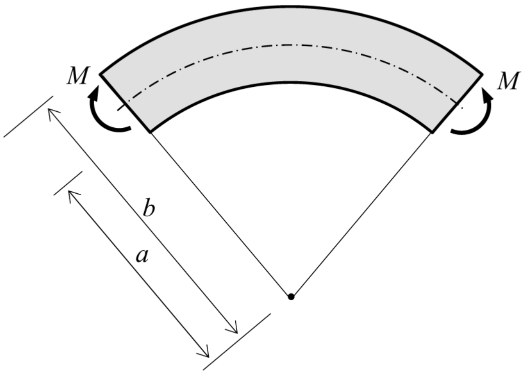

Figure 6.7 Curved beam subjected to constant moment.

6.1.3 Curved Beam Subjected to Constant Moment

Consider a curved beam with a constant radius of curvature as shown in Figure 6.7. The inner and outer radii are a and b, respectively, and the beam is subjected to a constant moment M. The stress state in this axisymmetric structure will also be axisymmetric. The beam has a rectangular cross-section of small width t. This curved bar is modelled as a thin plate with thickness t. This problem can be solved with the findings of Section 6.1. An alternative solution procedure is given in [7]. The stresses are

$$\sigma_{rr} = f_{rr}\frac{M}{C}; \quad \sigma_{\theta\theta} = f_{\theta\theta}\frac{M}{C} \tag{6.19}$$

where

$$C = t\left(\frac{1}{4}\left(b^2 - a^2\right) - \frac{a^2b^2}{b^2 - a^2}\left(\ln\frac{b}{a}\right)^2\right) \tag{6.20}$$

$$\begin{aligned} f_{rr} &= -\frac{ab}{b^2-a^2}\ln\left(\frac{b}{a}\right)\frac{ab}{r^2} + \frac{a^2}{b^2-a^2}\ln\left(\frac{r}{a}\right) - \frac{b^2}{b^2-a^2}\ln\left(\frac{r}{b}\right), \\ f_{\theta\theta} &= -1 + \frac{a\,b}{b^2-a^2}\ln\left(\frac{b}{a}\right)\frac{a\,b}{r^2} + \frac{a^2}{b^2-a^2}\ln\left(\frac{r}{a}\right) - \frac{b^2}{b^2-a^2}\ln\left(\frac{r}{b}\right) \end{aligned} \tag{6.21}$$

In these expressions, C depends only on a, b and the thickness t, the geometry data. The functions $f_{rr}(r)$ and $f_{\theta\theta}(r)$ are dimensionless and provide the distribution of stresses over the height of the cross-section. In Figure 6.8 this distribution is displayed for two different values of the ratio a/b, a value

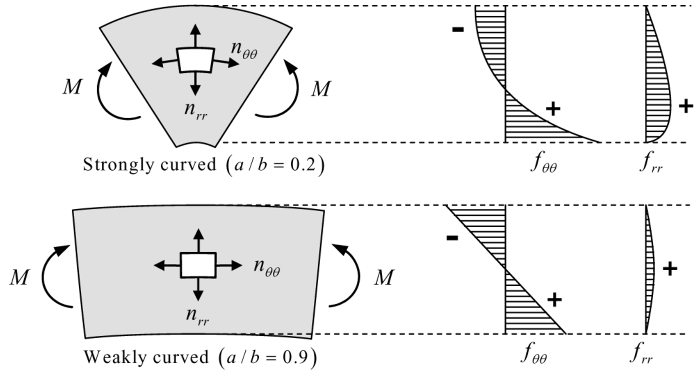

Figure 6.8 Stress distribution in curved beam for different curvatures.

that is small compared to unity (strong curvature) and a value close to unity (weak curvature). For a strong curvature, the bending stress distribution deviates severely from a linear distribution, irrespective of the fact that flat cross-sections remain flat.

