Goal Oriented Mesh Adaptivity for Mixed Control-State Constrained Elliptic Optimal Control Problems

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1 Introduction

Adaptive finite element methods for the numerical solution of partial differential equations consist of successive cycles of the loop

$$\mathsf{SOLVE} \Longrightarrow \mathsf{ESTIMATE} \Longrightarrow \mathsf{MARK} \Longrightarrow \mathsf{REFINE}.$$

Here, SOLVE stands for the finite element solution of the problem with respect to a given triangulation of the computational domain. The following step ESTIMATE is devoted to the estimation of the global discretization error in some appropriate norm or a user specified quantity of interest by a cheaply computable a posteriori error estimator. The estimator is assumed to consist of local contributions whose actual magnitude is then used in the step MARK to specify elements of the triangulation for refinement. The final step REFINE deals with the generation of a new triangulation based on the refinement of the elements selected in the previous step according to specific refinement rules. Adaptive finite elements are by now well established. There are various approaches such as residual-type a posteriori error estimators which rely on the proper evaluation of the residuals with respect to a computed approximation in the norm of the dual space and hierarchical type estimators where the equation satisfied by the error is suitably localized along with a solution of the local problems by higher order finite elements (cf., e.g. [1, 3, 35]). Averaging-type estimators typically use some sort of gradient recovery on element-related patches (cf., e.g. [1, 35]), whereas the theory of guaranteed error majorants provides reliable upper bounds for the error (see [31]). Finally, the goal oriented weighted dual approach extracts information on the error via the dual problem (cf. [4, 12]).

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As far as the optimal control of PDEs are concerned, the goal oriented dual weighted approach has been applied to unconstrained problems in [4, 5], to control constrained ones in [17, 36] and to state constrained problems in [16, 19]. Residual-type a posteriori error estimators for control constrained problems have been developed and analyzed in [13, 14, 18, 20, 23, 26, 27]. State constrained optimal control problems are more difficult to handle than control constrained ones, since the Lagrange multiplier for the state constraints typically lives in a measure space. An appropriate way to cope with this problem is to use a regularization of the state constrained problems by means of mixed control-state constraints (Lavrentiev regularization). With regard to numerical solution techniques the regularized problems can be formally treated as in the case of control constraints (cf., e.g. [2, 9, 29, 32–34]).

In this paper, we will develop, analyze and implement the goal oriented weighted dual approach to mixed control-state constrained distributed optimal control problems for linear second order elliptic boundary value problems. The paper is organized as follows: In Section 2, we consider a model distributed optimal control problem for a two-dimensional, second order elliptic PDE with a quadratic objective functional and mixed unilateral constraints on the state and on the control. The finite element discretization is based on standard P1 conforming finite elements with respect to simplicial triangulations of the computational domain and gives rise to a finite dimensional constrained minimization problem. In both the continuous and discrete regime, the optimality conditions are stated in terms of the associated Lagrangians. Section 3 is devoted to a representation of the error in the quantity of interest which is chosen as the objective functional. The error representation involves primal–dual residuals, a primal–dual mismatch in complementarity due to a possible mismatch between the continuous and discrete active and non-active sets, and data oscillation terms. In Section 4, we derive the goal oriented a posteriori error estimator based on appropriate upper bounds both for the primal–dual residuals and the primal–dual mismatch in complementarity. The final section, Section 5 contains a brief description of the marking and refinement strategy as well as numerical results for an example illustrating the performance of the error estimator.

2 The Mixed Control-State Elliptic Optimal Control Problem and Its Finite Element Approximation

We assume Ω to be a bounded domain in $\mathbb{R}^2$ with boundary $\Gamma := \overline{\Gamma}_D \cup \overline{\Gamma}_N$, $\Gamma_D \cap \Gamma_N = \emptyset$. We use standard notation from Lebesgue and Sobolev space theory. In particular, we refer to $L^2(\Omega)$ as the Hilbert space with inner product $(\cdot,\cdot)_{0,\Omega}$ and norm $\|\cdot\|_{0,\Omega}$ and to $H^k(\Omega)$, $k \in \mathbb{N}$, as the Sobolev space with norm $\|\cdot\|_{k,\Omega}$. The set $L^2_+(\Omega)$ stands for the positive cone in $L^2(\Omega)$ with respect to the canonical ordering.

Given a desired state $y^d \in L^2(\Omega)$, a shift control $u^d \in L^2(\Omega)$, regularization parameters $\alpha > 0$, $\varepsilon > 0$, and a function $\psi \in L^\infty(\Omega)$, we consider the mixed control-state constrained distributed optimal control problem:
Find $(y,u) \in V \times L^2(\Omega)$, where $V := \{v \in H^1(\Omega) \mid v|_{\Gamma_D} = 0\}$, such that

$$\inf_{y,u} J(y,u) := \frac{1}{2}\|y - y^d\|^2_{0,\Omega} + \frac{\alpha}{2}\|u - u^d\|^2_{0,\Omega}, \tag{1a}$$

$$\text{subject to} \quad a(y,v) = (u,v)_{0,\Omega}, \quad v \in V, \tag{1b}$$

$$\varepsilon u + y \in K := \{v \in L^2(\Omega) \mid v(x) \le \psi(x) \text{ f.a.a. } x \in \Omega\}. \tag{1c}$$

Here, $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ stands for the bounded, V-elliptic bilinear form

$$a(u,v) := \int_\Omega (\nabla u \cdot \nabla v + cuv)\, dx, \quad c \in \mathbb{R}_+.$$

