Chapter 13
FE Analysis for Different Supports

In Chapter 4 we became acquainted with various edge conditions in thin plate theory; in Chapter 5 we applied this knowledge to square plates with three different support conditions. Here we meet the three cases again, now they appear in a FE analysis. In Chapter 5 we considered a two-way sine load and a homogeneous distributed load. That was done in order to be able to solve the differential equation. Here we need not make that difference in load type and will subject the plate to a homogenous load in all cases, as we did earlier for the discrete model in Section 9.4. In Chapter7 we became acquainted with the behaviour of circular plates subjected to both distributed load and a point load. Here we consider the behaviour of a square plate due to a central point load. It will appear that the response near the point load is of the same nature as occurs for the point load on a circular plate. In all analyses we choose Kirchhoff theory.

13.1 Simply-Supported Plate

We consider a square simply-supported thin plate and choose a mesh of 20 elements in each direction. In thin plates the span is about 25 times thickness, so the chosen mesh size is in the order of the thickness of the plate. Poisson's ratio is 0.2. Section 13.1.1 is devoted to distributed load and Section 13.1.2 to a point load.

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13.1.1 Distributed Load

Figure 13.1 depicts FE results for a homogeneous distributed load. The distributions are very similar to, and a confirmation of, theoretical results in Fig-

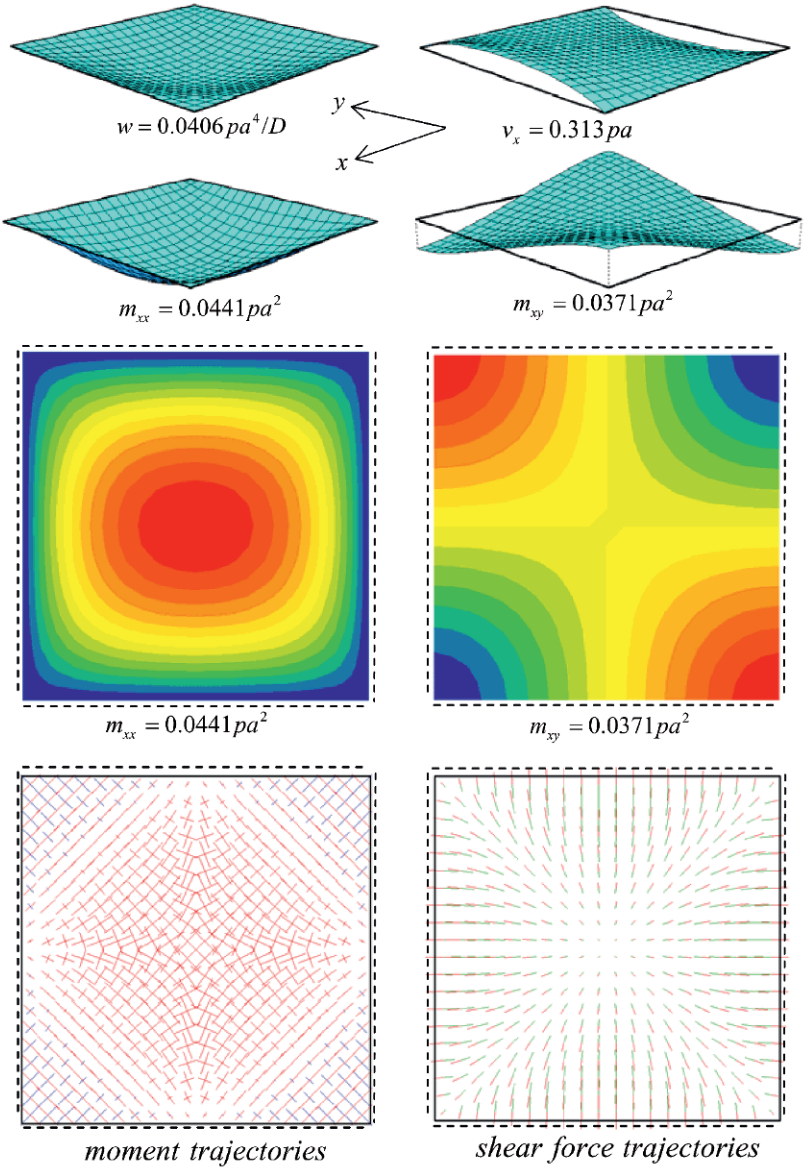

Figure 13.1 FE results. Square plate. Simply-supported. Distributed load.

ures 5.6 and 5.7 for the two-way sine load. In Figure 13.1 we show moments twice, as three-dimensional plots and as contour plots. Three-dimensional representation is preferred, for it highlights areas needing special care. This is particularly important when moment peaks occur. FE codes should offer this output option in any case.

