Invariants of identity-tangent diffeomorphisms expanded as series of multitangents and multizetas

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Abstract. We return to the subject of local, identity-tangent diffeomorphisms f of $\mathbb{C}$ and their analytic invariants $A_\omega(f)$, under the complementary viewpoints of effective computation and explicit expansions. The latter rely on two basic ingredients: the so-called multizetas (transcendental numbers) and multitangents (transcendental functions), with resurgence monomials and their monics providing the link between the two. We also stress the difference between the collectors (pre-invariant but of one piece) and the connectors (invariant but mutually unrelated).

Much attention has been paid to streamlining the nomenclature and notations. On the analysis side, resurgence theory rules the show. On the algebraic or combinatorial side, mould theory brings order and structure into the profusion of objects. Along the way, the paper introduces quite a few novel notions: new alien operators, new forms of resurgence, new symmetry types for moulds. It also broaches the subject of 'phantom dynamics' (dealing with formal diffeos that nonetheless possess invariants $A_\omega(f)$) and culminates in the comparison of arithmetical and dynamical monics, a distinction that reflects the dual nature of the $A_\omega(f)$ as Stokes constants and holomorphic invariants.

Contents **109**

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1 Setting and notations

1.1 Introduction

The holomorphic invariants of identity-tangent diffeomorphisms are a long-established subject. Awareness of their existence is as old as the hills. It goes back at least 120 years, to Fatou's geometric treatment [11]. The sharper-edged resurgent treatment, which yields a wealth of information denied to the geometric approach, is not exactly new either: it was laid out in full in [4] and [5], in the late seventies.

What is sorely lacking, however, is a realisation that these invariants can be accurately described and explicitely calculated. Indeed, the prevailing (if seldom clearly stated) opinion in the holomorphic dynamics community appears to be that *they cannot*. With a view to correcting this misapprehension, we posted in 2012 a short paper[1] that showed otherwise. Though it contained little that was strictly new (in the main, it restated results already extant in decades-old papers like [3] or [5], and referred for the computational programs to a recent PhD thesis [1]),

[1] *Invariants of identity-tangent diffeomorphisms: explicit formulae and effective computation*. The paper with the appended tables can be accessed online on
< http://www.math.u-psud.fr/∼ecalle/fichiersweb/WEB_iden_tang_0.pdf >.

such feedback as we received convinced us that these questions were still dimly understood, and in need of a more thorough exposition.

So, with something of a sinking heart, we set about re-revisiting the whole subject. Since we were at it, however, and given that *ter repetita non placent*, we felt that we might just as well insert some new material. These extras include:

(1) a procedure for the 'uniformisation' of convolution products and powers in the Borel plane, leading to optimal bounds;
(2) a new class of alien operators, the *medial operators* $\Delta^{\sharp}_{\omega}$ and $\Delta^{\sharp\sharp}_{\omega}$, which do not obey the Leibniz rule but make up for it by having a simpler definition and being easier to evaluate;
(3) the notion of *affiliates* of a diffeomorphism f, defined via the corresponding substitution operators F and their images $\gamma(F-1)$ under an analytic γ;
(4) a new class of mouldian *symmetry types*, of proven usefulness, and the rather intriguing combinatorics that goes with them;
(5) special classes of *multizetas* and *multitangents* well-suited for expressing the invariants $A_{\omega}(f)$ and bringing out their parity properties;
(6) the distinction between the semi-invariant *collectors*, which carry the multitangents, and the exactly invariant *connectors*, which carry the multizetas;
(7) the distinction between the full *arithmetical constraints* on the multizetas and the weaker *dynamical constraints*, which are responsible for *making the invariants invariant*;
(8) the complications specific to the *ramified case* (for diffeos f of tangency order $p \geq 2$), which call for new monics related to, yet distinct from, the rational-indexed multizetas;
(9) the subject of *phantom dynamics* which deals with groups of formal diffeos that nonetheless possess holomorphic invariants and for which many of the key notions familiar from holomorphic dynamics (sectorial models, connectors, Fourier analysis, etc) still make sense, albeit in a new setting, with *acceleration operators* replacing Laplace integration.

1.2 Classical results

We shall be concerned here with *local*[2] *identity-tangent diffeomorphisms* of $\mathbb{C}$, or *diffeos* for short, with the fixed-point located at ∞ for technical

[2] *I.e. analytic germs of –*

convenience:

$$f \ : \quad z \mapsto z + \sum_{1 \le s} f_s \, z^{1-s} \qquad\qquad f_s \in \mathbb{C}\,. \tag{1.1}$$

Unless f be the identity map, we can always subject it to an analytic (respectively formal) conjugation $f \mapsto f_1 = h \circ f \circ h^{-1}$, followed if necessary by an elementary ramification $\left(f_1(z^{1/p})\right)^p$, so as to give f the following *prepared* (respectively *normal*) form:

$$f_{\text{prep}} : \ \ z \mapsto z + 1 - \rho z^{-1} + \sum_{2<s_0\le s} f_{[s]}\, z^{1-s} \qquad \left(s \in \frac{1}{p}\mathbb{N}^*\right) \tag{1.2}$$

$$f_{\text{norm}} : \ \ z \mapsto z + 1 - \rho z^{-1} \tag{1.3}$$

where s_0 may be chosen as large as one wishes.[3]

The *tangency order* p and *iteration residue* ρ are the only *formal invariants* of identity-tangent diffeos. But our diffeos also possess countably many (independent) scalar *analytic invariants*, also known as *holomorphic invariants*,[4] which are best defined as the Fourier coefficients of the so-called *connectors*.[5] The connectors are pairs of germs of 1-periodic analytic mappings $\boldsymbol{\pi} = (\boldsymbol{\pi}_{\text{no}}, \boldsymbol{\pi}_{\text{so}})$ defined on the upper/lower half-planes $\pm\Im(z) \gg 1$. There are p such pairs, corresponding to the p-fold ramification of z in (1.2). Here, *no* and *so* stand for *north* and *south*, *i.e.* the upper and lower half-planes.

We shall throughout prioritise the *standard case* $p = 1$, $\rho = 0$, *i.e.* focus on diffeos of the form:

$$f := l \circ g \ \textit{with} \ l := z \mapsto z + 1 \ \textit{and} \ g \ : \ z \mapsto z + \sum_{3\le s} g_s\, z^{1-s} \tag{1.4}$$

and merely sketch the changes required to cover the general case.

Any standard f possesses two well-defined, mutually inverse so-called *iterators*, to wit $f^*_\pm$ (direct iterator) and ${}^*f_\pm$ (reciprocal iterator), defined

[3] After 'preparation', the diffeo acquires new coefficients denoted $f_{[s]}$ for distinctiveness.

[4] *Analytic invariants* means invariant relative to analytic changes of z-coordinate, whereas *holomorphic invariant* points to the holomorphic dependence of $A_\omega(f)$ in f – in contradistinction to cases like that of diffeos with Liouvillian multipliers λ. Such diffeos do possess non-trivial *analytic invariants*, but none with holomorphic dependence on f.

[5] In the context of identity-tangent diffeos, the connectors are sometimes referred to as *horn maps*, but the former notion is more general: in resurgent analysis (see Section 1.2 *infra*) the connectors are the operators that take us from one sectorial model to the next.

on U-shaped domains[6] by the limits:

$$f^*_\pm(z) = \lim_{k\to\pm\infty} l^{-k} \circ f^k \quad ; \quad {}^*f_\pm(z) = \lim_{k\to\pm\infty} f^{-k} \circ l^k. \tag{1.5}$$

The connectors $\boldsymbol{\pi}^{\pm 1}$, with their northern and southern components, are then defined on $\pm\Im(z) \gg 1$ by:

$$\boldsymbol{\pi} := f^*_+ \circ {}^*f_- \quad ; \quad \boldsymbol{\pi}^{-1} := f^*_- \circ {}^*f_+ \,. \tag{1.6}$$

For reasons that will soon become apparent, we must also consider the infinitesimal generators f_* and $\boldsymbol{\pi}_*$ of f and $\boldsymbol{\pi}$. These are formal, generically divergent power respectively Fourier series. Of course, $\boldsymbol{\pi}_*$ is not constructed directly from $\boldsymbol{\pi}$, but via its northern and southern components. We thus have the three pairs:

$$\boldsymbol{\pi} := (\boldsymbol{\pi}_{\text{no}}, \boldsymbol{\pi}_{\text{so}}) \;;\; \boldsymbol{\pi}^{-1} := (\boldsymbol{\pi}^{-1}_{\text{no}}, \boldsymbol{\pi}^{-1}_{\text{so}}) \;;\; \boldsymbol{\pi}_* := (\boldsymbol{\pi}_{*\text{no}}, \boldsymbol{\pi}_{*\text{so}}) \tag{1.7}$$

along with the relations

$$f(z) = \exp\big(f_*(z)\,\partial_z\big).\,z \qquad \big(\, f_*\,\partial_z f^* \equiv 1\,\big) \tag{1.8}$$

$$\boldsymbol{\pi}^{\pm 1}_{\text{no}}(z) = \exp\big(\,\pm\,\boldsymbol{\pi}_{*\text{no}}(z)\,\partial_z\big).\,z \tag{1.9}$$

$$\boldsymbol{\pi}^{\pm 1}_{\text{so}}(z) = \exp\big(\,\pm\,\boldsymbol{\pi}_{*\text{so}}(z)\,\partial_z\big).\,z\,. \tag{1.10}$$

