Remarks on the Controllability of Some Parabolic Equations and Systems

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Summary. This paper is devoted to present a review of recent results concerning the controllability of some (linear and nonlinear) parabolic systems. Among others, we will consider the classical heat equation, the Burgers, Navier–Stokes and Boussinesq equations, etc.

1 Introduction: Controllability and Observability

Let us first recall some general ideas. Suppose that we are considering an abstract *state equation* of the form

$$\begin{cases} y_t - A(y) = Bv, \quad t \in (0,T), \\ y(0) = y^0, \end{cases} \tag{1}$$

which governs the behavior of a physical system. It is assumed that

- $y : [0,T] \mapsto H$ is the *state*, i.e. the variable that serves to identify the physical properties of the system.
- $v : [0,T] \mapsto U$ is the *control*, i.e. the variable we can choose (for simplicity, we assume that U and H are Hilbert spaces).
- $A : D(A) \subset H \mapsto H$ is a (generally nonlinear) operator with $A(0) = 0$, $B \in \mathcal{L}(U;H)$ and $y^0 \in H$.

Suppose that (1) is well-posed in the sense that, for each $y^0 \in H$ and each $v \in L^2(0,T;U)$, it possesses exactly one solution. Then the *null controllability* problem for (1) can be stated as follows:

> *For each $y^0 \in H$, find $v \in L^2(0,T;U)$ such that the corresponding solution of (1) satisfies $y(T) = 0$.*

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More generally, the *exact controllability to the trajectories* problem for (1) is the following:

> *For each free trajectory* $\overline{y} : [0,T] \mapsto H$ *and each* $y^0 \in H$*, find* $v \in L^2(0,T;U)$ *such that the corresponding solution of (1) satisfies* $y(T) = \overline{y}(T)$*.*

Here, by a *free* or *uncontrolled* trajectory we mean any (sufficiently regular) function $\overline{y} : [0,T] \mapsto H$ satisfying $\overline{y}(t) \in D(A)$ for all t and

$$\overline{y}_t - A(\overline{y}) = 0, \quad t \in (0,T).$$

Notice that exact controllability to the trajectories is a very useful property from the viewpoint of applications: if we can find such a control, then after time T we can switch off the control and the system will follow the "ideal" trajectory $\overline{y}$.

For each system of the form (1), these problems lead to several interesting questions. Among them, let us mention the following:

- First, are there controls v such that $y(T) = 0$ and/or $y(T) = \overline{y}(T)$?
- Then, if this is the case, which is the *cost* we have to pay to drive y to zero and/or $\overline{y}(T)$? In other words, which is the minimal norm of a control $v \in L^2(0,T;U)$ satisfying these properties?
- How can these controls be computed?

The controllability of differential systems is a very relevant area of research and has been the subject of many papers the last years. In particular, in the context of partial differential equations, the null controllability problem was first analyzed in [26, 29–31, 33, 34]. For semilinear systems of this kind, the first contributions have been given in [9, 19, 35].

In this paper, we will be mainly concerned with the case of parabolic partial differential systems. The typical situation corresponds to the classical heat equation in a bounded N-dimensional domain, complemented with appropriate initial and boundary-value conditions; see Section 2.

The paper is organized as follows. In Section 2, we consider the heat equation and some linear variants. We explain the role of observability and Carleman estimates in control theory, we recall the main results in this framework and we mention some open problems. Section 3 deals with the viscous Burgers equation. We show that, for this equation, the null controllability problem (with distributed and locally supported control) is well understood.[1] In Sections 4 and 5, we consider the Navier–Stokes and Boussinesq equations and some other systems from mechanics. We recall several results concerning the local exact controllability to the trajectories and we explain how to deal with a reduced number of controls. Several open problems are also indicated.

[1] More precisely, if we denote by $T^*(r)$ the minimal time needed to drive any initial state with L^2 norm $\leq r$ to zero, we show that $T^*(r) > 0$, with explicit sharp estimates from above and from below.

