The resurgent approach to topological string theory

Ricardo Couso-Santamaría

Abstract. In these notes I describe practical applications of resurgence to topological strings, a theory that enjoys connections with matrix models, enumerative and complex geometry, and strong/weak dualities in Physics. Starting from the asymptotic series representation of the free energy I outline recent results which are first steps for arriving at a transseries, which should in principle contain all the nonperturbative information of the theory.

1 Introduction

The goal of these notes is to present an overview of the work developed in [7,8] about the applications of resurgence to topological string theory. These references are not pieces of mathematics but physics work so they sit on a lower step in the staircase of rigor that has the work of Écalle at the top [12]. The objective then is to introduce the ideas and techniques that have been quite useful in understanding and uncovering nonperturbative effects in physical theories, and to put part of the focus on issues that could be taken as working problems for the resurgent mathematician. A nice companion to this article is [21], which assumes no familiarity with resurgence or topological strings.

The ultimate application of resurgence to Physics would be the use resurgent techniques to define and compute nonperturbative observables of a given physical theory. See [11] for an overview of the role of resurgence in Physics. By nonperturbative I mean valid for any value of the interaction couplings, small or large. (In Mathematics we denote this coupling by x^{-1} while in Physics we may call it g_s.) However, such an ambitious goal could not completely work in general for physical and technical reasons. Resurgence can capture nonperturbative information about a system and store it in the form of a *transseries*. This is a *formal*

F. Fauvet et al. (eds.), *Resurgence, Physics and Numbers*

object in a variable x built out of exponentials and powers,

$$\varphi = \sum_{n=0}^{\infty} \sigma^n \mathrm{e}^{-nAx} \sum_{g=0}^{\infty} x^{-g} \varphi_g^{(n)}, \qquad \sigma, A, \varphi_g^{(n)} \in \mathbb{C}. \tag{1.1}$$

The notation here is the following: A is called the instanton action in physical contexts and I will stick to that name; n labels instanton numbers, where $n = 0$ denotes the perturbative sector; σ is called the transseries parameter and is not constrained at the formal level. Transseries can be more general including many instanton actions, A_α, and associated sectors labeled by vectors $\boldsymbol{n}$ with natural numbers as entries. They can also allow for other transseries monomials besides exponentials, such as logarithms, $\log x$. See for example [6,13].

The computation of the transseries can be quite challenging in practice. Even if we have surpassed that obstacle we still have to perform the task of resumming the trans series into a function of x that would eventually define the physical observable φ_{phys}. The resummation handles each asymptotic series $\sum_{g=0}^{\infty} x^{-g} \varphi_g^{(n)}$ for every n to produce a finite number for a given x. However we still have to determine the value of the transseries parameter σ. For this we need some *physical input*, such as a boundary condition for φ_{phys} at infinity. This last step is important because resurgence alone cannot choose between all the possible nonperturbative completions. When we lack knowledge about the nonperturbative regime of a physical theory we may not have a way to determine the right completion.

In sight of such a disheartening picture we could just turn to other techniques sometimes available in Physics such as the strong/weak coupling duality or integrability, but we would be missing on the information revealable by resurgence and displayed in the transseries. To illustrate what I mean by the *resurgent approach* let us have a look at Figure 1.1 where I show a practical route from an asymptotic series of perturbative nature, $\varphi^{(0)}$, to a full nonperturbative physical quantity, φ. This approach will not always be successful but it is always worth trying.

The starting point in most physical problems is a finite sequence of perturbative coefficients, $\varphi_g^{(0)}$. How one arrives at these quantities and how many of them can be computed depends very much on the problem. The asymptotic nature of the resurgent approach focuses strongly on behaviors at large order g, so the more coefficients we have the more precise our numerical results will be. For generic quantum field theories computing $\varphi_g^{(0)}$ for even moderate g requires calculating a large amount of Feynman diagrams — roughly $g!$ of them. For theories like topological strings we rely on the recursive properties of the coefficients — like

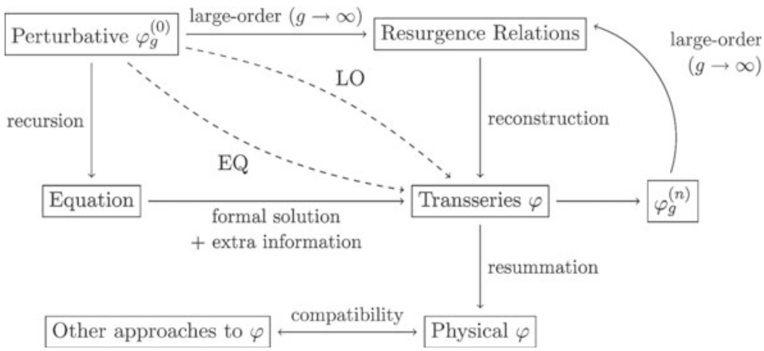

Figure 1.1. The resurgent diagram describes schematically the routes to the transseries starting from perturbation theory, namely EQ and LO, as well as the feedback triangle on the right-hand side and the resummation at the bottom.

the holomorphic anomaly equations, see later — to bypass Feynman diagrams altogether.