In (1.8) f^* and ${}^*f$ denote of course the *formal iterators*, *i.e.* the power series solutions of the equations

$$f^* \circ f = l \circ f^* \quad \textit{with} \quad f^*(z) = z + o(1) \tag{1.11}$$

$$f \circ {}^*f = {}^*f \circ l \quad \textit{with} \quad {}^*f(z) = z + o(1) \tag{1.12}$$

normalised by the condition of carrying no constant term. Anticipating on the sequel, here is how the scalar invariants can be read off the Fourier expansions of the connectors:

$$\boldsymbol{\pi}_{\text{no}}(z) = z + \sum_{\omega\in\Omega^-} A^+_\omega\, e^{-\omega z} \;;\; \boldsymbol{\pi}_{\text{so}}(z) = z + \sum_{\omega\in\Omega^+} A^-_\omega\, e^{-\omega z} \tag{1.13}$$

$$\boldsymbol{\pi}^{-1}_{\text{no}}(z) = z + \sum_{\omega\in\Omega^-} A^-_\omega\, e^{-\omega z} \;;\; \boldsymbol{\pi}^{-1}_{\text{so}}(z) = z + \sum_{\omega\in\Omega^+} A^+_\omega\, e^{-\omega z} \tag{1.14}$$

$$\boldsymbol{\pi}_{*\text{no}}(z) = +2\pi i \sum_{\omega\in\Omega^-} A_\omega\, e^{-\omega z} \;;\; \boldsymbol{\pi}_{*\text{so}}(z) = -2\pi i \sum_{\omega\in\Omega^+} A_\omega\, e^{-\omega z}\,. \tag{1.15}$$

[6] f^*_+ and ${}^*f_+$ are defined on a west-north-south domain, while f^*_- and ${}^*f_-$ are defined on an east-north-south domain.

Pay attention to the altered position of $\pm$ in 1.13 and 1.14; the reasons for this apparent incoherence shall become clear in due course. The indices ω run through $\Omega := 2\pi i \mathbb{Z}^*$ or $\Omega^\pm := \pm 2\pi i\, \mathbb{N}^*$, and each of the three systems

$$\{A^+_\omega\,,\ \omega\in\Omega\} \qquad , \qquad \{A^-_\omega\,,\ \omega\in\Omega\} \qquad , \qquad \{A_\omega\,,\ \omega\in\Omega\} \quad (1.16)$$

constitutes a *free* and *complete* system of analytic invariants.[7]

1.3 Affiliates. Generators and mediators

General affiliates. To each identity-tangent germ f and each power series $\gamma(t) = t + \sum \gamma_r\, t^{r+1}$ we associate the so-called γ-affiliate $f_\diamondsuit$ along with an infinite-order differential operator $F_\diamondsuit$. The correspondence $(f, F) \mapsto (f_\diamondsuit, F_\diamondsuit)$ goes like this:

$$f \mapsto f_\diamondsuit := F_\diamondsuit \,.\, z \qquad ; \qquad F \mapsto F_\diamondsuit := \gamma(F-1)\,. \qquad (1.17)$$

For a general γ, the operator $F_\diamondsuit$ has a non-elementary coproduct:

$$\mathrm{cop}(F_\diamondsuit) := F_\diamondsuit \oplus 1 + 1 \oplus F_\diamondsuit + \sum_{1\le p,q} \gamma^{[p,q]}\,(F_\diamondsuit)^p \oplus (F_\diamondsuit)^q\,. \quad (1.18)$$

As a consequence, the straightforward germ-to-operator correspondence:

$$f \mapsto F = 1 + \sum_{1\le n} (\underline{f})^n\, \frac{\partial^n}{n!} \qquad (\underline{f}(z) := f(z) - z) \qquad (1.19)$$

assumes a more intricate form for the affiliates:

$$f_\diamondsuit \mapsto F_\diamondsuit = \underline{f_\diamondsuit}\,\partial + \sum_{2\le r}\ \sum_{1\le n_i, 2\le n_r} \diamondsuit^{n_1,\dots,n_r}\, (\underline{f_\diamondsuit})^{n_1}\,\frac{\partial^{n_1}}{n_1!} \dots (\underline{f_\diamondsuit})^{n_r}\,\frac{\partial^{n_r}}{n_r!}\,. \quad (1.20)$$

Special affiliates: generators and mediators. The structure coefficients $\gamma^{[p,q]}$ and $\diamondsuit^{n_1,\dots,n_r}$ shall be investigated in Section 5-1, Section 5-2 and Section 5-4, but they assume a particularly simple form for three special types of affiliates:

(i) the infinitesimal *generator* (f_*, F_*) with $\gamma(t) = \log(1+t)$;
(ii) the main *mediator* $(f_\natural, F_\natural)$ with $\gamma(t) = 2\,\frac{(1+t)-1}{(1+t)+1} = \frac{t}{1+\frac{1}{2}t}$;
(iii) the second *mediator* $(f_{\natural\natural}, F_{\natural\natural})$ with $\gamma(t) = \frac{(1+t)^2-1}{(1+t)^2+1} = \frac{t+\frac{1}{2}t^2}{1+t+\frac{1}{2}t^2}$.

[7] With the minor qualifier that, under a conjugation by a shift h of the form $l^\alpha(z) := z+\alpha$, the periodic germs $\boldsymbol{\pi}^\pm$ also undergo conjugation by the same shift, with obvious repercussions for their Fourier coefficients.

The *generators* we have already mentioned. For them, the co-product and the germ-to-operator correspondence reduce to

$$\mathrm{cop}(F_*) = F_* \otimes 1 + 1 \oplus F_* \quad , \quad f \mapsto F_* = f_* \,\partial\,. \tag{1.21}$$

For the *mediators*, the formulae, while still relatively simple, become more interesting

$$\begin{aligned} \mathrm{cop}(F_\sharp) &= F_\sharp \otimes 1 + 1 \otimes F_\sharp \\ &\quad + \sum_{1\le n} \left(-\frac{1}{4}\right)^n \left(F_\sharp^{n+1} \otimes F_\sharp^n + F_\sharp^n \otimes F_\sharp^{n+1}\right) \qquad (1.22) \\ \mathrm{cop}(F_{\sharp\sharp}) &= F_{\sharp\sharp} \otimes 1 + 1 \otimes F_{\sharp\sharp} \\ &\quad + \sum_{1\le n} (-1)^n \left(F_{\sharp\sharp}^{n+1} \otimes F_{\sharp\sharp}^n + F_{\sharp\sharp}^n \oplus F_{\sharp\sharp}^{n+1}\right). \qquad (1.23) \end{aligned}$$

Relating F and $F_\sharp$, $F_{\sharp\sharp}$. As *operators*, the mediators $F_\sharp$ and $F_{\sharp\sharp}$ admit three distinct types of expansions, each with its own merits and drawbacks:

$$F_\sharp = 2\,\frac{F-1}{F+1} = 2\,\mathcal{C}_\sharp\,\mathcal{D}_\sharp^{-1} = 2\,\mathcal{D}_{[\sharp]}^{-1}\,\mathcal{C}_{[\sharp]} \tag{1.24}$$

$$F_{\sharp\sharp} = \frac{F-F^{-1}}{F+F^{-1}} = \mathcal{C}_{\sharp\sharp}\,\mathcal{D}_{\sharp\sharp}^{-1} = \mathcal{D}_{[\sharp\sharp]}^{-1}\,\mathcal{C}_{[\sharp\sharp]}\,. \tag{1.25}$$

The operators $\mathcal{C}_\sharp, \mathcal{D}_\sharp, \mathcal{C}_{\sharp\sharp}, \mathcal{D}_{\sharp\sharp}$ are defined as follows:

$$\begin{aligned} \mathcal{C}_\sharp &= \sum_{1\le n}^{n \text{ odd}} 2^{-n} f_\sharp^n \frac{\partial^n}{n!} \;\; || \; \mathcal{C}_\sharp : \varphi(z) \mapsto \tfrac{1}{2}\Big(\varphi(z+\tfrac{1}{2}f_\sharp(z)) - \varphi(z-\tfrac{1}{2}f_\sharp(z))\Big) \\ \mathcal{D}_\sharp &= 1 + \sum_{1\le n}^{n \text{ even}} 2^{-n} f_\sharp^n \frac{\partial^n}{n!} \;\; || \; \mathcal{C}_\sharp : \varphi(z) \mapsto \tfrac{1}{2}\Big(\varphi(z+\tfrac{1}{2}f_\sharp(z)) + \varphi(z-\tfrac{1}{2}f_\sharp(z))\Big) \\ \mathcal{C}_{\sharp\sharp} &= \sum_{1\le n}^{n \text{ odd}} f_\sharp^n \frac{\partial^n}{n!} \;\; || \mathcal{C}_{\sharp\sharp} : \varphi(z) \mapsto \tfrac{1}{2}\Big(\varphi(z+f_{\sharp\sharp}(z)) - \varphi(z-f_{\sharp\sharp}(z))\Big) \\ \mathcal{D}_{\sharp\sharp} &= 1 + \sum_{1\le n}^{n \text{ even}} f_{\sharp\sharp}^n \frac{\partial^n}{n!} \;\; || \mathcal{C}_{\sharp\sharp} : \varphi(z) \mapsto \tfrac{1}{2}\Big(\varphi(z+f_{\sharp\sharp}(z)) + \varphi(z-f_{\sharp\sharp}(z))\Big)\,. \end{aligned}$$