Denoting by $A : V \to V^*$ the operator associated with $a(\cdot,\cdot)$, we introduce the Lagrangian $\mathcal{L} : V \times L^2(\Omega) \times V \times L^2_+(\Omega) \to \mathbb{R}$ according to

$$\mathcal{L}(y,u,p,\sigma) := J(y,u) + \langle Ay - u, p\rangle + (\varepsilon u + y - \psi, \sigma)_{0,\Omega}, \tag{2}$$

where $\langle\cdot,\cdot\rangle$ denotes the dual pairing between V^* and V. Then, the minimization problem (1a)–(1c) can be equivalently stated as the saddle point problem

$$\inf_{y,u} \sup_{p,\sigma} \mathcal{L}(y,u,p,\sigma). \tag{3}$$

Setting $x := (y,u,p) \in X := V \times L^2(\Omega) \times V$, the optimality conditions read as follows:

$$\nabla_x \mathcal{L}(x,\sigma) = 0, \tag{4a}$$

$$\nabla_\sigma \mathcal{L}(x,\sigma)(\mu - \sigma) \le 0, \quad \mu \in L^2_+(\Omega), \tag{4b}$$

where $\nabla_x \mathcal{L}(x,\sigma)$ and $\nabla_\sigma \mathcal{L}(x,\sigma)$ stand for the derivatives of $\mathcal{L}$ with respect to x and σ in (x,σ). The multiplier p is referred to as the adjoint state. We note that (4a) gives rise to the state equation (1b), the adjoint state equation

$$a(p,v) = (y^d - y - \sigma, v)_{0,\Omega}, \quad v \in V, \tag{5}$$

and the equation

$$p = \alpha(u - u^d) + \varepsilon\sigma, \tag{6}$$

whereas the variational inequality (4b) can be equivalently written in terms of the complementarity conditions

$$\sigma \in L^2_+(\Omega), \quad \psi - (\varepsilon u + y) \in L^2_+(\Omega), \quad (\varepsilon u + y - \psi, \sigma)_{0,\Omega} = 0. \tag{7}$$

We define the active set $\mathcal{A}$ as the maximal open set $A \subset \Omega$ such that $\varepsilon u(x) + y(x) = \psi(x)$ f.a.a. $x \in A$ and the inactive set $\mathcal{I}$ according to $\mathcal{I} := \bigcup_{\kappa>0} B_\kappa$, where B_κ is the maximal open set $B \subset \Omega$ such that $\varepsilon u(x) + y(x) \leq \psi(x) - \kappa$ for almost all $x \in B$.

For the finite element discretization of (1a)–(1c) we consider a family $\{\mathcal{T}_\ell(\Omega)\}$ of shape-regular simplicial triangulations of Ω which align with Γ_D, Γ_N on Γ. We denote by $\mathcal{N}_\ell(D)$ and $\mathcal{E}_\ell(D)$, $D \subseteq \overline{\Omega}$, the sets of vertices and edges of $\mathcal{T}_\ell(\Omega)$ in $D \subseteq \overline{\Omega}$, and we refer to h_T and $|T|$ as the diameter and the area of an element $T \in \mathcal{T}_\ell(\Omega)$, whereas h_E stands for the length of an edge $E \in \mathcal{E}_\ell(D)$. For $E \in \mathcal{E}_\ell(\Omega)$ such that $E = T_+ \cap T_-$, $T_\pm \in \mathcal{T}_\ell(\Omega)$, we define $\omega_E := T_+ \cup T_-$. Further, we denote by $S_\ell := \{v_\ell \in C_0(\Omega) \mid v_\ell|_T \in P_1(T),\ T \in \mathcal{T}_\ell(\Omega)\}$ the finite element space of continuous, piecewise linear finite elements and we refer to V_ℓ as its subspace $V_\ell := \{v_\ell \in S_\ell \mid v_\ell|_{\Gamma_D} = 0\}$. We will also use the following notation: If A and B are two quantities, then $A \preceq B$ means that there exists a positive constant C such that $A \leq CB$, where C only depends on the shape regularity of the triangulations, but not on their granularities.

Then, given approximations $y_\ell^d \in S_\ell$, $u_\ell^d \in S_\ell$ and $\psi_\ell \in S_\ell$ of y^d, u^d and ψ, the finite element approximation of (1a)–(1c) is given by
Find $(y_\ell, u_\ell) \in V_\ell \times S_\ell$ such that

$$\inf_{y_\ell, u_\ell} J_\ell(y_\ell, u_\ell) := \frac{1}{2}\|y_\ell - y_\ell^d\|_{0,\Omega}^2 + \frac{\alpha}{2}\|u_\ell - u_\ell^d\|_{0,\Omega}^2, \tag{8a}$$

$$\text{subject to} \quad a(y_\ell, v_\ell) = (u_\ell, v_\ell)_{0,\Omega}, \quad v_\ell \in V_\ell, \tag{8b}$$

$$\varepsilon u_\ell + y_\ell \in K_\ell := \{v_\ell \in S_\ell \mid v_\ell \leq \psi_\ell \quad \text{in } \Omega\}. \tag{8c}$$

We proceed as in the continuous regime and introduce the Lagrangian $\mathcal{L}_\ell$: $V_\ell \times S_\ell \times V_\ell \times (S_\ell \cap L_+^2(\Omega))$ by

$$\mathcal{L}_\ell(y_\ell, u_\ell, p_\ell, \sigma_\ell) := J_\ell(y_\ell, u_\ell) + \langle Ay_\ell - u_\ell, p_\ell\rangle + (\varepsilon u_\ell + y_\ell - \psi_\ell, \sigma_\ell)_{0,\Omega} \tag{9}$$

such that (8a)–(8c) is equivalent to the saddle point problem

$$\inf_{y_\ell, u_\ell} \sup_{p_\ell, \sigma_\ell} \mathcal{L}_\ell(y_\ell, u_\ell, p_\ell, \sigma_\ell). \tag{10}$$