If our goal is to build a transseries by studying perturbative data alone we have two routes, which generically complement each other. They are labeled LO and EQ in Figure 1.1.

Route EQ, if available, is the fastest way to the transseries. We may take it if we can find an equation of some type that the perturbative asymptotic series, $\varphi^{(0)}(x)$, satisfies. The easiest way is to find a recursion relation between the coefficients $\varphi_g^{(0)}$ and build an equation from it. The next step is to promote this equation to be valid not only perturbatively but also for a transseries like (1.1). After this, computing coefficients $\varphi_g^{(n)}$ is basically a mechanical task.

To give a somewhat trivial example consider the list perturbative coefficients

$$0, 1, -1, 3, -11, 51, -283, \dots \tag{1.2}$$

that come from solving a Riccati equation in power series. Even if we did not know that such equation was behind these numbers it would not be too unlikely to find a (nonlinear) relation for them, namely

$$\varphi_0^{(0)} = 0, \quad \varphi_1^{(0)} = 1, \quad \varphi_g^{(0)} = -g\varphi_{g-1}^{(0)} + \sum_{h=0}^{g} \varphi_h^{(0)}\varphi_{g-1-h}^{(0)}, \tag{1.3}$$

and eventually arrive at the equation

$$\varphi^{(0)\prime}(x) = \varphi^{(0)}(x) - \varphi^{(0)}(x)^2/x - 1/x. \tag{1.4}$$

After this we can just drop the perturbative superscript and plug in a transseries ansatz. This would tell us that the *instanton action* in (1.1) is $A = -1$ and that the following coefficients are

$$\text{one-instanton}, \varphi_g^{(1)}: \quad 1, 2, 1, \frac{4}{3}, \dots \tag{1.5}$$

$$\text{two-instanton}, \varphi_g^{(2)}: \quad -1, -5, -14, -\frac{122}{3}, \dots \tag{1.6}$$

and so on.

The route labeled LO (after large order) deals with the perturbative coefficients and nothing else. The goal is to determine, with as much detail as possible, how the coefficients $\varphi_g^{(0)}$ grow with g, because in those details are hidden the nonperturbative coefficients. This is what the theory of resurgence tells us that generically happens. So, following with the example, if we take the Riccati numbers (1.2) and pretend for a moment to forget their origin, we can analyze numerically their dependence in the index g. After some numerical computations we would find

$$\begin{aligned}
\varphi_g^{(0)} &\sim \frac{\sinh(\pi)}{\pi}\left[\frac{g!}{(-1)^g}1 + \frac{(g-1)!}{(-1)^{g-1}}2 + \frac{(g-2)!}{(-1)^{g-2}}1 + \frac{(g-3)!}{(-1)^{g-3}}\frac{4}{3} + \dots\right] \\
&\quad + \left(\frac{\sinh(\pi)}{\pi}\right)^2\left[\frac{g!}{(2(-1))^g}(-1) + \frac{(g-1)!}{(2(-1))^{g-1}}(-5)\right. \\
&\quad \left. + \frac{(g-2)!}{(2(-1))^{g-2}}(-14) + \frac{(g-3)!}{(2(-1))^{g-3}}\left(-\frac{122}{3}\right) + \dots\right] + \dots \\
&= \sum_{n=1}^{\infty}\frac{S_1^n}{2\pi i}\sum_{h=0}^{\infty}\frac{(g-h)!}{(nA)^{g-h}}\varphi_h^{(n)} \qquad \text{as } g \to \infty.
\end{aligned} \tag{1.7}$$

Thus we find, in a very neat and organized fashion, all the nonperturbative information we were looking for. As an extra bit we obtain the Stokes constant S_1, a quantity intrinsic to the problem that dictates how different resummations are related to each other. See [**?**] for a rigorous resurgent treatment of the Riccati equation. Let us stress now two important facts about the LO route.

The first is that the relation between $\varphi_g^{(0)}$ and $\varphi_h^{(n)}$ displayed in (1.7) for the Riccati example is generic. The presence of factorials of decreasing intensity and the instanton action in the denominator is a general consequence of resurgence and we expect to find relations similar to these for other problems. That a relation exists between perturbative and nonperturbative data is not that surprising given the existence of route EQ. What may be regarded as unexpected and useful is that the form of (1.7) is valid for a large class of problems.

The second point is that the route LO is essentially a numeric approach to the problem of finding the transseries, but one that is always available if we can work with enough perturbative coefficients and precision. Practical concerns in this area include the use of convergence acceleration techniques like Richardson extrapolation, see [5].