2 The Classical Heat Equation: Observability and Carleman Estimates

Let us consider the following control system for the heat equation:

$$\begin{cases} y_t - \Delta y = v 1_\omega, & (x,t) \in \Omega \times (0,T), \\ y(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ y(x,0) = y^0(x), & x \in \Omega. \end{cases} \tag{2}$$

Here (and also in the following sections), $\Omega \subset \mathbb{R}^N$ is a nonempty regular and bounded domain, $\omega \subset\subset \Omega$ is a (small) nonempty open subset (1_ω is the characteristic function of ω) and $y^0 \in L^2(\Omega)$.

It is well known that, for every $y^0 \in L^2(\Omega)$ and every $v \in L^2(\omega \times (0,T))$, there exists a unique solution y to (2), with $y \in L^2(0,T;H_0^1(\Omega)) \cap C^0([0,T];L^2(\Omega))$.

In this context, the null controllability problem reads:

For each $y^0 \in L^2(\Omega)$, find $v \in L^2(\omega \times (0,T))$ such that the associated solution of (2) satisfies $y(x,T) = 0$ in Ω.

Since the state equation (2) is linear, null controllability is equivalent in this case to *exact controllability to the trajectories.* This means that, for any uncontrolled solution $\overline{y}$ and any $y^0 \in L^2(\Omega)$, there exists $v \in L^2(\omega \times (0,T))$ such that the associated state y satisfies

$$y(x,T) = \overline{y}(x,T) \quad \text{in } \Omega.$$

A related notion is *approximate controllability.* It is said that (2) is approximately controllable in $L^2(\Omega)$ at time T if, for any $y^0, y^1 \in L^2(\Omega)$ and any $\varepsilon > 0$, there exist controls $v \in L^2(\omega \times (0,T))$ such that the solutions to (2) associated to these v and the initial state y^0 satisfy

$$\|y(\cdot\,,T) - y^1\|_{L^2} \le \varepsilon. \tag{3}$$

It is not difficult to prove that this is weaker notion: the null controllability of (2) at any time T implies the approximate controllability of (2) in $L^2(\Omega)$ at any T. On the other hand, since $\omega \subset\subset \Omega$, in view of the regularizing effect of the heat equation, *exact controllability*, i.e. approximate controllability with $\varepsilon = 0$, does not hold.

Together with (2), for each $\varphi^1 \in L^2(\Omega)$, we can introduce the associated adjoint system

$$\begin{cases} -\varphi_t - \Delta\varphi = 0, & (x,t) \in \Omega \times (0,T), \\ \varphi(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ \varphi(x,T) = \varphi^1(x), & x \in \Omega. \end{cases} \tag{4}$$

Then, it is well known that the null controllability of (2) is equivalent to the following property:

There exists $C > 0$ *such that*

$$\|\varphi(\cdot,0)\|_{L^2}^2 \le C \iint_{\omega\times(0,T)} |\varphi|^2 \, dx \, dt \qquad \forall \varphi^1 \in L^2(\Omega). \tag{5}$$

This is called an observability estimate for the solutions of (4). We thus find that, in order to solve the null controllability problem for (2), it suffices to prove (5).

The estimate (5) is implied by the so called global Carleman inequalities. These have been introduced in the context of the controllability of PDEs by Fursikov and Imanuvilov, see [19, 26]. When they are applied to the solutions of the adjoint system (4), they take the form

$$\iint_{\Omega\times(0,T)} \rho^2 \, |\varphi|^2 \, dx \, dt \le K \iint_{\omega\times(0,T)} \rho^2 \, |\varphi|^2 \, dx \, dt \quad \forall \varphi^1 \in L^2(\Omega), \tag{6}$$

where $\rho = \rho(x,t)$ is an appropriate weight depending on Ω, ω and T and the constant K only depends on Ω and ω.[2]

Combining (6) and the dissipativity of the backwards heat equation (4), it is not difficult to deduce (5) for some C only depending on Ω, ω and T.

