Guaranteed Error Bounds for Conforming Approximations of a Maxwell Type Problem

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Summary. This paper is concerned with computable error estimates for approximations to a boundary-value problem

$$\operatorname{curl}\mu^{-1}\operatorname{curl}u+\kappa^2 u=j\quad\text{in }\Omega,$$

where $\mu>0$ and κ are bounded functions. We derive a posteriori error estimates valid for any conforming approximations of the considered problems. For this purpose, we apply a new approach that is based on certain transformations of the basic integral identity. The consistency of the derived a posteriori error estimates is proved and the corresponding computational strategies are discussed.

Key words: A posteriori estimates, the Maxwell equation, guaranteed bounds of approximation errors

1 Introduction

Boundary-value problems related to the Maxwell equation are interesting from the mathematical viewpoint and arise in numerous applications. Existence and regularity properties of solutions and viable methods of approximation are well investigated and presented in the literature. Approximation methods for the Maxwell equation were investigated in, e.g. [5, 6, 8, 11]. A posteriori estimates were obtained in [1] in the framework of the residual approach and in [2] with the help of equilibrated approach. A posteriori estimates for nonconforming approximations of $H(\text{curl})$-elliptic partial differential equations were studied in [7].

In this paper, we derive consistent a posteriori estimates by a different method, which is based upon purely functional analysis of the problem in question and do not attract specific properties of approximations or exact solutions. Earlier, such type of methods were applied to many other classes

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of boundary-value problems (see [10, 12, 13, 16] and the references therein). The so-called functional error majorants derived by this techniques are able to estimate the error for any conforming approximation of the exact solution. We show that for the Maxwell type problem (1) such estimates follow from the corresponding generalized statement (integral identity), which defines a weak solution to the problem. The integral identity can be transformed in various ways. The more sophisticated methods of transforming (3) we apply the better estimates of the difference between an approximate solution v and the exact one u we obtain.

The outline of the paper is as follows. Section 2 contains definitions and the generalized statement of the primal problem. In Section 3, we derive a posteriori error estimate of the first type using the simple modus operandi. For problems with $\kappa > 0$ the respective estimate is presented in Proposition 1. This estimate consists of two parts related to errors in the duality relations and in the differential equation and contains no geometrical constants. An important property of this estimate is that it gives a guaranteed upper bound of the error, which is as close to the exact error as it is required provided that the parameters of the majorant are properly selected. However, as the estimates derived for the reaction-diffusion problem the estimate looses the efficiency for small κ. In Section 4, we derive another upper bound of the error, which is insensitive with respect to small values of the coefficients. This estimate contains global constants that depend only on Ω. Regrettably, we cannot prove that error majorants established in Propositions 2 and 3 are equal to the corresponding error norms if the "free" function y is properly selected. Thus, the estimates exposed in Sections 3 and 4 has certain drawbacks that may affect practical efficiency of error estimation. A way out is presented in Section 5, which is devoted to establishing a more general error majorant. The latter encompasses majorants derived in the previous sections as special cases. The majorants defined in Propositions 4 and 5 are also insensitive with respect to small values of the coefficients and as the estimate obtained in Section 3 have no gap between its right- and left-hand sides (so that a computable upper bound of the error can be as close to the exact error as it is required).

2 Notation and Basic Relations

We consider the simplest version of the Maxwell equation

$$\operatorname{curl} \mu^{-1} \operatorname{curl} u + \kappa^2 u = j \quad \text{in } \Omega, \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^d$, j is a given current density, and μ is the permeability of a medium (may be a positive constant or a positive bounded function). The case $\kappa = 0$ corresponds to stationary transverse magnetic (TM) or transverse electric (TE) equations that arise if one of the components of the electromagnetic field is excluded (e.g., see [8, 11]). The equation (1) with positive κ arises in semidiscrete approximations of the evolutionary Maxwell problem.

