Chapter 5
Rectangular Plate Examples

We focus on special aspects of the theory of thin plates by discussing a state of constant bending curvature in Section 5.1 and a panel of constant torsion in Section 5.2. In Section 5.3 we show the effectiveness of a square simply-supported plate subject to a distributed load. In Section 5.4 we discuss the special case of a twist-less plate. Finally, we devote Section 5.5 to a viaduct subject to an edge load.

5.1 Basic Bending Cases

5.1.1 Cylindrical Deflection

We consider a cylindrical deflection (see Figure 5.1) with shape

$$w = Cx\,(a - x) \tag{5.1}$$

for a plate with a non-zero Poisson's ratio. Substitution of this expression into the bi-harmonic equation (4.7) gives $p = 0$. This means that the function w is a solution to the differential equation in the absence of a distributed load p. The deflection is zero along the straight edges $x = 0$ and $x = a$. This is where the supports can be thought to be. All lines that run parallel to the supports remain straight. The formulas in (4.7) imply

$$m_{xx} = 2DC, \quad m_{yy} = 2\nu DC, \quad m_{xy} = 0 \tag{5.2}$$

We conclude that there is a bending moment in the y-direction, Poisson's ratio times the moment in the x-direction

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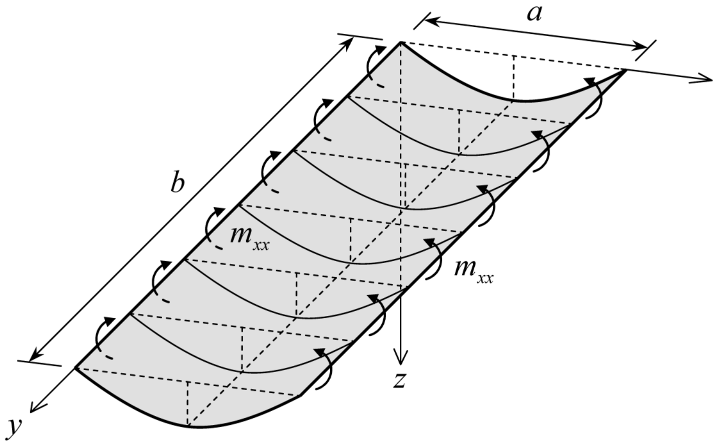

Figure 5.1 Cylindrical deflection plane.

$$m_{yy} = \nu m_{xx} \tag{5.3}$$

Furthermore Eq. (4.4) implies

$$v_x = 0, \quad v_y = 0 \tag{5.4}$$

There is a constant bending moment m_{xx} in the plate, which is caused by an externally applied moment of the same size along the straight edges $x = 0$ and $x = a$. In the direction of the straight generating lines, there is a constant moment m_{yy} of magnitude νm_{xx}. For steel, with a value $\nu = 0.3$, this will lead to $m_y = 0.3 m_{xx}$; for concrete with the value $\nu = 0.2$, $m_y = 0.2 m_{xx}$. Twenty percent of the reinforcement in the span direction is necessary in the lateral direction, even when there is no curvature there!

5.1.2 Cylindrical Deflection of Arbitrary Shape

Now we consider the general shape of the deflection $w = f(x)$ due to a distributed load p. Substitution into the bi-harmonic equation in (4.7) shows that the load is

$$p = D \frac{d^4}{dx^4} f(x) \tag{5.5}$$

For the moments we find, see Eq. (4.7)

$$m_{xx} = -D \frac{d^2}{dx^2} f(x), \quad m_{yy} = \nu m_{xx}, \quad m_{xy} = 0 \tag{5.6}$$

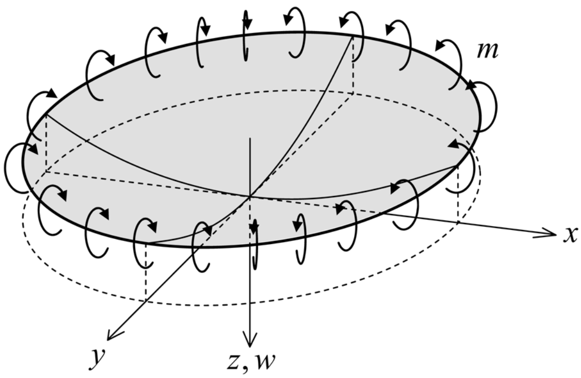

Figure 5.2 Omni-directional bending.

We infer again that if the deflection is constant in the y-direction, a moment m_{yy} is generated even though there is no curvature in the y-direction. This confirms our finding for the clamped edge in Section 4.4.1.