Here we are not interested in values but rather in distributions and ways of load transfer. Therefore we skip legend scales. Maximum values are expressed in terms of the homogeneous load p and the span a. Figure 13.1 shows that the maximum values of m_{xx} occur in the centre of the plate, and of m_{xy} in the corners. These values are $0.0441pa^2$ and $0.0371pa^2$ respectively. They are identical to the theoretical values $0.0442pa^2$ and $0.0371pa^2$, borrowed from Timoshenko and Geere, converted from Poisson's ratio 0.3 to 0.2 [16]. Most plots for the moments and shear forces need no comment. We draw attention to the trajectories of the moments and shear force. For the support condition and load under consideration the shear trajectories approach the plate edges normal to the edge, because the shear force parallel to the edge is zero; Eq. (4.23) leads to a trajectory angle $\beta_o = \pi/2$.

13.1.2 Point Load

Figure 13.2 shows the results for a point load on a square simply-supported plate for the 20×20 mesh. The plots for the trajectories of the principal moments and shear force show that the state of moments and shears is almost perfectly axisymmetric near the point load. Note that the trajectories for the shear force are depicted for a part of the plate with sizes $a/2$ around the point load. This is done because of the large gradient in that region. In the contour plot of the twisting moment we recognise the horizontal and vertical lines of plate symmetry. In the considered example the twisting moments increase in the direction of the corners. The maximum value is of red colour. In other examples the contour plot of the twisting moment may appear as a cloverleaf. At the lines of symmetry m_{xy} is zero (yellow zones). This always holds true for twisting moments and shear forces for symmetrical loads. They are antisymmetric quantities in contrast to bending moments which are symmetric, and have maximum values on lines of plate symmetry.