The optimality conditions turn out to be

$$\nabla_x \mathcal{L}_\ell(x_\ell, \sigma_\ell) = 0, \tag{11a}$$

$$\nabla_\sigma \mathcal{L}_\ell(x_\ell, \sigma_\ell)(\mu_\ell - \sigma_\ell) \leq 0, \quad \mu_\ell \in S_\ell \cap L_+^2(\Omega), \tag{11b}$$

where $x_\ell := (y_\ell, u_\ell, p_\ell) \in X_\ell := V_\ell \times S_\ell \times V_\ell$. Again, (11a) comprises the discrete state equation (8b), the discrete adjoint state equation

$$a(p_\ell, v_\ell) = (y_\ell^d - y_\ell - \sigma_\ell, v_\ell)_{0,\Omega}, \quad v_\ell \in V_\ell, \tag{12}$$

and the equation

$$p_\ell = \alpha(u_\ell - u_\ell^d) + \varepsilon\sigma_\ell. \tag{13}$$

On the other hand, (11b) represents the discrete complementarity conditions

$$\sigma_\ell \in S_\ell \cap L^2_+(\Omega), \quad \psi_\ell - (\varepsilon u_\ell + y_\ell) \in S_\ell \cap L^2_+(\Omega), \quad (\varepsilon u_\ell + y_\ell - \psi_\ell, \sigma_\ell)_{0,\Omega} = 0. \tag{14}$$

We define the discrete active set $\mathcal{A}_\ell$ according to $\mathcal{A}_\ell := \{x \in \overline{\Omega} \mid \varepsilon u_\ell(x) + y_\ell(x) = \psi_\ell(x)\}$ and refer to $\mathcal{I}_\ell := \overline{\Omega} \setminus \mathcal{A}_\ell$ as the discrete inactive set.

3 Error Representation in the Quantity of Interest

We derive an error representation in the quantity of interest which involves the second derivative of the Lagrangian $\mathcal{L}$ with respect to x. Since this second derivative does depend neither on x nor on σ, we simply write $\nabla_{xx}\mathcal{L}(z, z')$, $z, z' \in X$, instead of $\nabla_{xx}\mathcal{L}(x, \sigma)(z, z')$. We will use the same simplifying notation for the second derivative of $\mathcal{L}_h$.

Theorem 1. *Let $(x, \sigma) \in X \times L^2_+(\Omega)$ and $(x_\ell, \sigma_\ell) \in X_\ell \times (S_\ell \cap L^2_+(\Omega))$ be the solutions of (3) and (10), respectively. Then there holds*

$$J(y, u) - J_\ell(y_\ell, u_\ell) = -\frac{1}{2}\nabla_{xx}\mathcal{L}_\ell(x_\ell - x, x_\ell - x) + (\varepsilon u_\ell + y_\ell - \psi, \sigma)_{0,\Omega} + \mathrm{osc}^{(1)}_\ell, \tag{15}$$

where $\mathrm{osc}^{(1)}_\ell$ stands for the data oscillations

$$\mathrm{osc}^{(1)}_\ell := \sum_{T \in \mathcal{T}_\ell(\Omega)} \mathrm{osc}^{(1)}_T, \tag{16}$$

$$\mathrm{osc}^{(1)}_T := (y_\ell - y^d_\ell, y^d_\ell - y^d)_{0,T} + \alpha(u_\ell - u^d_\ell, u^d_\ell - u^d)_{0,T} + \frac{1}{2}\|y^d - y^d_\ell\|^2_{0,T} + \frac{\alpha}{2}\|u^d - u^d_\ell\|^2_{0,T}.$$

Proof. We note that for $z_\ell = (\delta y_\ell, \delta u_\ell, \delta p_\ell) \in X_\ell$ there holds

$$\mathcal{L}(x, \sigma_\ell) = \mathcal{L}(x, \sigma) + (\varepsilon u + y - \psi, \sigma_\ell - \sigma)_{0,\Omega}, \tag{17a}$$

$$\nabla_x\mathcal{L}(x_\ell, \sigma_\ell)(z_\ell) = \nabla_x\mathcal{L}(x_\ell, \sigma_\ell)(z_\ell) + (\varepsilon\delta u_\ell + \delta y_\ell, \sigma_\ell - \sigma)_{0,\Omega}. \tag{17b}$$

Using the optimality conditions (4a), (4b) and (11a), (11b) as well as (17a), (17b), Taylor expansion yields

$$\begin{aligned}
&J(y, u) - J_\ell(y_\ell, u_\ell)\mathcal{L}(x, \sigma) - \mathcal{L}_\ell(x_\ell, \sigma_\ell) \\
&= \mathcal{L}(x, \sigma) - \mathcal{L}_\ell(x, \sigma_\ell) - \nabla_x\mathcal{L}_\ell(x, \sigma_\ell)(x_\ell - x) - \frac{1}{2}\nabla_{xx}\mathcal{L}_\ell(x_\ell - x, x_\ell - x) \\
&= J(y, u) - J_\ell(y, u) - (\varepsilon u + y - \psi_\ell, \sigma_\ell)_{0,\Omega} \\
&\quad - \nabla_x\mathcal{L}_\ell(x, \sigma_\ell)(x_\ell - x) - \frac{1}{2}\nabla_{xx}\mathcal{L}_\ell(x_\ell - x, x_\ell - x)
\end{aligned}$$