5.1.3 Omni-Directional Bending

We consider the superposition of the solutions $w = -Cx^2$ and $w = -Cy^2$. With $x^2 + y^2 = r^2$ this leads to

$$w = -C(x^2 + y^2) = -Cr^2 \tag{5.7}$$

This is a paraboloid of revolution (see Figure 5.2). The moments are

$$m_{xx} = 2DC(1+\nu), \quad m_{yy} = 2DC(1+\nu), \quad m_{xy} = 0 \tag{5.8}$$

Using the transformation formulas (4.17) we find

$$\begin{aligned} m_{nn} &= m_{xx}\cos^2\alpha + m_{yy}\sin^2\alpha = 2DC\,(1+\nu) \\ m_{tt} &= 2DC\,(1+\nu) \\ m_{nt} &= 0 \end{aligned} \tag{5.9}$$

The bending moment is equal in all directions. Torsional moments do not appear. Here we have the case of pure bending due to a constant moment m along the perimeter of a circular plate. The constant C follows from

$$m = 2DC\,(1+\nu) \;\rightarrow\; C = \frac{m}{2D\,(1+\nu)} \tag{5.10}$$

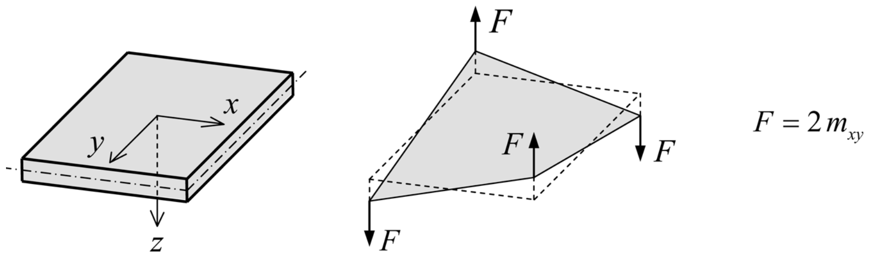

Figure 5.3 Panel with constant torsion.

and the formula for the plane of deflection is

$$w = \frac{-m}{2D\,(1+\nu)} r^2 \tag{5.11}$$

5.2 Torsion Panel

We give a rectangular plate a deflection of the shape (see Figure 5.3)

$$w = -Cxy \tag{5.12}$$

This gives a mixed second derivative

$$\frac{\partial^2 w}{\partial x\ \partial y} = -C \tag{5.13}$$

The other two derivatives are zero. The moments are

$$m_{xy} = (1-\nu)\,DC, \quad m_{xx} = 0, \quad m_{yy} = 0 \tag{5.14}$$

Both bending moments are zero and the torsion is constant and positive. According to Eq. (4.4) the derivative of the moments provides the shear forces. The shear forces are zero:

$$v_x = 0, \quad v_y = 0 \tag{5.15}$$

According to Eq. (4.5) we can compute the load from the second derivatives of the moments. As a result the load p is also zero. On the edge $x =$ constant, the following support reaction is obtained:

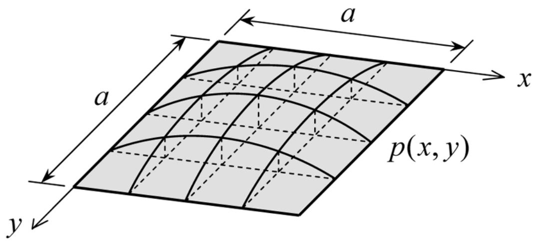

Figure 5.4 Two-way sine load on a simply-supported square plate.

$$f = v_x + \frac{\partial m_{xy}}{\partial y} = 0 \tag{5.16}$$

which is also zero. This is also the case for the other edges. So, no load occurs at the edges and no load over the area of the plate. Yet a twisting moment is present. The twisting moments in the four corners of the plate generate a concentrated support reaction of $2m_{xy}$, as shown in Figure 5.3. The load consists of two couples of point loads in opposite direction. In literature, this plate case is known as the *Nadai's plate*. This stress state may be used to experimentally determine the plate flexural rigidity D. The panel with a constant twisting moment and four corner forces will play a role in Chapter 9 on approximating computational methods in pre-FE days.

5.3 Two-Way Sine Load on Square Plate

A square plate with dimensions a is simply supported along its four edges. The origin of the coordinate system is chosen in a corner, and the axes coincide with the sides of the square, see Figure 5.4. A distributed load is applied of the form

$$p = \hat{p} \sin \frac{\pi x}{a} \sin \frac{\pi y}{a} \tag{5.17}$$

This two-way sine load may be considered to be the first term of a Fourier series of a homogeneously distributed load. The amplitude $\hat{p}$ is the value of the load at the plate centre ($x = y = a/2$).