The operators $\mathcal{C}_{[\sharp]}, \mathcal{D}_{[\sharp]}, \mathcal{C}_{[\sharp\sharp]}, \mathcal{D}_{[\sharp\sharp]}$ are defined in exactly the same way, but relative to inputs $f_{[\sharp]}$, $f_{[\sharp\sharp]}$ with $f_\sharp(z) \sim f_{\sharp\sharp}(z) \sim f_{[\sharp]}(z) \sim f_{[\sharp\sharp]}(z) \sim f(z)-z$. As operators acting on formal germs, $\mathcal{D}_\sharp^{-1}$ and $\mathcal{D}_{\sharp\sharp}^{-1}$ have to be expanded in the predictable way, leading to formulae such as:

$$f_\sharp \mapsto F_\sharp = f_\sharp\,\partial + \sum_{\substack{1\leq r\\ \left(n_1\ odd \atop n_2,..,n_r\ even\right)}} (-1)^{r-1} 2^{1-\sum n_i} f_\sharp^{n_1} \frac{\partial^{n_1}}{n_1!} f_\sharp^{n_2} \frac{\partial^{n_2}}{n_2!} \dots f_\sharp^{n_r} \frac{\partial^{n_r}}{n_r!} \quad (1.26)$$

$$f_{\sharp\sharp} \mapsto F_{\sharp\sharp} = f_{\sharp\sharp}\,\partial + \sum_{\substack{1\leq r\\ \left(n_1\ odd \atop n_2,..,n_r\ even\right)}} (-1)^{r-1}\ f_{\sharp\sharp}^{n_1} \frac{\partial^{n_1}}{n_1!} f_{\sharp\sharp}^{n_2} \frac{\partial^{n_2}}{n_2!} \dots f_{\sharp\sharp}^{n_r} \frac{\partial^{n_r}}{n_r!}. \quad (1.27)$$

Let us focus on the *second* mediator $F_{\sharp\sharp}$, to avoid the nuisance of the factors $(1/2)^n$. Its first expansion $F_{\sharp\sharp} = \frac{F-F^{-1}}{F+F^{-1}}$ is wholly unproblematic, with a commuting numerator and denominator, and simply reflects the definition of $F_{\sharp\sharp}$. The *existence* of parallel expansions $\mathcal{C}_{\sharp\sharp}\,\mathcal{D}_{\sharp\sharp}^{-1}$ and $\mathcal{D}_{[\sharp\sharp]}^{-1}\,\mathcal{C}_{[\sharp\sharp]}$ follows, to put it briefly, from the fact that the operators

$$\mathcal{C}_{\sharp\sharp}\ and\ \mathcal{C}_{[\sharp\sharp]}\ ,\ \mathcal{D}_{\sharp\sharp}\ and\ \mathcal{D}_{[\sharp\sharp]}\ ,\ \mathcal{D}_{\sharp\sharp}^{-1}\ and\ \mathcal{D}_{[\sharp\sharp]}^{-1}\ ,\ \mathcal{C}_{\sharp\sharp}\mathcal{D}_{\sharp\sharp}^{-1}\ and\ \mathcal{D}_{[\sharp\sharp]}^{-1}\mathcal{C}_{[\sharp\sharp]}$$

verify exactly the same types of co-product as, respectively, the operators

$$\sinh(\partial)\quad,\quad\cosh(\partial)\quad,\quad\cosh(\partial)^{-1}\quad,\quad\tanh(\partial)$$

and from the fact that $\tanh(\partial)$ has precisely a co-product of type (1.23). But since numerators and denominators no longer commute, we should expect the inputs $f_{\sharp\sharp}$ and $f_{[\sharp\sharp]}$ to differ, in a way that remains to elucidate.

For the moment, let us observe that, of the latter two expansions, $F_{\sharp\sharp} = \mathcal{C}_{\sharp\sharp}\,\mathcal{D}_{\sharp\sharp}^{-1}$ is the more useful, since it allows us to express the *operatorial* mediator $F_{\sharp\sharp}$ directly in terms of the germ $f_{\sharp\sharp} := F_{\sharp\sharp}.z$. But the other expansion, namely $F_{\sharp\sharp} = \mathcal{D}_{[\sharp\sharp]}^{-1}\,\mathcal{C}_{[\sharp\sharp]}$, has its merits too, since it relies on a germ $f_{[\sharp\sharp]}$ which, as we shall see in a moment, is 'closer' than $f_{\sharp\sharp}$ to the original f and, unlike $f_{\sharp\sharp}$, converges whenever f does. It is also more economical than the first expansion $F_{\sharp\sharp} = \frac{F-F^{-1}}{F+F^{-1}}$, in the sense of concentrating all the odd or even terms respectively in the numerator and denominator.

Relating $f_\sharp$, $f_{\sharp\sharp}$ to f. Equating the first two expansions of the mediators, we get

$$(F+1)\,\mathcal{C}_\sharp\,\mathcal{D}_\sharp{-1} = F-1 \quad \text{an} \quad (F^2+1)\,\mathcal{C}_\sharp\,\mathcal{D}_\sharp{-1} = F^2-1\,.$$

Letting these operators act on z, we find the sought-for relations

$$f_\sharp(f(z)) + f_\sharp(z) = f(z) - z \quad (1.28)$$

$$f_{\sharp\sharp}(f(z)) + f_{\sharp\sharp}(f^{-1}(z)) = f(z) - f^{-1}(z)\,. \quad (1.29)$$

Relating $f_{[\sharp]}$, $f_{[\sharp\sharp]}$ to f. Inverting the definition-based expansion of the mediators, we get successively

$$F-1=(1-(1/2)\,F_{\sharp})^{-1}\,F_{\sharp} \text{ and } F^2-1=2(1-F_{\sharp\sharp})^{-1}\,F_{\sharp\sharp}$$
$$(1-(1/2)\,F_{\sharp})(F-1)=F_{\sharp} \text{ and } (1-F_{\sharp\sharp})(F^2-1)=2\,F_{\sharp\sharp}$$
$$(1-\mathcal{D}_{[\sharp]}^{-1}\mathcal{C}_{[\sharp]})(F-1)=2\mathcal{D}_{[\sharp]}^{-1}\mathcal{C}_{[\sharp]} \text{ and } (1-\mathcal{D}_{[\sharp\sharp]}^{-1}\mathcal{C}_{[\sharp\sharp]})(F^2-1)=2\mathcal{D}_{[\sharp\sharp]}^{-1}\mathcal{C}_{[\sharp\sharp]}$$
$$(\mathcal{D}_{[\sharp]}-\mathcal{C}_{[\sharp]})\,F=(\mathcal{D}_{[\sharp]}+\mathcal{C}_{[\sharp]}) \text{ and } (\mathcal{D}_{[\sharp\sharp]}-\mathcal{C}_{[\sharp\sharp]})\,F^2=(\mathcal{D}_{[\sharp\sharp]}+\mathcal{C}_{[\sharp\sharp]}).$$

Finally, letting the operators act on z, we get:

$$f\left(z-\frac{1}{2}\,f_{[\sharp]}\right) = z+\frac{1}{2}\,f_{[\sharp]} \tag{1.30}$$
$$f^{\circ 2}(z-f_{[\sharp\sharp]}) = z+f_{[\sharp\sharp]}. \tag{1.31}$$

This implies, first, that the germs $z \mapsto z-\frac{1}{2}f_{[\sharp]}$ and $z \mapsto z-f_{[\sharp\sharp]}$ are respectively reciprocal to the germs $z \mapsto \frac{1}{2}(z+f(z))$ and $z \mapsto \frac{1}{2}(z+f^{\circ 2}(z))$ and, second, that $f_{[\sharp]}$ and $f_{[\sharp\sharp]}$ are convergent if and only if f is.