We note that for pure bending, there are also stresses σ_{rr} in the height direction. This can be made clear if we consider the equilibrium of the part of the beam below the neutral line. Integration of the tensile stresses $\sigma_{\theta\theta}$ over the height of the beam part leads to a tensile force. The two tensile forces acting on both ends of the beam part have different directions and work line. Equilibrium is possible only if there is a radial outward-pointing stress σ_{rr} in the neutral line, acting over the whole length of the beam part. Therefore, it can be concluded that σ_{rr} is a tensile stress. The same conclusion is obtained if we consider the part of the beam outside the neutral line, where compressive stresses $\sigma_{\theta\theta}$ are present. If we translate this finding to a curved reinforced concrete beam, we conclude that stirrups are needed in a curved beam even when there is no shear force. Figure 6.9 demonstrates this by drawing struts and ties in the curved beam. Red lines are in tension and green lines in compression.

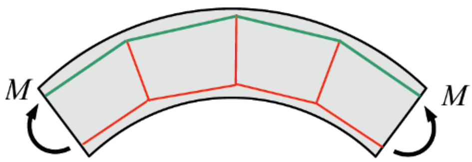

Figure 6.9 A constant moment in concrete curved beam may ask for stirrups.

6.2 Non-Axisymmetric Circular Membrane Problems

If the stress state is not axisymmetric, we have to account for three stresses σ_{rr}, $\sigma_{\theta\theta}$ and $\sigma_{r\theta}$. It appears advantageous to switch to the force method for this type of problem. We will discuss problems where load is applied only on the edges. We start the discussion in the orthogonal set of axes x, y. In Chapter 1 we derived the kinematic equations (1.9), the constitutive equations (1.13) and the equilibrium equations (1.14), and we substituted them in each other, starting from the kinematic equations and ending with the equilibrium equations. In the force method we make use of these three set of equations in the opposite order. We first construct a stress solution that satisfies equilibrium, after that we use the constitutive relations, to end up with expressions for strains and finally we construct a compatibility condition for the strains from the kinematic relations. In Eq. (1.14) we have three unknown stresses in two equilibrium equations. Therefore the stress state is statically indeterminate and we introduce a stress function $\varphi(x, y)$, as initially proposed by Airy [13]. The following set of stresses satisfies the two equilibrium equations in (1.6) for zero distributed area forces p_x and p_y

$$n_{xx} = \frac{\partial^2 \varphi}{\partial y^2}; \quad n_{yy} = \frac{\partial^2 \varphi}{\partial x^2}; \quad n_{xy} = -\frac{\partial^2 \varphi}{\partial x \partial y} \quad \textit{Equilibrium} \tag{6.22}$$

The constitutive relations are now used in the shape of Eq. (1.12)

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{Et} \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{Bmatrix} n_{xx} \\ n_{yy} \\ n_{xy} \end{Bmatrix} \quad \textit{Constitutive} \tag{6.23}$$

The required compatibility relation for the strains is derived from the kinematic relations in (1.9). There three strain relations are expressed in terms of in two degrees of freedom u_x and u_y. Elimination of these two displacements leads to one relation between the three strains

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} - \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = 0 \quad \textit{Compatibility} \tag{6.24}$$

Substitution of the three relationships (6.22) and (6.23) into Eq. (6.24) leads to a differential equation for the stress function φ. Again we find the biharmonic equation which was obtained in Chapter 1.

$$\left(\frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial x^2} + \frac{\partial^4}{\partial y^4} \right) \varphi = 0, \quad \nabla^2 \nabla^2 \phi = 0 \tag{6.25}$$