5.3.1 Displacement

We assume

$$w = \hat{w} \sin \frac{\pi x}{a} \sin \frac{\pi y}{a} \tag{5.18}$$

where $\hat{w}$ is the centre value. This choice satisfies the boundary conditions along the simple supports as discussed in Section 4.4.2.

$$x = 0 \quad \text{and} \quad x = a \rightarrow \begin{cases} w = 0 \\ \dfrac{\partial^2 w}{\partial x^2} = 0 \end{cases} \tag{5.19}$$

$$y = 0 \quad \text{and} \quad y = a \rightarrow \begin{cases} w = 0 \\ \dfrac{\partial^2 w}{\partial y^2} = 0 \end{cases} \tag{5.20}$$

Substitution of this trial solution in the differential equation (4.7) yields

$$\left[\frac{\pi^4}{a^4} + 2\frac{\pi^4}{a^4} + \frac{\pi^4}{a^4}\right]\hat{w} = \frac{\hat{p}}{D} \rightarrow \hat{w} = \frac{\hat{p}a^4}{4\pi^4 D} \tag{5.21}$$

The solution is therefore

$$w = \frac{\hat{p}a^4}{4\pi^4 D} \sin\frac{\pi x}{a} \sin\frac{\pi y}{b} \tag{5.22}$$

This is a particular solution that satisfies the boundary conditions of the simply-supported plate. Then the particular solution is the complete solution and we do not need find a homogenous solution of the differential equation. The plane of deflection is similar in shape to the load distribution. This is visualized in Figure 5.5. For the maximum deflection of the square plate we find

$$\hat{w} = \frac{\hat{p}a^4}{4\pi^4 D} \tag{5.23}$$

For the maximum deflection of a beam with unit width, flexural rigidity D and the same span and subjected to a one-way sine line load with maximum $\hat{p}$, we find

$$\hat{w} = \frac{\hat{p}a^4}{\pi^4 D} \tag{5.24}$$

The deflection of the plate is a quarter of the deflection of the beam. The beam solution applies for a very wide plate that spans in one direction; that plate is only a quarter as stiff as the square plate.

One might expect a square plate to be twice as stiff as a beam, at a first look, noticing that a plate can transfer loads in two directions, so beam-action in the x-direction and beam-action in the y-direction may cooperate. However, the square plate receives additional stiffness by two other ‘beams’, which act in the diagonal direction. The length of these ‘beams’ is longer, but

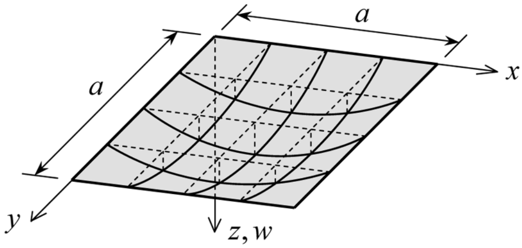

Figure 5.5 Deflection of the square plate.

the ends act as clamped ends in the corners. Because of the straight edges, both $\partial w/\partial x$ and $\partial w/\partial y$ are zero at the corners. Therefore, the derivative of w must be zero in all directions at that position, which explains the apparent clamped ends of the diagonal 'beams' and their important contribution to the stiffness.

Effectiveness of plate

The middle term in the bi-harmonic differential equation due to torsion contributes to the same extent as the first and last term due to bending. This shows that a square or nearly square plate is a very effective load-carrying structure. A factor of about four in effectiveness is also to be expected for a homogeneously distributed load.

5.3.2 Moments and Shear Forces

The formulas in (4.7) lead to the moments

$$\begin{aligned} m_{xx} &= \frac{1+\nu}{4\pi^2}\hat{p}a^2 \sin\frac{\pi x}{a} \sin\frac{\pi y}{a} \\ m_{yy} &= \frac{1+\nu}{4\pi^2}\hat{p}a^2 \sin\frac{\pi x}{a} \sin\frac{\pi y}{a} \\ m_{xy} &= -\frac{1-\nu}{4\pi^2}\hat{p}a^2 \cos\frac{\pi x}{a} \cos\frac{\pi y}{a} \end{aligned} \tag{5.25}$$

The distributions of these moments are drawn in Figure 5.6. The solution for the moments confirms that the boundary conditions are satisfied. The distributions of the bending moments have the same shape as the deflection and

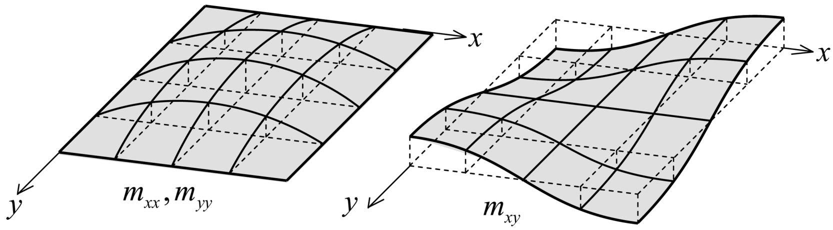

Figure 5.6 Distribution of moments under two-way sine loading.