Relating $f_{\sharp}$, $f_{\sharp\sharp}$ and $f_{[\sharp]}$, $f_{[\sharp\sharp]}$. Post-composing the identies (1.28)-(1.29) by the germs $z-(1/2)\,f_{[\sharp]}(z)$ or $z-f_{[\sharp\sharp]}(z)$ and using (1.30)-(1.31) to eliminate f, we find:

$$2\,f_{[\sharp]}(z) = f_{\sharp}\left(z+\frac{1}{2}\,f_{[\sharp]}(z)\right)+f_{\sharp}\left(z-\frac{1}{2}\,f_{[\sharp]}(z)\right) \tag{1.32}$$
$$2\,f_{[\sharp\sharp]}(z) = f_{\sharp\sharp}(z+f_{[\sharp\sharp]}(z))+f_{\sharp\sharp}\big(z-f_{[\sharp\sharp]}(z)\big). \tag{1.33}$$

After some non-commutative manipulations on differential operators and their generating series, this yields:

$$f_{[\sharp]} = f_{\sharp}+\sum_{1\leq s}\sum_{1\leq m_i}\frac{(\sum 2m_i)!4^{-\sum m_i}}{s!(1-s+\sum 2m_i)!}\,f_{\sharp}^{1-s+2\sum m_i}\prod_{1\leq i\leq s}f_{\sharp}^{(2m_i)} \tag{1.34}$$
$$f_{[\sharp\sharp]} = f_{\sharp\sharp}+\sum_{1\leq s}\sum_{1\leq m_i}\frac{(\sum 2m_i)!}{s!(1-s+\sum 2m_i)!}\,f_{\sharp\sharp}^{1-s+2\sum m_i}\prod_{1\leq i\leq s}f_{\sharp\sharp}^{(2m_i)} \tag{1.35}$$

1.4 Brief reminder about resurgent functions

We will have to be content here with a very sketchy presentation. The algebra of *resurgent fonctions* admits three different realisations or models:

(i) the *formal model*, consisting of formal power series $\widetilde{\varphi}(z)$ in z^{-1} or of more general *transseries*;[8]
(ii) the *convolutive model*, consisting of microfunctions[9] at $\zeta = 0$, whose *majors* $\check{\varphi}(\zeta)$ are defined at the origin only and constraint-free but whose *minors* $\hat{\varphi}(\zeta)$ have the property of endless continuation[10] and exponential growth;[11]
(iii) the *geometric model(s)*, consisting of analytic germs $\varphi_\theta(z)$ defined on sectorial neighbourhoods of ∞ of bisectrix $arg(z^{-1}) = \theta$ and aperture at least π.

The natural algebra product in the z-models (i) and (iii) is of course multiplication. In the ζ-model (ii) it is convolution, defined first *locally*[12] by

$$(\hat{\varphi}_1 * \hat{\varphi}_2)(\zeta) := \int_0^\zeta \hat{\varphi}_1(\zeta_1)\,\hat{\varphi}_2(\zeta - \zeta_1)\,d\zeta_1 \qquad (\zeta \sim 0) \tag{1.36}$$

and then *in the large* by analytic continuation.

In practice, one starts with elements $\widetilde{\varphi}$ of model (i) obtained as formal solutions of differential or functional equations, and the aim is to resum them, *i.e.* to go to model (iii). Generally speaking, this is possible only over the detour through model (ii), with the *formal Borel tranform* $\mathcal{B}$

$$z^{-\sigma} \mapsto \frac{\zeta^{\sigma-1}}{\Gamma(\sigma)} \quad ; \quad (\partial_\sigma)^n z^{-\sigma} \mapsto (\partial_\sigma)^n \frac{\zeta^{\sigma-1}}{\Gamma(\sigma)} \quad ; \quad etc \tag{1.37}$$

taking us from (i) to (ii), and the *polarised Laplace transform* $\mathcal{L}_\theta$

$$\varphi_\theta(z) = \int_{\arg(\zeta)=\theta} \widehat{\varphi}(\zeta)\, e^{-\zeta z}\, d\zeta \tag{1.38}$$

taking us from (ii) to (iii).

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[8] The tilda stands for 'formal', but will be omitted in contexts where everything is formal.

[9] *I.e.* minor-major pairs $(\hat{\varphi}(\zeta), \check{\varphi}(\zeta))$. The *majors* are defined up to regular germs at the origin, and the *minors* are related to them under $2\pi i\,\hat{\varphi}(\zeta) \equiv \check{\varphi}(\zeta\, e^{-\pi i}) - \check{\varphi}(\zeta\, e^{+\pi i})$ for $\zeta \sim 0$. In the present paper, we shall almost entirely dispense with majors, since we shall mostly be dealing with so-called *integrable* microfunctions, whose minors carry the complete information.

[10] Laterally along any *finite and finitely punctured* broken lines.

[11] *I.e. at most exponential*, along *infinite but finitely punctured broken lines*, with a suitable uniformity condition.

[12] When the *minors* $\hat{\varphi}$ are not integrable at the origin, one must modify the definition and draw in the *majors* $\check{\varphi}$. Convolution is then defined on loop integrals that avoid the origin.

The most outstanding feature of the resurgence algebras is the existence on them of a rich array of so-called *alien operators* Δ_ω and $\Delta^{\pm}_\omega$, with indices ω running through $\mathbb{C}_\bullet := \widetilde{\mathbb{C} - \{0\}}$. These operators act on all three models[13], but are first defined in the convolutive model, where they have the effect of measuring the singularities of the (often highly ramified) minors $\hat{\varphi}$ *at* or rather *over* ω. Here is how they act:

$$(\widehat{\Delta}_\omega \hat{\varphi})(\zeta) := \sum_{\epsilon_1,\ldots,\epsilon_r} \frac{\epsilon_r}{2\pi i}\, \lambda_{\epsilon_1,\ldots,\epsilon_{r-1}}\; \widehat{\varphi}^{\binom{\epsilon_1\ ,\ldots,\ \epsilon_r}{\omega_1\ ,\ldots,\ \omega_r}}(\omega+\zeta) \tag{1.39}$$

$$(\widehat{\Delta}^{\pm}_\omega \hat{\varphi})(\zeta) := \sum_{\epsilon_1,\ldots,\epsilon_r} \pm\epsilon_r\, \lambda^{\pm}_{\epsilon_1,\ldots,\epsilon_{r-1}}\; \widehat{\varphi}^{\binom{\epsilon_1\ ,\ldots,\ \epsilon_r}{\omega_1\ ,\ldots,\ \omega_r}}(\omega+\zeta) \tag{1.40}$$

with $\omega_r = \omega$, with signs $\epsilon_j \in \{+,-\}$, with weights $\lambda_{\boldsymbol{\epsilon}}, \lambda^{+}_{\boldsymbol{\epsilon}}, \lambda^{-}_{\boldsymbol{\epsilon}}$ defined by

$$\lambda_{\epsilon_1,\ldots,\epsilon_{r-1}} := \frac{p!\,q!}{r!} \quad \textit{with} \quad p := \sum_{\epsilon_i=+} 1\ ,\ q := \sum_{\epsilon_i=-} 1 \tag{1.41}$$

$$\begin{aligned} \lambda^{\epsilon}_{\epsilon_1,\ldots,\epsilon_{r-1}} &:= 1 \quad \textit{if} \quad \epsilon_1 = \cdots = \epsilon_{r-1} = \epsilon \\ &:= 0 \quad \textit{otherwise} \end{aligned} \tag{1.42}$$

and with $\widehat{\varphi}^{\binom{\epsilon_1\ ,\ldots,\ \epsilon_r}{\omega_1\ ,\ldots,\ \omega_r}}(\omega+\zeta)$ denoting the analytic continuation of $\widehat{\varphi}$ from ζ to $\omega+\zeta$ under right (respectively left) circumvention of each intervening singularity ω_j if $\epsilon_j = +$ (respectively $\epsilon_j = -$). We start of course with a point ζ close enough to 0 on the axis $arg(\zeta) = arg(\omega)$, and extend the definition in the large by analytic continuation. The operators $\widehat{\Delta}_\omega$ and their pull-backs Δ_ω in the formal model are *derivations*. This means that in the convolutive or formal models the Leibniz identities hold:

$$\widehat{\Delta}_\omega(\widehat{\varphi}_1 * \widehat{\varphi}_2) = \widehat{\Delta}_\omega(\widehat{\varphi}_1) * \widehat{\varphi}_2 + \hat{\varphi}_1 * \widehat{\Delta}_\omega(\widehat{\varphi}_2) \tag{1.43}$$

$$\Delta_\omega(\widetilde{\varphi}_1\ .\ \widetilde{\varphi}_2) = \Delta_\omega(\widetilde{\varphi}_1)\ .\ \widetilde{\varphi}_2 + \widetilde{\varphi}_1\ .\ \Delta_\omega(\widetilde{\varphi}_2) \tag{1.44}$$

When working in any one of the multiplicative models (formal or geometric), it is often convenient to phase-shift the alien operators, and to set:

$$\boldsymbol{\Delta}_\omega := e^{-\omega z}\Delta_\omega \qquad (\ [\partial_z, \boldsymbol{\Delta}_\omega] \equiv 0\ ) \tag{1.45}$$

$$\boldsymbol{\Delta}^{\pm}_\omega := e^{-\omega z}\Delta^{\pm}_\omega \qquad (\ [\partial_z, \boldsymbol{\Delta}^{\pm}_\omega] \equiv 0\ ) \tag{1.46}$$

The gain here is that the new operators commute with ∂_z. These phase-shifted operators are also the natural ingredients of the *axial operators*

[13] With the same symbols doing service in all three, since no confusion is possible.