The roads labeled LO and EQ describe the square in Figure 1.1. There is also a triangle between the transseries, the nonperturbative coefficients $\varphi_g^{(n)}$ (read out from the transseries), and new resurgence relations for $\varphi_g^{(n)}$ when g is large. The latter are quite similar in form to that in (1.7): the lhs is $\varphi_g^{(n)}$ and the rhs involves factorials, instanton actions, Stokes constants, and other coefficients $\varphi_h^{(m)}$. Since we have one such resurgence relation for each instanton number n the complete set of equations imposes quite a constraint on the coefficients of the transseries. This is a property of resurgence that we can take advantage of in problems where the roads LO and EQ do not yield as much information as we wanted (*e.g.*, due to numerical obstacles) or when that information is incomplete (*e.g.*, we cannot determine the coefficients completely). That will be the case in topological string theory.

The bottom part of the diagram deals with the resummation of the transseries, the determination of the transseries parameter σ, and the related issue of Stokes phenomena. I will not cover these topics here because for topological string theory the problem is still under investigation (see however the recent article [9]). I will just mention that to transform a formal transseries into an actual function we use Borel resummation on each of the asymptotic series $\varphi^{(n)}(x)$, see for example the second part of [20]. This resummation process can lead to an ambiguous answer at each instanton sector n. The cure to this problem comes from applying Borel resummation to the complete transseries, that is, Borel–Écalle resummation. It can be a nontrivial step to prove that the resummation is free of the ambiguities.

In the lucky cases in which we have access to alternative descriptions of our theories we can compare the resummed transseries with these other predictions. This can be a crucial step in determining the correct element in the family of transseries parametrized by σ.

2 Basics of topological strings

The first question that comes to mind is why apply the resurgent approach to topological string theory. The answer is twofold.

First of all, topological string theory is a subject of interest by theoretical physicists and mathematicians alike due to its central role in mirror symmetry, in understanding questions of the full string theory and M-theory and their dualities, as well as the connections with random matrix theory/matrix models. See [17] for details. Topological string theory is defined from first principles as a perturbative expansion in a small parameter, g_s, called the string coupling constant. This series turns out to be asymptotic due to the factorial growth of the coefficients (Gevrey-1). Although the nonperturbative nature of the theory has been probed through several avenues, a general nonperturbative definition of topological strings is lacking. Nevertheless some proposals have appeared, at least for large classes of theories, that could fill this void, see for example [18] for a recent approach. This problem of finding a nonperturbative completion for a theory is one for which resurgence can provide valuable insight.

This leads us to the second part of our answer. The perturbative topological string coefficients can be computed very efficiently in some cases and that is the starting point we need for the resurgent approach along the LO route. Moreover I will show that the EQ road is also available. That is, there is an equation that generalizes the perturbative recursion — the *holomorphic anomaly equation* — and that computes many ingredients of the transseries.

Perturbative topological string theories are defined on top of topological field theories of maps from Riemann surfaces to Calabi–Yau manifolds (CY). They come in two kinds, A and B, defined on different CYs but related by mirror symmetry. This means that the two free energies, on the A and B side, will agree as formal asymptotic series once we have figured out the relation between the two geometries, also known as the mirror map,

$$F^{(0),A}(g_s,t) \underset{\text{map}}{\overset{\text{mirror}}{\longleftrightarrow}} F^{(0),B}(g_s,z), \qquad t=t(z) \tag{2.1}$$

The dependence on the geometries appears through moduli, which are variables that capture the Kähler structure (parametrized by t) of the A-model CY, or the complex structure (parametrized by z) on the B-model CY. This means that these asymptotic series come in families labeled by t or z, and we have to regard the coefficients $F_g^{(0)}$ as functions, not just numbers. Moreover, the dependence on the moduli is not holomorphic, so $\bar{t}$ and $\bar{z}$ also appear. Taking $\bar{z}\to 0$ we obtain a holomorphic limit[1] of the free energies. For the A model this limit has the form

$$\mathcal{F}_g^{(0),A}(g_s,t)=\sum_{g=0}^{\infty} g_s^{2g-2}\sum_{d=1}^{\infty} N_{g,d}\,\mathrm{e}^{-dt}, \tag{2.2}$$

[1] The holomorphic limit is not unique but attached to the notion of frame which will be ignored for the sake of clarity and brevity. We use curly $\mathcal{F}$ to indicate holomorphic, while roman F to denote holomorphic and nonholomorphic dependence.

where $N_{g,d} \in \mathbb{Q}$ are the famous Gromov–Witten invariants. These count, in the appropriate sense, holomorphic maps from complex curves of genus g into the CY with fixed degree or homology class d.

The A-side of things is not kind towards the resurgent approach because the calculations there are hard. The B-model is far more welcoming thanks to the existence of the holomorphic anomaly equations that relate free energies of different orders, [3, 4]. They are roughly of the form

$$\partial_{\overline{z}} F_g^{(0)} \simeq \partial_z^2 F_{g-1}^{(0)} + \sum_{h=1}^{g-1} \partial_z F_h^{(0)} \, \partial_z F_{g-h}^{(0)}, \tag{2.3}$$

and they can be solved recursively in g. The expressions for $F_g^{(0)}$ can be expressed quite compactly if we use an auxiliary (set of) variable(s), $S(z, \overline{z})$, rather than $\overline{z}$. In these variables the perturbative free energies have a dependence that is polynomial in S and rational in z with coefficients in $\mathbb{Q}$.