On Γ the condition

$$n \times u = 0 \tag{2}$$

is stated. Here, n denotes the unit outward normal to Γ. By $V(\Omega)$ we denote the space $H(\Omega, \mathrm{curl})$, which is a Hilbert space endowed with the norm

$$\|w\|_{\mathrm{curl}} := \left(\|w\|^2 + \|\,\mathrm{curl}\, w\|^2\right)^{1/2}.$$

Here and later on, the symbol := means 'equals by definition' and $\|\cdot\|$ stands for L^2-norm of scalar- and vector-valued functions.

The generalized solution $u \in V_0$ is defined by the integral relation

$$\int_\Omega \mu^{-1}\,\mathrm{curl}\, u \cdot \mathrm{curl}\, w + \kappa^2 u \cdot w \, dx = \int_\Omega j \cdot w \, dx, \tag{3}$$

where $u \cdot w$ means scalar product of vector-valued functions u and w and

$$V_0 := \{w \in V \mid w \times n = 0 \text{ on } \partial\Omega\}.$$

Henceforth, we assume that j satisfies the condition

$$\int_\Omega j \cdot \nabla\phi \, dx = 0 \quad \forall \phi \in \overset{\circ}{H}{}^1(\Omega) \tag{4}$$

and

$$0 < \mu_\ominus \le \mu(x) \le \mu_\oplus, \tag{5}$$
$$0 < \kappa_\ominus \le \kappa(x) \le \kappa_\oplus. \tag{6}$$

By scaling arguments, we can set $\mu_\oplus = 1$ without a loss of generality.

Our goal is to derive computable estimates of the difference $u - v$ where $v \in V_0$ is a function viewed as an approximation of u. Estimates are obtained for the weighted energy norm defined by the relation

$$|[w]|^2_{(\gamma,\delta)} := \int_\Omega (\gamma|\,\mathrm{curl}\, w|^2 + \delta|w|^2)\, dx.$$

The derivation method is based on transformations of the integral relation (3). It does not use specific properties of the exact solution or its approximation v (e.g., Galerkin orthogonality). Therefore, the estimates are valid for conforming approximations of all types regardless of the numerical method applied for their construction. These estimates belong to the class of functional a posteriori error estimates that has been derived for some other elliptic and parabolic problems (see [10, 13, 16] and the references therein).

3 A Posteriori Error Estimates of the First Type

3.1 Upper Bound of the Error

Proposition 1. *Assume that $\kappa > 0$ and $v \in V_0$ is an approximation of u. For any $y \in H(\Omega, \mathrm{curl})$ the following estimate holds:*

$$|[u-v]|^2_{(\mu^{-1},\kappa^2)} \leq \mathcal{M}_1^2(v,y) := \left\| \frac{1}{\kappa} \mathrm{r}(v,y) \right\|^2 + \|\mu^{1/2} \mathrm{d}(v,y)\|^2, \tag{7}$$

where

$$\mathrm{r}(v,y) := j - \mathrm{curl}\, y - \kappa^2 v,$$
$$\mathrm{d}(v,y) := y - \mu^{-1}\, \mathrm{curl}\, v.$$

Proof. From (3) it follows that

$$\int_\Omega \left(\mu^{-1}\, \mathrm{curl}(u-v) \cdot \mathrm{curl}\, w + \kappa^2 (u-v) \cdot w \right) dx$$
$$= \int_\Omega (j \cdot w - \mu^{-1}\, \mathrm{curl}\, v \cdot \mathrm{curl}\, w - \kappa^2 v \cdot w)\, dx. \tag{8}$$

Take $y \in H(\Omega, \mathrm{curl})$ and use the identity

$$(\mathrm{curl}\, y) \cdot w = \mathrm{div}(y \times w) + y \cdot \mathrm{curl}\, w. \tag{9}$$

Since

$$\int_\Omega \mathrm{div}(y \times w)\, dx = \int_{\partial\Omega} n \cdot (y \times w)\, ds = \int_{\partial\Omega} y \cdot (w \times n)\, ds = 0,$$

we find that

$$\int_\Omega (\mathrm{curl}\, y \cdot w - y \cdot \mathrm{curl}\, w)\, dx = 0 \quad \forall w \in V_0. \tag{10}$$

