Chapter 15
Sense and Nonsense of Mindlin

In Chapters 3 and 4 we presented the theories of Mindlin (more properly Mindlin and Reissner) and Kirchhoff, without explaining which theory must be used in a particular practical case. Commercially provided FEM software usually offers both options and even may have chosen one of them as the default option. The goal of this chapter is to help users make a proper choice.

15.1 Result Dependence on Analyst and Program

We start with the comparison of plate bending results obtained by different programs and analysts. Four providers of commercial software accepted the invitation to participate in the computation of a plate structure. They were asked to perform a linear-elastic computation. The plate, shown in the left top part of Figure 15.1, has the shape of a carpenter's square. The two long edges have a length of 4 m and the short edges 2 m. The modulus of elasticity is 40×10^6 kN/m^2 and Poisson's ratio is 0.2. The plate is subjected to a distributed load of 1 kN/m^2. We choose a set of orthogonal axes x, y along the long edges. The z-axis is normal to the plate. The corner is supported by a ball support which prevents the displacement w and permits free rotations ψ_x and ψ_y. The two short edges of length 2 m are simply-supported. All other edges are free.

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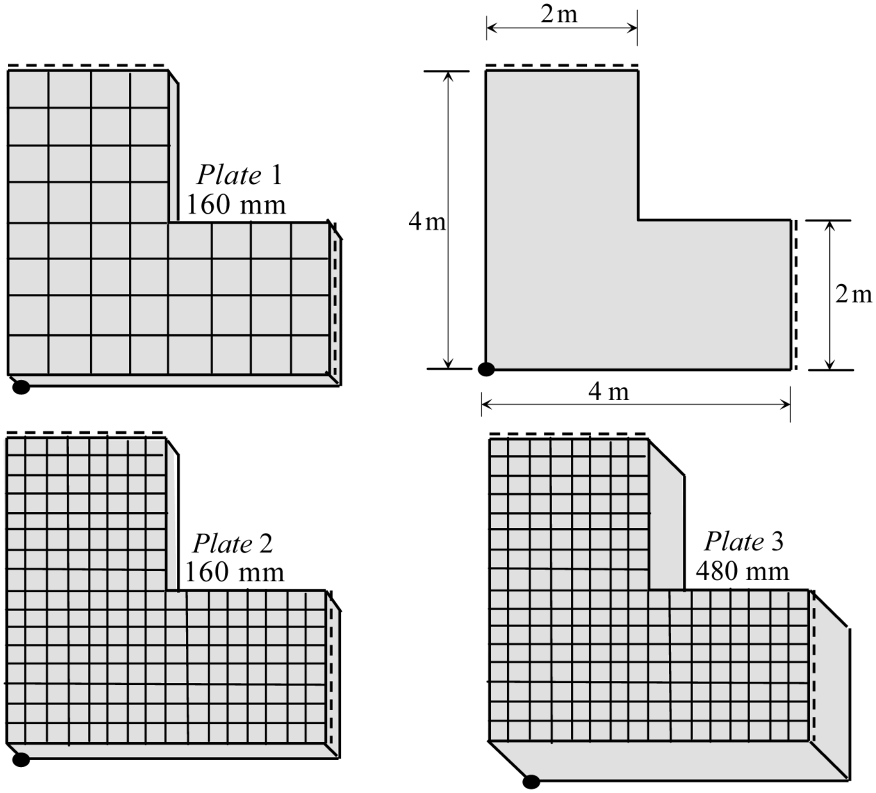

Figure 15.1 Three plates. Mesh fineness and thickness varied.

15.1.1 Invitation

The software houses were invited to perform three calculations with square elements for three versions of the plate.

- Plate 1. Mesh 8 × 8 elements (spacing 500 mm). Thickness 160 mm.
- Plate 2. Mesh 16 × 16 elements (spacing 250 mm). Thickness 160 mm.
- Plate 3. Mesh 16 × 16 elements (spacing 250 mm). Thickness 480 mm.

These different plates are shown in Figure 15.1. The required output should include

- a plot of the shear force v_x in section $x = 0.5$ m.
- a plot of the moment m_{xx} in the section $x = 2.0$ m.
- a plot of the twisting moment m_{xy} along the edge $x = 0$.

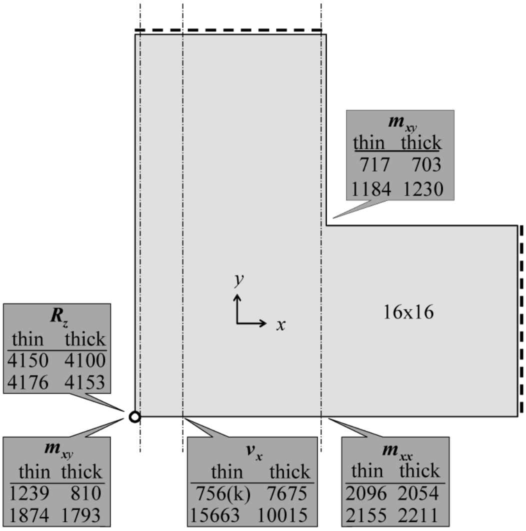

Figure 15.2 Big scatter in submitted results for twisting moment and shear force. Units in N and m.