$\mathcal{D}_\theta$ and $\mathcal{D}_\theta^\pm$:

$$\mathcal{D}_\theta = \sum_{\arg(\omega)=\theta} \mathbf{\Delta}_\omega \tag{1.47}$$

$$\mathcal{D}_\theta^\pm = 1 + \sum_{\arg(\omega)=\theta} \mathbf{\Delta}_\omega^\pm = \exp\big(\pm 2\pi i\, \mathcal{D}_\theta\big) \tag{1.48}$$

which are the key to the *axis-crossing* identities :

$$\varphi_{\theta-\epsilon} = (\mathcal{D}_\theta^+ \varphi)_{\theta+\epsilon} \quad ; \quad (\varPhi \,.\, \mathcal{D}_\theta^+)_{\theta-\epsilon} = (\mathcal{D}_\theta^+ . \varPhi)_{\theta+\epsilon} \tag{1.49}$$

$$\varphi_{\theta+\epsilon} = (\mathcal{D}_\theta^- \varphi)_{\theta-\epsilon} \quad ; \quad (\varPhi \,.\, \mathcal{D}_\theta^-)_{\theta-\epsilon} = (\mathcal{D}_\theta^- . \varPhi)_{\theta+\epsilon} \tag{1.50}$$

that connect two sectorial germs $\varphi_{\theta-\epsilon}$ and $\varphi_{\theta+\epsilon}$ relative to Laplace integration right and left of any given singularity-carrying axis θ in the ζ-plane.[14]

1.5 Alien derivations as a tool for uniformisation

Convolution domains. A Riemann surface $\mathcal{R}$ is said to be *unobstructed* if, for any point ζ on it, the set S_ζ of all singular points *seen* or *half-seen* from ζ has a discrete projection $\dot{S}_\zeta$ on $\mathbb{C}$.

A ramified analytic germ $\widehat{\varphi}(\zeta)$ at the origin $0_\bullet$ of $\mathbb{C}_\bullet$ is said to be *endlessly continuable* if under analytic continuation it extends to an *unobstructed* Riemann surface.

Endlessly continuable germs are stable under convolution.

A *convolution domain* is an unobstructed Riemann surface $\underline{\mathcal{R}}$ for which the space $Hol(\underline{\mathcal{R}})$ of all holomorphic functions on $\underline{\mathcal{R}}$ is closed under convolution.

Any unobstructed Riemann surface $\mathcal{R}$ can, in a unique way, through the adjunction of a suitable set of singular points, be turned into a minimally ramified convolution domain $\underline{\mathcal{R}}$ – the so-called convolution completion, or stabilisation, of $\mathcal{R}$.

Fine convolution domains. We shall introduce a notion of *fine* Riemann surface and *fine* convolution domain which is hardly restrictive (all resurgent functions encountered in practice have Borel transforms that naturally extend to *fine* surfaces) and has the merit of greatly facilitating the proofs of all the statements to follow in this section.[15]

[14] In (1.43), (1.44), φ denotes any *resurgent function* and $\varPhi$ any *resurgent operator* (such as multiplication or postcomposition by a resurgent function etc).

[15] Let us stress that *fineness* is by no means necessary for the statements in question to hold. It simply makes life easier and costs nothing.

For any $\rho > 0$ and $\theta_1 < \theta_2$ in $\mathbb{R}$, let $\mathcal{D}^{\pm}_{\rho,\theta_1,\theta_2}$ denote the sets of all alien operators Δ of the form:

$$\mathcal{D}^{+}_{\rho,\theta_1,\theta_2} := \left\{\Delta = \Delta^{+}_{\omega_r} \dots \Delta^{+}_{\omega_1} \, ; \sum |\omega_i| \leq \rho \, , \theta_1 \leq \arg \omega_r \leq \dots \leq \arg \omega_1 \leq \theta_2 \right\}$$
$$\mathcal{D}^{-}_{\rho,\theta_1,\theta_2} := \left\{\Delta = \Delta^{-}_{\omega_r} \dots \Delta^{-}_{\omega_1} \, ; \sum |\omega_i| \leq \rho \, , \theta_1 \leq \arg \omega_1 \leq \dots \leq \arg \omega_r \leq \theta_2 \right\}.$$

Note that the number r of factors in the decomposition of Δ is not bounded.

Let us say that an (unobstructed) Riemann surface $\mathcal{R}$ is *fine* if, for any $(\rho, \theta_1, \theta_2)$, the number of operators Δ in $\mathcal{D}^{\pm}_{\rho,\theta_1,\theta_2}$ such that $\Delta.Hol(\mathcal{R}) \neq \emptyset$ is finite. This amounts to an extremely weak condition on the distribution of $\mathcal{R}$'s ramification points.

Any fine *Riemann surface $\mathcal{R}$ can, in a unique way, through the adjunction of a suitable set of singular points, be turned into a minimally ramified* fine *convolution domain $\underline{\mathcal{R}}$ – the completion, or stabilisation, of $\mathcal{R}$.*

Atomic alien operators. Any ramification point η of a fine convolution domain $\underline{\mathcal{R}}$ is connected with the origin $0_\bullet$ by a well-defined *taut broken line* Γ_η or TT-path, which in turn can be uniquely represented by a sequence $(\omega_1, \dots, \omega_r)$ whose elements $\omega_i \in \mathbb{C}_\bullet$ represent the successive intervals of Γ_η. Inequalities of the form

$$0 < \pi(2n-1) < \arg \omega_{i+1} - \arg \omega_i < \pi(2n+1)$$
$$\textit{respectively} \quad -\pi(2n+1) < \arg \omega_{i+1} - \arg \omega_i < -\pi(2n-1) < 0$$

signal that Γ_η makes n positive (respectively negative) turns round its i^{th} summit. Between any two aligned[16] ω_i, ω_{i+1} we must insert a sign $\epsilon_i \in \{+, -\}$ to indicate whether Γ_η circumvents the i^{th} 'summit' to the right or to the left.

To each ramification point η of a fine convolution domain $\underline{\mathcal{R}}$ there also correspond two 'ramified shifts' S^+_η, S^-_η and an alien operator $\widehat{D}_\eta$.

Each $S^{\pm}_\eta$ is defined locally, near $0_\bullet$. In projection on $\mathbb{C}$, it amounts to an ordinary $\dot{\eta}$-shift but it takes $0_\bullet$ to the end-point of Γ_η in such as way as to map the small intervals issuing from $0_\bullet$ in the direction $\arg \omega \mp \pi$ onto the small interval of same length that ends the broken line Γ_η.

The atomic alien operators $\widehat{D}_\eta$ (so-called because they measure *the* singularity at the end-point of Γ_η rather than a *superposition* of singularities,

[16] *I.e.* when $\arg \omega_i = \arg \omega_{i+1}$.

as the alien derivations do) are then defined by:

$$\begin{aligned} \widehat{D}_\eta \ &: \ \mathrm{Hol}(\underline{\mathcal{R}}) \to \mathrm{Hol}(\underline{\mathcal{R}}_\eta) \\ \widehat{D}_\eta \, \widehat{\varphi}(\zeta) &:= \widehat{\varphi}(S^+_\eta(\zeta)) - \widehat{\varphi}(S^-_\eta(\zeta)) \end{aligned} \tag{1.51}$$

first for ζ near $0_\bullet$, and then continued in the large, on a fine convolution domain $\underline{\mathcal{R}}_\eta$ that may, and often is, *more* (never *less*) ramified than $\underline{\mathcal{R}}$.

There is a natural order $\prec$ on the ramification set $\underline{\mathcal{R}}_{ram}$ of any fine convolution domain $\underline{\mathcal{R}}$, along with a natural co-product on its atomic operators:

$$\widehat{D}_\eta(\widehat{\varphi}_1 * \widehat{\varphi}_2) \equiv \sum_{\eta_1,\eta_2 \prec \eta} H^{\eta_1,\eta_2}_\eta \ \left(R^{\,P^{\eta_1,\eta_2}_\eta} \, \widehat{D}_{\eta_1}\widehat{\varphi}_1\right) * \left(R^{\,Q^{\eta_1,\eta_2}_\eta} \, \widehat{D}_{\eta_2}\widehat{\varphi}_2\right) \tag{1.52}$$

(i) *with R denoting the one-turn rotation operator round* $0_\bullet$,

(ii) *with a sum $\sum_{\eta_1,\eta_2 \prec \eta}$ that is always finite,*

(iii) *with integers H^{η_1,η_2}_η, P^{η_1,η_2}_η, Q^{η_1,η_2}_η that reflect the self-intersection pattern of the broken line Γ_η.*

The structure tensor H^{η_1,η_2}_η turns $\mathcal{C}(\underline{\mathcal{R}}_{ram})$ into a commutative algebra with its own discretised convolution

$$(h_1 * h_2)(\eta) := \sum_{\eta_1,\eta_2 \prec \eta} H^{\eta_1,\eta_2}_\eta \, h_1(\eta_1) \, h_2(\eta_2) \quad \big(h_1, h_2 \in \mathcal{C}(\underline{\mathcal{R}}_{ram})\big) \tag{1.53}$$

The convolution algebra $\mathcal{C}(\underline{\mathcal{R}}_{ram})$ may be viewed as the discrete scaffolding of the convolution algebra $Hol(\underline{\mathcal{R}})$. In fact, $\mathcal{C}(\underline{\mathcal{R}}_{ram})$ is isomorphic to the quotient[17] $Hol_{polar}(\underline{\mathcal{R}})/Hol_{subpolar}(\underline{\mathcal{R}})$.