$$
\begin{aligned}
&= -\nabla_x \mathcal{L}(x,\sigma_\ell)(x_\ell - x) - \frac{1}{2}\nabla_{xx}\mathcal{L}_\ell(x_\ell - x, x_\ell - x) \\
&\quad - (\varepsilon u + y - \psi_\ell, \sigma_\ell)_{0,\Omega} + \mathrm{osc}_\ell^{(1)} \\
&= -\frac{1}{2}\nabla_{xx}\mathcal{L}_\ell(x_\ell - x, x_\ell - x) - (\varepsilon u + y - (\varepsilon u_\ell + y_\ell), \sigma_\ell)_{0,\Omega} \\
&\quad + (\varepsilon u_\ell + y_\ell - (\varepsilon u + y), \sigma - \sigma_\ell)_{0,\Omega} + \mathrm{osc}_\ell^{(1)} \\
&= -\frac{1}{2}\nabla_{xx}\mathcal{L}_\ell(x_\ell - x, x_\ell - x) + (\varepsilon u_\ell + y_\ell - \psi, \sigma)_{0,\Omega} + \mathrm{osc}_\ell^{(1)},
\end{aligned}
$$

from which we conclude. □

Remark 1. We note that the error representation (15) reduces to the result from [5] in the unconstrained case, i.e. when $\sigma = \sigma_\ell = 0$.

For a further evaluation of the error, we introduce interpolation operators

$$
i_\ell^y : V \to V_\ell, \quad i_\ell^p : V \to V_\ell, \quad i_\ell^u : L^2(\Omega) \to S_\ell, \quad i_\ell^\sigma : L^2(\Omega) \to S_\ell, \tag{18}
$$

such that for all $y, p \in V$ and $u \in L^2(\Omega)$ there holds

$$
\begin{aligned}
&\|i_\ell^y y - y\|_{0,T}^2 + h_T^{1/2}\|i_\ell^y y - y\|_{0,\partial T}^2 \preceq h_T \|y\|_{1,D_T}, \\
&\|i_\ell^p p - p\|_{0,T}^2 + h_T^{1/2}\|i_\ell^p p - p\|_{0,\partial T}^2 \preceq h_T \|p\|_{1,D_T}, \\
&\|i_\ell^u u - u\|_{0,T}, \quad \|i_\ell^\sigma \sigma - \sigma\|_{0,T} \to 0 \quad \text{as } h_T \to 0.
\end{aligned}
$$

where $D_T := \{T' \in \mathcal{T}_\ell(\Omega) \mid \mathcal{N}_\ell(T') \cap \mathcal{N}_\ell(T) \neq \emptyset\}$. We may choose, for instance, Clément-type quasi-interpolation operators (cf., e.g. [35]) or the Scott–Zhang interpolation operators (cf., e.g. [8]).

Theorem 2. *In addition to the assumptions of Theorem 1, let $i_\ell^x = (i_\ell^y, i_\ell^u, i_\ell^p)$ be the interpolation operators as given by (18). Then there holds*

$$
J(y,u) - J_\ell(y_\ell, u_\ell) = -r(i_\ell^y y - y) - r(i_\ell^p p - p) + \mu_\ell(x,\sigma) + \mathrm{osc}_\ell^{(1)} + \mathrm{osc}_\ell^{(2)}, \tag{19}
$$

where $r(i_\ell^y y - y)$ and $r(i_\ell^p p - p)$ stand for the primal–dual residuals

$$
r(i_\ell^y y - y) := \frac{1}{2}\left((y_\ell - y_\ell^d + \sigma_\ell, i_\ell^y y - y)_{0,\Omega} + (\nabla p_\ell, \nabla(i_\ell^y y - y))_{0,\Omega}\right), \tag{20a}
$$

$$
r(i_\ell^p p - p) := \frac{1}{2}\left((\nabla y_\ell, \nabla(i_\ell^p p - p))_{0,\Omega} - (u_\ell, i_\ell^p p - p)_{0,\Omega}\right), \tag{20b}
$$

Moreover, $\mu_\ell(x,\sigma)$ is the primal–dual mismatch in complementarity and $\mathrm{osc}_\ell^{(2)}$ *a further data oscillation term given by*

$$
\mu_\ell(x,\sigma) := \frac{1}{2}\left((\varepsilon u_\ell + y_\ell - \psi, \sigma)_{0,\Omega} + (\psi_\ell - (\varepsilon u + y), \sigma_\ell)_{0,\Omega}\right), \tag{21a}
$$

$$
\begin{aligned}
\mathrm{osc}_\ell^{(2)} &:= \frac{1}{2}(y^d - y_\ell^d, y_\ell - i_\ell^y y)_{0,\Omega} + \frac{1}{2}(y^d - y_\ell^d, i_\ell^y y - y)_{0,\Omega} \\
&\quad + \frac{\alpha}{2}(u^d - u_\ell^d, u_\ell - i_\ell^u u)_{0,\Omega} + \frac{\alpha}{2}(u^d - u_\ell^d, i_\ell^u u - u)_{0,\Omega}.
\end{aligned} \tag{21b}
$$

Proof. Using (11a) and (17b), for $z_\ell = (\delta y_\ell, \delta u_\ell, \delta p_\ell) \in X_\ell$ we find

$$\begin{aligned}
0 &= \nabla_x \mathcal{L}(x,\sigma)(z_\ell) \\
&= \nabla_x \mathcal{L}(x_\ell,\sigma_\ell)(z_\ell) + \nabla_{xx}\mathcal{L}(x - x_\ell, z_\ell) + (\varepsilon\delta u_\ell + \delta y_\ell, \sigma - \sigma_\ell)_{0,\Omega} \\
&= \nabla_{xx}\mathcal{L}(x - x_\ell, z_\ell) + (\varepsilon\delta u_\ell + \delta y_\ell, \sigma - \sigma_\ell)_{0,\Omega} + (y_\ell^d - y^d, \delta y_\ell)_{0,\Omega} \\
&\quad + \alpha(u_\ell^d - u^d, \delta u_\ell)_{0,\Omega},
\end{aligned}$$

from which we deduce

$$\nabla_x \mathcal{L}(x_\ell,\sigma)(x - x_\ell - z_\ell) = \nabla_{xx}\mathcal{L}(x_\ell - x, x - x_\ell - z_\ell), \tag{22a}$$

$$\begin{aligned}
\nabla_{xx}\mathcal{L}(x_\ell - x, x_\ell - x) &= \nabla_{xx}\mathcal{L}(x_\ell - x, x_\ell - x + z_\ell) \\
&\quad - (\varepsilon\delta u_\ell + \delta y_\ell, \sigma - \sigma_\ell)_{0,\Omega}.
\end{aligned} \tag{22b}$$