The ultimate goal was to investigate which plate theory should be used: Kirchhoff or Mindlin.

A span of 4,000 mm and a thickness of 160 mm imply a span-depth ratio of 25. Without any doubt, this is a thin plate. This holds for the first two analyses; any differences are due to the mesh fineness. The element size 500 mm is about three times the thickness, and the element size of 250 mm about one and a half times. In both analyses the element size is not smaller than the plate thickness. If related to the longest span of 4,000 mm, the element size 500 mm means that only eight elements occur between the ball support and the simply-supported edge; this must be considered a coarse mesh, at least in the neighbourhood of the ball support. The element size 250 mm means the application of 16 elements, which sounds acceptable for the case.

The second and third analyses have the mesh fineness in common. Here the difference is in thickness. In the third analysis the span-thickness ratio is slightly greater than eight; to the perception of structural engineers, this plate is thick.

In conclusion, the thin plate 2 may be considered representative for the choice of structural engineers in practice. Variant plate 1 is intended to show the effect of mesh change and variant plate 3 the effect of slenderness.

15.1.2 Submitted Results

Three software providers performed the analysis primarily with Mindlin theory and one chose Kirchhoff theory. Figure 15.2 depicts the scatter of results. Each time, we report the smallest and largest submitted value, and do it for both the thin plate 2 and the thick plate 3. The support reaction R at the ball support is insensitive to the plate thickness. Differences are less than 1%. The differences in the bending moment m_{xx} are not significant too. However, large differences appear in the values of the twisting moment m_{xy}, both at the re-entrant corner and at the ball support, and also in the shear force v_x. Except in one case, the lowest and largest values come from a Mindlin analysis. The exception is marked with (k) of Kirchhoff.

We computed the same three plates with the program *Kola*, switching on the Reissner theory (as the program calls Mindlin theory). Figure 15.3 is the result for the moment m_{xx} in section $x = 2.0$ m. This moment is insensitive to mesh fineness or slenderness, and to the applied theory. Values vary within 2%. Note that the bending moment near the re-entrant corner is not correctly zero at the free edge. Only at a couple of elements distant from the corner does the moment become zero. This is in agreement with the expectation, as explained in Section 11.4.

Figure 15.4 is the result for the twisting moments at the edge $x = 0$, and Figure 15.5 for the shear force in the section $x = 0.5$ m. Now large differences are exposed in a convincing way. The values of the twisting moment differ by a factor of 1.3. The shear forces at the edge differ by a factor of 2.0. The output submitted by the participating providers is very similar to the results of the program *Kola*.

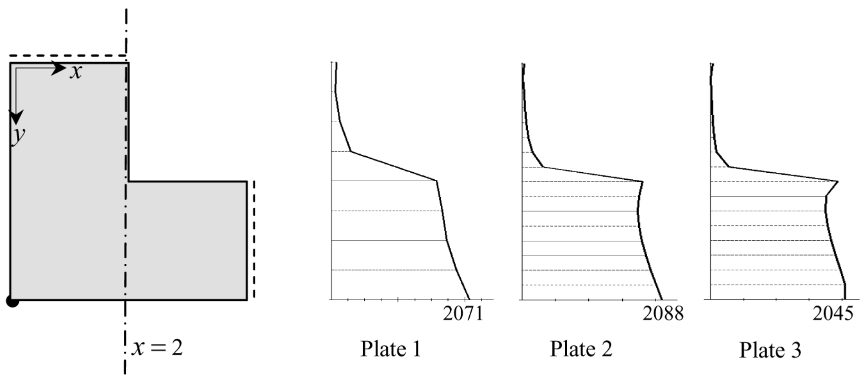

Figure 15.3 Bending moments m_{xx} (Nm/m) in section $x = 2$ m for Mindlin theory.

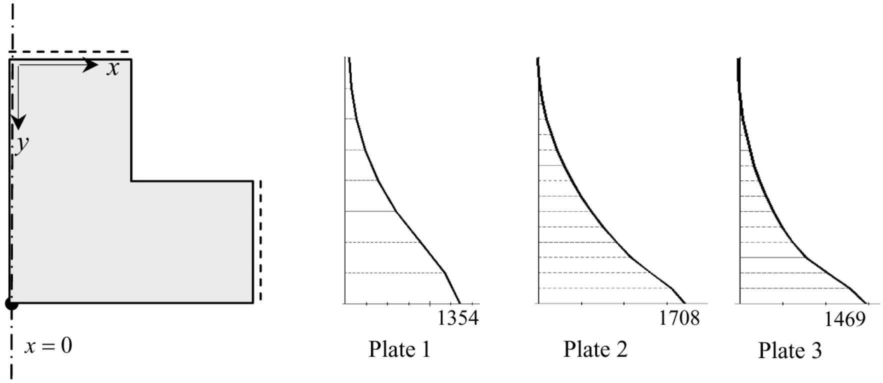

Figure 15.4 Twisting moments m_{xy} (Nm/m) at edge $x = 0$ for Mindlin theory.