About solving the holomorphic anomaly equations I only want to mention that integration produces a constant, or rather a function of z but not of $\overline{z}$. This is called the *holomorphic ambiguity*. Finding what it is requires nontrivial knowledge about $\mathcal{F}_g^{(0)}$ at particular values of z called the large-radius point and the conifold point. For certain CY geometries, or toric type, this knowledge is believed to be enough to fix the ambiguity for all orders [1,15]. A particularly simple geometry in this class is called local $\mathbb{P}^2$ (along with its mirror) for which over a hundred perturbative free energies were computed in [8] and were used for the resurgent analysis.

Recall that our goal is to exploit these perturbative coefficients to uncover the underlying transseries. That means finding instanton actions, A_α (one or several, and their modulus dependence) and higher instanton coefficients $F_g^{(\boldsymbol{n})}(z, S)$, for $\boldsymbol{n} \neq \boldsymbol{0}$. Taking the LO route alone would be hopeless because it is mainly a numerical enterprise and we have to keep track of two variables, z and S. Fortunately the path along EQ will be opened once we extend the validity of the holomorphic anomaly equations past perturbation theory.

3 Resurgent approach to topological strings

At this point of the discussion we are sitting on the upper left corner of the diagram in figure 1.1. We need to make progress in making EQ available while already cranking the numerical machine of LO. We also have to start making predictions of what we should find. What particular functions of z and S should the instanton actions be? How many of them are there? What about higher instanton coefficients? The first two questions

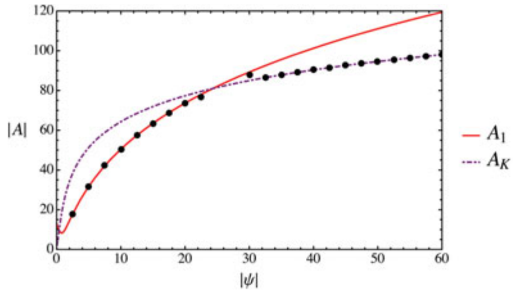

Figure 3.1. Dominant instanton actions obtained from large order growth of perturbative free energies for different values of the complex modulus $\psi = z^{-1/3}$. A_1 dominates for small ψ (around conifold point at $\psi = 1$), and A_{K} dominates for large ψ (around large-radius point at $\psi = \infty$). Since the actions are holomorphic these results can be obtained for any value of the propagator. In the transition region, around $|\psi| \simeq 25$, the two actions have similar weights in the large-order growth of $F_g^{(0)}$ and the numerics get worse.

can be guessed from previous experience in other theories, particularly from matrix models. There it was understood that instanton actions, A_α, are periods of the underlying geometry (the so called spectral curve), that is, integrals along cycles of the relevant differential form in the theory. Moreover, the periods are holomorphic and can be computed right after we know what CY we are working with [10]. We only need to find what particular linear combination of periods is realized as an instanton action, since only a small number of cycles are independent. The LO approach can tell us this numerically.

For local $\mathbb{P}^2$ there are three independent periods that are computed from the so-called Picard–Fuchs equations. A possible basis for the periods is

$$(t(z), t_{\mathrm{c}}(z), 1) \tag{3.1}$$

where t is the Kähler modulus and t_{c} is the called the flat coordinate around the conifold point. They can be written in terms of hypergeometric functions with respect to z. From a LO analysis of the free energies we find two instanton actions as shown in Figure 3.1.

$$A_{\mathrm{K}} = 4\pi^2 i\, t(z), \qquad A_1 = \frac{2\pi i}{\sqrt{3}} t_{\mathrm{c}}(z). \tag{3.2}$$

Due to their geometric origin as periods they are labeled Kähler and conifold. But we should not be too confident that our job finding actions is over because some, with a larger absolute value, could be lurking behind the dominant ones. To understand this remark notice that when we have several instanton actions and their corresponding sectors the large-order growth of $F_g^{(0)}$ will include all of them. However, the order in which they appear will depend on the relative size of the actions because A_α enters the resurgence relation as A_α^{-g}, so the smallest it is in absolute value the more it contributes as $g \to \infty$. Some actions might always be larger than the dominant ones so they will only be seen as exponentially suppressed contributions in g.

Let us now deal with the one-instanton sectors associated to the actions we have found. Now we have no external insight as to what they should be so it is time to explore EQ. The logic to use here is the same as that in the Riccati example: take your perturbative recursion and make it into a single equation for $F^{(0)}(g_s, z, S^{zz})$; then drop the perturbative superscript, plug in your transseries ansatz and solve for the coefficients.