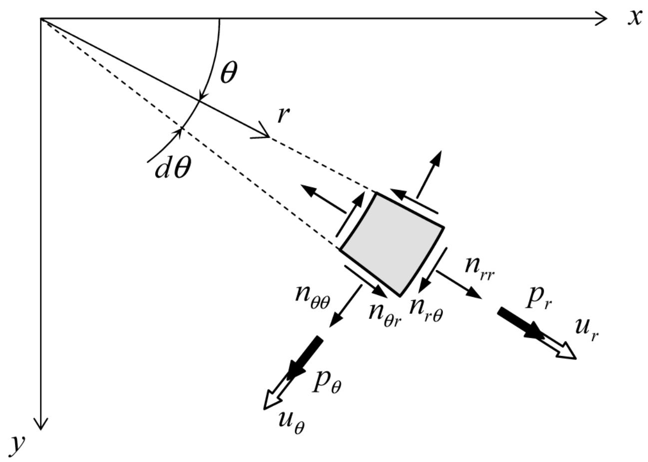

Figure 6.10 Displacements and stresses in polar coordinates.

Due to Eq. (6.22), the quantity $\nabla^2\varphi$ is the sum of the two normal forces. Figure 6.10 shows which displacements and stresses are involved for the description in polar coordinates. The three second derivatives in (6.22) need be transformed to derivatives with respect to r and θ. More precisely, we have to change from the orthogonal set of axis (x, y) to the orthogonal set (r, t) of directions, where t is the direction of the tangent line to a circle of radius r, see Figure 6.10. Formally the transformation is done with aid of the chain rule for derivatives; This leads to

$$\begin{aligned}
\frac{\partial^2\varphi}{\partial x^2} &\to \frac{\partial^2\varphi}{\partial r^2} \\
\frac{\partial^2\varphi}{\partial y^2} &\to \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} \\
\frac{\partial^2\varphi}{\partial x\partial y} &\to \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\varphi}{\partial\theta}\right)
\end{aligned} \tag{6.26}$$

The transfer from the second derivative with respect to x to the second derivative with respect to r is simple. We just replace x by r. The mixed second derivative with respect to x and y is also simple, if we notice that dy is equal to $r\,d\theta$. However, the transformation of the second derivative with respect to y needs more explication. The result consists of two contributions. The last one, which has $r^2\,d\theta^2$ in the numerator, is expected; but the first one may be a surprise. This term is independent of θ. Figure 6.11 helps to explain this term. To understand the second derivative in t-direction we consider the

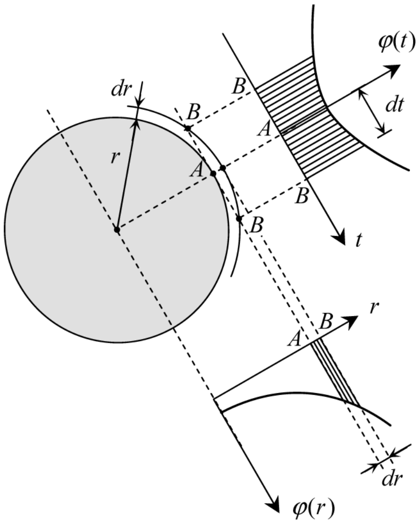

Figure 6.11 Second derivative in tangent line in axisymmetric state.

value of φ at point A on the circle with radius r and the two points B at the circle with radius $r + dr$. If $d\varphi/dr$ is not zero the value of φ in the points B will be different from the value in point A, and therefore a non-zero second derivative exists in the tangent line at point A. This second derivative always occurs, in axisymmetric as well as non-axisymmetric cases.

Now it is clear how we must replace Eq. (6.22). For polar coordinates we use

$$n_{rr} = \frac{1}{r}\frac{\partial \varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \varphi}{\partial \theta^2}; \quad n_{\theta\theta} = \frac{\partial^2 \varphi}{\partial r^2}; \quad n_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \varphi}{\partial \theta}\right) \tag{6.27}$$

The Laplace operator ∇^2 can be determined from the sum of n_{rr} and $n_{\theta\theta}$. The equation of Airy (6.25) then becomes:

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\varphi = 0 \tag{6.28}$$

6.2.1 Point Load on a Half Plane

The obtained differential equation (6.28) can be used to find the stress distribution in a half plane due to a point load F on the edge, as shown in