the load. The shape of the twisting moments is different. Where the bending moment is at a maximum, the twisting moment is zero, and where the bending moment is zero the twisting moment is at a maximum. The maximum bending moment in the plate is

$$\hat{m}_{xx} = \hat{m}_{yy} = \frac{1+\nu}{4\pi^2}\hat{p}a^2 \tag{5.26}$$

For a very wide plate that is supported only in one direction (x-direction) we find

$$\hat{m}_{xx} = \frac{1}{\pi^2}\hat{p}a^2 \tag{5.27}$$

This is also the moment in a beam with a unit width under a comparable load. The maximum moment in the plate is a factor $(1+\upsilon)/4$ smaller than in a beam. Again, the force action in a square plate is very effective. The largest twisting moments arise in the four corners. In the corner $x = 0$, $y = 0$ of the plate the twisting moment is

$$\hat{m}_{xy} = -\frac{(1-\nu)}{4\pi^2}\hat{p}a^2 \tag{5.28}$$

This moment is of the same order of magnitude as the maximum bending moment in the centre of the plate. For a zero Poisson's ratio it is even equal. The shear forces can be derived from the moments by applying Eq. (4.4)

$$\begin{aligned} v_x &= \frac{1}{2\pi}\hat{p}a\cos\frac{\pi x}{a}\sin\frac{\pi y}{a} \\ v_y &= \frac{1}{2\pi}\hat{p}a\sin\frac{\pi x}{a}\cos\frac{\pi y}{a} \end{aligned} \tag{5.29}$$

Their distribution over the plate area is depicted in Figure 5.7. The correctness of the shear forces can be checked as follows. We can compute the total shear force that flows to the edges. Along edge $x = 0$ the shear force is

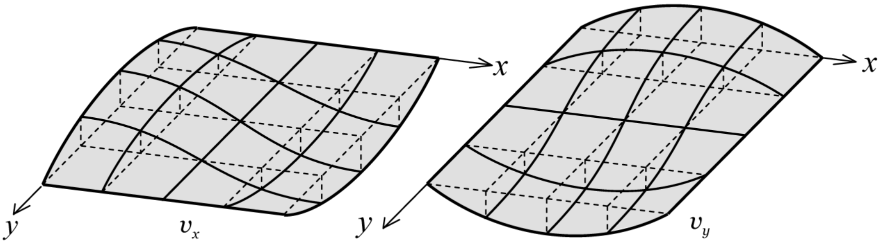

Figure 5.7 Distribution of shear forces under two-way sine load.

$$v_x = \frac{1}{2\pi}\hat{p}a \sin\frac{\pi y}{a} \tag{5.30}$$

and the total shear force S along this edge is

$$S = \frac{1}{2\pi}\hat{p}a \int_0^a \sin\frac{\pi y}{a}\,dy = \frac{1}{2\pi}\hat{p}a \cdot \frac{2a}{\pi} = \frac{1}{\pi^2}\hat{p}a^2 \tag{5.31}$$

For reasons of symmetry the total shear force which flows to the four edges is four times S

$$4S = \frac{4}{\pi^2}\hat{p}a^2 \tag{5.32}$$

This total shear force should be equal to the total load P, which is applied to the plate

$$P = \hat{p}\int_0^a \sin\frac{\pi x}{a}\,dx \int_0^a \sin\frac{\pi y}{a}\,dy = \hat{p}\frac{2a}{\pi} \cdot \frac{2a}{\pi} = \frac{4}{\pi^2}\hat{p}a^2 \tag{5.33}$$

Indeed $4S$ equals P correctly.

5.3.3 Support Reactions

We continue the analysis of the square plate by computing the distributed support reactions. Along the edge $x = 0$ the formula is, see Eq. (4.33)

$$f = -\left(v_x + \frac{\partial m_{xy}}{\partial y}\right)_{x=0} \tag{5.34}$$

The earlier results for v_x and m_{xy} lead to

$$f = -\left(\frac{3-\nu}{4\pi}\hat{p}a\cos\frac{\pi x}{a}\sin\frac{\pi y}{a}\right)_{x=0} = -\frac{3-\nu}{4\pi}\hat{p}a\sin\frac{\pi y}{a} \tag{5.35}$$

The support reaction is negative, so its direction will be opposite to the direction of w and the load p (compressive reactions).

Surprising support reaction

The support reaction f is larger than the shear force v_x. For zero Poisson's ratio the difference is a factor of 1.5.