What worked like a charm for Riccati is going to fall short for topological strings and the holomorphic anomaly equations. First of all we are going to inherit the holomorphic ambiguity problem at every instanton level $\boldsymbol{n}$ and order g. For Riccati the only ambiguity lied in the first coefficient $\varphi_0^{(1)}$ but it was conventionally set to 1, transferring the ambiguity to the transseries parameter σ. The difference between both examples has to do with the nature of the equations. While Riccati is a differential equation in x, the resurgent variable, the extended holomorphic anomaly equations are only algebraic in g_s (and differential in z and S). This also means that the number of transseries parameters is not determined by the equations.

We find that the EQ route is not as powerful as we would have wanted and leaves several parts of the transseries undetermined:

- Number of transseries parameters: several transseries *ansätze* can solve the equations, with or without logarithmic transmonomials.
- Holomorphic ambiguities: recursive integration produces ambiguities that need to be fixed.
- Instanton actions: equations only impose that they are holomorphic, what at least is in agreement with their interpretation as periods.

To arrive at the box ‘Transseries F’ in the resurgent diagram we need to complement EQ with the LO path and some amount of guess work on numerical results. Also, as we start making progress with this strategy we can activate the feedback triangle

$$F_g^{(\boldsymbol{n})} \longrightarrow \text{Resurgence Relations} \longrightarrow \text{Transseries } F \longrightarrow F_g^{(\boldsymbol{n})}. \qquad (3.3)$$

This loop will act both as a source of information and as a consistency check because transseries coefficients appear multiple times in resurgence relations, as we mentioned in the introduction.

There is a big caveat here, though, one that we have not talked about yet. We do not know what the resurgence relations look like exactly for topological string theory. It is not an option here to derive a bridge equation that links alien and ordinary derivatives, as is done many examples such as Riccati, due to the nature of the holomorphic anomaly equations in relation to g_s. So we work on the assumption that the relations will be similar to those derived from a bridge equation because the bridge equation does appear in closely related theories like Painlevé equations and matrix models. See [19] for a review on the relation between topological strings and matrix models, and [2, 14] for a resurgent treatment of Painlevé I equation and the quartic matrix model.

3.1 Main results

Here are the main results we find for the CY geometry of local $\mathbb{P}^2$. They start to paint the picture of what the transseries for the free energy looks like. Any attempt to obtain a nonperturbative value of the free energy from resurgence will have to use the results described below. In particular it will be crucial to know which actions and their corresponding transseries sector can contribute to a Borel-Écalle resummation of the transseries.

Holomorphic ambiguities. We can solve the extended holomorphic anomaly equations up to the holomorphic ambiguity and understand the dependence of the solutions on z and S, although the resulting expressions are not very illuminating.

To fix the ambiguities we look at what happens in the holomorphic limit near the large-radius and conifold points. More precisely we take the holomorphic limit of a resurgent relation linking perturbative and nonperturbative free energies.

$$\mathcal{F}_g^{(0)} \sim \frac{\Gamma(2g-1)}{A^{2g-1}} \frac{S_1}{2\pi i} \mathcal{F}_0^{(1)} \Rightarrow \frac{S_1}{2\pi i} \mathcal{F}_0^{(1)} = \lim_{g\to\infty} \frac{A^{2g-1}}{\Gamma(2g-1)} \mathcal{F}_g^{(0)}. \quad (3.4)$$

Since we know how holomorphic perturbative free energies behave at these special points we can extract results for the nonperturbative free energies using large-order limits. This can sometimes be done analytically and others numerically, but it is enough to fix ambiguities. We will comment on the fact that the Stokes constant cannot be disentangled from the ambiguity later.

Further instanton actions. We find two other instanton actions, A_2 and A_3, besides A_{K} and A_1. See Figure 3.2. They are also related to the

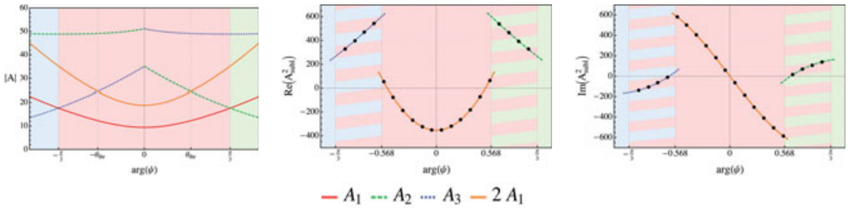

Figure 3.2. Instanton actions A_2, A_3, associated to conifold points, lie almost always behind A_1 and can only be seen as subleading contributions to $F_g^{(0)}$ when g is large. In these figures we fix $|\psi| = 2$ and vary $\arg(\psi)$. We also check that a 2-instanton sector contributes as well with action $2A_1$.