By (8) and (10) we obtain

$$\int_\Omega \left(\mu^{-1}\, \mathrm{curl}(u-v) \cdot \mathrm{curl}\, w + \kappa^2 (u-v) \cdot w \right) dx$$
$$= \int_\Omega (j - \mathrm{curl}\, y - \kappa^2 v) \cdot w\, dx + \int_\Omega (y - \mu^{-1}\, \mathrm{curl}\, v) \cdot \mathrm{curl}\, w\, dx. \tag{11}$$

Set $w = u - v$ and estimate two integrals in the right-hand side by the Hölder inequality. We have

$$|[u-v]|^2_{(\mu^{-1},\kappa^2)} \leq \left\| \frac{1}{\kappa} \mathrm{r}(v,y) \right\| \|\kappa(u-v)\| + \|\mu^{1/2} \mathrm{d}(v,y)\| \|\mu^{-1/2}\, \mathrm{curl}(u-v)\|,$$

which implies (7).

The estimate (7) shows that the distance between u and v measured in terms of the weighted norm $|[u-v]|_{(\mu^{-1},\kappa^2)}$ is bounded from above by the sum of two residuals $\mathrm{r}(v,y)$ and $\mathrm{d}(v,y)$ that are associated with the decomposition of (1), which has the form

$$\operatorname{curl} p + \kappa^2 u - j = 0,$$
$$p = \mu^{-1} \operatorname{curl} u.$$

We note that the estimate (7) has no gap between its left- and right-hand sides. Indeed, if we set $y = \mu^{-1} \operatorname{curl} u$ then

$$\|\mu^{1/2}\mathrm{d}(v,y)\| = \|\mu^{-1/2} \operatorname{curl}(u-v)\|$$

and

$$\left\|\frac{1}{\kappa}\mathrm{r}(v,y)\right\| = \|\kappa(u-v)\|$$

so that (7) holds as the equality. However, for small κ the estimate becomes sensitive with respect to $\mathrm{r}(v,y)$ and may loose practical efficiency if the value of this residual is not much smaller than $\mathrm{r}(v,y)$.

Remark 1. If $\kappa > 0$ only in $\Omega_+ \subset \Omega$, then (11) can be transformed as follows:

$$\int_\Omega \left(\mu^{-1} \operatorname{curl}(u-v) \cdot \operatorname{curl} w + \kappa^2 (u-v) \cdot w\right) dx$$
$$= \int_{\Omega_+} (j - \operatorname{curl} y - \kappa^2 v) \cdot w \, dx + \int_\Omega (y - \mu^{-1} \operatorname{curl} v) \cdot \operatorname{curl} w \, dx,$$

which implies the estimate

$$|[u-v]|^2_{(\mu^{-1},\kappa^2)} \leq \left\|\frac{1}{\kappa}\mathrm{r}(v,y)\right\|^2_{\Omega_+} + \|\mu^{1/2}\mathrm{d}(v,y)\|^2.$$

3.2 Lower Bound of the Error

A lower bound of the error norm is derived by the following arguments. First, we note that

$$\sup_{w \in V_0} \int_\Omega \Bigg(\mu^{-1} \operatorname{curl}(u-v) \cdot \operatorname{curl} w$$
$$+ \kappa^2 w \cdot (u-v) - \frac{1}{2}(\mu^{-1} \operatorname{curl} w \cdot \operatorname{curl} w + \kappa^2 w \cdot w) \Bigg) dx$$
$$\leq \sup_{\substack{\tau \in L^2(\Omega,\mathbb{R}^d) \\ w \in L^2(\Omega,\mathbb{R}^d)}} \int_\Omega \Bigg(\mu^{-1} \operatorname{curl}(u-v) \cdot \tau - \frac{1}{2}\mu^{-1}\tau \cdot \tau$$
$$+ \kappa^2 w \cdot (u-v) - \frac{1}{2}\kappa^2 w \cdot w \Bigg) dx = \frac{1}{2}|[u-v]|^2_{(\mu^{-1},\kappa^2)}.$$