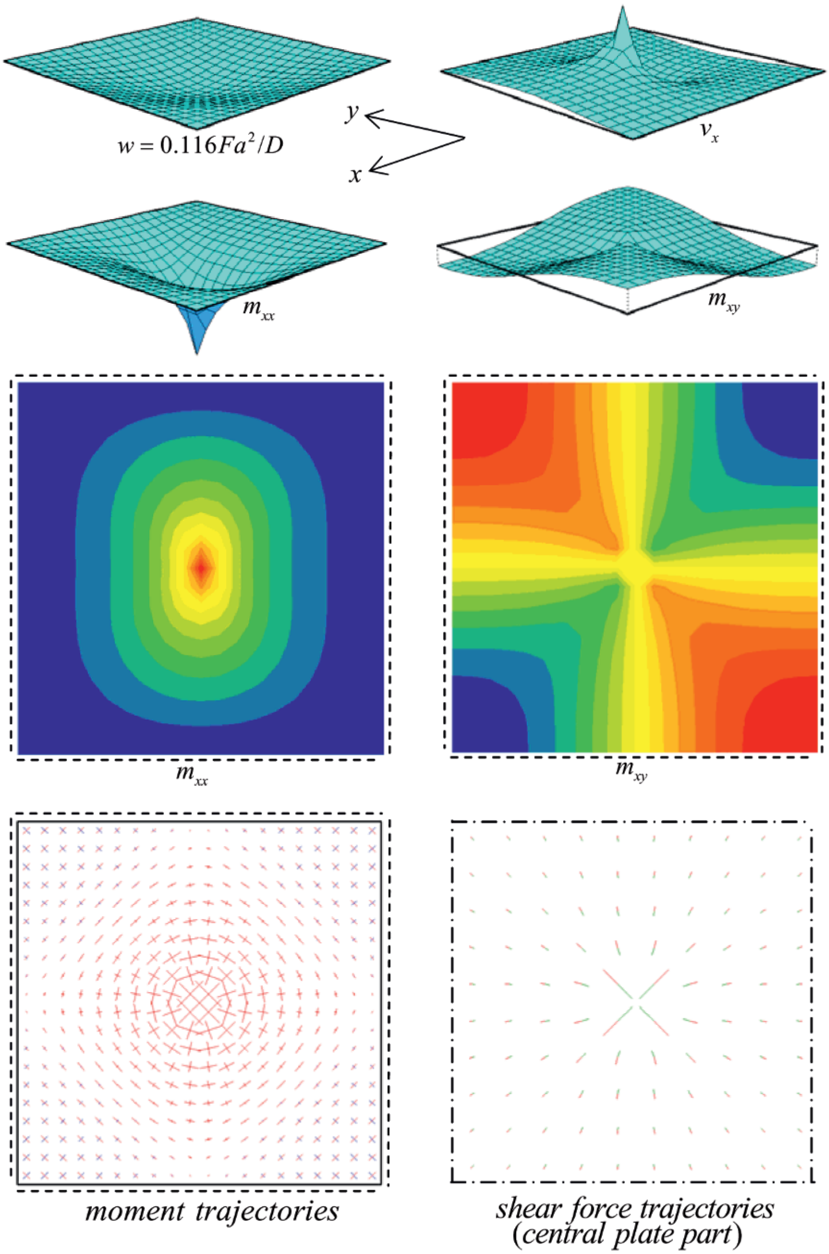

Figure 13.2 FE results. Square plate. Simply-supported. Central point load.

13.2 Corner Supports

We repeat the analyses for a plate with free edges and corner supports. In Section 13.2.1 we consider distributed load, in Section 13.2.2 a point load.

13.2.1 Distributed Load

Figure 13.3 shows the results for the distributed load. The corners are supported by balls, which permit rotations but prevent vertical displacements. Now the maximum moment does not occur in the plate centre, but mid-span of the free edge. The maximum twisting moments again occur in the corners, however with opposite signs compared to the simply-supported case. There a tensile corner reaction occurs, here a compressive one.

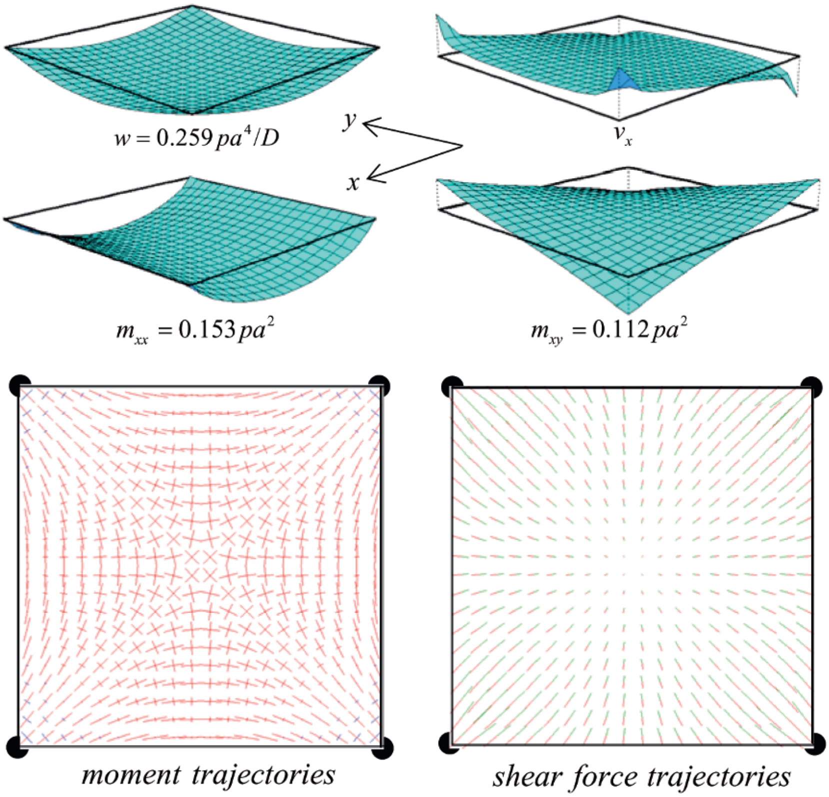

Figure 13.3 FE results. Square plate. Corner supports. Distributed load.