The sum of the total support reaction along the four edges is

$$4R = 4\left(-\frac{3-\nu}{4\pi}\hat{p}a\int_0^a \sin\frac{\pi y}{a}dy\right) = -\left(\frac{6-2\nu}{\pi^2}\right)\hat{p}a^2 \tag{5.36}$$

The absolute value of this is evidently much larger than the total load given in Eq. (5.33); again a factor of 1.5 exists between load and support reactions for zero Poisson's ratio. This difference is fully explained by the existence of balancing concentrated reactions in the four plate corners. In the left-upper corner ($x = 0$, $y = 0$) the value of the twisting moment is

$$m_{xy} = -\frac{1-\nu}{4\pi^2}\hat{p}a^2 \tag{5.37}$$

This is a negative value, so the direction of the shear stresses in sections perpendicular to the edges is as shown in Figure 5.8. Therefore, the two concentrated edge shear forces V_x and V_x are directed upward. For vertical equilibrium, a downward lumped corner reaction F is needed

$$F = |m_{xy} + m_{yx}| = 2\left|-\frac{1-\nu}{4\pi^2}\hat{p}a^2\cos\frac{\pi x}{a}\cos\frac{\pi y}{a}\right|_{\substack{x=0\\y=0}} = \frac{1-\nu}{2\pi^2}\hat{p}a^2 \tag{5.38}$$

Apparently a local downward force is needed to keep the square plate on the simple support. If the plate is not fixed to the support, it will lift up in the corner. To fix the corner, a tensile reaction force is needed. The same force occurs in all corners. Now we should compare $4R + 4F$ to the applied load P

$$4R + 4F = -\frac{6-2\nu}{\pi^2}\hat{p}a^2 + \frac{2-2\nu}{\pi^2}\hat{p}a^2 = \frac{-4}{\pi^2}\hat{p}a^2 = -P \tag{5.39}$$

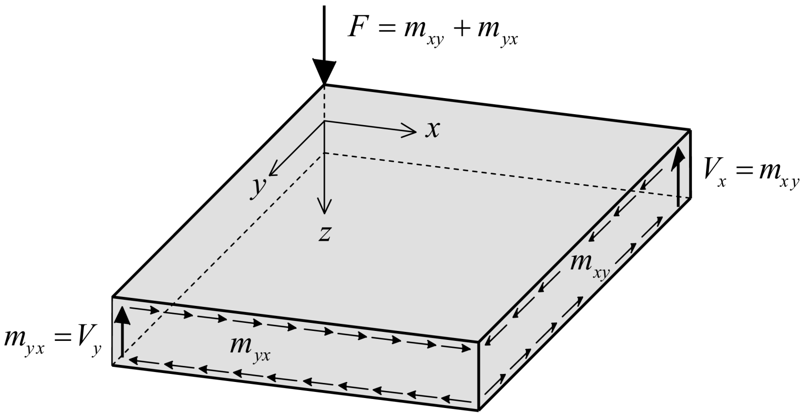

Figure 5.8 Direction of shear stresses for negative m_{xy} in left-upper corner

The sum of all reactions is equal to the load. The sign has become negative because of the sign convention for support reactions.

It is interesting to examine how the shower analogy must be interpreted in this case. Figure 5.9 is a picture of the trajectories. Let us consider the 'hill' as a roof. The diagonals and the horizontal and vertical lines through the middle of the roof are lines of symmetry and therefore trajectories. For this combination of load and boundary conditions the trajectories between these lines of symmetry end perpendicular to the edges. We may consider the edges as open gutters that are perforated at their lower side over their full length

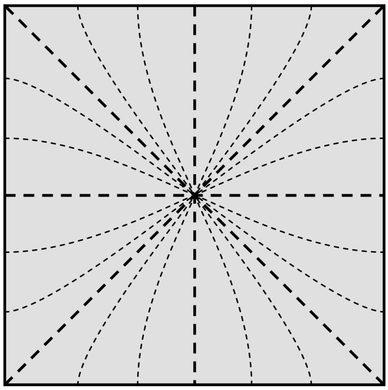

Figure 5.9 Trajectories for shear forces.

in order to let the rain that flows from the roof through immediately. At the same time a lumped well in each corner is bringing up water. This additional water flows through the gutters off the corner and also disappears through the perforations. In this way the water that flows through the perforations is more than the water that falls upon the roof in a rain shower. The additional part comes from the wells in the corners.

Remark

If the simply-supported plate is not rigidly connected to the support, and tensile forces cannot be carried, corners will lift and tilt, which makes the plate less stiff and leads to higher bending moments. A known example of the tilting of a corner of the plate occurs at lock gates produced as double mitre gate. There is leakage at the lower corners of each single gate, because no tensile reaction force can occur.

5.3.4 Stiff Edge Beams

We have seen that the support reaction f is a factor of 1.5 larger than the shear force v_x near the edge for zero Poisson's ratio. We could imagine that the simple support is realized by edge beams of infinite flexural and shear rigidity and zero torsion rigidity, supported by columns at the corners. These beams are subjected to higher loads than might be expected at first glance, and this needs to be kept in mind when detailing such beams. We will now elaborate on the maximum moment and shear force in the beams.