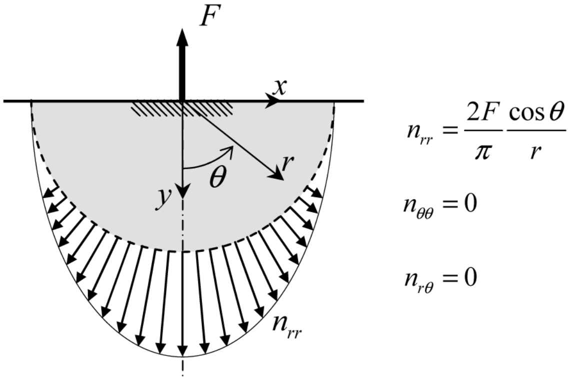

Figure 6.12 Stresses in half-plane with vertical point load.

Figure 6.12. The boundary conditions are zero stresses $\sigma_{\theta\theta}$ and $\sigma_{r\theta}$ at the free edge and zero stresses for infinitely large radius r. It can be shown that the trial function

$$\varphi = Cr\theta\sin\theta \tag{6.29}$$

satisfies Airy's biharmonic equation and the boundary conditions. By application of Eq. (6.27) the membrane forces become

$$n_{rr} = 2C\frac{1}{r}\cos\theta; \quad n_{\theta\theta} = 0; \quad n_{r\theta} = 0 \tag{6.30}$$

This result is very special. The stresses $\sigma_{\theta\theta}$ and $\sigma_{r\theta}$ are not zero just at the edge ($\theta = \pi/2$), but for any value of the angle θ. At a half circle in the half plane the shear stress $\sigma_{r\theta}$ and tangential stress $\sigma_{\theta\theta}$ are zero. Just a membrane force n_{rr} is present. The value of C can be calculated from the condition that vertical equilibrium must exist between the point load F and the membrane forces n_{rr}. This condition leads to

$$n_{rr} = \frac{2F}{\pi}\frac{\cos\theta}{r} \tag{6.31}$$

Boussinesq [14] even found such a type of solution for a compressive point-load F on an infinite 3D half-space, from which Flamant [15] obtained the stated solution. Therefore, the solution for a point-load on a half-plane is also called Boussinesq's solution. In each point (r, θ) a transformation can be made from the stresses σ_{rr}, $\sigma_{\theta\theta}$ and $\sigma_{r\theta}$ to the stresses σ_{xx}, σ_{yy} and σ_{xy}. These three stresses are all different from zero. Figure 6.13 shows the distribution of the vertical membrane force n_{yy}. The deeper the section, the more n_{yy} is spread.

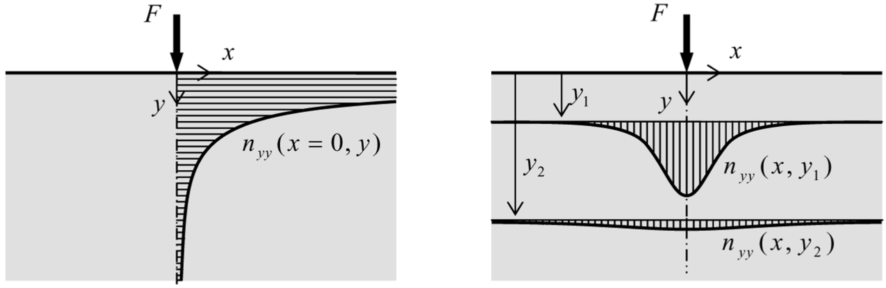

Figure 6.13 Vertical stress under point load on half-space.