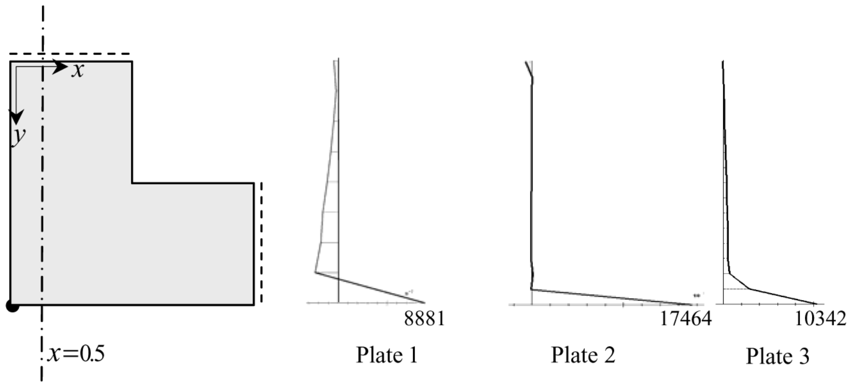

Figure 15.5 Shear force v_x (N/m) in section $x = 0.5$ m for Mindlin theory.

15.2 Explanation of the Differences

In order to understand the large differences in the twisting moment and shear force we must recall the starting points of the Kirchhoff and Mindlin theories. Classically, plates have been analyzed by the Kirchhoff theory for thin plates. Only since the broad availability of FE codes has Mindlin theory come on the scene. The user of commercial software is expected to make a choice from the two theories, but often one of them is a default option without the user being aware of this. Kirchhoff theory holds for plates in which the deformation by shear forces can be neglected, which is the case for a sufficiently large span-thickness ratio l/t. The slenderness $l/t > 10$ is sufficient, and most slabs will satisfy $l/t \geq 20$.

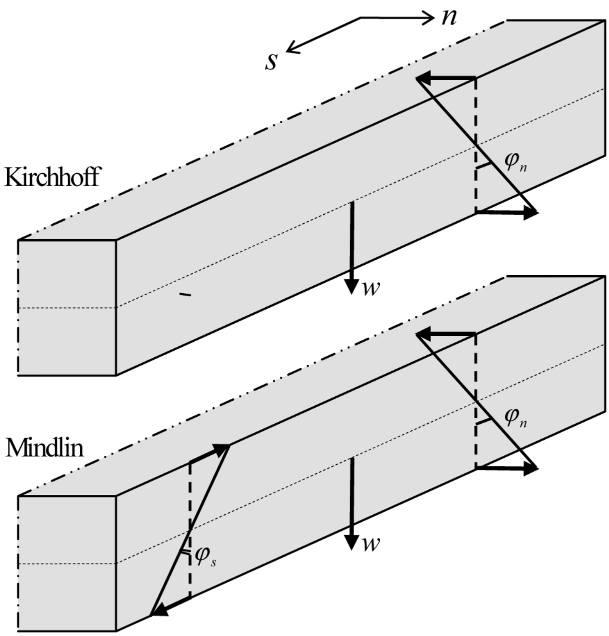

Figure 15.6 Different boundary conditions for Kirchhoff and Mindlin.

We must investigate which circumstances justify or even command the choice of Mindlin theory. We have presented his theory in Section 3.2 to which we refer. For the purpose of explaining differences in computational results it is helpful to repeat the discussion about boundary conditions at a free edge. For convenience we choose axes n normal to and s parallel to the edge. In Mindlin theory there are three independent degrees of freedom at the edge, the displacement w, the rotations φ_n normal to the edge and φ_s in the plane of the edge. This is visualized in Figure 15.6. Note that we return tot the rotation φ as used in Chapter 3 for the derivation of the differential equations. In general there are three edge load components: a distributed force f in the direction of w, a distributed torque t_n in the direction of φ_n and a distributed torque t_s in the direction of φ_s. These edge loads are one-to-one equal to the shear force v_n, the bending moment m_{nn} and the twisting moment m_{ns}, respectively. Usually t_n and t_s are zero, therefore both the bending moment m_{nn} and the twisting moment m_{ns} are zero. Eq. (15.1) summarizes this

$$\begin{Bmatrix} w \\ \varphi_n \\ \varphi_s \end{Bmatrix} \rightarrow \begin{Bmatrix} f \\ t_n \\ t_s \end{Bmatrix} = \begin{Bmatrix} v_n \\ m_{nn} \\ m_{ns} \end{Bmatrix} \rightarrow \begin{matrix} v_n = f \\ m_{nn} = 0 \\ m_{ns} = 0 \end{matrix} \tag{15.1}$$