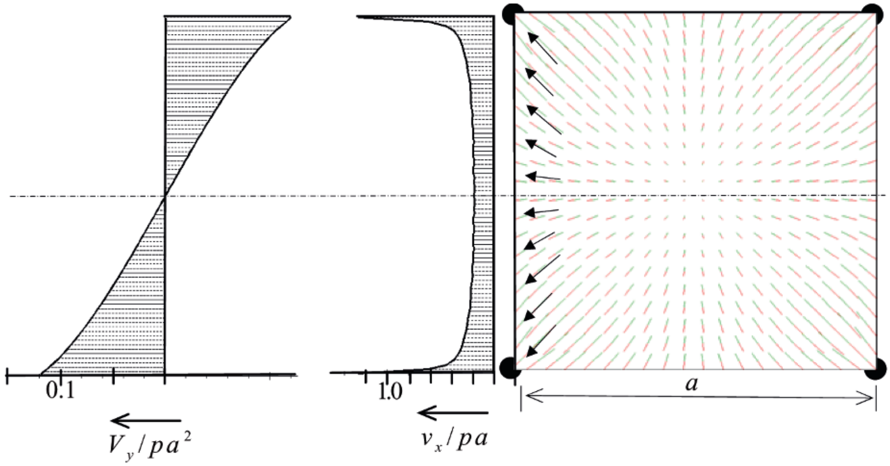

Figure 13.4 Load flow in shower analogy.

The engineer's feeling is that the load will flow in the direction of the corners, and high transverse shear forces occur in a quarter of a circle around the ball support. The structural engineer expects that all trajectories will be directed to the corner in a different way from the simply-supported case. However, reality is different. Figure 13.4 is very instructive about the transfer of loads to the supports. The shear trajectories are hardly different from the trajectories for the simply-supported structure in Figure 13.1. The load p flows as shear force v_x to the depicted edge. The value of v_x is almost constant along the major part of the edge. A load flow occurs along the free edge in the direction of the support, which is the concentrated force V_y in the y-direction. This force has the value of m_{xy} at the free edge. Starting from the middle of the edge, the concentrated shear force increases from zero to its maximum at the ball support. V_y is the integral of the shear force v_x. The shower analogy of Section 4.3 fully applies. The square plate is the m-hill, the free edges are gullies, and the ball support is a drain pipe. The distributed p-load is the shower. The water flows in the direction of the deepest slope to the gully, and from there through the gully to the drain pipe at the corner. The flow in the gully represents the concentrated shear force V_y.

The support reaction in the ball support is twice the maximum value of the concentrated shear force V_y. Therefore the expected maximum value of this shear force is $pa^2/8$. The FE analysis value for the reported mesh fineness 20×20 is $0.117pa^2$ instead of $0.125pa^2$. Continuous refinement to 40×40,

80×80 and 120×120 leads to coefficients 0.117, 0.120 and 0.121. The convergence to the exact value 0.125 is slow, but appears to be obtained. Therefore, all the distributed load p is transferred to the corner supports by the concentrated edge shear force. The distributed shear force v_y parallel to the edge is not zero at the edge and increases from zero at mid-span of the edge to its maximum at the ball support, but it remains finite as the element mesh is refined. The same holds for the principle shear force along the diagonal of the plate.

In reality corner supports will have some size. Let us assume that the support covers an area of a quarter of a circle with a radius of the order of the plate thickness t. Then the boundary between support and plate has a length of the order of $\pi t/2$, say $2t$. In Section 3.6 we learned that the concentrated edge shear force attenuates over a length of the order t. This means that the two concentrated forces which arrive at a corner do not spread nicely over boundary of length $2t$, but rather remain concentrated at the two ends of the boundary.

13.2.2 Point Load

The results for the point load are assembled in Figure 13.5. After the discussion of the point load for the simply-supported edge and the distributed load for the ball support, no further comment is needed. In the plate centre the correspondence with a circular plate is seen again. Figure 13.5 confirms the shower analogy Distributed shear forces v_x transfer the point load to the free edge and a concentrated shear force V_y (equal to m_{xy}) carries the distributed shear forces v_x to the ball supports.