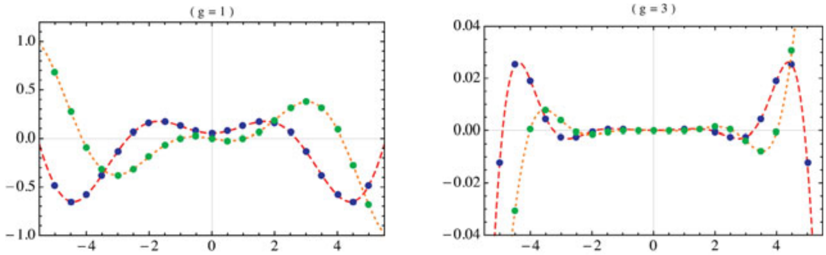

Figure 3.3. Representation of the real (red, dashed) and imaginary (orange, pointed) parts of $F_g^{(1)}(\psi, S^{zz})$ (wrt action A_1 and $g = 1, 3$) for $\psi = 2$ and free S. The blue and green plots show the numerical checks from large-order lying on top of the analytic expressions computed from the extended holomorphic anomaly equations. The dependence in the propagator is exponential (hence the oscillations) times polynomial (hence the changes in amplitude).

conifold point which, using the right coordinate $\psi = z^{-1/3}$, becomes three conifold points, one for each instanton action.

Since there can only be three independent cycles for this geometry there is a relation between all actions. It is $A_1 + A_2 + A_3 + A_K = 0$. This suggests that the transseries is resonant, provided that all actions here mentioned give rise to transseries sectors of their own.

Numerical checks. Inasmuch as we can carry out LO we find that all free energies $F_g^{(n)}$ we come across can be computed, up to ambiguity, from EQ. See an example of the numerical checks in Figure 3.3. This gives credit to the extended holomorphic anomaly equations we obtained out of the perturbative regime alone.

3.2 Open issues

Along the resurgent path we encountered several problems, some technical and some genuinely interesting from the resurgent viewpoint. I think they are both worth describing.

Numerical constraints. The numerical approach has an expiration date from the start. A finite amount of data can only give a finite number results of approximate precision. Taking large-order limits imposes a toll on precision that can only be kept at bay if we identify in closed form the numerical approximations we obtain for A_α, $F_h^{(\boldsymbol{n})}$, $h = 0, 1, 2, \ldots$

On the other hand we also want to compute nonperturbative energies from the extended holomorphic anomaly equations and analyze their own resurgent properties (c.f., triangle in the resurgent diagram), but this turns out to be computationally more demanding than perturbation theory was. Eventually one is forced to perform a numerical integration of the equations around particular values of z.

Unfamiliar resurgence relation. Let us think about Riccati again for a moment, and in particular about the asymptotics of the perturbative and one-instanton sectors. They have the form, suppressing Stokes constants,

$$\varphi_g^{(0)} \sim \sum_{h=0}^{\infty} \frac{\Gamma(g+1-h)}{A^{g+1-h}}\, \varphi_h^{(1)} + \sum_{h=0}^{\infty} \frac{\Gamma(g-h)}{(2A)^{g-h}}\, \varphi_h^{(2)} + \cdots \tag{3.5}$$

$$\varphi_g^{(1)} \sim \sum_{h=0}^{\infty} \frac{\Gamma(g+1-h)}{A^{g+1-h}}\, \varphi_h^{(2)} + \cdots \tag{3.6}$$

Note how $\varphi_0^{(2)}$ appears on both equations. This is a consequence of the bridge equation and quite a natural one because no other ingredients are available to play with. For topological strings and local $\mathbb{P}^2$ in particular, we find[2]

$$F_g^{(\mathbf{0})} \sim \sum_{h=0}^{\infty} \frac{\Gamma(2g-1-h)}{A_1^{2g-1-h}}\, F_h^{(1\boldsymbol{e}_1)} + \sum_{h=0}^{\infty} \frac{\Gamma(2g-1-h)}{(2A_1)^{2g-1-h}}\, \widetilde{F}_h^{(2\boldsymbol{e}_1)} + \cdots \tag{3.7}$$

$$F_g^{(1\boldsymbol{e}_1)} \sim \sum_{h=0}^{\infty} \left\{ \frac{\Gamma(g+1-h)}{(+A_1)^{g+1-h}}\, \widehat{F}_h^{(2\boldsymbol{e}_1)} + \frac{\Gamma(g+1-h)}{(-A_1)^{g+1-h}}\, \widehat{F}_h^{(\boldsymbol{e}_{1,1})} \right\} + \cdots \tag{3.8}$$

The two 2-instanton coefficients in the analogous slots are in fact distinct. Their being different comes from the way their ambiguities are fixed, either imposing that the holomorphic limit is zero (for $\widetilde{F}$) or that it is a particular and natural quantity that generalizes the 1-instanton case (for $\widehat{F}$). Both equations can be checked numerically. The question is

[2] Notation: $(1\boldsymbol{e}_1) = (1, 0, 0, \ldots)$, $(2\boldsymbol{e}_1) = (2, 0, 0, \ldots)$, $(\boldsymbol{e}_{1,1}) = (1, 1, 0, \ldots)$, where the first entry corresponds to A_1 and the second to $-A_1$.

then, why are there two types of 2-instanton free energy and how can we interpret them? Do both of them appear in the transseries?