Taking advantage of (22a),(22b) in (15), it follows that

$$\begin{aligned}
&J(y,u) - J_\ell(y_\ell,u_\ell) \\
&= \frac{1}{2}\nabla_{xx}\mathcal{L}(x,\sigma_\ell)(x - x_\ell, x_\ell - x + z_\ell) \\
&\quad + \frac{1}{2}(\varepsilon\delta u_\ell + \delta y_\ell, \sigma - \sigma_\ell)_{0,\Omega} + \frac{1}{2}(y_\ell^d - y^d, \delta y_\ell)_{0,\Omega} \\
&\quad + \frac{\alpha}{2}(u_\ell^d - u^d, \delta u_\ell)_{0,\Omega} + (\varepsilon\delta u_\ell + y_\ell - \psi, \sigma)_{0,\Omega} + \mathrm{osc}_\ell^{(1)} \\
&= -\frac{1}{2}\nabla_x \mathcal{L}(x_\ell,\sigma_\ell)(x_\ell - x + z_\ell) + \frac{1}{2}(\varepsilon u_\ell + y_\ell - (\varepsilon u + y), \sigma_\ell + \sigma)_{0,\Omega} \\
&\quad + \frac{1}{2}(y^d - y_\ell^d, y_\ell - y)_{0,\Omega} + \frac{\alpha}{2}(u_\ell^d - u^d, \delta u_\ell)_{0,\Omega} + \mathrm{osc}_\ell^{(1)}.
\end{aligned}$$

We conclude by choosing $z_\ell = (i_\ell^y y - y_\ell, i_\ell^p p - p_\ell, i_\ell^u - u_\ell)$ and observing (7) and (14). □

Remark 2. The primal–dual residuals $r(i_\ell^y y - y)$ and $r(i_\ell^p p - p)$ will be further estimated in the following section and will be made fully a posteriori in a standard way (cf., e.g. [4]). The term $\mu_\ell(x,\sigma)$ as given by (21a) represents the primal–dual mismatch in complementarity due to a possible mismatch in the approximation of the active and inactive sets $\mathcal{A}$ and $\mathcal{I}$ by their discrete counterparts $\mathcal{A}_\ell$ and $\mathcal{I}_\ell$. In its present form it is not yet a posteriori. In the subsequent section, we will show how $\mu_\ell(x,\sigma)$ can be made fully a posteriori and thus be included in the refinement strategy. A similar remark applies to the term $\mathrm{osc}_\ell^{(2)}$ which is essentially a data oscillation term, but as given by (21b) not a posteriori due to the occurrence of y. It will be made fully a posteriori as well.

4 Weighted Primal–Dual A Posteriori Error Estimator

By straightforward estimation of the right-hand sides in the representations (20a), (20b) of the primal–dual residuals the following result can be easily established.

Theorem 3. *The primal–dual residuals can be estimated according to*

$$|r(i_\ell^y y - y)| \preceq \sum_{T \in \mathcal{T}_\ell(\Omega)} \omega_T^y \rho_T^y, \tag{23a}$$

$$|r(i_\ell^p p - p)| \preceq \sum_{T \in \mathcal{T}_\ell(\Omega)} \omega_T^p \rho_T^p. \tag{23b}$$

Here, ρ_T^y and ρ_T^p are the L^2-norms of the residuals associated with the state and the adjoint state equation

$$\rho_T^y := \left(\|u_\ell\|_{0,T}^2 + h_T^{-1} \|\frac{1}{2}\nu \cdot [\nabla y_\ell]\|_{0,\partial T}^2 \right)^{1/2}, \tag{24a}$$

$$\rho_T^p := \left(\|y_\ell - y_\ell^d - \sigma_\ell\|_{0,T}^2 + h_T^{-1} \|\frac{1}{2}\nu \cdot [\nabla p_\ell]\|_{0,\partial T}^2 \right)^{1/2}. \tag{24b}$$

The corresponding dual weights ω_T^y and ω_T^p are given by

$$\omega_T^y := \left(\|i_\ell^p p - p\|_{0,T}^2 + h_T \|i_\ell^p p - p\|_{0,\partial T}^2 \right)^{1/2}, \tag{25a}$$

$$\omega_T^p := \left(\|i_\ell^y y - y\|_{0,T}^2 + h_T \|i_\ell^y y - y\|_{0,\partial T}^2 \right)^{1/2}. \tag{25b}$$

Remark 3. If the state y of the purely state constrained problem (i.e. $\varepsilon = 0$) is in $W^{1,r}(\Omega)$ for some $r > 2$ and hence represents a continuous function, the adjoint state p lives in $W^{1,s}(\Omega)$ with s being conjugate to r. The multiplier σ turns out to be a bounded Borel measure, and the discrete multipliers σ_ℓ are chosen as a linear combination of Dirac delta functionals associated with the nodal points of the triangulation. In this case, the primal–dual residuals have to be estimated in the respective L^r- and L^s-norms and the multipliers have to be treated separately (cf. [19]).