Figure 5.10c shows the plan of a square plate on edge beams. The edge beams are supported by ball supports at the four corners. An edge beam is supposed to be an I-section. A side view is made in section A-A, shown in Figure 5.10b. At both ends of the section, other edge beams are crossed. The plate (thickness t) fits nicely between the flanges of the I-section and is perfectly glued to the web of the edge beam. The connection is able to transfer the shear force v_x and twisting moment m_{xy} from plate to web. These plate actions are the loading of the edge beam. No concentrated vertical shear force need occur in the plate edge zone, as is the case at a simply-supported edge. At the end of the section A-A, there are shear forces V_b in the edge beams which are crossed by the section. We show such a force in Figure 5.10b in the web of the I-section.

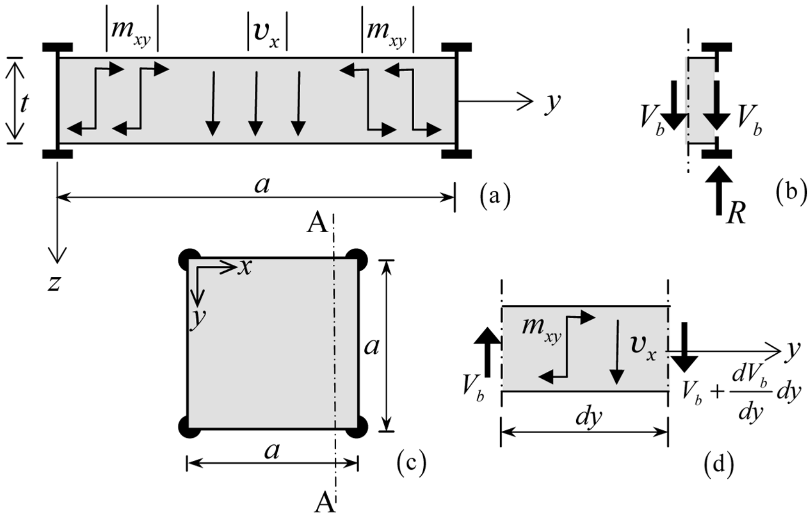

Figure 5.10 Loading of rigid edge beam due to plate.

The maximum bending moment M_b in the edge beam occurs at mid-span, and the maximum shear force V_b at the beam ends. The beam moment is due to both the shear force v_x and twisting moment m_{xy}. The shear force in the beam is due to the shear force v_x only. Accounting for the cosine shape of v_x and the sine shape of m_{xy} in Eqs. (5.28) and (5.29) we obtain

$$\begin{aligned} M &= \frac{\hat{v}_x a^2}{\pi^2} + \frac{1}{\pi}\hat{m}_{xy} a = \frac{\hat{p}a^3}{2\pi^3} + \frac{(1-\nu)\,\hat{p}a^3}{4\pi^3} = \frac{(3-\nu)\,\hat{p}a^3}{4\pi^3} \\ V &= \frac{1}{\pi}\hat{v}_x a = \frac{\hat{p}a^2}{2\pi^2} \end{aligned} \tag{5.40}$$

Surprising large beam moment

In the expression for the beam moment M we notice the factor $(3-\nu)/4$ again as seen earlier in Eq. (5.35) for the support reaction! For zero Poisson's ratio the moment is 50% larger than expected on the basis of the shear force that acts on the beam. For $\nu = 0.2$ it is 40%.

The column reaction R is computed as follows:

$$R = -2V = -\frac{\hat{p}a^2}{\pi^2} \tag{5.41}$$

which indeed is one quarter of the total load P on the plate, and it is a compressive force. No corner tensile force occurs if the simple support is realized through a flexure-rigid edge beam.

The introduction of the edge beam provides an alternative way to derive the boundary condition at a free edge. For that purpose we have to consider an elementary beam part of length dy as depicted in Figure 5.10d. For force equilibrium in the z-direction and moment equilibrium about the x-axis, respectively, we obtain

$$\frac{dV_b}{dy} - v_x = 0, \qquad V_b + m_{xy} = 0 \tag{5.42}$$

From the second equation we learn $V_b = -m_{xy}$. Substitution in the first equation and sign change leads to

$$\frac{\partial m_{xy}}{\partial y} + v_x = 0 \tag{5.43}$$

which, for zero load f, is identical to the condition we derived earlier in Eq. (4.34).

5.4 Twist-Less Plate

In the preceding section we considered a square plate subjected to distributed load and supported by flexure-stiff and torsion-weak edge beams. For the two-way sine load we found maximum bending moments in the plate centre and maximum twisting moments at the corners. The values of these moments are of about the same size and for zero Poisson's ratio exactly equal. If the flexural rigidity of the edge beams decreases, the deflections will increase and the distribution of moments will change. The bending moments will become larger and the twisting moments smaller. For a sufficiently small flexural rigidity the twisting moments become zero. This can best be shown for a homogeneously distributed load p. No twisting moments in the plate means that the middle term in the bi-harmonic differential equation can be skipped. Then the plate behaves as a grid of orthogonal strips in which only bending occurs. The displacement field in this case becomes

$$w(x, y) = \hat{w}[f(x) + f(y)] \tag{5.44}$$

Here $\hat{w}$ is the maximum deflection of the edge beams; the shape function $f(x)$, with maximum 1, is the deflected shape of a simply-supported beam