On the other hand,

$$\sup_{w\in V_0}\int_\Omega\Big(\mu^{-1}\operatorname{curl}(u-v)\cdot\operatorname{curl}w + \kappa^2 w\cdot(u-v)-\frac{1}{2}(\mu^{-1}\operatorname{curl}w\cdot\operatorname{curl}w+\kappa^2 w\cdot w)\Big)\,dx$$
$$\geq \int_\Omega\Big(\mu^{-1}\operatorname{curl}(u-v)\cdot\operatorname{curl}(u-v) + \kappa^2(u-v)\cdot(u-v)-\frac{1}{2}(\mu^{-1}|\operatorname{curl}(u-v)|^2+\kappa^2|u-v|^2)\Big)\,dx$$
$$=\frac{1}{2}|[u-v]|^2_{(\mu^{-1},\kappa^2)}.$$

Thus, we conclude that

$$\frac{1}{2}|[u-v]|^2_{(\mu^{-1},\kappa^2)} = \sup_{w\in V_0}\int_\Omega\Big(\mu^{-1}\operatorname{curl}(u-v)\cdot\operatorname{curl}w + \kappa^2 w\cdot(u-v)-\frac{1}{2}(\mu^{-1}\operatorname{curl}w\cdot\operatorname{curl}w+\kappa^2 w\cdot w)\Big)\,dx.$$

By (3), we obtain

$$|[u-v]|^2_{(\mu^{-1},\kappa^2)} \geq \mathcal{M}^2_\ominus(v,w), \tag{12}$$

where

$$\mathcal{M}^2_\ominus(v,w) := \int_\Omega\big(2j\cdot w-\mu^{-1}|\operatorname{curl}w|^2-\kappa^2|w|^2 - 2\mu^{-1}\operatorname{curl}v\cdot\operatorname{curl}w - 2\kappa^2 v\cdot w\big)\,dx.$$

For any $w\in V_0$ the quantity $\mathcal{M}^2_\ominus(v,w)$ provides a lower bound of the error. Certainly, the sharpest bound is given by the quantity

$$M^2_\ominus(v) := \sup_{w\in V_0}\mathcal{M}^2_\ominus(v,w).$$

It is not difficult to prove that this quantity coincides with the squared error (to prove that it suffices to set $w=u-v$).

3.3 Practical Implementation

Practically computable upper (lower) bounds can be determined if minimization of the majorant (maximization of the minorant) is performed over a finite-dimensional subspace $V_k\subset V$, $\dim V_k = k$ ($V_{0m}\subset V_0$, $\dim V_{0m}=m$). Then, finding the quantities

$$M_{k\oplus}(v) := \inf_{y\in V_k}\mathcal{M}^2_\oplus(v,y), \tag{13}$$

$$M_{m\ominus}(v) := \sup_{w \in V_{0m}} \mathcal{M}^2_{\ominus}(v, w) \tag{14}$$

requires solving quadratic type minimization (maximization) problems what can be done by standard methods.

We note that conforming approximations in V (and in V_0) are usually constructed by the Nédélec elements (see [9]), which are also natural to use for the construction of V_k (and V_{0m}). If V_k and V_{0m} are limit dense in V and V_0, respectively (for $k, m \to +\infty$), then it is easy to prove that

$$M_{k\oplus}(v) \to |[u - v]|_{(\mu^{-1},\kappa^2)} \quad \text{and} \quad M_{m\ominus}(v) \to |[u - v]|_{(\mu^{-1},\kappa^2)}.$$

The ratio

$$i_{km} := \frac{M_{k\oplus}(v)}{M_{m\ominus}(v)}$$

is, indeed, computable. It shows the efficiency of the error estimation.