As a consequence, we have:

Theorem 1. *The linear system (2) is null controllable. In other words, for each* $y^0 \in L^2(\Omega)$*, there exists* $v \in L^2(\omega \times (0,T))$ *such that the corresponding solution of (2) satisfies*

$$y(x,T) = 0 \quad \text{in } \Omega. \tag{7}$$

Remark 1. Notice that Theorem 1 ensures the null controllability of (2) for *any* ω and T. This is a consequence of the fact that, in a parabolic equation, the transmission of information is instantaneous. For instance, this is not the case for the transport equation. Thus, let us consider the control system

$$\begin{cases} y_t + y_x = v 1_\omega, & (x,t) \in (0,L) \times (0,T), \\ y(0,t) = 0, & t \in (0,T), \\ y(x,0) = y^0(x), & x \in (0,L), \end{cases} \tag{8}$$

with $\omega = (a,b) \subset\subset (0,L)$. Then, if $0 < T < a$, null controllability does not hold, since the solution always satisfies

$$y(x,T) = y^0(x-T) \quad \forall x \in (T,a),$$

independently of the choice of v; see [7] for more details and similar results concerning other control systems for the wave, Schrödinger and Korteweg–De Vries equations. ■

[2] In order to prove (6), we have to use a weight ρ decreasing to zero, as $t \to 0$ and also as $t \to T$, for instance, exponentially.

There are many generalizations and variants of Theorem 1 that provide the null controllability of other similar linear (parabolic) state equations:

- Time–space dependent (and sufficiently regular) coefficients can appear in the equation, other boundary conditions can be used, boundary control (instead of distributed control) can be imposed, etc.; see [19]. For a review of recent applications of Carleman inequalities to the controllability of parabolic systems, see [11].
- The null controllability of Stokes-like systems can also be analyzed with these techniques. This includes systems of the form

$$y_t - \Delta y + (a\cdot\nabla)y + (y\cdot\nabla)b + \nabla p = v1_\omega, \quad \nabla\cdot y = 0, \tag{9}$$

  where a and b are regular enough. See, for instance, [14]; see also [8] for other controllability properties.
- Other linear parabolic (non-scalar) systems can also be considered, etc.

However, there are several interesting problems related to the controllability of linear parabolic systems that still remain open. Let us mention two of them.

First, let us consider the controlled system

$$\begin{cases} y_t - \nabla\cdot(a(x)\nabla y) = v1_\omega, & (x,t)\in\Omega\times(0,T),\\ y(x,t)=0, & (x,t)\in\partial\Omega\times(0,T),\\ y(x,0)=y^0(x), & x\in\Omega, \end{cases} \tag{10}$$

where y^0 and v are as before and the coefficient a is assumed to satisfy

$$a\in L^\infty(\Omega), \quad 0<a_0\le a(x)\le a_1<+\infty \quad \text{a.e.} \tag{11}$$

It is natural to consider the null controllability problem for (10). Of course, this is equivalent to the observability of the associated adjoint system

$$\begin{cases} -\varphi_t - \nabla\cdot(a(x)\nabla\varphi) = 0, & (x,t)\in\Omega\times(0,T),\\ \varphi(x,t)=0, & (x,t)\in\partial\Omega\times(0,T),\\ y\varphi(x,T)=\varphi^1(x), & x\in\Omega, \end{cases} \tag{12}$$

that is to say, to the fact that an inequality like (5) holds for the solutions to (12).

To our knowledge, it is at present unknown whether (10) is null controllable. In fact, it is also unknown whether approximate controllability holds.

Remark 2. Recently, some partial results have been obtained in this context. Thus, when $N=1$, the null controllability of (10) has been established in [1]. When $N\ge 2$, the best known result up to now is that this property holds under the following assumption:

$$\exists \textit{ smooth open set } \Omega_0\subset\subset\Omega \textit{ such that } a \textit{ is } C^1 \textit{ in } \overline{\Omega_0} \textit{ and } \overline{\Omega\setminus\Omega_0}. \tag{13}$$

This has been proved in [28]. In both cases, the proofs use that a is independent of t in an essential way. In fact, it is an open question whether a Carleman estimate like (6) holds for the solutions to (12) even if $N = 1$ or (13) holds. ■