With respect to this last question we can suspect that the bottom part of the resurgent diagram could be useful to provide some insight. If we manage to resum the transseries we may find that one or both of these sectors are needed to reproduce the full physical observable.

Dissappearance of A_{K} in the Borel plane. Instanton actions depend on the complex structure z and so their strength (measured in absolute value) varies as we move in the z-plane (or the ψ-plane). This means that in one place A_{K} can be dominant over (less strong than) A_1 or the other way around. The dominant one controls the large-order growth of $F_g^{(0)}$ as $g \to \infty$. Since we have analytic expressions for the actions we can predict which one will be dominant in which areas. For small ψ or large z, it should be A_{K} the dominant action but we find numerically that $F_g^{(0)}$ is controlled by A_1. How can this be? We look at the Borel plane of $F^{(0)}$, that is, we plot the singularities of the Borel transform of $F^{(0)}$ with respect to g_s (numerically using Padé approximants). Singularities are related to instanton actions. There we can see how, as we vary ψ, the singularities move. The closer to the origin the more dominant they are. We find that the singularity for A_{K} disappears when we dial ψ towards 0 even before it has the chance to become dominant. See Figure 3.4.[3] That is why we do not find it in the numerical analysis. However, we do not yet understand the mechanism controlling the disappearance, though it may be related to higher order Stokes phenomenon [16].

Stokes constants. A natural problem in resurgence is the computation of Stokes constants. For the Riccati equation one can guess the value for S_1 from the numerics or formally prove what this value is. In the approach to topological strings we have to rely on numerics alone. However, we have already used the information contained in the large-order numerics to fix the holomorphic ambiguity and there is none left to find S_1 and other Stokes constants. Equivalently, we can only have expressions for the product of the ambiguity and the Stokes constants but not for the two separately. What we can show using LO and EQ is that S_1 does not carry dependence on z or S.

[3] The perturbative free energy is an asymptotic series in g_s^2, while higher instanton sectors depend on g_s. This implies that the instanton actions come in pairs of opposite signs.

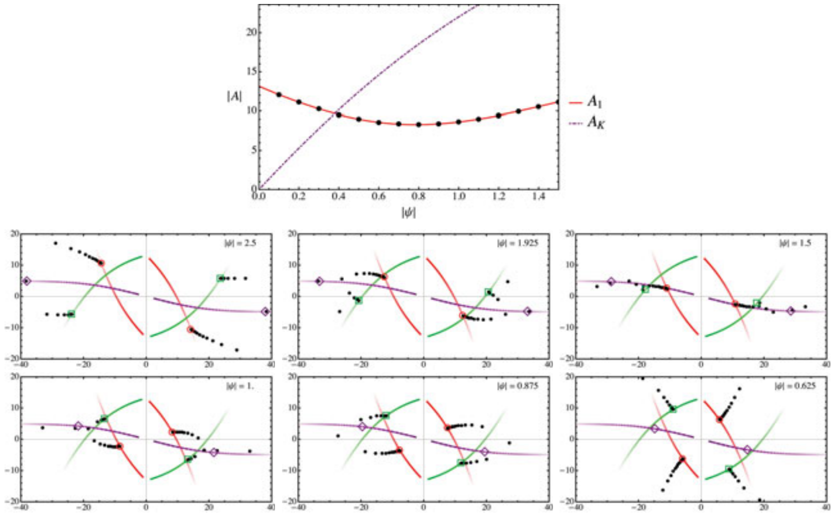

Figure 3.4. Above: the analytic expressions for A_1 and A_K, in absolute value, showing the would be transsition in dominance if A_K did not dissappear as a Borel singularity. Below: a sequence of frames showing the dissappearance of the singularity as ψ decreases. We show A_K as a purple diamond, A_1 as a red circle, and A_3 as a green square, as well as their opposite values.

4 Conclusions

In the pages above I have described the basics of what resurgence can tell us about topological string theory in the nonperturbative regime. Many details have been simplified or skipped altogether. They can be found in [7,8]. The main goal has been to discuss the ideas summarized in the resurgent approach diagram in Figure 1.1, as they are applied to topological strings.

If we had to assign just one label for this work it would probably be 'experimental mathematics', because although our target theory is topological strings and the mathematical framework is resurgence, the techniques we use can be aimed at other targets and the results obtained are suggestive rather than rigorous. Formal theorems and proofs should follow to set on firm grounds the evidence here exposed and shed light on the issues yet to be understood. And while in other parts of Physics not everything can be made rigorous the case of topological strings is singular because it sits comfortably on the boundary of Physics and Mathematics.