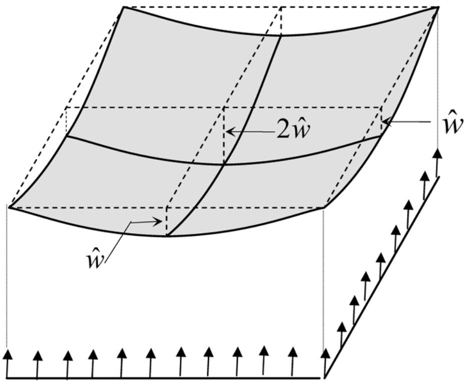

Figure 5.11 Deflection and support reactions in twist-less plate.

subjected to a homogeneously distributed load. The same applies for $f(y)$. The displacement field is shown in Figure 5.11. The consequence of this choice is a zero torsion deformation ρ_{xy} and zero twisting moments m_{xy}. Furthermore the curvature κ_{xx} depends only on x and the curvature κ_{yy} only on y. This leads to moments m_{xx} and shear forces v_x which are constant in the y-direction, and moments m_{yy} and shear forces v_y which are constant in the x-direction.

So, each edge beam is loaded by a homogeneously distributed load $pa/4$ and must have the same deflected shape as the adjacent plate. This can be the case only when the flexural rigidity EI of the edge beam equals the bending stiffness of a plate strip of width $a/2$. Therefore $EI = aD/2$. If we choose this beam rigidity, no twisting moments will occur in the plate.

Twistless slab

For a proper choice of the edge beam stiffness no twisting moments will occur. Then the distributed load p is half transferred in the x-direction and half in the y-direction.

5.5 Edge Load on Viaduct

Consider a bridge slab with span a, and width b. The bridge is simply supported at $x = a/2$, and $x = -a/2$, and has free edges at $y = b/2$ and

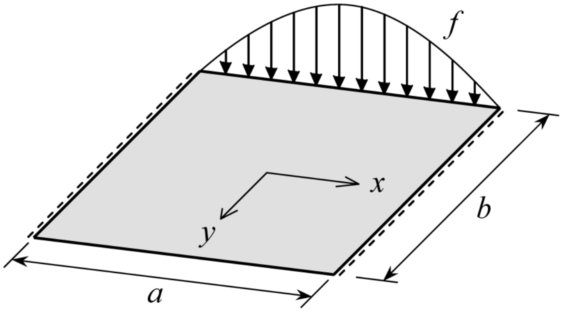

Figure 5.12 Bridge with load on edge.

$y = -b/2$. A cosine-shaped load is applied on the edge $y = -b/2$, see Figure 5.12. This load can be considered to be an approximation to a distributed line load with some heavy vehicles in the middle part of the span. In this case, a distributed load p is not taken into account. The distributed edge load is

$$f(x) = \hat{f} \cos \alpha x, \quad \alpha = \pi / a \tag{5.45}$$

The boundary conditions require

$$x = \pm \frac{1}{2} a \rightarrow \begin{cases} w = 0 \\ m_{xx} = 0 \end{cases} \tag{5.46}$$

$$y = -\frac{1}{2} b \rightarrow \begin{cases} m_{yy} = 0 \\ -\left(v_x + \dfrac{\partial m_{xy}}{\partial y} \right) = f \end{cases} \tag{5.47}$$

$$y = +\frac{1}{2} b \rightarrow \begin{cases} m_{yy} = 0 \\ v_x + \dfrac{\partial m_{xy}}{\partial y} = 0 \end{cases} \tag{5.48}$$

Applying the method of separation of variables, we can describe w as a product of two functions, $w = w(y) \cos \alpha x$. The function $w(y)$ is the distribution of the deflection along the vertical line at mid-span. This choice for w satisfies the boundary conditions at the supports. Substitution into the biharmonic equation (4.7) delivers an ordinary differential equation for $w(y)$ of the fourth order.

$$D \left(\alpha^4 w - 2\alpha^2 \frac{d^2 w}{dy^2} + \frac{d^4 w}{dy^4} \right) = 0 \tag{5.49}$$

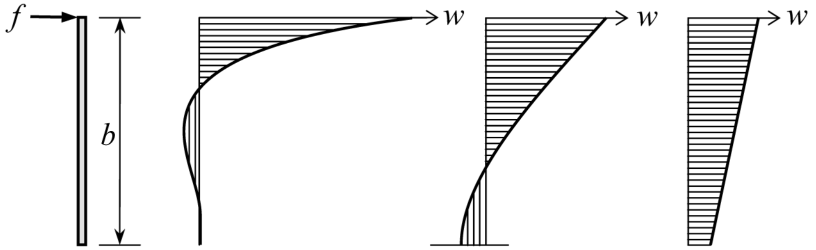

Figure 5.13 Deflection curve at mid-span.