4 A Posteriori Error Estimate of the Second Type

In this section, we derive a posteriori estimates of a more general type assuming that κ is a positive constant. By the Helmholtz decomposition of a vector-valued function, we represent the exact solution u

$$u = u_0 + \nabla \psi,$$

where u_0 is a solenoidal vector-valued function and $\psi \in \overset{\circ}{H}{}^1(\Omega)$. Since $\operatorname{curl} \nabla \psi = 0$, we rewrite (3) as follows:

$$\int_\Omega \mu^{-1} \operatorname{curl} u_0 \cdot \operatorname{curl} w + \kappa^2 (u_0 + \nabla \psi) \cdot w \, dx = \int_\Omega j \cdot w \, dx. \tag{15}$$

Next, we make the same decomposition for the trial function and set $w = w_0 + \nabla \phi$. Recall that

$$\int_\Omega j \cdot \nabla \phi \, dx = \int_\Omega u_0 \cdot \nabla \phi \, dx = \int_\Omega w_0 \cdot \nabla \psi \, dx = 0.$$

We observe that

$$\int_\Omega (\mu^{-1} \operatorname{curl} u_0 \cdot \operatorname{curl} w_0 + \kappa^2 u_0 \cdot w_0 + \kappa^2 \nabla \psi \cdot \nabla \phi) \, dx = \int_\Omega j \cdot w_0 \, dx. \tag{16}$$

In (16), we set $w_0 = 0$ and $\phi = \psi$. We find that $\|\nabla \psi\| = 0$. Hence, u is a divergence-free function.

We use this fact to rearrange (11) in a different way. We have

$$\int_\Omega \mathrm{r}(v, y) \cdot w \, dx = \int_\Omega \mathrm{r}(v, y) \cdot (w_0 + \nabla \phi) \, dx \le \|\mathrm{r}(v, y)\| \left(\|w_0\| + \|\nabla \phi\| \right). \tag{17}$$

Note that ϕ satisfies the relation

$$\int_\Omega \nabla\phi \cdot \nabla\widetilde{\phi}\, dx = \int_\Omega w \cdot \nabla\widetilde{\phi}\, dx = -\int_\Omega (\operatorname{div} w)\widetilde{\phi}\, dx \quad \forall\widetilde{\phi} \in \mathring{H}^1(\Omega), \tag{18}$$

which implies the estimate

$$\|\nabla\phi\| \le C_1(\Omega)\|\operatorname{div} w\|, \tag{19}$$

where $C_1(\Omega)$ is the constant in the Friedrich inequality for the domain Ω. For solenoidal fields we also have the estimate (see, e.g. [4, 8, 18])

$$\|w_0\| \le C_2(\Omega)\|\operatorname{curl} w_0\| = C_2(\Omega)\|\operatorname{curl} w\|. \tag{20}$$

Hence,

$$\int_\Omega \mathrm{r}(v,y) \cdot w\, dx \le \|\mathrm{r}(v,y)\| \left(C_1(\Omega)\|\operatorname{div} w\| + C_2(\Omega)\|\operatorname{curl} w\|\right) \tag{21}$$

and we arrive at the estimate

$$\begin{aligned}\int_\Omega &(\mu^{-1}|\operatorname{curl}(u-v)|^2 + \kappa^2|u-v|^2)\, dx \\ &\le (\|\mathrm{d}(v,y)\| + C_2(\Omega)\mathrm{r}(v,y))\,\|\operatorname{curl}(u-v)\| \\ &+ C_1(\Omega)\|\mathrm{r}(v,y)\|\|\operatorname{div}(u-v)\| \le \frac{\alpha}{4}\left(\|\mathrm{d}(v,y)\| + C_2(\Omega)\mathrm{r}(v,y)\|\right)^2 \\ &+ \frac{1}{\alpha}\|\operatorname{curl}(u-v)\|^2 + C_1(\Omega)\|\mathrm{r}(v,y)\|\|\operatorname{div} v\|,\end{aligned} \tag{22}$$

where $\alpha \ge \mu$.

Hence, we arrive at the following result:

Proposition 2. *If κ is a positive constant and $v \in V_0 \cap H(\Omega, \operatorname{div})$ then for any $y \in H(\Omega, \operatorname{curl})$*

$$\begin{aligned}|[u-v]|^2_{((\frac{1}{\mu}-\frac{1}{\alpha}),\kappa^2)} \le \frac{\alpha}{4}\left(\|\mathrm{d}(v,y)\| + C_2(\Omega)\|\mathrm{r}(v,y)\|\right)^2 \\ + C_1(\Omega)\|\mathrm{r}(v,y)\|\|\operatorname{div} v\|.\end{aligned} \tag{23}$$