In this way it should be shown if the free energy is resurgent and of which kind. The resurgent relations between different coefficients should be completely understood and the Stokes constants identified. We should also aspire to understand the physical interpretation or realization of the nonperturbative data in the transseries. Avenues to explore, some of them under development, include the eventual resummation of the transseries, the relation with other nonperturbative approach to topological strings, and the possible consequences of resurgence in enumerative geometry.

The applications of resurgence in Physics have increased considerably in the past decade, but this trend will only finally take off when physicists see clear and accessible examples of the usefulness of resurgence. The exchange of ideas that took place at the conference 'Resurgence, Physics and Numbers' is a big step towards this goal.

References

[1] M. ALIM, J. D. LANGE and P. MAYR, *Global properties of topological string amplitudes and orbifold invariants*, JHEP **1003** (2010), 113.

[2] I. ANICETO, R. SCHIAPPA and M. VONK, *The resurgence of instantons in string theory*, Commun. Num. Theor. Phys. **6** (2012), 339–496.

[3] M. BERSHADSKY, S. CECOTTI, H. OOGURI and C. VAFA, "Holomorphic anomalies in topological field theories," Nucl.Phys. **B405** (1993), 279–304.

[4] M. BERSHADSKY, S. CECOTTI, H. OOGURI and C. VAFA, *Kodaira –Spencer theory of gravity and exact results for quantum string amplitudes*, Commun. Math. Phys. **165** (1994), 311–428.

[5] E. CALICETI, M. MEYER-HERMANN, P. RIBECA, A. SURZHYKOV and U. JENTSCHURA, *From useful algorithms for slowly convergent series to physical predictions based on divergent perturbative expansions*, Physics Reports **446** (2007), 1 – 96.

[6] O. COSTIN, "Asymptotics and Borel Summability", Monographs and Surveys in Pure and Applied Mathematics 141. CRC Press, 2012.

[7] R. COUSO-SANTAMARÍA, J. D. EDELSTEIN, R. SCHIAPPA and M. VONK, *Resurgent transseries and the holomorphic anomaly*, Annales Henri Poincaré (2015), 1–69, in press. arXiv:1308.1695

[8] R. COUSO-SANTAMARÍA, J. D. EDELSTEIN, R. SCHIAPPA and M. VONK, *Resurgent transseries and the holomorphic anomaly: nonperturbative closed strings in local* $\mathbb{CP}^2$, Commun. Math. Phys. **338** (2015), 285–346.

[9] R. COUSO-SANTAMARÍ A, M. MARIÑO and R. SCHIAPPA, *Resurǵence matches quantization*, J. Physics A: Math. Theor. **50** (2017), 14, 145402.

[10] N. DRUKKER, M. MARIÑO and P. PUTROV, *Nonperturbative aspects of ABJM theory*, JHEP **1111** (2011), 141.

[11] G. V. DUNNE, "Resurgence and Non-Perturbative Physics.". Lectures at CERN Winter School on Supergravity, Strings, and Gauge Theory, 2014.

[12] J. ÉCALLE, *Les fonctions résurgentes*, Publ. Math. d'Orsay, **81-05** (1981), **81-06** (1981), **85-05** (1985).

[13] G. A. EDGAR , *Transseries for beginners*, Real Anal. Exchange **35** (2009), 253–310.

[14] S. GAROUFALIDIS, A. ITS, A. KAPAEV and M. MARIÑO, *Asymptotics of the instantons of Painlevé I*, Int. Math. Res. Not. **2012** (2012), 561–606.

[15] B. HAGHIGHAT, A. KLEMM and M. RAUCH, *Integrability of the holomorphic anomaly equations*, JHEP **0810** (2008), 097.

[16] C. J. HOWLS, P. J. LANGMAN and A. B. OLDE DAALHUIS, *On the higher-order Stokes phenomenon*, In: "Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences", The Royal Society, Vol. 460, 2004, 2285–2303.

[17] K. HORI, S. KATZ, A. KLEMM, R. PANDHARIPANDE, R. THOMAS, C. VAFA, R. VAKIL and E. ZASLOW, "Mirror Symmetry" Clay Mathematics Monographs, Vol. 1, 2003.

[18] M. MARIÑO, *Spectral theory and mirror symmetry*, arXiv:1506.07757v4.

[19] M. MARINO, *Les Houches lectures on matrix models and topological strings*, arXiv:hep-th/0410165.

[20] C. MITSCHI and D. SAUZIN, "Divergent series, summability and resurgence. I. Monodromy and resurgence", with a foreword by Jean-Pierre Ramis and a preface by Éric Delabaere, Michèle Loday-Richaud, Claude Mitschi and David Sauzin, Lecture Notes in Mathematics, Vol. 2153, Springer, [Cham], 2016, xxi + 298 pp.

[21] M. VONK, *Resurgence and topological strings*, In: "String Math.", 2014 Edmonton, Alberta, Canada, June 9-13, 2014, 221.