In Section 2.2.2 we have already determined the solution of this homogenous equation. It follows that $w(x, y)$ becomes

$$w = \left(A_1 e^{\alpha y} + A_2 \alpha y e^{\alpha y} + A_3 e^{-\alpha y} + A_4 \alpha y e^{-\alpha y}\right) \cos \alpha x \tag{5.50}$$

The four constants A_1 to and included A_4 follow from two boundary conditions in the edge $y = -b/2$ and two in the edge $y = b/2$. At $y = -b/2$ the moment in y-direction is zero and the Kirchhoff shear force is $-f$. At $y = b/2$ both the bending moment and the Kirchhoff shear force are zero. We will now outline the solution for w on the line $x = 0$ (see Figure 5.13) for three special cases. This solution is closely related to the distribution of the bending moment over the width of the bridge at the middle of the span.

Case 1

The plate is supposed infinitely long in the y-direction, so the viaduct is very wide. Then A_1 and A_2 must be zero, for fading away of the first two terms in Eq. (5.50) to take place. As stated, the picture for the displacement is also a measure for the bending moment in the span direction. At sufficient distance from the loaded edge there is no deflection and bending moment.

Case 2

The plate is a square. This is a practical shape, as could appear in a viaduct. All four constants now are involved, and therefore all four terms $e^{\alpha y}$, $\alpha y e^{\alpha y}$,

$e^{-\alpha y}$ and $\alpha y e^{-\alpha y}$ are present in the solution. The plate sags at the loaded edge and lifts at the opposite edge.

Case 3

The plate (viaduct) is so narrow that it turns into a strip-shaped beam. The plate contributes over all its width in carrying the load, though the part close to the load carries most. In Section 2.2.2 we showed the solution in another form (Taylor expansion).

$$w = \left\{ B_1 + B_2\alpha y + B_3(\alpha y)^2 + B_4(\alpha y)^3 + \cdots \right\} \cos \alpha x \tag{5.51}$$

We shall limit ourselves to the case that $\nu = 0$. The curvature is the second derivative, so the moment m_{yy} is linear in y; only the coefficients B_1, B_2, B_3 and B_4 need be considered. Because the moment m_{yy} has to be zero on both edges, it is zero everywhere. This means that B_3 and B_4 are zero. As a result, the deflection becomes linear in y. The two constants B_1 and B_2 follow from Kirchhoff shear force at the two edges, $-f$ and 0 respectively. The same solution would follow from beam theory. The beam is subjected to bending in x-direction by a line load f and to torsion about the x-axis by a distributed torque $bf/2$. The line load causes the constant deflection B_1 and the torque load the rotated part $B_2\alpha y$.

Effective Width

For the convenience of structural design the concept of *effective width* is introduced in codes of practice, because designers normally prefer to do a beam analysis. Suppose that Figure 5.14 is the distribution of the bending

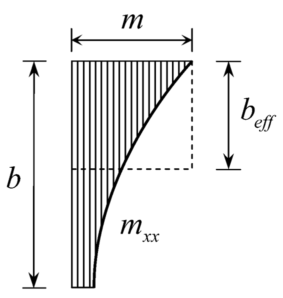

Figure 5.14 Definition of the effective width b_{eff}.

moment m_{xx} over the width b and the maximum value at the edge is m. If the sine-shaped load were applied to a beam with the same span a and a rectangular cross-section with the same depth as the plate, the maximum moment at mid-span would be

$$M = \frac{\hat{f}\, a^2}{\pi^2} \tag{5.52}$$

The ratio of M and m is the effective width b_{eff} of the plate. If engineers can make a good guess for this width, it suffices to do a beam analysis and to spread the total moment over the effective width in order to calculate the edge moment m. Codes of practice offer practical rules for the determination of the effective width.

5.6 Message of the Chapter

- A bending moment can occur for zero curvature. This is due to the effect of a non-zero Poisson's ration and a curvature in the transverse direction.

- A rectangular plate can be brought in a state of constant twisting moment by a set of four equilibrating corner forces.

- A simply-supported square plate under distributed load is about four times more effective than a one-way plate for the same load. Torsion in the corner zones takes care of half the load.

- The support reaction in a simply-supported plate under distributed load is about 50% higher than we expect on the basis of the shear force. The too-large compressive support reactions are balanced by concentrated tensile forces at the four corners. A large concentrated shear force occurs along the edges.

- If the simple support is materialized by stiff edge beams, the moment in the edge beams is about 50% larger than expected on the basis of the load that flows to the edge.

- The flexural stiffness of edge beams can be chosen such that zero twisting moments occur. In a twist-less square plate the bending moments in the edge beams must be calculated on the basis of one quarter of the load on the plate.
- Plate theory helps us to make estimates of the effective width in case of line edge loadings.