If $\operatorname{div} v = 0$*, then the estimate is simplified and has the form*

$$|[u-v]|_{((\frac{1}{\mu}-\frac{1}{\alpha}),\kappa^2)} \le \frac{\sqrt{\alpha}}{2}\left(\|\mathrm{d}(v,y)\| + C_2(\Omega)\|\mathrm{r}(v,y)\|\right). \tag{24}$$

We can use a somewhat different way and estimate the first term in the right-hand side of (11) as follows:

$$\int_\Omega \mathrm{r}(v,y) \cdot w\, dx \le \|\mathrm{r}(v,y)\| \left(C_1(\Omega)\|\operatorname{div} w\| + C_2(\Omega)\mu_\oplus^{1/2}\|\mu^{-1/2}\operatorname{curl} w\|\right). \tag{25}$$

Set $w = u - v$ and note that $\operatorname{div}(u - v) = \operatorname{div} v$. Then we obtain

$$\begin{aligned}
|[u-v]|^2_{(\mu^{-1},\kappa^2)} &\leq C_1(\Omega)\|\operatorname{div} v\|\|\mathrm{r}(v,y)\| \\
&\quad + \left(C_2(\Omega)\mu_\oplus^{1/2}\|\mathrm{r}(v,y)\| + \|\mu^{1/2}\mathrm{d}(v,y)\|\right)\|\mu^{-1/2}\operatorname{curl}(u-v)\| \\
&\leq C_1(\Omega)\|\operatorname{div} v\|\|\mathrm{r}(v,y)\| \\
&\quad + \left(C_2(\Omega)\mu_\oplus^{1/2}\|\mathrm{r}(v,y)\| + \|\mu^{1/2}\mathrm{d}(v,y)\|\right)|[u-v]|_{(\mu^{-1},\kappa^2)} \qquad (26)
\end{aligned}$$

and arrive at the following result.

Proposition 3. *If κ is a positive constant and $v \in V_0 \cap H(\Omega, \operatorname{div})$ then for any $y \in H(\Omega, \operatorname{curl})$*

$$|[u-v]|_{(\mu^{-1},\kappa^2)} \leq \mathcal{M}_2(v,y) := \frac{R_2}{2} + \sqrt{R_1 + \frac{R_2^2}{4}}, \qquad (27)$$

where

$$R_1 = C_1(\Omega)\|\operatorname{div} v\|\|\mathrm{r}(v,y)\|$$

and

$$R_2 := C_2(\Omega)\mu_\oplus^{1/2}\|\mathrm{r}(v,y)\| + \|\mu^{1/2}\mathrm{d}(v,y)\|.$$

If, in addition, $\operatorname{div} v = 0$*, then*

$$|[u-v]|_{(\mu^{-1},\kappa^2)} \leq R_2. \qquad (28)$$

Remark 2. If $\kappa = 0$, then (28) has the form

$$\|\mu^{-1}\operatorname{curl}(u-v)\| \leq R_2. \qquad (29)$$

For $\kappa = 0$, this estimate was earlier derived in [14, 15].

The estimates (23) and (27) are insensitive with respect to small values of κ (what differs them from (7)). However, we made a certain overestimation of the right-hand side in the last transformation of (22). Therefore, we cannot guarantee that this upper bound has no gap (substitution of $y = \mu^{-1}\operatorname{curl} u$ does not make the respective right-hand sides equal to the error).

5 A Posteriori Estimate of the Third Type

5.1 An Advanced Form of the Error Majorant

To derive upper bounds that possess all positive features of the estimates of the first and second types we use a more sophisticated method.