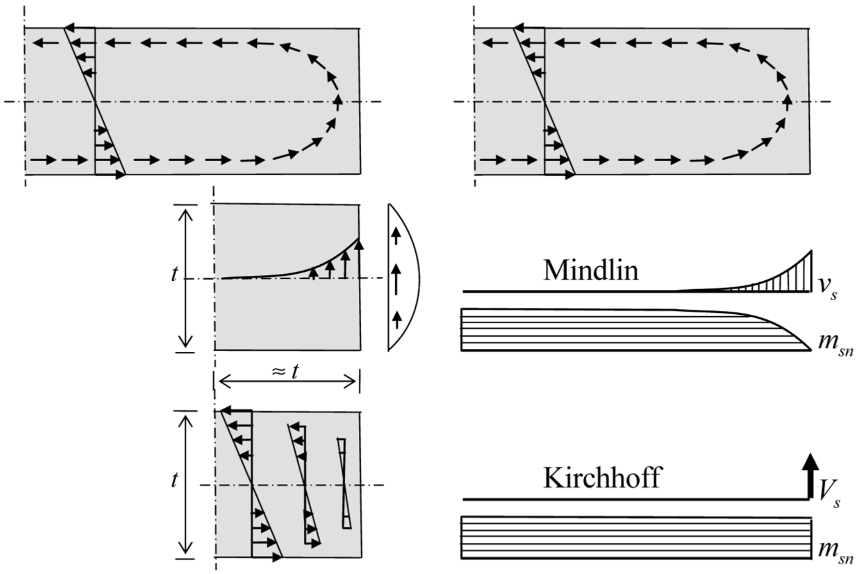

Figure 15.7 Close look at stress state near free edge.

In Kirchhoff theory there are only two degrees of freedom at the free edge: the displacement w and the rotation φ_n normal to the edge. The rotation φ_s in the plane of the edge is a slave of the displacement w because of the relation $\varphi_s = \partial w / \partial s$. Now only two edge loads can be applied, f in the direction of w, and t_n in the direction of φ_n. Yet, in general all three plate quantities v_n, m_{nn} and m_{ns} can occur at the edge and may be non-zero. In Section 4.4 we have seen which relations exist between these three quantities and the two edge loads. We repeat them here.

$$\left\{ \begin{matrix} w \\ \varphi_n \end{matrix} \right\} \rightarrow \left\{ \begin{matrix} f \\ t_n \end{matrix} \right\} = \left\{ \begin{matrix} v_n + \dfrac{\partial m_{sn}}{\partial s} \\ m_{nn} \end{matrix} \right\} \rightarrow \begin{matrix} v_n \neq f; \; m_{ns} \neq 0 \\ m_{nn} = 0 \end{matrix} \tag{15.2}$$

A zero edge load t_n will lead to a zero bending moment m_{nn}, however a zero edge load f does not in general lead to a zero shear force v_n and zero twisting moment m_{ns}. At an unloaded free edge ($f = 0, t = 0$) there will be both a twisting moment and a shear force. This phenomenon is closely related with the concentrated edge shear force V_s in sections normal to the edge, as we found in Section 4.4. This concentrated shear force in Kirchhoff theory does not appear in Mindlin theory. That theory is able to compute distributed shear forces v_s in a narrow edge zone. In order to understand what makes the difference, we repeat in Figure 15.7 the shear stress flow due to a twisting moment in a section normal to the free edge. At a sufficient distance from the edge, the twisting moment causes a linear distribution of shear stresses over the depth of the plate, with zero value in the mid plane. Close to the edge, the shear flow must turn around within the section, because the edge face is stress free. This happens over a plate part with a length of about plate thickness t. We want to describe the stress state in this small end part of the plate in terms of plate quantities v_s and m_{ns}. At the mid plane of the plate the shear stresses have a vertical direction, and a distribution which increases from zero to a maximum value at the edge. The distribution is not linear. At the edge, the shear stress is vertical over the full thickness of the plate, with a distribution which is (close to) parabolic, becoming zero at the top and bottom of the edge. Outside the mid plane and at some distance from the edge we can decompose the shear stress into vertical and horizontal components. The integral of the vertical components delivers a vertical shear force v_s, which is zero at a distance from the edge of about one thickness, and becomes maximum at the edge. The integral of all horizontal components leads to a twisting moment m_{ns}, which decreases in the opposite direction and becomes zero at the free edge. Here the principal difference between Mindlin theory and Kirchhoff theory becomes apparent. Mindlin is able to describe the discussed distribution of the shear force and twisting moment and Kirchhoff is not. In Mindlin theory we can handle the boundary condition $m_{ns} = 0$, whereas we cannot in Kirchhoff theory. Instead, Kirchhoff determines the integral of all the local vertical stress components and concentrates them into one shear force V_s located at the very edge. At the same time, Kirchhoff is not able to have the twisting moment diminish to zero, and instead keeps it constant up to the edge, see Figure 15.7.