Our second open problem concerns the system

$$\begin{cases} y_t - D\Delta y = Ay + Bv1_\omega, & (x,t) \in \Omega \times (0,T), \\ y(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ y(x,0) = y^0(x), & x \in \Omega, \end{cases} \tag{14}$$

where $y = (y_1, \ldots, y_n)$ is the state, $v = (v_1, \ldots, v_m)$ is the control and D, A and B are constant matrices, with $D, A \in \mathcal{L}(\mathbb{R}^n; \mathbb{R}^n)$ and $B \in \mathcal{L}(\mathbb{R}^m; \mathbb{R}^n)$. It is assumed that D is definite positive, that is,

$$D\xi \cdot \xi \geq d_0 |\xi|^2 \quad \forall \xi \in \mathbb{R}^n,\ d_0 > 0. \tag{15}$$

When D is diagonal (or similar to a diagonal matrix), the null controllability problem for (14) is well understood. In view of the results in [2], (14) is null controllable if and only if

$$\text{rank}[(-\lambda_i D + A); B] = n \quad \forall i \geq 1, \tag{16}$$

where the λ_i are the eigenvalues of the Dirichlet–Laplace operator and, for any matrix $H \in \mathcal{L}(\mathbb{R}^n; \mathbb{R}^n)$, $[H; B]$ stands for the $n \times nm$ matrix

$$[H; B] := [B|HB| \cdots |H^{n-1}B].$$

Therefore, it is natural to search for (algebraic) conditions on D, A and B that ensure the null controllability of (14) in the general case. But, to our knowledge, this is unknown.

Remark 3. The results in [2] have been extended recently to the case of any D having no eigenvalue of geometric multiplicity > 3; see [10]. ■

Remark 4. As we have said, global Carleman estimates are the main tool we can use to establish the observability property (5). These two open questions can be viewed as consequences of the limitations of Carleman estimates: first, they need regular coefficients; then, they are, in fact, a tool proper of *scalar* equations. ■

As mentioned above, an interesting question related to Theorem 1 concerns the cost of null controllability. One has the following result from [16]:

Theorem 2. *For each $y^0 \in L^2(\Omega)$, let us denote by $C(y^0)$ the minimal norm in $L^2(\omega \times (0,T))$ of a control v such that the associated solution of (2) satisfies (7). Then, for some C only depending on Ω and ω, the following estimate holds:*

$$C(y^0) \leq \exp\left[C\left(1 + \frac{1}{T}\right)\right] \|y^0\|_{L^2}. \tag{17}$$

Remark 5. We can be more explicit on the way C depends on Ω and ω: there exist "universal" constants $C_0 > 0$ and $m \geq 1$ such that C can be taken of the form

$$C = \exp\left(C_0 \|\psi\|_{C^2}^m\right),$$

where $\psi \in C^2(\overline{\Omega})$ is any function satisfying $\psi > 0$ in Ω, $\psi = 0$ on $\partial\Omega$ and $\nabla\psi \neq 0$ in $\overline{\Omega} \setminus \omega$. All this is a consequence of the particular form that must have ρ in order to ensure (6); see [16] for more details. ■

3 Positive and Negative Controllability Results for the One-Dimensional Burgers Equation

In this section, we will be concerned with the null controllability of the following system for the viscous Burgers equation:

$$\begin{cases} y_t - y_{xx} + yy_x = v1_\omega, & (x,t) \in (0,1) \times (0,T), \\ y(0,t) = y(1,t) = 0, & t \in (0,T), \\ y(x,0) = y^0(x), & x \in (0,1). \end{cases} \tag{18}$$

Recall that some controllability properties of (18) have been studied in [19, Chapter 1, Theorems 6.3 and 6.4]. There, it is shown that, in general, a stationary solution of (18) with large L^2-norm cannot be reached (not even approximately) at any time T. In other words, with the help of one control, the solutions of the Burgers equation cannot go anywhere at any time.

For each $y^0 \in L^2(0,1)$, let us introduce

$$T(y^0) = \inf\{T > 0 : \text{ (18) is null controllable at time } T\}.$$

Then, for each $r > 0$, let us define the quantity

$$T^*(r) = \sup\{T(y^0) : \|y^0\|_{L^2} \leq r\}.$$

Our main purpose is to show that $T^*(r) > 0$, with explicit sharp estimates from above and from below. In particular, this will imply that (global) null controllability at any positive time does not hold for (18).