Proposition 4. *Let v and y satisfy the assumptions of Proposition 2. Then*

$$|[u-v]|^2_{\gamma,\delta} \leq \mathcal{M}^2_3(\lambda,\alpha_1,\alpha_2,v,y), \tag{30}$$

where

$$\mathcal{M}^2_3(\lambda,\alpha_1,\alpha_2,v,y) := R_1(\lambda,v,y) + \frac{\alpha_1}{4}R^2_2(\lambda,v,y) + \frac{\alpha_2}{4}R^2_3(\lambda,v,y),$$

α_1 and α_2 are arbitrary numbers in $[1,+\infty)$,

$$\gamma = \left(1-\frac{1}{\alpha_1}\right)\mu^{-1}, \quad \delta = \left(1-\frac{1}{\alpha_2}\right)\kappa^2,$$
$$\lambda \in I_{[0,1]} := \{\lambda \in L^\infty(\Omega) \mid \lambda(x) \in [0,1] \text{ for a.e. } x \in \Omega\},$$

and R_i, $i=1,2,3$, are defined by (33)–(35).

Proof. With the help of λ we decompose the integral identity (11) as follows (in [17], a similar method was used for the decomposition of the reaction-diffusion equation):

$$\int_\Omega \left(\mu^{-1}\operatorname{curl}(u-v)\cdot\operatorname{curl} w + \kappa^2(u-v)\cdot w\right)\,dx$$
$$= \int_\Omega \lambda(j-\operatorname{curl} y-\kappa^2 v)\cdot w\,dx + \int_\Omega (1-\lambda)(j-\operatorname{curl} y - \kappa^2 v)\cdot w\,dx$$
$$+ \int_\Omega (y-\mu^{-1}\operatorname{curl} v)\cdot\operatorname{curl} w\,dx, \tag{31}$$

where $\lambda \in I_{[0,1]}$. Since

$$\int_\Omega \lambda \mathrm{r}(v,y)\cdot(u-v)\,dx \leq \left\|\frac{\lambda}{\kappa}\mathrm{r}(v,y)\right\| \|\kappa(u-v)\|$$

and

$$\int_\Omega (1-\lambda)\mathrm{r}(v,y)\cdot(u-v)\,dx$$
$$\leq \|(1-\lambda)\mathrm{r}(v,y)\| \left(C_1(\Omega)\|\operatorname{div} v\| + C_2(\Omega)\mu_\oplus^{1/2}\|\mu^{-1/2}\operatorname{curl}(u-v)\|\right),$$

we obtain

$$\int_\Omega \left(\mu^{-1}|\operatorname{curl}(u-v)|^2 + \kappa^2|u-v|^2\right)\,dx$$
$$\leq R_1 + R_2\|\mu^{-1/2}\operatorname{curl}(u-v)\| + R_3\|\kappa(u-v)\|, \tag{32}$$

where

$$R_1(\lambda,v,y) = C_1(\Omega)\|(1-\lambda)\mathrm{r}(v,y)\|\|\operatorname{div} v\|, \tag{33}$$

$$R_2(\lambda, v, y) = C_2(\Omega)\mu_\oplus^{1/2}\|(1-\lambda)\mathrm{r}(v,y)\| + \|\mu^{1/2}\mathrm{d}(v,y)\|, \tag{34}$$

$$R_3(\lambda, v, y) = \left\|\frac{\lambda}{\kappa}\mathrm{r}(v,y)\right\|. \tag{35}$$

By applying the Young inequality to the right-hand side of (32), we obtain

$$\int_\Omega \left(1 - \frac{1}{\alpha_1}\right)\mu^{-1}|\operatorname{curl}(u-v)|^2\,dx + \int_\Omega \left(1-\frac{1}{\alpha_2}\right)\kappa^2|u-v|^2\,dx \\ \leq R_1 + \frac{\alpha_1}{4}R_2^2 + \frac{\alpha_2}{4}R_3^2, \tag{36}$$

which implies (30).

Corollary 1. *If $\alpha_1 = \alpha_2 = 2$ then (30) comes in the form*

$$|[u-v]|^2_{(\mu^{-1},\kappa^2)} \leq 2R_1(\lambda, v, y) + R_2^2(\lambda, v, y) + R_3^2(\lambda, v, y). \tag{37}$$

The proposition below shows that the estimate (30) possesses the same principal property as (7): it has no gap between the left- and right-hand sides.