There are several ways to provide approximations of the weights ω_T^y and ω_T^p, $T \in \mathcal{T}_\ell(\Omega)$. We refer to [4] for a detailed discussion. Here, we use piecewise quadratic interpolations $i_{\ell,2}^y y_\ell$ and $i_{\ell,2}^p p_\ell$ of the computed P1 approximations y_ℓ and p_ℓ of the state y and the adjoint state p with respect to the coarser triangulation $\mathcal{T}_{\ell-1}(\Omega)$. This results in the computable weights

$$\hat{\omega}_T^y := \left(\|i_{\ell,2}^p p_\ell - p_\ell\|_{0,T}^2 + h_T \|i_{\ell,2}^p p_\ell - p_\ell\|_{0,\partial T}^2 \right)^{1/2}, \tag{26a}$$

$$\hat{\omega}_T^p := \left(\|i_{\ell,2}^y y_\ell - y_\ell\|_{0,T}^2 + h_T \|i_{\ell,2}^y y_\ell - y_\ell\|_{0,\partial T}^2 \right)^{1/2}. \tag{26b}$$

We now concentrate on the primal–dual mismatch in complementarity $\mu_\ell(x, \sigma)$ where for notational simplicity we drop the argument (x, σ). Taking the complementarity conditions (7) and (14) into account, we find

$$\mu_\ell|_{\mathcal{I}\cap\mathcal{I}_\ell} = 0, \tag{27a}$$

$$\begin{aligned}\mu_\ell|_{\mathcal{A}\cap\mathcal{I}_\ell} &= \frac{1}{2}\big((\varepsilon u_\ell + y_\ell - \psi, i_\ell^\sigma \sigma)_{0,\mathcal{A}\cap\mathcal{I}_\ell} \\ &\quad + (\varepsilon u_\ell + y_\ell - \psi, \sigma - i_\ell^\sigma \sigma)_{0,\mathcal{A}\cap\mathcal{I}_\ell}\big),\end{aligned} \tag{27b}$$

$$\begin{aligned}\mu_\ell|_{\mathcal{I}\cap\mathcal{A}_\ell} &= \frac{1}{2}(\psi_\ell - (\varepsilon u + y), \sigma_\ell)_{0,\Omega} \\ &= \frac{1}{2}\big((\varepsilon(u_\ell - i_\ell^u u) + y_\ell - i_\ell^y y, \sigma_\ell)_{0,\mathcal{I}\cap\mathcal{A}_\ell} \\ &\quad + (\varepsilon(i_\ell^u u - u) + i_\ell^y y - y, \sigma_\ell)_{0,\mathcal{I}\cap\mathcal{A}_\ell}\big),\end{aligned} \tag{27c}$$

$$\begin{aligned}\mu_\ell|_{\mathcal{A}\cap\mathcal{A}_\ell} &= \frac{1}{2}\left((\varepsilon u_\ell + y_\ell - \psi, \sigma)_{0,\mathcal{A}\cap\mathcal{A}_\ell} + (\psi_\ell - (\varepsilon u + y), \sigma_\ell)_{0,\mathcal{A}\cap\mathcal{A}_\ell}\right) \\ &= \frac{1}{2}\big((\psi_\ell - \psi, i_\ell^\sigma \sigma + \sigma_\ell)_{0,\mathcal{A}\cap\mathcal{A}_\ell} + (\psi_\ell - \psi, \sigma - i_\ell^\sigma \sigma)_{0,\mathcal{A}\cap\mathcal{A}_\ell}\big).\end{aligned} \tag{27d}$$

We further need to provide computable approximations of the sets $\mathcal{A}$ and $\mathcal{I}$. We use a modification of the approximation of the indicator function $\chi(\mathcal{A})$ of the continuous coincidence set $\mathcal{A}$ from [26] (cf. also [17]) according to

$$\chi_\ell^{\mathcal{A}} := 1 - \frac{\psi - (\varepsilon i_{\ell,2}^u u_\ell + i_{\ell,2}^y y_\ell)}{\gamma h_\ell^r + \psi - (\varepsilon i_{\ell,2}^u u_\ell + i_{\ell,2}^y y_\ell)}, \tag{28}$$

where $0 < \gamma \le 1$ and $r > 0$ are fixed and $i_{\ell,2}^u u_\ell$ is defined in the same way as $i_{\ell,2}^y y_\ell$. Indeed, for $T \subset \mathcal{A}$ we find

$$\|\chi(\mathcal{A}) - \chi_\ell^{\mathcal{A}}\|_{0,T} \le \min(|T|^{1/2}, \gamma^{-1} h_\ell^{-r} \|\varepsilon u + y - (\varepsilon i_{\ell,2}^u u_\ell + i_{\ell,2}^y y_\ell)\|_{0,T})$$

which converges to zero whenever $\|\varepsilon u + y - (\varepsilon i_{\ell,2}^u u_\ell + i_{\ell,2}^y y_\ell)\|_{0,T} = O(h_\ell^q)$, $q > r$. By the same arguments, for $T \subset \mathcal{I}$ one can show as well that $\|\chi(\mathcal{A}) - \chi_\ell^{\mathcal{A}}\|_{0,T} \to 0$ as $h_\ell \to 0$. Now, for fixed $0 < \kappa \le 1$ and $0 < s \le r$ we provide approximations $\hat{\mathcal{A}}_\ell$ of $\mathcal{A}$ and $\hat{\mathcal{I}}_\ell$ of $\mathcal{I}$ according to

$$\hat{\mathcal{A}}_\ell := \bigcup \{T \in \mathcal{T}_\ell(\Omega) \mid \chi_\ell^{\mathcal{A}}(x) \ge 1 - \kappa h_\ell^s \text{ for all } x \in T\}, \tag{29a}$$

$$\hat{\mathcal{I}}_\ell := \bigcup \{T \in \mathcal{T}_\ell(\Omega) \mid \chi_\ell^{\mathcal{A}}(x) < 1 - \kappa h_\ell^s \text{ for some } x \in T\}. \tag{29b}$$