6.2.2 Brazilian Splitting Test

A well-known method for determining the splitting strength of brittle materials like concrete is the so called Brazilian splitting test. In this test a circular cylinder is loaded by two opposite line loads as shown in Figure 6.14. Direct tensile tests on concrete are difficult to perform, because a special tensile test set-up needs to be available. Fortunately, there is a simple relation between the vertical line load and the tensile stress in a Brazilian splitting test. It is assumed that the stresses do not vary along the axial direction of the cylinder so that a slice of unit thickness can be considered. The solution of the point load on an half space of Section 6.2.1 can be used to determine the stress state in the cylinder. The solution for a compressive force F on a half-plane becomes very simple when it is presented by eccentric circles as done in Figure 6.15. For all points on a circle $r = d\cos\theta$. Then in each circle the stress σ_{rr} is constant while the other stress components $\sigma_{\theta\theta}$ and $\sigma_{r\theta}$ are zero. The constant value of σ_{rr} is $-\sigma_o$ in which $\sigma_o = 2F/\pi d$ is positive. In the vertical line of symmetry, the horizontal stress, $\sigma_{\theta\theta}$, is zero. The material outside

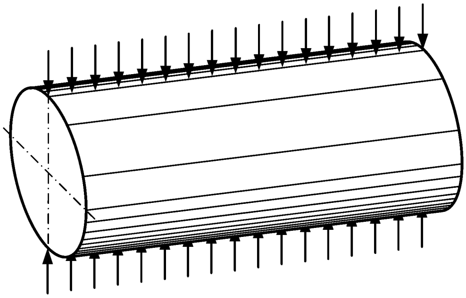

Figure 6.14 Loading scheme on solid cylinder in Brazilian splitting test.

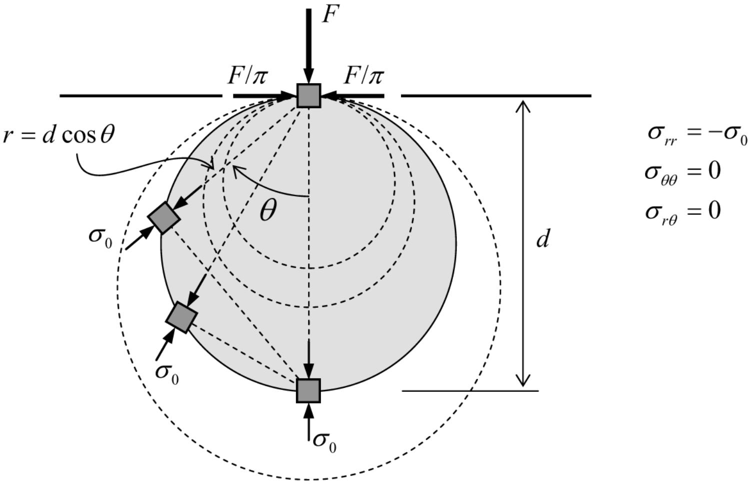

Figure 6.15 Alternative presentation of stresses in half-space.

the circle has been removed and replaced by the edge loading σ_o. In Figure 6.16 the same figure is presented together with the mirror image of the solution. When both solutions are superimposed a circular disk is obtained that is loaded by two concentrated forces F and a radial edge stress $-\sigma_o$. Note that no horizontal stresses are present in the vertical line of symmetry.

The final step in the derivation is to remove the edge stress by adding the axisymmetric solution of a disk with a constant tensile stress σ_o on the edge depicted in Figure 6.3. In this load case there is a hydrostatic stress state with a tensile stress σ_o acting in all directions. Therefore, the horizontal stress, $\sigma_{\theta\theta}$, in the vertical line of symmetry is σ_o. The result of the superposition is