Uniformisation of convolution products or powers. Similar formulae (of which there exist several variants) hold for ordinary points ζ of $\underline{\mathcal{R}}$.

The following variant involves the standard alien derivations and has the advantage of uniqueness:

$$\widehat{\varphi}(\zeta) \equiv \sum_s K^\zeta_{\zeta_s} \widehat{\varphi}(\zeta_s) + \sum_r \sum_{\omega_i} \sum_s (2\pi i)^r K^\zeta_{\zeta_s,\boldsymbol{\omega}} \ \widehat{\Delta}_{\omega_r} \ldots \widehat{\Delta}_{\omega_1} \widehat{\varphi}(\zeta_{s,\boldsymbol{\omega}}) \tag{1.54}$$

[17] A function $\widehat{\varphi}$ in $Hol(\underline{\mathcal{R}})$ is said to be of *polar* respectively *subpolar* type if it behaves like $\frac{h(\eta)}{2\pi i\,(\zeta-\dot{\eta})} + o(\frac{1}{(\zeta-\dot{\eta})})$ respectively $o(\frac{1}{(\zeta-\dot{\eta})})$ in the ramified vicinity of any given $\eta \in \underline{\mathcal{R}}_{ram}$. The space $Hol_{polar}(\underline{\mathcal{R}})$ is clearly closed under convolution, with $Hol_{subpolar}(\underline{\mathcal{R}})$ as an ideal.

with a finite number of points ζ_s (respectively $\zeta_{s,\boldsymbol{\omega}}$) located over $\dot{\zeta}$ (respectively $\dot{\zeta} - \sum \dot{\omega}_i$) but lying within the holomorphy star of $\widehat{\varphi}$ (respectively $\widehat{\Delta}_{\omega_r} \dots \widehat{\Delta}_{\omega_1} \widehat{\varphi}$), and with entire (respectively rational) structure coefficients $K^{\zeta}_{\zeta_s}$ (respectively $K^{\zeta}_{\zeta_s,\boldsymbol{\omega}}$).

Here is a second variant that relies on the operators $\widehat{\Delta}^{+}_{\omega}$ and $\widehat{\Delta}^{-}_{\omega}$ of (1.40). It is not unique, but can always be adjusted so as to involve only entire coefficients $H^{\zeta}_{\zeta_s}$ and $H^{\zeta}_{\zeta_s,\boldsymbol{\omega},\boldsymbol{\epsilon}}$.

$$\widehat{\varphi}(\zeta) \equiv \sum_{s} H^{\zeta}_{\zeta_s}\, \widehat{\varphi}(\zeta_s) + \sum_{r} \sum_{\omega_i,\epsilon_i} \sum_{s} H^{\zeta}_{\zeta_s,\boldsymbol{\omega},\boldsymbol{\epsilon}}\, \widehat{\Delta}^{\epsilon_r}_{\omega_r} \dots \widehat{\Delta}^{\epsilon_1}_{\omega_1} \widehat{\varphi}(\zeta_{s,\boldsymbol{\omega},\boldsymbol{\epsilon}}) \quad (1.55)$$

Both variants reduce the evaluation of any convolution product or power, at any given point ζ *of* $\underline{\mathcal{R}}$*, on any Riemann sheet, however distant from* $0_\bullet$*, to a finite number of convolution integrals to be calculated on straight intervals joining* $0_\bullet$ *to points* ζ_i *or* $\zeta_{i,\boldsymbol{\omega}}$, $\zeta_{i,\boldsymbol{\omega},\boldsymbol{\epsilon}}$ *that lie on the main Riemann sheet.*

For instance, if we apply (1.54) to the evaluation of the convolution power $\widehat{\varphi}^{*n}(\zeta)$, for any $\zeta \in \underline{\mathcal{R}}$, any $\widehat{\varphi} \in Hol(\underline{\mathcal{R}})$, and $n \to \infty$, we find that everything reduces to finitely many terms of the form

$$\widehat{\Delta}_{\boldsymbol{\omega}} \widehat{\varphi}^{*n}(\zeta_{s,\boldsymbol{\omega}}) = \sum_{\boldsymbol{\omega} \in \mathrm{sha}(\boldsymbol{\omega}^1,\dots,\boldsymbol{\omega}^k)}^{1 \le k \le r} \frac{n!}{k!(n-k)!} \left(\widehat{\varphi}^{*(n-k)} * \widehat{\Delta}_{\boldsymbol{\omega}^1}\widehat{\varphi} * \dots \widehat{\Delta}_{\boldsymbol{\omega}^k}\widehat{\varphi}\right)(\zeta_{s,\boldsymbol{\omega}}) \quad (1.56)$$

with s and k bounded, so that in the end the asymptotics is dominated by trite convolution integrals $\widehat{\varphi}^{*(n-k)}(\zeta_{s,\boldsymbol{\omega}})$ evaluated on simple intervals $(0_\bullet, \zeta_{s,\boldsymbol{\omega}}]$ safely located within the main Riemann sheet (or its boundary).

This *uniformising virtue* of alien derivations (by which we mean their power to reduce complicated operations on ramified, multivalued functions to simple operations on their, and their alien derivatives', uniform restrictions to the holomorphy star) is one of the main justifications (though not the topmost) of alien calculus.

Remark. Alongside the *TT-paths*[18] that connect any $\zeta \in \underline{\mathcal{R}}$ to the origin $0_\bullet$, we must also consider two classes of more convolution-friendly, but also more complex paths: the wildly contorted *SS-paths*[19] and the even more intricate *ZZ-paths*[20]. The SS-paths are useful for establishing the

[18] *"Taut broken lines"*.

[19] *"Self-symmetrical and self-symmetrically shrinkable paths"*.

[20] *"Self-symmetrical, self-symmetrically shrinkable, and self-replicating paths"*.

stability under convolution of *endless continuability*, and the ZZ-paths for illustrating the formulae (1.52)-(1.56).

Where these paths fail miserably, though, is in providing decent estimates for convolution products or powers on far-flung Riemann sheets. For the convolution powers[21], SS-path considerations lead to asymptotically correct estimates

$$\left|\widehat{\varphi}^{*n}(\zeta)\right| \leq c_0(\zeta)\,\frac{c_1(\zeta)^n}{n!} \qquad \big(c_0(\zeta), c_1(\zeta) > 0\big).$$

However, for points $\zeta \in \underline{\mathcal{R}}$ whose TT-path has k summits, the bounds derivable in this way (especially c_0) become hopelessly suboptimal as k increases. Even for values as small as $k = 20$, c_0 can fall off the mark by something like a factor 10^{10}.

The convolution domains $\underline{\mathcal{R}} := \widetilde{\mathbb{C} - \Omega}$ with Ω a lattice. For any discrete lattice $\Omega = \tau_1\mathbb{Z}$ or $\tau_1\mathbb{Z} + \tau_2\mathbb{Z}$ ($\tau_i \in \mathbb{C}^*$, $\Im(\tau_1/\tau_2) \neq 0$), the surface $\underline{\mathcal{R}} := \widetilde{\mathbb{C} - \Omega}$ is an – obviously *fine* – convolution domain with a particularly simple structure: its ramified shifts $S_\eta^{\pm}$ form a group which contains the one-turn rotation R and is generated by just two elements (whether Ω is one- or two-dimensional!). There is even an elementary algorithm for finding all the $\prec$-antecedents of any ramification point $\eta \in \underline{\mathcal{R}}_{ram}$, as well as all the structure coefficients featuring in (1.52) and (1.54). This applies in particular for the surface $\underline{\mathcal{R}} := \widetilde{\mathbb{C} - 2\pi i\mathbb{Z}}$, which is the natural surface of practically all the resurgent functions to appear in this investigation.