13.3 Edge Beams

In Section 5.3 we touched on the subject of flexurally rigid, torsionally weak beams as a way to simulate simply-supported edges. It was found that for zero Poisson's ratio the bending moment M in the edge beam is 50% higher than expected on the basis of the load which flows to the edge. In Section 5.3 this study was done on the basis of a two-way sine load. In the discrete model of Section 9.4 we touched on the same subject for a homogeneously distributed load p. In the present section we again use a homogeneous distribution and check whether similar results are produced in a FE analysis.

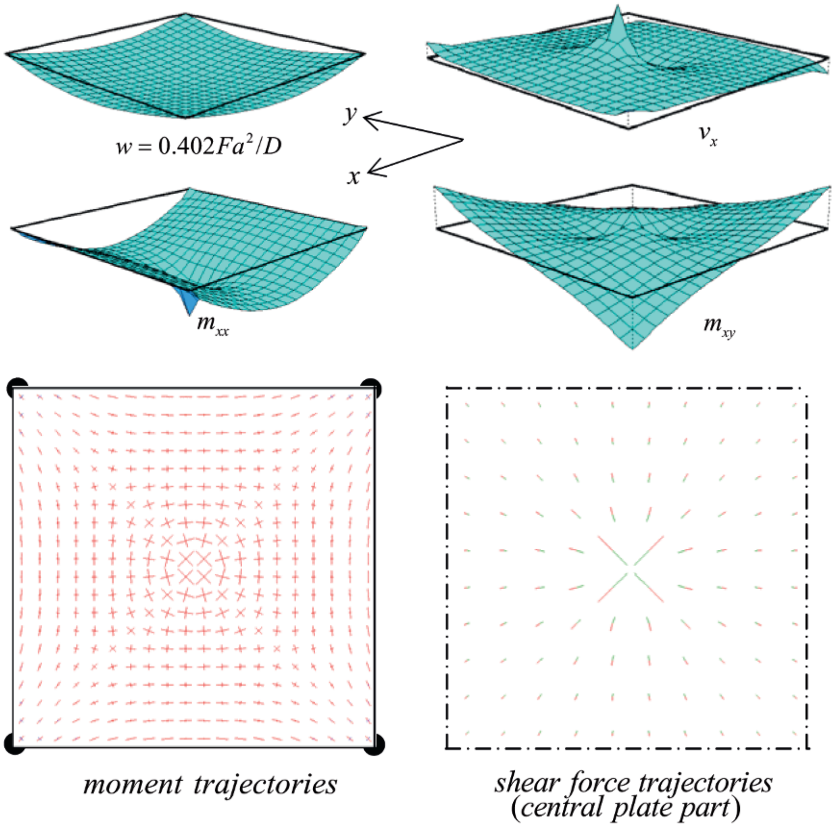

Figure 13.5 FE results. Square plate. Corner supports. Central point load.

13.3.1 Rigid Beams

In this section the four edge beams have infinitely large flexural rigidity, but zero torsion rigidity. The flexural stiffness in the FE analysis is chosen a thousand times larger than the total plate stiffness aD. The beams are supported by balls at the corners. We do not repeat plots for moments and shear forces, for they are precisely the same as for the simply-supported plate in Figure 13.1. Here we are interested in the bending moment M at mid-span of the beam, and the maximum shear force V at the beam end. In Figure 13.6 we have plotted the shear force v_x and twisting moment m_{xy} in the plate along the edge beam, and the moment M and shear V of the beam itself. We can check what moment and shear force in the beam occur if we load it by the shear force v_x only. For this purpose we first adapt the values