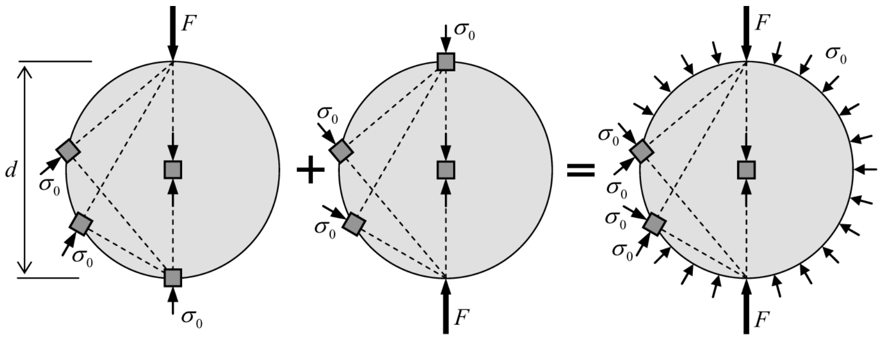

Figure 6.16 Sum of half-space solution and its mirror image.

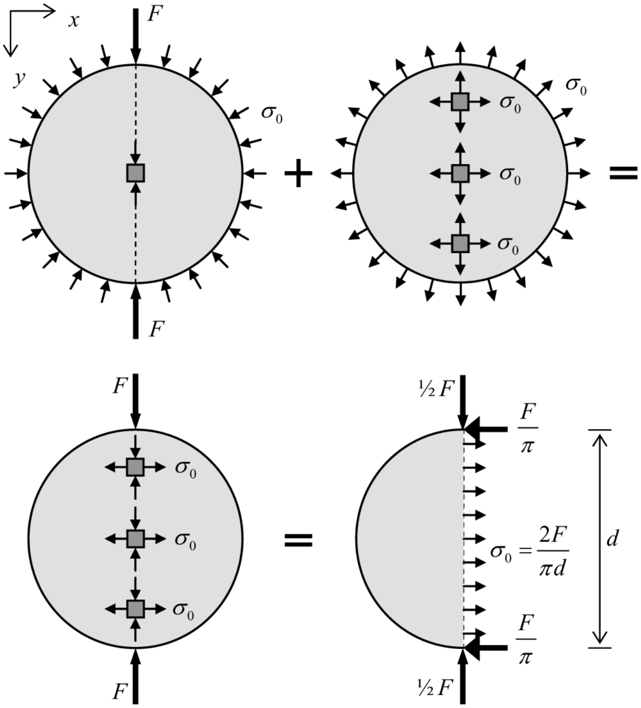

Figure 6.17 Superposition of two stress states yields the final solution.

a circular disk subjected to two diametrically opposite point-loads F with a free unloaded circular edge, see Figure 6.17. On the vertical line of symmetry there is a constant tensile stress σ_o. Therefore, the ultimate result is a homogeneous tensile stress along the vertical line of symmetry, $\sigma_{xx} = 2F/\pi d$. The total horizontal force on the line of symmetry has to be zero. Therefore, there must be local horizontal compressive forces at the point of action of the forces F, equal to $\frac{1}{2}\sigma_o d = F/\pi$.

In this linear-elastic solution the homogeneous tensile stress in the vertical line of symmetry is balanced by a concentrated horizontal force F/π at each end of the line of symmetry. This implies infinitely large compressive stresses at those positions. In reality the elasticity limits will be surpassed and nonlinear material effects will enter in the zones where the loads are applied.

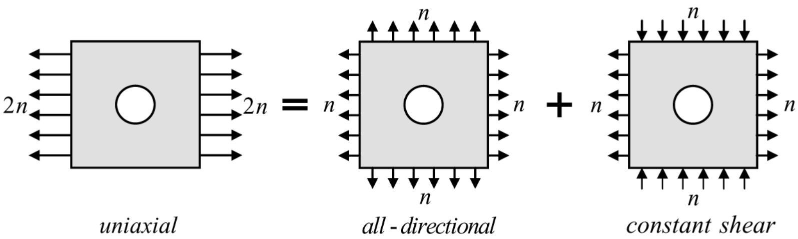

Figure 6.18 Uniaxial stress state as combination of two basic cases.