More precisely, let us set $\phi(r) = (\log \frac{1}{r})^{-1}$. We have the following result from [13]:

Theorem 3. *One has*

$$C_0\phi(r) \leq T^*(r) \leq C_1\phi(r) \quad \text{as } r \to 0, \tag{19}$$

for some positive constants C_0 and C_1 not depending of r.

Remark 6. The same estimates hold when the control v acts on system (18) through the boundary *only* at $x = 1$ (or only at $x = 0$). Indeed, it is easy to transform the boundary controlled system

$$\begin{cases} y_t - y_{xx} + yy_x = 0, & (x,t) \in (0,1) \times (0,T), \\ y(0,t) = 0, \quad y(1,t) = w(t), & t \in (0,T), \\ y(x,0) = y^0(x), & x \in (0,1) \end{cases} \tag{20}$$

into a system of the kind (18). The boundary controllability of the Burgers equation with *two* controls (at $x = 0$ and $x = 1$) has been analyzed in [23]. There, it is shown that even in this more favorable situation null controllability does not hold for small time. It is also proved in that paper that exact controllability does not hold for large time.[3] ■

The proof of the estimate from above in (19) can be obtained by solving the null controllability problem for (18) via a (more or less) standard fixed point argument, using global Carleman inequalities to estimate the control and energy inequalities to estimate the state and being very careful with the role of T in these inequalities.

The proof of the estimate from below is inspired by the arguments in [3] and is implied by the following property: there exist positive constants C_0 and C_0' such that, for any sufficiently small $r > 0$, we can find initial data y^0 and associated states y satisfying $\|y^0\|_{L^2} \le r$ and

$$|y(x,t)| \ge C_0' r \quad \text{for some } x \in (0,1) \text{ and any } t : 0 < t < C_0 \phi(r).$$

For more details, see [13].

4 The Navier–Stokes and Boussinesq Systems

There are a lot of more realistic nonlinear equations and systems from mechanics that can also be considered in this context. First, we have the well known Navier–Stokes equations:

$$\begin{cases} y_t + (y \cdot \nabla) y - \Delta y + \nabla p = v 1_\omega, \quad \nabla \cdot y = 0, & (x,t) \in Q, \\ y = 0, & (x,t) \in \Sigma, \\ y(x,0) = y^0(x), & x \in \Omega. \end{cases} \tag{21}$$

Here and below, Q and Σ respectively stand for the sets $Q = \Omega \times (0,T)$ and $\Sigma = \partial\Omega \times (0,T)$, where $\Omega \subset \mathbb{R}^N$ is a nonempty regular and bounded domain, $N = 2$ or $N = 3$ and (again) $\omega \subset\subset \Omega$ is a nonempty open set.

[3] Let us remark that the results in [23] do not allow to estimate $T(r)$; in fact, the proofs are based in contradiction arguments.

In (21), (y, p) is the state (the velocity field and the pressure distribution) and v is the control (a field of external forces applied to the fluid particles located at ω). To our knowledge, the best results concerning the controllability of this system have been given in [14, 15].[4] Essentially, these results establish the local exact controllability of the solutions of (21) to uncontrolled trajectories.

In order to be more specific, let us recall the definition of some usual spaces in the context of Navier–Stokes equations:

$$V = \left\{ y \in H_0^1(\Omega)^N : \nabla \cdot y = 0 \text{ in } \Omega \right\}$$

and

$$H = \{ y \in L^2(\Omega)^N : \nabla \cdot y = 0 \text{ in } \Omega,\ y \cdot n = 0 \text{ on } \partial\Omega \}.$$