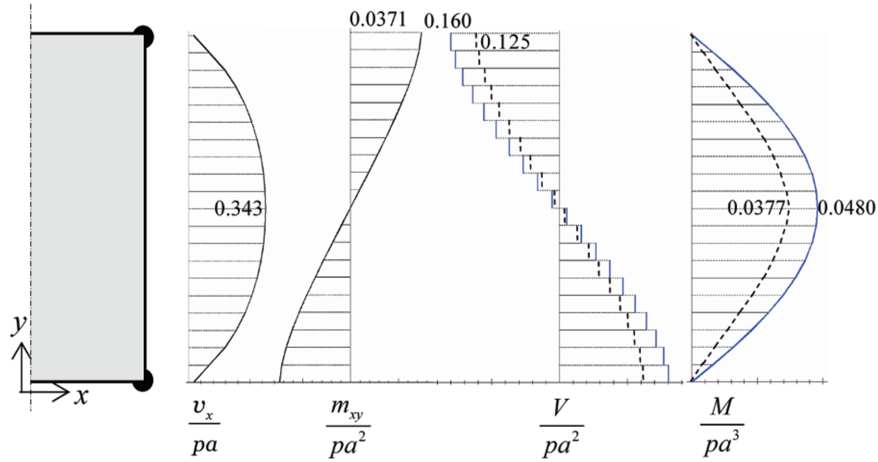

Figure 13.6 Check on excessive moment in rigid edge beam.

of v_x because shear forces are less accurate than moments; the reason is that shear forces are calculated in the centre line of elements. We know that the area of the v_x-diagram must be $0.250pa^2$. Assuming that the distribution shape is correct, we increase the maximum value of v_x shown in Figure 13.1 from $0.313pa$ to $0.343pa$. Numerical integration of the shear force over the half span of the beam then leads to $0.125pa^2$. Integrating twice leads to the bending moment $0.0377pa^3$ in the beam. The integration results are the dashed lines in the moment and shear diagram of the beam. We see that the FE analysis confirms the theory that the beam moment is larger than would be expected on the basis of the distributed load v_x only. The difference is caused by the twisting moments, which are an additional load on the edge beam. The area of the twisting moment diagram over half the span is $0.0106pa^3$. Adding this to $0.0377pa^3$ we obtain $0.0483pa^3$ which is very close to the FE result $0.0480pa^3$. For the two-way sine load and Poisson's ratio 0.2, the beam moment is 1.4 times the expected value. In the present case of homogeneously distributed load, a factor $0.0480/0.0377 = 1.27$ applies.

In Section 5.3.4 we expressed the expectation that the twisting moment will not influence the shear force in the beam, based on the assumption that the twisting moment can be carried to the web of the edge beam. This is not confirmed by the analysis. The analysis leads to a beam shear force which is a factor $0.160/0.125 = 1.28$ too large, practically the same amplification factor as for the moment. Apparently, a concentrated force of opposite sign

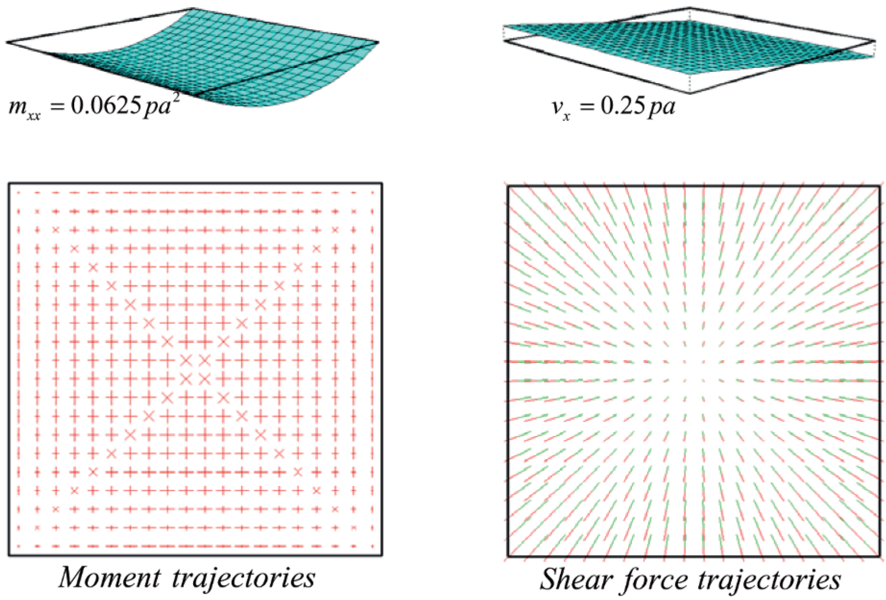

Figure 13.7 FE results. Square plate. Twistless case.

still remains in the plate edge, which balances the overestimated shear force in the beam. The reason is the use of Kirchhoff theory. Here the thick plate theory of Mindlin performs better.