1.6 Medial operators

Their definition resembles that of the alien derivations

$$(\widehat{\Delta}^{\sharp}_{\omega}\hat{\varphi})(\zeta) := \sum_{\epsilon_1,\dots,\epsilon_r} \frac{\epsilon_r}{2\pi i}\,\lambda^{\sharp}_{\epsilon_1,\dots,\epsilon_{r-1}}\ \hat{\varphi}^{\left(\substack{\epsilon_1\ ,\dots,\ \epsilon_r\\ \omega_1\ ,\dots,\ \omega_r}\right)}(\omega+\zeta) \tag{1.57}$$

$$(\widehat{\Delta}^{\sharp\sharp}_{\omega}\hat{\varphi})(\zeta) := \sum_{\epsilon_1,\dots,\epsilon_r} \frac{\epsilon_r}{2\pi i}\,\lambda^{\sharp\sharp}_{\epsilon_1,\dots,\epsilon_{r-1}}\ \hat{\varphi}^{\left(\substack{\epsilon_1\ ,\dots,\ \epsilon_r\\ \omega_1\ ,\dots,\ \omega_r}\right)}(\omega+\zeta) \tag{1.58}$$

with $\omega_r = \omega$ and the usual signs $\epsilon_j \in \{+,-\}$ but with simpler weights $\lambda^{\sharp}_{\boldsymbol{\epsilon}}, \lambda^{\sharp\sharp}_{\boldsymbol{\epsilon}}$, still independent of the intervals ω_i:

$$\lambda^{\sharp}_{\epsilon_1,\dots,\epsilon_{r-1}} = \lambda^{[p,q]}_{\sharp} := 2^{-p-q} = 2^{1-r} \tag{1.59}$$

$$\lambda^{\sharp\sharp}_{\epsilon_1,\dots,\epsilon_{r-1}} = \lambda^{[p,q]}_{\sharp\sharp} := \varrho(p-q)\ 2^{-int(\frac{p+q+1}{2})} \tag{1.60}$$

[21] Of a function $\widehat{\varphi}(\zeta)$ regular at $0_\bullet$.

As usual, p and q denote the numbers of $+$ and $-$ signs in $\{\epsilon_1, \dots, \epsilon_{r-1}\}$. As for the elementary factor $\varrho(p-q) \equiv \varrho(q-p)$, it assumes only three values, 0, 1, –1, and displays a remarkable 8-periodicity :

$$\varrho(k+8) \equiv \varrho(k) \ , \ \varrho : [0,1,2,3,4,5,6,7] \mapsto [1,1,0,-1,-1,-1,0,1] \quad (1.61)$$

Like the earlier weights λ_ϵ in (1.41) attached to the standard alien derivations, the new weights $\lambda^\sharp_\epsilon, \lambda^{\sharp\sharp}_\epsilon$ add up to 1:

$$\sum_{\epsilon_i \in\{+,-\}} \lambda_{\epsilon_1,\dots,\epsilon_{r-1}} = \sum_{\epsilon_i \in\{+,-\}} \lambda^\sharp_{\epsilon_1,\dots,\epsilon_{r-1}} = \sum_{\epsilon_i \in\{+,-\}} \lambda^{\sharp\sharp}_{\epsilon_1,\dots,\epsilon_{r-1}} = 1 \quad (\forall r)$$

The simplest way to express the relations between the new operators and the classical ones is via the generating series:

$$\mathcal{D}^\sharp = \sum_{\arg(\omega)=0} \boldsymbol{\Delta}^\sharp_\omega \quad , \quad \mathcal{D}^{\sharp\sharp} = \sum_{\arg(\omega)=0} \boldsymbol{\Delta}^{\sharp\sharp}_\omega \tag{1.62}$$

The relations read:

$$\mathcal{D}^\sharp = \frac{1}{\pi}\tan(\pi\mathcal{D}) = \frac{1}{\pi i}\,\frac{\mathcal{D}^+ - 1}{\mathcal{D}^+ + 1} = \frac{1}{\pi i}\,\frac{1-\mathcal{D}^-}{1+\mathcal{D}^-} \tag{1.63}$$

$$\mathcal{D}^{\sharp\sharp} = \frac{1}{2\pi}\tan(2\pi\mathcal{D}) = \frac{1}{2\pi i}\,\frac{\mathcal{D}^+ - \mathcal{D}^-}{\mathcal{D}^+ + \mathcal{D}^-} \tag{1.64}$$

As pointed out at the outset, the new operators are neither derivations nor automorphisms. They possess co-products *sui generis* which, once again, are best expressed in terms of the generating series:

$$\begin{aligned}\mathcal{D}^\sharp \mapsto\ & \mathcal{D}^\sharp \otimes \mathbf{1} + \mathbf{1} \otimes \mathcal{D}^\sharp \\ & + \sum_{1\le n} (\pi)^{2n} \Big[(\mathcal{D}^\sharp)^{n+1} \otimes (\mathcal{D}^\sharp)^n + (\mathcal{D}^\sharp)^n \otimes (\mathcal{D}^\sharp)^{n+1} \Big] \\ \mathcal{D}^{\sharp\sharp} \mapsto\ & \mathcal{D}^{\sharp\sharp} \otimes \mathbf{1} + \mathbf{1} \otimes \mathcal{D}^{\sharp\sharp} \\ & + \sum_{1\le n} (2\pi)^{2n} \Big[(\mathcal{D}^{\sharp\sharp})^{n+1} \otimes (\mathcal{D}^{\sharp\sharp})^n + (\mathcal{D}^{\sharp\sharp})^n \otimes (\mathcal{D}^{\sharp\sharp})^{n+1} \Big]\end{aligned}$$

Short proofs. The quickest way to prove all the above relations at one go is to start with the axis $\arg\zeta = 0$ punctured over $\mathbb{N}$. Denoting σ and τ the non-commuting "shifts" that take ζ small (with $\arg\zeta = 0$) to $\zeta + 1$ after circumventing the point at 1 respectively to the right or to the left (and then extending the action of σ and τ in the large), we find that

$$\mathcal{D}^+ = (1-\tau)\,(1-\sigma)^{-1} \quad , \quad \mathcal{D}^- = (1-\sigma)\,(1-\tau)^{-1} \tag{1.65}$$

Next, proceeding backwards, we define $\boldsymbol{\Delta}^{\sharp}_{\omega}$, $\boldsymbol{\Delta}^{\sharp\sharp}_{\omega}$ via (1.62) in terms of $\mathcal{D}^{\sharp}$, $\mathcal{D}^{\sharp\sharp}$; then $\mathcal{D}^{\sharp}$, $\mathcal{D}^{\sharp\sharp}$ via (1.63)-(1.64) in terms of $\mathcal{D}^{\pm}$; then $\mathcal{D}^{\pm}$ via (1.65) in terms of the elementary shits σ, τ. After some rather easy calculations in the non-commutative variables σ, τ, we find the expressions (1.59), (1.60) for the weights $\lambda^{\sharp}_{\boldsymbol{\omega}}, \lambda^{\sharp\sharp}_{\boldsymbol{\omega}}$, though at first only for the case when $\{\omega_1, \omega_2, \omega_3 \ldots\} = \{1, 2, 3 \ldots\}$. But we clearly have

$$\begin{aligned}
\sum_{\epsilon_{i_0}=\pm} \lambda^{\sharp}_{\epsilon_1,\ldots,\epsilon_{r-1}} &= \lambda^{\sharp}_{\epsilon_1,\ldots,[\epsilon_{i_0}],\ldots,\epsilon_{r-1}}\,, \\
\sum_{\epsilon_{i_0}=\pm} \lambda^{\sharp\sharp}_{\epsilon_1,\ldots,\epsilon_{r-1}} &= \lambda^{\sharp\sharp}_{\epsilon_1,\ldots,[\epsilon_{i_0}],\ldots,\epsilon_{r-1}} \qquad (\forall i_0 < r)
\end{aligned}$$

with the notation $[\epsilon_{i_0}]$ signaling the omission of ϵ_{i_0}. It follows that the weights $\lambda^{\sharp}_{\bullet}, \lambda^{\sharp\sharp}_{\bullet}$ retain their expression (1.59),(1.60) for all sequences $\{\omega_i\}$ over $\mathbb{N}$ and, in fact, over $\mathbb{R}^+$.

1.7 Resurgence of the iterators and generators

The iterator f^* and ${}^*\!f$, characterised by the relations (1.11)-(1.12), and the (infinitesimal) generator f_*, characterised by the relation (1.8), verify the following resurgence equations

$$\Delta_\omega\, {}^*\!f(z) = +A_\omega\, \partial_z\, {}^*\!f(z) \qquad (\forall \omega \in \Omega) \tag{1.66}$$

$$\Delta_\omega\, f^*(z) = -A_\omega\, e^{-\omega\,(f^*(z)-z)} \qquad (\forall \omega \in \Omega) \tag{1.67}$$

$$\Delta_\omega\, f_*(z) = -\omega\, A_\omega\, f_*(z)\, e^{-\omega\,(f^*(z)-z)} \tag{1.68}$$

with the very same scalar coefficients A_ω as in (1.15). For all values of ω not in Ω, the alien derivatives are $\equiv 0$. If we now introduce the differential operators:

$$\mathbb{A}_\omega := A_\omega\, e^{-\omega z}\, \partial_z \qquad (\forall \omega \in \Omega) \tag{1.69}$$

the resurgence equations assume the form of the Bridge equation:[22]

$$\boldsymbol{\Delta}_\omega\, {}^*\!f(z) = +\mathbb{A}_\omega\, {}^*\!f(z) \tag{1.70}$$

$$\boldsymbol{\Delta}_\omega\, f^*(z) = -(\mathbb{A}_\omega\,.\,z) \circ f^*(z)\,. \tag{1.71}$$

[22] So-called because it relates *ordinary* and *alien* derivatives of one and the same resurgent function. The Bridge equation has in fact much wider applications, and extends, in one form or another, to practically all *resonant* local objects, of which *identity-tangent diffeos* are but a special case. An entire book [6] has been devoted to the subject.