Once more we want to stress the fact that these differences happen in a plate length of about one plate thickness. In this domain a big gradient occurs in the n-direction. To cover this in a FE analysis needs a big number of elements over a short distance.

15.3 Supporting Side Study

What we have explained on the basis of the theory can be supported by a case study. We computed a simply-supported square plate by both Kirchhoff and Mindlin, each for a thin and a thick plate. The length l of the edges is 9 m. The thickness t is 200 mm for the thin plate, and 2,250 mm for the thick plate. The span-thickness ratios are 45 and 4 respectively. The first is

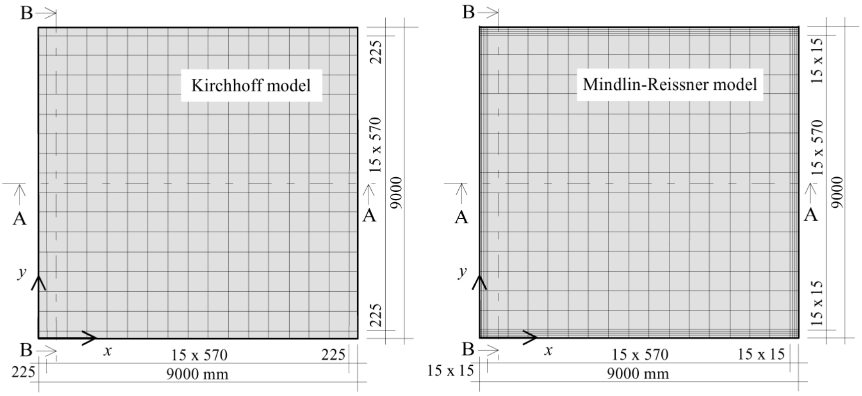

Figure 15.8 Mesh for thin plate analyses.

clearly a thin plate, and the second a thick plate. In all calculations the same distributed load $p = 3$ kN/m^2 is applied. In the plots to follow, Kirchhoff is depicted on the left and Mindlin on the right. The chosen meshes are displayed in Figure 15.8. An element size of 570 mm is used over the main area of the plate outside the edge zone. For the thin plate this is almost three times the plate thickness and for the thick plate it is about a quarter of the plate thickness. In the edge zone one element of 225 mm width is used in the Kirchhoff analysis. A smaller element size has no effect because the localized shear force in the edge zone anyhow is replaced by a concentrated shear force at the edge. This is different for the Mindlin analysis; there this 225 mm edge zone has been divided into 15 very small elements of 15 mm width each. Usually we would avoid a large aspect ratio, but we can use it for the purpose of this study, as no large gradients are expected in the direction parallel to the edges.

15.3.1 Thin Plate Results

We start with the analysis for the thin plate. Figure 15.9 shows the displacement and bending moment. The difference between Kirchhoff and Mindlin is of the order of 2%, both for displacement and moment. A difference occurs in the corners, where Mindlin leads to an isolated peak, which is absent for Kirchhoff. The twisting moments and the shear forces are given in Figure 15.10. The twisting moment in the Kirchhoff-analysis is non-zero at the edge, whereas the Mindlin analysis manages to make the twisting moment

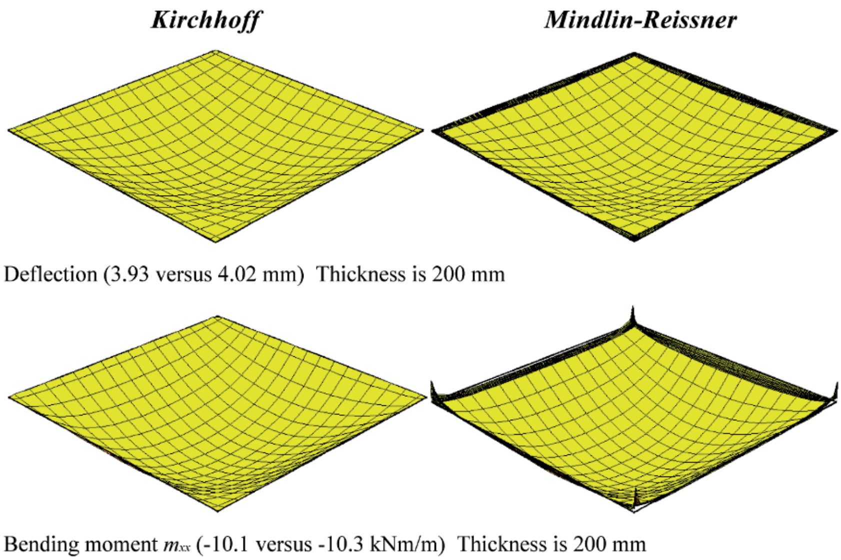

Figure 15.9 Deflection and bending moment in thin plate.

practically zero. However, this must be obtained in a very small edge zone. The maximum value occurs very close to the corner and is only 3% smaller than in the Kirchhoff analysis, where the maximum value is found exactly in the corner. Remembering that the Mindlin analysis requires an impractically fine mesh to provide the correct solution, our conclusion is justified that Kirchhoff in combination with a practical mesh delivers a good result. The shear force distributions are at first glance very different. However, we must bear in mind that the Mindlin analysis is supposed to reproduce the local shear force distribution at the edge due to the returning twisting moment shear flow, and that these local shear stresses very much dominate the plot.