Of course, it will be said that (21) is *exactly controllable to the trajectories* if, for any trajectory $(\overline{y}, \overline{p})$, i.e. any solution of the uncontrolled Navier–Stokes system

$$\begin{cases} \overline{y}_t + (\overline{y} \cdot \nabla)\overline{y} - \Delta \overline{y} + \nabla \overline{p} = 0, \quad \nabla \cdot \overline{y} = 0, & (x,t) \in Q, \\ \overline{y} = 0, & (x,t) \in \Sigma \end{cases} \tag{22}$$

and any $y^0 \in H$, there exist controls $v \in L^2(\omega \times (0,T))^N$ and associated solutions (y, p) such that

$$y(x,T) = \overline{y}(x,T) \quad \text{in } \Omega. \tag{23}$$

At present, we do not know any global result concerning exact controllability to the trajectories for (21). However, the following local result holds:

Theorem 4. *Let* $(\overline{y}, \overline{p})$ *be a strong solution of (22), with*

$$\overline{y} \in L^\infty(Q)^N, \quad \overline{y}(\cdot\,, 0) \in V. \tag{24}$$

Then, there exists $\delta > 0$ *such that, for any* $y^0 \in H \cap L^{2N-2}(\Omega)^N$ *satisfying* $\|y^0 - \overline{y}^0\|_{L^{2N-2}} \leq \delta$, *we can find a control* $v \in L^2(\omega \times (0,T))^N$ *and an associated solution* (y, p) *to (21) such that (23) holds.*

In other words, the local exact controllability to the trajectories holds for (21) in the space $X = L^{2N-2}(\Omega)^N \cap H$; see [14] for a slightly stronger result. Similar questions were addressed (and solved) in [17, 18]. The fact that we consider here Dirichlet boundary conditions and locally supported distributed control increases a lot the mathematical difficulty of the control problem.

Remark 7. It is clear that we cannot expect exact controllability for the Navier–Stokes equations with an arbitrary target function, because of the dissipative and non reversible properties of the system. On the other hand,

[4] The main ideas come from [20,27]; some additional results will appear soon in [21].

approximate controllability is still an open question for this system. Some results in this direction have been obtained in [6] for different boundary conditions (Navier slip boundary conditions) and in [8] for a different nonlinearity. However, the notion of approximate controllability does not appear to be optimal from a practical viewpoint. Indeed, even if we could reach an arbitrary neighborhood of a given target y^1 at time T by the action of a control, the question of what to do after time T to stay in the same neighbourhood would remain open. ■

The proof of Theorem 4 can be obtained as an application of *Liusternik's inverse mapping theorem* in an appropriate framework.

A key point in the proof is a related null controllability result for the linearized Navier–Stokes system at $(\overline{y}, \overline{p})$, that is to say

$$\begin{cases} y_t + (\overline{y} \cdot \nabla) y + (y \cdot \nabla)\overline{y} - \Delta y + \nabla p = v 1_\omega, & (x,t) \in Q, \\ \nabla \cdot y = 0, & (x,t) \in Q, \\ y = 0, & (x,t) \in \Sigma, \\ y(x,0) = y^0(x), & x \in \Omega. \end{cases} \tag{25}$$

This control result is a consequence of a global Carleman inequality of the kind (6) that can be established for the solutions to the adjoint of (25), which is the following:

$$\begin{cases} -\varphi_t - (\nabla\varphi + \nabla\varphi^t)\,\overline{y} - \Delta\varphi + \nabla\pi = g, & (x,t) \in Q, \\ \nabla \cdot \varphi = 0, & (x,t) \in Q, \\ \varphi = 0, & (x,t) \in \Sigma, \\ \varphi(T) = \varphi^0, & x \in \Omega. \end{cases} \tag{26}$$

The details can be found in [14].

Similar results have been given in [22] for the Boussinesq equations

$$\begin{cases} y_t + (y \cdot \nabla) y - \Delta y + \nabla p = v 1_\omega + \theta\, e_N, \quad \nabla \cdot y = 0 & (x,t) \in Q, \\ \theta_t + y \cdot \nabla\theta - \Delta\theta = h 1_\omega, & (x,t) \in Q, \\ y = 0, \quad \theta = 0, & (x,t) \in \Sigma, \\ y(x,0) = y^0(x), \quad \theta(x,0) = \theta^0(x), & x \in \Omega. \end{cases} \tag{27}$$

Here, the state is the triplet (y, p, θ) (θ is interpreted as a temperature distribution) and the control is (v, h) (as before, v is a field of external forces; h is an external heat source).