Proposition 5. *If $\alpha_1 = \alpha_2 = 2$, then*

$$|[u-v]|^2_{(\mu^{-1},\kappa^2)} = M_\oplus(v), \tag{38}$$

where

$$M_\oplus(v) := \inf_{\substack{\lambda\in I_{[0,1]},\\ y\in H(\Omega,\mathrm{curl})}} \left\{2R_1(\lambda, v, y) + R_2^2(\lambda, v, y) + R_3^2(\lambda, v, y)\right\}. \tag{39}$$

Proof. Obviously,

$$M_\oplus \leq 2R_1(1, v, p) + R_2^2(1, v, p) + R_3^2(1, v, p),$$

where $p = \mu^{-1}\operatorname{curl} u$. Note that

$$R_1(1, v, p) = 0,$$
$$R_2(1, v, p) = \|\mu^{-1/2}\operatorname{curl}(u-v)\|,$$
$$R_3(1, v, p) = \|\kappa(u-v)\|.$$

Therefore,

$$M_\oplus = \|\mu^{-1/2}\operatorname{curl}(u-v)\|^2 + \|\kappa(u-v)\|^2 = |[u-v]|^2_{(\mu^{-1},\kappa^2)}.$$

5.2 Optimal Form of $\boldsymbol{\lambda}$

Now our goal is to derive the sharpest upper bound by defining the function $\lambda(x)$ in an "optimal" way. For this purpose, we first reform (36) by introducing positive parameters α_3 and α_4 and noting that

$$\begin{aligned} R_1(\lambda, v, y) &\leq \frac{\alpha_3}{2} C_1^2(\Omega) \|(1-\lambda)\mathrm{r}(v,y)\|^2 + \frac{1}{2\alpha_3} \| \operatorname{div} v\|^2, \\ R_2^2(\lambda, v, y) &\leq (1+\alpha_4) C_2^2(\Omega) \mu_\oplus \|(1-\lambda)\mathrm{r}(v,y)\|^2 \\ &\quad + \left(1 + \frac{1}{\alpha_4}\right) \|\mu^{1/2} \mathrm{d}(v,y)\|^2. \end{aligned}$$

Therefore, (36) implies

$$\begin{aligned} &\int_\Omega \left(1 - \frac{1}{\alpha_1}\right) \mu^{-1} |\operatorname{curl}(u-v)|^2 \, dx + \int_\Omega \left(1 - \frac{1}{\alpha_2}\right) \kappa^2 |u-v|^2 \, dx \\ &\leq \int_\Omega \left((1-\lambda)^2 P + \lambda^2 Q\right) \mathrm{r}^2(v,y) \, dx + \left(1 + \frac{1}{\alpha_4}\right) \frac{\alpha_1}{4} \|\mu^{1/2} \mathrm{d}(v,y)\|^2 + \frac{1}{2\alpha_3} \| \operatorname{div} v\|^2, \end{aligned} \tag{40}$$

where

$$\begin{aligned} P &= \frac{\alpha_3}{2} C_1^2(\Omega) + (1+\alpha_4) \frac{\alpha_1}{4} C_2^2(\Omega) \mu_\oplus, \\ Q &= \frac{\alpha_2}{4\kappa^2}. \end{aligned}$$

Optimal λ is defined by the relation

$$\lambda = \frac{P}{P+Q} \in [0,1]$$

and the respective estimate reads

$$\begin{aligned} &\int_\Omega \left(1 - \frac{1}{\alpha_1}\right) \mu^{-1} |\operatorname{curl}(u-v)|^2 \, dx + \int_\Omega \left(1 - \frac{1}{\alpha_2}\right) \kappa^2 |u-v|^2 \, dx \\ &\leq \int_\Omega \frac{PQ}{P+Q} \mathrm{r}^2(v,y) \, dx + \left(1 + \frac{1}{\alpha_4}\right) \frac{\alpha_1}{4} \|\mu^{1/2} \mathrm{d}(v,y)\|^2 + \frac{1}{2\alpha_3} \| \operatorname{div} v\|^2. \end{aligned} \tag{41}$$

Remark 3. Note that

$$\frac{PQ}{P+Q} \leq \min\{P, Q\}.$$

Therefore, the estimate (41) is insensitive to small values of κ^2.

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