We define approximations $\mathcal{T}_{\mathcal{A}\cap\mathcal{A}_\ell}$, $\mathcal{T}_{\mathcal{I}\cap\mathcal{A}_\ell}$ and $\mathcal{T}_{\mathcal{A}\cap\mathcal{I}_\ell}$ of $\mathcal{A}\cap\mathcal{A}_\ell$, $\mathcal{I}\cap\mathcal{A}_\ell$ and $\mathcal{A}\cap\mathcal{I}_\ell$ by means of

$$\mathcal{T}_{\mathcal{A}\cap\mathcal{A}_\ell} := \hat{\mathcal{A}}_\ell \cap \mathcal{A}_\ell, \quad \mathcal{T}_{\mathcal{I}\cap\mathcal{A}_\ell} := \hat{\mathcal{I}}_\ell \cap \mathcal{A}_\ell, \quad \mathcal{T}_{\mathcal{A}\cap\mathcal{I}_\ell} := \hat{\mathcal{A}}_\ell \cap \mathcal{I}_\ell.$$

We further define

$$\begin{aligned}\tilde{\omega}_T^y &:= \|i_{\ell,2}^y y_\ell - y_\ell\|_{0,T}, \\ \tilde{\omega}_T^u &:= \|i_{\ell,2}^u u_\ell - u_\ell\|_{0,T}, \\ \tilde{\omega}_T^\sigma &:= \|i_{\ell,2}^\sigma \sigma_\ell - \sigma_\ell\|_{0,T},\end{aligned}$$

where $i^{\sigma}_{\ell,2}\sigma_\ell$ is also given by piecewise quadratic interpolation. Then, we can estimate the contributions to the primal–dual mismatch in complementarity in (27b)–(27d) according to

$$|\mu_\ell|_{\mathcal{A}\cap\mathcal{I}_\ell}| \le \sum_{T\in\mathcal{T}_{\mathcal{A}\cap\mathcal{I}_\ell}} \bar{\mu}_T^{(1)}, \tag{30a}$$

$$\bar{\mu}_T^{(1)} := \frac{1}{2}\|\varepsilon u_\ell + y_\ell - \psi\|_{0,T}(\|i^{\sigma}_{\ell,2}\sigma_\ell\|_{0,T} + \tilde{\omega}_T^{\sigma}),$$

$$|\mu_\ell|_{\mathcal{I}\cap\mathcal{A}_\ell}| \le \sum_{T\in\mathcal{T}_{\mathcal{I}\cap\mathcal{A}_\ell}} \bar{\mu}_T^{(2)}, \tag{30b}$$

$$\bar{\mu}_T^{(2)} := \|\sigma_\ell\|_{0,T}(\varepsilon\tilde{\omega}_T^{u} + \tilde{\omega}_T^{y}),$$

$$|\mu_\ell|_{\mathcal{A}\cap\mathcal{A}_\ell}| \le \sum_{T\in\mathcal{T}_{\mathcal{A}\cap\mathcal{A}_\ell}} \bar{\mu}_T^{(3)}, \tag{30c}$$

$$\bar{\mu}_T^{(3)} := \frac{1}{2}\|\psi_\ell - \psi\|_{0,T}(\|i^{\sigma}_{\ell,2}\sigma_\ell + \sigma_\ell\|_{0,T} + \tilde{\omega}_T^{\sigma}).$$

This leads to the following upper bound for the primal–dual mismatch in complementarity:

$$|\mu_\ell(x,\sigma)| \le \sum_{T\in\mathcal{T}_\ell(\Omega)} \bar{\mu}_T, \tag{31}$$

where

$$\bar{\mu}_T := \begin{cases} 0, & T\in\mathcal{T}_{\mathcal{I}\cap\mathcal{I}_\ell}, \\ \bar{\mu}_T^{(1)}, & T\in\mathcal{T}_{\mathcal{A}\cap\mathcal{I}_\ell}, \\ \bar{\mu}_T^{(2)}, & T\in\mathcal{T}_{\mathcal{I}\cap\mathcal{A}_\ell}, \\ \bar{\mu}_T^{(3)}, & T\in\mathcal{T}_{\mathcal{A}\cap\mathcal{A}_\ell}. \end{cases}$$

The oscillation term $\mathrm{osc}_\ell^{(2)}$ as given by (21b) is treated analogously which results in

$$|\mathrm{osc}_\ell^{(1)} + \mathrm{osc}_\ell^{(2)}| \le \sum_{T\in\mathcal{T}_\ell(\Omega)} \mathrm{osc}_T, \quad \mathrm{osc}_T := \mathrm{osc}_T^{(1)} + \mathrm{osc}_T^{(2)}, \tag{32}$$

where $\mathrm{osc}_T^{(1)}$ is given by (16) and $\mathrm{osc}_T^{(2)}$ by

$$\mathrm{osc}_T^{(2)} := \tilde{\omega}_T^{y}\|y^d - y_\ell^d\|_{0,T} + \tilde{\omega}_T^{u}\|u^d - u_\ell^d\|_{0,T}.$$

Hence, we end up with the computable upper bound

$$|J(y,u) - J_\ell(y_\ell,u_\ell)| \preceq \sum_{T\in\mathcal{T}_\ell(\Omega)} (\hat{\omega}_T^{y}\rho_T^{y} + \hat{\omega}_T^{p}\rho_T^{p} + \bar{\mu}_T + \mathrm{osc}_T)\,. \tag{33}$$