An interesting question concerning both (21) and (27) is whether we can still get local exact controllability to the trajectories with a reduced number of scalar controls. This is partially answered in [15], where the following results are proved:

Theorem 5. *Assume that the following property is satisfied:*

$$\exists x^0 \in \partial\Omega,\ \exists \varepsilon > 0 \ \text{ such that } \overline{\omega} \cap \partial\Omega \supset B(x^0; \varepsilon) \cap \partial\Omega. \tag{28}$$

Here, $B(x^0; \varepsilon)$ is the ball centered at x^0 of radius ε. Then, for any $T > 0$, (21) is locally exactly controllable at time T to the trajectories satisfying (24) with controls $v \in L^2(\omega \times (0, T))^N$ having one component identically zero.

Theorem 6. *Assume that ω satisfies (28) with $n_k(x^0) \neq 0$ for some $k < N$. Then, for any $T > 0$, (27) is locally exactly controllable at time T to the trajectories $(\overline{y}, \overline{p}, \overline{\theta})$ satisfying (24) and*

$$\overline{\theta} \in L^\infty(Q), \quad \overline{\theta}(\cdot\,, 0) \in H^1_0(\Omega). \tag{29}$$

with controls $v \in L^2(\omega \times (0, T))^N$ and $h \in L^2(\omega \times (0, T))$ such that $v_k \equiv v_N \equiv 0$. In particular, if $N = 2$, we have local exact controllability to these trajectories with controls $v \equiv 0$ and $h \in L^2(\omega \times (0, T))$.

The proofs of Theorems 5 and 6 are similar to the proof of Theorem 4. We have again to rewrite the controllability property as a nonlinear equation in a Hilbert space. Then, we have to check that the hypotheses of Liusternik's theorem are fulfilled.

Again, a crucial point is to prove the null controllability of certain linearized systems, this time with *reduced* controls. For instance, when dealing with (21), the task is reduced to prove that, for some $\rho = \rho(x, t)$ and $K > 0$, the solutions to (25) satisfy the following Carleman-like estimates:

$$\iint_{\Omega \times (0,T)} \rho^2 |\varphi|^2 \, dx \, dt \leq K \iint_{\omega \times (0,T)} \rho^2 (\varphi_1^2 + \varphi_2^2) \, dx \, dt \quad \forall \varphi^1 \in L^2(\Omega). \tag{30}$$

This inequality can be proved using the assumption (28) and the incompressibility identity $\nabla \cdot \varphi = 0$; see [15].

5 Some Other Nonlinear Systems from Mechanics

The previous arguments can be applied to other similar partial differential systems arising in mechanics. For instance, this is made in [12] in the context of micro-polar fluids.

To fix ideas, let us assume that $N = 3$. The behavior of a micro-polar three-dimensional fluid is governed by the following system:

$$\begin{cases} y_t - \Delta y + (y \cdot \nabla) y + \nabla p = \nabla \times w + v 1_\omega, \quad \nabla \cdot y = 0, & (x, t) \in Q, \\ w_t + (y \cdot \nabla) w - \Delta w - \nabla(\nabla \cdot w) = \nabla \times y + u 1_\omega, & (x, t) \in Q, \\ y = 0, \quad w = 0 & (x, t) \in \Sigma, \\ y(x, 0) = y^0(x), \quad w(x, 0) = w^0(x) & x \in \Omega. \end{cases} \tag{31}$$

Here, the state is (y, p, w) and the control is (v, u). As usual, y and p stand for the velocity field and pressure and w is the microscopic velocity of rotation of the fluid particles. Then, the following result holds:

Theorem 7. *Let* $(\overline{y}, \overline{p}, \overline{w})$ *be such that*

$$\overline{y},\, \overline{w} \in L^\infty(Q) \cap L^2(0,T; H^2(\Omega)), \quad \overline{y}_t,\, \overline{w}_t \in L^2(Q) \tag{32}$$

and

$$\begin{cases} \overline{y}_t - \Delta\overline{y} + (\overline{y}\cdot\nabla)\overline{y} + \nabla\overline{p} = \nabla\times\overline{w}, \quad \nabla\cdot\overline{y} = 0, & (x,t) \in Q, \\ \overline{w}_t + (\overline{y}\cdot\nabla)\overline{w} - \Delta\overline{w} - \nabla(\nabla\cdot\overline{w}) = \nabla\times\overline{y}, & (x,t) \in Q, \\ \overline{y} = 0, \quad \overline{w} = 0 & (x,t) \in \Sigma. \end{cases} \tag{33}$$

Then, for each $T > 0$, *(31) is locally exactly controllable to* $(\overline{y}, \overline{p}, \overline{w})$ *at time* T. *In other words, there exists* $\delta > 0$ *such that, for any initial data* $(y^0, w^0) \in (H^2(\Omega) \cap V) \times H_0^1(\Omega)$ *satisfying*

$$\|(y^0, w^0) - (\overline{y}(\cdot\,,0), \overline{w}(\cdot\,,0))\|_{H^2\times H_0^1} \le \delta, \tag{34}$$

there exist L^2 *controls* u *and* v *and associated solutions* (y, p, w) *satisfying*

$$y(x,T) = \overline{y}(x,T), \quad w(x,T) = \overline{w}(x,T) \quad \text{in } \Omega. \tag{35}$$

Notice that this case involves a nontrivial difficulty. Indeed, w is a non-scalar variable and the equations satisfied by its components w_i are coupled through the second-order terms $\partial_i(\nabla\cdot w)$. This is a serious inconvenient. An appropriate strategy has to be applied in order to deduce the required Carleman estimates.

Let us also mention [4, 24, 25], where the controllability of the MHD and other related equations has been analyzed.

For all these systems, the proof of the controllability can be achieved arguing as in the first part of the proof of Theorem 4. This is the general structure of the argument:

- First, rewrite the original controllability problem as a nonlinear equation in a space of admissible "state-control" pairs.
- Then, prove an appropriate global Carleman inequality and a regularity result and deduce that the linearized equation possesses at least one solution. This provides a controllability result for a related linear problem.
- Check that the hypotheses of a suitable implicit function theorem are satisfied and deduce a local result.

Remark 8. Recall that an alternative strategy was introduced in [35] in the context of the semilinear wave equation: first, consider a linearized similar problem and rewrite the original controllability problem in terms of a fixed point equation; then, prove a global Carleman inequality and deduce an observability estimate for the adjoint system and a controllability result for the linearized problem; finally, prove appropriate estimates for the control and the state (this usually needs some kind of *smallness* of the data), prove an appropriate compactness property of the state and deduce that there exists at least one fixed point. This method has been used in [21] to prove a result similar to Theorem 4.

Remark 9. Observe that all these results are positive, in the sense that they provide local controllability properties. At present, no negative result is known to hold for these nonlinear systems (except for the already considered one-dimensional Burgers equation).

To end this section, let us mention another system from fluid mechanics, apparently not much more complex than (21), for which local exact controllability (and even local null controllability) is an open question:

$$\begin{cases} y_t + (y\cdot\nabla)y - \nabla\cdot(\nu(|Dy|)Dy) + \nabla p = v1_\omega, & (x,t)\in Q,\\ \nabla\cdot y = 0, & (x,t)\in Q,\\ y = 0, & (x,t)\in \Sigma,\\ y(x,0) = y^0(x), & x\in\Omega. \end{cases} \tag{36}$$

Here, $Dy = \frac{1}{2}(\nabla y + \nabla y^t)$ and $\nu : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is a regular function (for example, we can take $\nu(s) \equiv a + bs^{r-1}$ for some $a, b, r > 0$).

This system models the behavior of a *quasi-Newtonian* fluid; for a mathematical analysis, see [5, 32]. In view of the new nonlinear diffusion term $\nabla\cdot(\nu(|Dy|)Dy)$, its control properties are much more difficult to analyze than for (21).

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