We have repeated the Mindlin analysis for the practical mesh (in fact still a fine mesh) of the Kirchhoff analysis. The result is presented in Figure 15.11. Now the maximum appears at a distance of about one tenth of the span from the edge near the corner and is 9% smaller than the correct value. Mindlin in combination with a practical mesh does apparently not offer an advantage, but rather decreases the accuracy of the computation. As the saying goes, it is ‘Neither fish, flesh nor good red herring’. We recommend using Kirchhoff; this theory gives good results for both moments and shear forces. The structural engineer must be aware that there is a local concentrated shear force V_s along the edge, and must be able accounting for it.

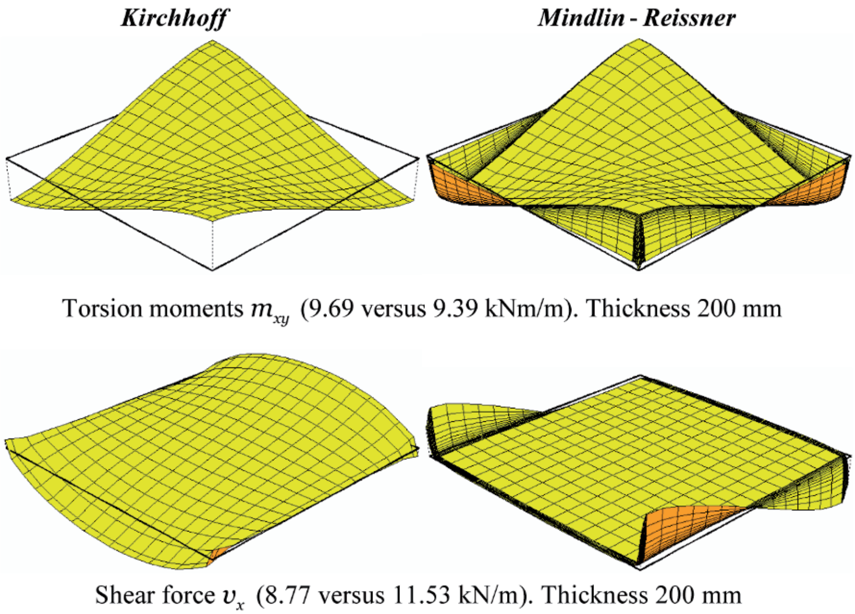

Figure 15.10 Twisting moment and shear force in thin plate.

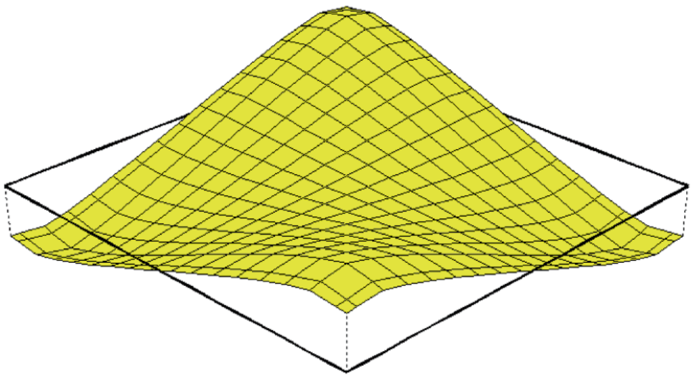

Figure 15.11 Mindlin analysis for thin plate with practical mesh. Neither fish, nor flesh, nor good red herring.

We stress that there is no point in choosing smaller elements than plate thickness. Edge zone effects are disturbed anyhow. At simple supports and clamped edges, there is always an edge disturbance with a three-dimensional stress state. The same holds at columns and at intermediate supports of continuous plates. Point loads actually apply over non-zero areas. Therefore,