6.2.3 Hole in Plates with Shear and Uniaxial Stress

In Section 6.1 for axisymmetric problems we were able to determine the stress peak near a hole in a plate in a radially homogeneous (is hydrostatic) stress state. Now we will investigate the case of a hole in a plate with a constant shear stress. We do it for a plate which has a tensile stress in the x-direction and a compressive stress in the y-direction. After we have solved that problem we also can determine the stress concentration in an uniaxially stressed plate. That case is a superposition of the hydrostatic case of Section 6.1 and the present constant shear case. Figure 6.18 shows this superposition.

Shear Stress

We consider a large plate with a hole, in which equal principal stresses of opposite sign occur, see Figure 6.19. We choose the origin of the axes r and θ at the centre of the hole. The value of the principal membrane forces is n. It can be verified from Mohr's circle or the transformation rules that the homogeneous membrane forces at each position in the plate in absence of the hole would be

$$\begin{aligned} n_{rr} &= n \cos 2\theta \\ n_{\theta\theta} &= -n \cos 2\theta \\ n_{r\theta} &= -n \sin 2\theta \end{aligned} \tag{6.32}$$

If a hole is created, the membrane forces n_{rr} and $n_{r\theta}$ have to be made zero on the edge of the hole. This means that an edge loading has to be superimposed, which causes the same membrane forces but with an opposite sign:

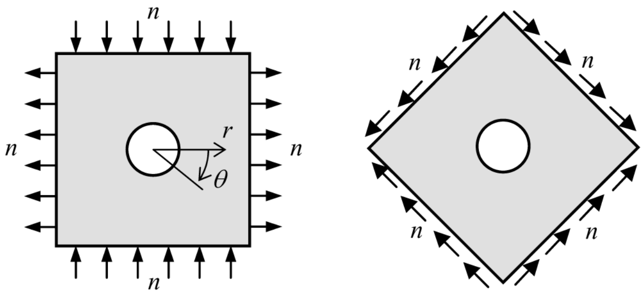

Figure 6.19 Circular hole in constant shear field.

$$
\begin{aligned}
n_{rr} &= -n\cos 2\theta \\
n_{r\theta} &= n\sin 2\theta
\end{aligned}
\tag{6.33}
$$

The bi-harmonic differential equation (6.28) has to be solved. This can be done by choosing a solution for φ of the form

$$
\varphi = \varphi(r)\cos 2\theta \tag{6.34}
$$

This means that the variables r and θ are separated. Substitution in the differential equation yields an ordinary fourth-order differential equation for $\varphi(r)$:

$$
\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{4}{r^2}\right)\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{4}{r^2}\right)\varphi = 0 \tag{6.35}
$$

The general solution of this fourth-order differential equation will have four constants, to be determined from the boundary conditions. Substitution of the trial function Cr^m leads to four roots $m = -2$, $m = 0$, $m = 2$ and $m = 4$, therefore the solution is

$$
\varphi = \left(C_1 r^4 + C_2 r^2 + C_3 + C_4\frac{1}{r^2}\right)\cos 2\theta \tag{6.36}
$$

From Eq. (6.27) we derive the expressions for the membrane forces. We must determine the four coefficients from the boundary conditions, two on the edge of the hole, see Eq. (6.33), and two from the condition that all membrane forces vanish for large r. The result for the membrane forces is

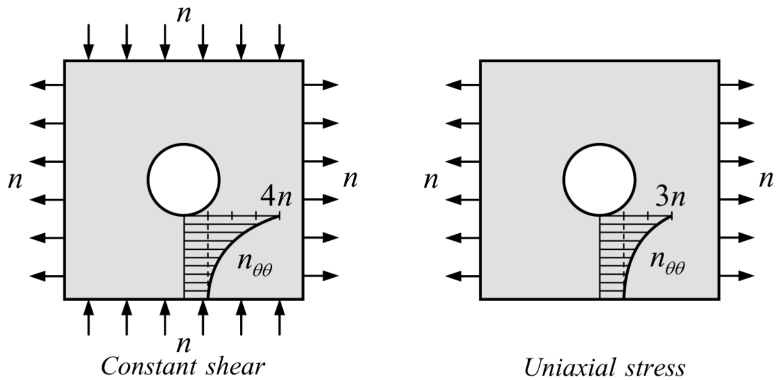

Figure 6.20 Stress concentration factors for constant shear and uniaxial stress.