5 Numerical Results

The marking strategy for selection of elements of the triangulation for refinement is based on a bulk criterion (cf. [11,30]) where we select a set $\mathcal{M}_\ell \subset \mathcal{T}_\ell(\Omega)$ of elements such that with respect to a given constant $0 < \Theta < 1$ there holds

$$\Theta \sum_{T \in \mathcal{T}_\ell(\Omega)} (\hat{\omega}_T^y \rho_T^y + \hat{\omega}_T^p \rho_T^p + \bar{\mu}_T + \mathrm{osc}_T) \leq \sum_{T \in \mathcal{M}} (\hat{\omega}_T^y \rho_T^y + \hat{\omega}_T^p \rho_T^p + \bar{\mu}_T + \mathrm{osc}_T)\,.$$

The bulk criterion is realized by a greedy algorithm (cf., e.g. [23]). The refinement is realized by newest vertex bisection.

We conclude this section with the results for an example which was chosen as a test case in [28]. The data of the problem are as follows:

$$\begin{aligned}
&\Omega := B(0,1), \quad \Gamma_D = \emptyset, \quad \alpha := 1.0, \quad c = 1.0,\\
&y^d(r) := 4 + \frac{1}{\pi} - \frac{1}{4\pi} r^2 + \frac{1}{2\pi}\ln(r),\\
&u^d(r) := 4 + \frac{1}{4\pi} r^2 - \frac{1}{2\pi}\ln(r), \quad \psi(r) := r + 4.
\end{aligned}$$

The optimal solution in the pure state constrained case is given by

$$\begin{aligned}
y(r) &\equiv 4, \qquad p(r) = \frac{1}{4\pi} r^2 - \frac{1}{2\pi}\ln(r),\\
u(r) &\equiv 4, \qquad \sigma = \delta_0.
\end{aligned}$$

As regularization parameter ε for the Lavrentiev regularization we have chosen $\varepsilon = 10^{-4}$. The finite element discretized optimal control problem has been solved by the Moreau–Yosida based active set strategy from [6]. Moreover, $\Theta = 0.4$ has been used for the bulk criterion in the step MARK of the adaptive loop.

Figure 1 shows the computed optimal state (left) and optimal control (right). We note that the peaks at the origin are numerical artefacts due

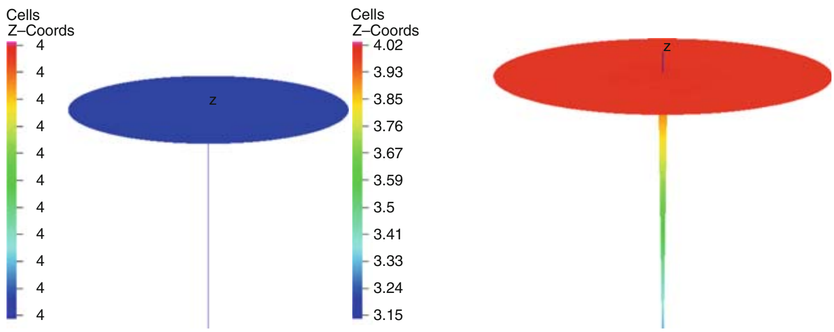

Fig. 1. Optimal state (left) and optimal control (right).

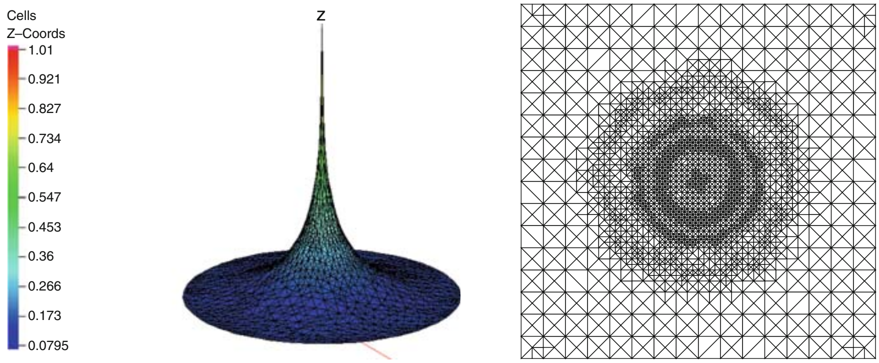

Fig. 2. Optimal adjoint state (left) and adaptively refined triangulation after 14 refinement steps of the adaptive loop (right).

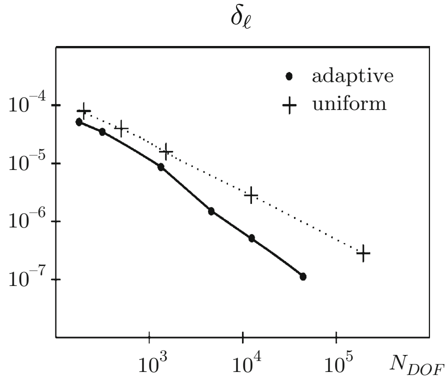

Fig. 3. Decrease of the quantity of interest $\delta_\ell := |J(y, u) - J_\ell(y_\ell, u_\ell)|$ as a function of the total number of degrees of freedom for adaptive and uniform refinement.

to the singularity of the optimal adjoint state in the origin (see Figure 2 (left)). Figure 2 (right) displays the computed adaptively refined mesh after 14 refinement steps of the adaptive loop. Finally, Figure 3 shows the decrease of the error $\delta_\ell := |J(y, u) - J_\ell(y_\ell, u_\ell)|$ measured in the quantity of interest as a function of the total number of degrees of freedom on a logarithmic scale both for adaptive and uniform refinement.

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