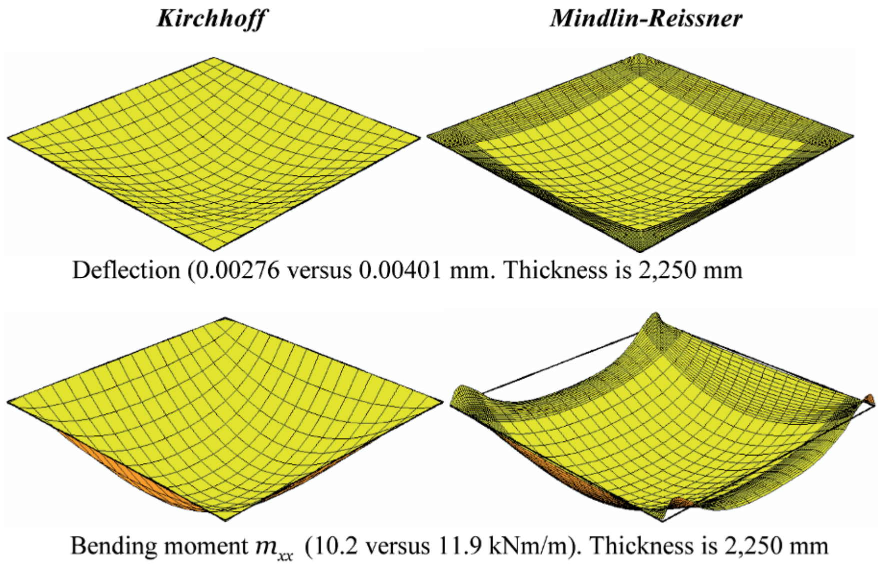

Figure 15.12 Mesh, deflection and bending moment for thick plate.

considering all these comments, we justify adopting the rule: never use an element size smaller than the order of magnitude of the plate thickness.

> ***Lesson***
>
> In a thin plate analysis we must use Kirchhoff. The Mindlin analysis requires a senseless fine mesh to produce practically the same results. Choosing Kirchhoff, we need never use element sizes smaller than the plate thickness.

15.3.2 Thick Plate Results

We now turn to the results for the thick plate with slenderness 4. A mesh of 17 × 17 elements is used in the Kirchhoff analysis. The mesh in the Mindlin analysis is 15 × 15 for the regular area outside the edge zones. In the edge zones again a fine mesh is used. Displacements and bending moments are presented in Figure 15.12. Now substantial differences are seen. The deflection in the Mindlin analysis is about 1.5 larger than in the Kirchhoff-analysis.

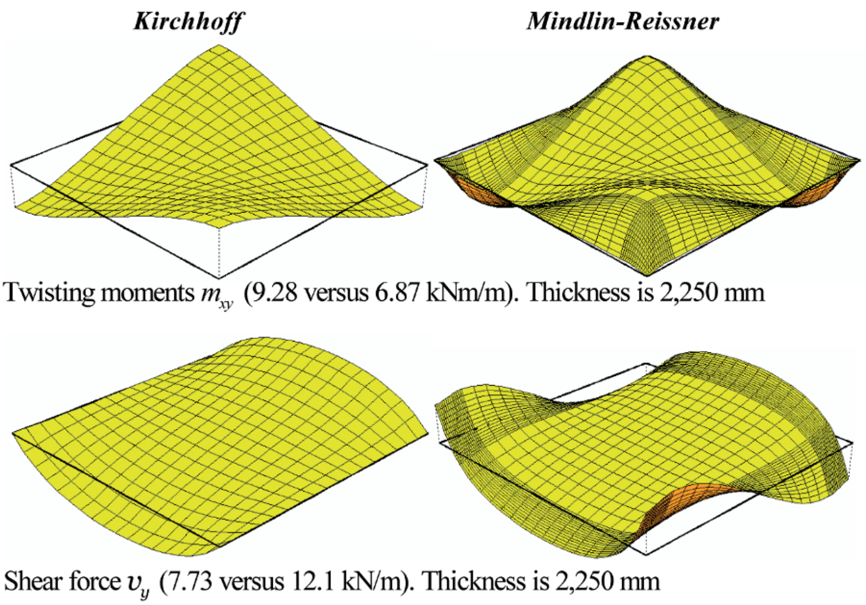

Figure 15.13 Twisting moment and shear force in thick plate.

Moment values are 17% larger in the Mindlin analysis. A Kirchhoff analysis would seriously underestimate the bending moment. Figure 15.13 shows the results for the twisting moment and shear force. Now the edge zone in the Mindlin analysis is a noticeable part of the plate and the peak value of the twisting moment occurs at a point of the plate far away from the boundary. The value is about three quarter of the corner value in the Kirchhoff analysis. Apparently, a thick plate reduces the twisting moment at the cost of a higher bending moment. The shear force plot in the Mindlin analysis is more regular than for the thin plate and needs no further explication. The concentrated edge shear force V_s in a thin plate is nicely spread over a wide zone in a thick plate and is just part of the overall distribution of the shear force.

Lesson

For thick plates we must use Mindlin. The mesh must be refined in an edge zone of width equal to about the plate thickness. It is sufficient to take five elements over the edge zone.