$$
\begin{aligned}
n_{rr} &= n\left(-4\frac{a^2}{r^2} + 3\frac{a^4}{r^4}\right)\cos 2\theta \\
n_{\theta\theta} &= n\left(-3\frac{a^4}{r^4}\right)\cos 2\theta \\
n_{r\theta} &= n\left(-2\frac{a^2}{r^2} + 3\frac{a^4}{r^4}\right)\sin 2\theta
\end{aligned}
\tag{6.37}
$$

This solution still has to be superimposed on the homogenous stresses of Eq. (6.33) for the situation without hole. The final result is

$$
\begin{aligned}
n_{rr} &= n\left(1 - 4\frac{a^2}{r^2} + 3\frac{a^4}{r^4}\right)\cos 2\theta \\
n_{\theta\theta} &= n\left(-1 - 3\frac{a^4}{r^4}\right)\cos 2\theta \\
n_{r\theta} &= n\left(-1 - 2\frac{a^2}{r^2} + 3\frac{a^4}{r^4}\right)\sin 2\theta
\end{aligned}
\tag{6.38}
$$

The maximum tensile stress $n_{\theta\theta}$ at the hole edge in peripheral direction appears for $r = a$, $\theta = \pm\pi$ and is equal to $4n$. This value is four times the applied principal membrane stresses; the stress concentration factor is 4, see Figure 6.20.

Uni-Axial Stress

The uni-axial stress state is found from the superposition of solution (6.39) and the solution for the axisymmetric case in Section 6.1.2, divided by 2 in order to relate it to an applied stress of the magnitude n

$$\begin{aligned}
n_{rr} &= \frac{n}{2}\left\{\left(1-\frac{a^2}{r^2}\right)+\left(1-4\frac{a^2}{r^2}+3\frac{a^4}{r^4}\right)\cos 2\theta\right\} \\
n_{\theta\theta} &= \frac{n}{2}\left\{\left(1+\frac{a^2}{r^2}\right)-\left(1+3\frac{a^4}{r^4}\right)\cos 2\theta\right\} \\
n_{r\theta} &= \frac{n}{2}\left\{-1-2\frac{a^2}{r^2}+3\frac{a^4}{r^4}\right\}\sin 2\theta
\end{aligned} \tag{6.39}$$

For this stress state the maximum tensile membrane force $n_{\theta\theta}$ is three times the value of the uniaxial membrane force. The stress concentration factor is 3. The distribution of the stresses is shown in the right-hand part of Figure 6.20.

6.3 Message of the Chapter

- In thick-walled tubes under internal pressure the stress in tangential direction is not constant over the thickness. There is a nonlinear distribution.
- In thick-walled tubes under internal pressure we cannot neglect the stresses in thickness direction.
- A constant moment in a curved beam evokes tensile stresses in the depth direction of the beam. For a reinforced beam, stirrups may be needed in absence of a shear force.
- At a round hole in a homogeneous (hydrostatic) stress field the stress concentration factor is 2.

- At a round hole in a uniaxial stress field the stress concentration factor is 3.

- For a constant shear stress field the stress concentration factor even gets the value 4.

- Stresses in a half plane due to a point load normal to the free edge are very special. In polar coordinates just normal stresses in radial direction occur. The tangential stress and shear stress are zero.

- In the Brazilian splitting test there is a homogeneous tensile stress over the vertical plane of symmetry of the cylinder. These stresses are accompanied by balancing compressive lumped forces, close to the applied loads.