13.3.2 Flexible Beams

For flexural edge beams leading to zero twisting moments we must choose the flexural rigidity $aD/2$, where D is the plate rigidity and a the plate span, corresponding with the discussion in Section 5.4. We expect that half the distributed load p is transferred in the x-direction and half in the y-direction. The bending moment m_{xx} is independent of y, and m_{yy} is independent of x. Similar considerations hold for the shear forces. The total load pa^2 is homogeneously distributed over the circumference $4pa$. The support reaction is $0.25pa$. The maximum moment and shear force in the plate are $0.0625pa^2$ and $0.25pa$, respectively. FE results for a 20×20 mesh are shown in Figure 13.7. The bending moment m_{xx} and shear force v_x indeed are clearly constant in the y-direction. The moment trajectories are different from Figure 13.1 for rigid edge beams. The direction is parallel to x- and y-direction all over the plate. The moments m_{xx} and m_{yy} have become principal moments because the twisting moment is zero. At the plate diagonals, the bending moments are equal and Mohr's circle becomes a point. Then the direction of the trajectories is indeterminate. The FE program computes them parallel and normal to the diagonal. The shear trajectories have also changed compared to the case of rigid edge beams. Similar to what we have seen for corner supports in Figure 13.3, they are no longer normal to the edge. For symmetry reasons the moment m_{yy} and shear force v_y need not be shown. The twisting moment is zero at each position. This plot is also skipped.

We now consider the shear trajectories more closely. In a twist-less slab they are straight lines, originating from the plate centre. The explanation is straight forward. Consider a set of x- and y-axes with origin in the plate centre. Then $v_x = \frac{1}{2}px$ and $v_y = \frac{1}{2}py$. According to Eq. (4.23) the trajectory direction is calculated from $\beta_o = \arctan(y/x)$. In a straight line, starting in the centre, y/x is constant, so a constant trajectory direction β is obtained along the straight line.

When we consider a length ds along the edge beam, the load p on a triangle with area $\frac{1}{2} \times a/2 \times ds$ flows to this edge part. The total load on this triangle is $\frac{1}{4}pa \times ds$, and this flows to an edge part of length ds. Therefore the shear force per unit length is $\frac{1}{4}pa$, which we had decided on earlier on other grounds.

According to classical beam theory the maximum moment in the edge beam is $M = \frac{1}{8} \times \frac{1}{4}pa \times a^2 = 0.03125pa^3$ and the maximum shear force $V = \frac{1}{4}pa \times a/2 = 0.125pa^2$. The value of the beam moment $M$is due to the load v_x only on the one edge and to v_y on the other edge. The FE analysis with the 20 × 20 mesh delivers $M = 0.03125pa^3$ and $V = 0.11875pa^2$ respectively. The bending moment is exact. The shear force is 5% less than the exact value. The difference is easily explained. The shear force in the FE analysis is constant over each beam element, so in fact holds true in the middle of the element. In reality the shear force increases linearly from the beam centre to the column. We have used 10 elements over the half beam length, therefore we are missing a half element size over 10 elements. This explains the 5% error. The shear force will approach the exact value $0.125pa^2$ for increasing mesh fineness.