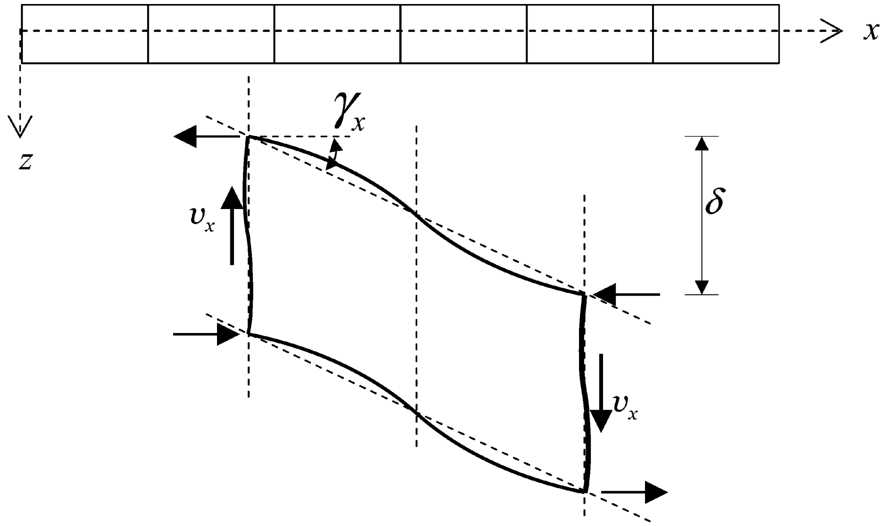

Figure 15.14 For slender multi-cell plates Mindlin theory must be applied.

Thin Orthotropic Plates

All conclusions in this chapter refer to plates of isotropic homogenous material. There is an exception, in which we always must analyze a thin plate with Mindlin theory. That is for a plate of relative small depth and small shear rigidity, like a multi-cell slab. The distortion of the cells in the x-direction, as shown in Figure 15.14, is interpreted as a shear deformation γ_x. For the shear rigidity of such plates, we refer to Section 21.3.

15.4 Comparison in Hindsight

The knowledge obtained in the preceding sections helps us understand why there was so much scatter in the submitted results for the three different plate analyses in Section 15.1. Return to Figure 15.4 for the twisting moment at the free edge. In a Mindlin analysis the twisting moment must become zero. In plate 1 the thickness is 160 mm and the element size 500 mm. There is no chance at all that a Mindlin analysis will lead to a reasonable result. At least five elements must be chosen in an edge zone of 160 mm. The value of the twisting moment m_{xy} at the ball support in a Kirchhoff analysis is half the support reaction, so 2,025 N. In the Mindlin analysis of plate 1 we should obtain $m_{xy} = 0$, however the moment becomes 1,448 N. It is less than 2,025 N but not zero. Again, 'neither one thing nor the other'. In plate 2 the element size is 250 mm, still about a factor 1.5 larger than thickness (160 mm), instead of a factor five smaller. Plate 3 has a three times larger thickness (480 mm). The edge zone in which we should refine the mesh is about twice the element size (250 mm). We still must refine the row of elements along the edge substantially in order to get reasonable results. As a matter of fact, we must conclude that plate 3 is hardly a thick plate; the edge zone of 480 mm is about one eighth of the span (4,000 mm), still a rather small zone, the more so if one remembers that the concentration of the edge shear force in fact occurs in the half of the disturbed edge zone. Mindlin has no chance to perform well with the applied mesh, and the result proves it. The corner value of the twisting moment, which should be zero for Mindlin and 2,025 N for Kirchhoff, is somewhere in between (1,212 N). Again the shear force is 'neither one thing nor the other'.

The story for the shear force in the section $x = 500$ mm can be short. For the explanation we refer to Figure 15.5. Near the free edge there is a concentrated shear force of the size of the twisting moment at that point. This value will be a little smaller than the corner value 2,025 N. In the Mindlin analysis this concentrated value is part of the smeared shear force v_x. The concentrated shear force acts in an edge zone of about 160 mm in plate 1 and 2 and in an edge zone of about 480 mm in plate 3. The Mindlin analysis cannot predict these localized shear forces with the applied large element sizes. Comparison of plate 1 to plate 2 shows immense element size dependence. The smoothest result is reached in plate 3, but even this is misleading: the computed maximum shear force has no physical meaning.

15.5 Message of the Chapter

Thin Plates

- Thin plates should preferably be calculated with Kirchhoff theory.
- If Mindlin theory is used for thin plates, this must be done at the cost of a very fine mesh, with results hardly different from Kirchhoff.

- Thin plates with Mindlin and a practical mesh are 'neither fish, flesh nor good red herring'.

- Application of Kirchhoff theory requires an element size not smaller than about plate thickness.

- If Kirchhoff theory is chosen and the FE-program offers the option of a graph for the shear force diagram across a section, also the concentrated edge shear force should be shown.

- If Kirchhoff theory is chosen and the FE-program is able to determine the resultant of shear forces and twisting moments (total force, total torque) over a section, also concentrated edge shear forces must be accounted for. Otherwise equilibrium is violated. This also holds at plate boundaries with edge beams.

- If Kirchhoff theory is chosen and edge beams are applied, the bending moment in the beam is correct, but the shear force must be obtained as the sum of the concentrated edge shear force V_{edge} and the beam shear force V_{beam}.

Thick Plates

- A thick homogeneous isotropic plate must be analyzed by Mindlin theory.

- An edge zone must be chosen of a width equal to about plate thickness, in which a sufficiently fine mesh is applied.

- Sufficiently fine is five or more elements over the edge zone.