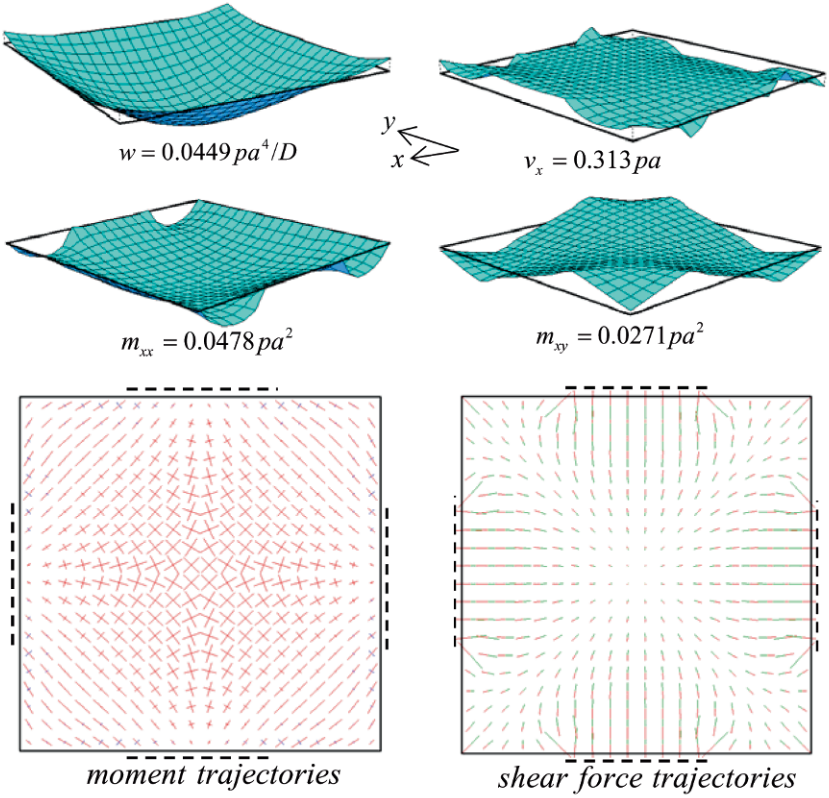

Figure 13.8 FE results. Square plate. Pressure only. Distributed load.

13.4 Pressure-Only Support

In Section 13.1 we considered a simply-supported plate subjected to a homogeneously distributed load. It was tacitly assumed that the support is able to transfer both pressure and tension reactions. And indeed the result showed that concentrated tensile reactions do occur in the corners. Physically this presupposes either a very good fixing to, say, a wall below the slab, or the presence of a wall on top of the slab edge, which provides the needed downward vertical reaction force in the corner. When neither the one nor the other is secured, we must reckon with another moment distribution. In Chapter 5 we noticed the high efficiency of the simply-supported plate, and that the diagonal beam action takes care for it to a large extent. This contribution will be reduced if no tensile reaction forces can occur. We must expect lower twisting moments at the cost of higher bending moments in the plate centre.

If we use a linear-elastic program we must do the analysis in an iterative way. In the first run, all edge nodes are fixed ($w = 0$). Then the computation is repeated with the nodes released where a tensile force occurs. This must be done until there are no tensile reactions. Figure 13.8 shows results of such an iterative analysis. If we compare the results with moments for the ideal simple support in Figure 13.1 the following conclusions hold. The plate corners lift from the supports. This occurs over about 30% of the edge length near each corner. Only 40% of the edge remains where we find compressive reactions. The maximum deflection increases 11%. The twisting moment m_{xy} at the plate edge is zero in the uplifted corner as it should be according to theory. Its maximum has shifted to a position at some distance from the corner. The maximum value has become 27% smaller. The bending moment m_{xx} in the plate centre has increased about 8%. Most probably not many structural engineers are aware of this phenomenon. It is always wise to make allowance for unexpected loadings and imperfect support conditions.

13.5 Message of the Chapter

- Whatever boundary conditions, we find a nearly axi-symmetric state in the neighbourhood of a point load.
- Twisting moments and shear forces are zero on lines of symmetry at symmetric loading. Bending moments are maximal on the lines of symmetry.
- For simply-supported and corner-supported plates, there are large twisting moments at the plate corners. Their signs are different, and so are the signs of the corner reactions. In the corner supported plate the reaction is compressive; in the simply-supported plate it is tensile.
- The two limit cases, simple supports and corner supports, can be simulated by edge beams with flexural rigidities, which are infinite or zero, respectively. An ideal twist-less case can be obtained by a proper in-between choice of the flexural rigidity of the edge beam.

- If tensile support reactions cannot occur, the twisting moments in the corner reduce substantially at the cost of higher bending moments and a larger deflection.
- FE codes should not only offer the output option of contour plots, but also three-dimensional presentations of deflections, moments and shear forces. The latter are more appealing to structural engineers, and concentrate attention to spots with large moments and shear forces.