When expressed in terms of the subsitution operators F^* and ${}^*F$ associated with ${}^*f$, f^*, the Bridge equation takes an even more pleasant form

$$\left[\mathbf{\Delta}_\omega, F^*\right] = -F^*\, \mathbb{A}_\omega \qquad (F^*\varphi := \varphi \circ f^*) \tag{1.72}$$

$$\left[\mathbf{\Delta}_\omega, {}^*F\,\right] = +\mathbb{A}_\omega\, {}^*F \qquad ({}^*F\ \varphi := \varphi \circ {}^* f)\,. \tag{1.73}$$

Likewise, with the (operatorial) generator $F_* := f_*\,\partial = F^*.\partial.F^*$, we get:

$$[\mathbf{\Delta}_\omega, F_*] = F^*\,[\partial, \mathbb{A}_\omega]\,{}^*F\,. \tag{1.74}$$

But whichever variant we may care to consider, the commutation identities $\left[\mathbf{\Delta}_{\omega_1}, \mathbb{A}_{\omega_2}\right] = 0$ make it easy to iterate the above resurgence equations. Thus from (1.70) we straightaway derive

$$\mathbf{\Delta}_{\omega_r} \ldots \mathbf{\Delta}_{\omega_1}\, {}^*f(z) = \mathbb{A}_{\omega_1} \ldots \mathbb{A}_{\omega_r}\, {}^*f(z) \qquad (\textit{order reversion}!)\,. \tag{1.75}$$

As a consequence, the effect on ${}^*f$ and f^* of the alien operators $\mathbf{\Delta}^{\pm}_\omega$ and of the axial operators $\mathcal{D}_\theta$ is easy to calculate. It is best written in terms of the substitution operators ${}^*F$ and F^* associated with ${}^*f$, f^*, and results in the so-called *axial* Bridge equation:

$$\mathcal{A}_\theta = \mathcal{D}_\theta \ - \ {}^*F\ \ \mathcal{D}_\theta\ \ F^* \tag{1.76}$$

$$\mathcal{A}^+_\theta = \mathcal{D}^+_\theta\ \ {}^*F\ \ \mathcal{D}^-_\theta\ \ F^* = {}^*F\ \ \mathcal{D}^-_\theta\ \ F^*\ \ \mathcal{D}^+_\theta \tag{1.77}$$

$$\mathcal{A}^-_\theta = \mathcal{D}^-_\theta\ \ {}^*F\ \ \mathcal{D}^+_\theta\ \ F^* = {}^*F\ \ \mathcal{D}^+_\theta\ \ F^*\ \ \mathcal{D}^-_\theta\,. \tag{1.78}$$

The axial Bridge equation[23] involves differential (respectively substitution) operators $\mathcal{A}_\theta$ (respectively $\mathcal{A}^{\pm}_\theta$):

$$\mathcal{A}_\theta = \sum_{\arg(\omega)=\theta} \mathbb{A}_\omega \tag{1.79}$$

$$\mathcal{A}^{\pm}_\theta = 1 + \sum_{\arg(\omega)=\theta} \mathbb{A}^{\pm}_\omega = \exp\big(\pm 2\pi i\, \mathcal{A}_\theta\big) \tag{1.80}$$

that are simply related to the differential (respectively substitution) operators $\mathbf{\Pi}_*$ (respectively $\mathbf{\Pi}^{\pm}$) associated with the connectors of Section 1.1:

$$\mathbf{\Pi}_{\boldsymbol{no}} := \mathcal{A}^+_{-\frac{\pi}{2}} \quad ; \quad \mathbf{\Pi}_{\boldsymbol{so}} := \mathcal{A}^-_{+\frac{\pi}{2}} \tag{1.81}$$

$$\mathbf{\Pi}^{-1}_{\boldsymbol{no}} := \mathcal{A}^-_{-\frac{\pi}{2}} \quad ; \quad \mathbf{\Pi}^{-1}_{\boldsymbol{so}} := \mathcal{A}^+_{+\frac{\pi}{2}} \tag{1.82}$$

$$\mathbf{\Pi}_{*\boldsymbol{no}} := +2\pi i\, \mathcal{A}_{-\frac{\pi}{2}} \quad ; \quad \mathbf{\Pi}_{*\boldsymbol{so}} := -2\pi i\, \mathcal{A}_{+\frac{\pi}{2}}\,. \tag{1.83}$$

[23] We say *Bridge equation* in the singular since (1.77) and (1.78) are merely exponential variants of (1.76). The commutation of the three automorphisms $\mathcal{A}^{\pm}_\theta, \mathcal{D}^{\pm}_\theta, {}^*F\ \ \mathcal{D}^{\mp}_\theta\ \ F^*$ is itself a consequence of the commutation of the three derivations $\mathbb{A}_\theta, \mathcal{D}_\theta, {}^*F\ \mathcal{D}_\theta\ F^*$.

The first identity (1.81) results from applying the direct axis-crossing formula (1.49) with $\theta = -\frac{\pi}{2}$ and $\varphi = {}^{*}\!f$ or $\Phi = {}^{*}\!F$, since ${}^{*}\!f_{\theta\pm\epsilon} = {}^{*}\!f_{\pm}$. The second identity (1.81) results from applying the inverse axis-crossing formula (1.50) with $\theta = +\frac{\pi}{2}$ and $\varphi = {}^{*}\!f$ or $\Phi = {}^{*}\!F$, since in that case ${}^{*}\!f_{\theta\pm\epsilon} = {}^{*}\!f_{\mp}$ (inversion!). The identities (1.82) and (1.82) immediately follow.

Direct access to the generators and mediators of π. Consider now the mediators $\pi_{\sharp}, \pi_{\sharp\sharp}$ of the connector π, with their northern/southern components and their formal Fourier expansions. They run parallel to those (see (1.68)) of the infinitesimal generator π_*:

$$\boldsymbol{\pi}_{\sharp,\mathrm{no}}(z) = +2\pi i \sum_{\omega\in\Omega^-} A^{\sharp}_{\omega}\, e^{-\omega z} \;;\; \boldsymbol{\pi}_{\sharp,\mathrm{so}}(z) = -2\pi i \sum_{\omega\in\Omega^+} A^{\sharp}_{\omega}\, e^{-\omega z} \tag{1.84}$$

$$\boldsymbol{\pi}_{\sharp\sharp,\mathrm{no}}(z) = +2\pi i \sum_{\omega\in\Omega^-} A^{\sharp\sharp}_{\omega}\, e^{-\omega z} \;;\; \boldsymbol{\pi}_{\sharp\sharp,\mathrm{so}}(z) = -2\pi i \sum_{\omega\in\Omega^+} A^{\sharp\sharp}_{\omega}\, e^{-\omega z} \tag{1.85}$$

Based on (1.67) and (1.57)-(1.58), we see that we can access the Fourier coefficients of $\pi_*, \pi_{\sharp}, \pi_{\sharp\sharp}$, or indeed those of the general affiliate $\pi_{\diamondsuit}$, *directly* from one and the same resurgent function, namely f^*:

$$\begin{aligned} \boldsymbol{\Delta}_{\omega}\, f^* &= -A_{\omega}\, e^{-\omega f^*}, \\ \boldsymbol{\Delta}^{\sharp}_{\omega}\, f^* &= -A^{\sharp}_{\omega}\, e^{-\omega f^*}, \\ \boldsymbol{\Delta}^{\sharp\sharp}_{\omega}\, f^* &= -A^{\sharp\sharp}_{\omega}\, e^{-\omega f^*} \end{aligned} \tag{1.86}$$

without bothering about the corresponding affiliates of f, *i.e.* f_*, $f_{\sharp}$, $f_{\sharp\sharp}$, $f_{\diamondsuit}$. Though it is true, as we shall aver in the next section, that $f_{\sharp}$, $f_{\sharp\sharp}$ etc. verify their own interesting resurgence equations with a mixture of invariant and non-invariant resurgence constants from which, after some sifting, all the Fourier coefficients $A^{\sharp}_{\omega}$, $A^{\sharp\sharp}_{\omega}$ etc. can be reconstructed, the fact remains that the f-affiliates have no particular closeness to the corresponding π-affiliates.

1.8 Resurgence of the mediators

The relations (1.28)-(1.29), which may be viewed as perturbed difference equations, determine $f_{\sharp}$ and $f_{\sharp\sharp}$ in terms of f. A standard argument shows that $f_\sharp(z)$ and $f_{\sharp\sharp}(z)$ are resurgent in z, with first-order alien derivatives verifying the homogeneous equation:

$$(\boldsymbol{\Delta}_{\omega_0} f_\sharp) \circ f + \boldsymbol{\Delta}_{\omega_0} f_\sharp = 0 \qquad (\forall \omega_0 \in \pi i \mathbb{Z} - 2\pi i \mathbb{Z}) \tag{1.87}$$

$$(\boldsymbol{\Delta}_{\omega_0} f_{\sharp\sharp}) \circ f^{\circ 2} + \boldsymbol{\Delta}_{\omega_0} f_{\sharp\sharp} = 0 \qquad (\forall \omega_0 \in \frac{1}{2}\pi i \mathbb{Z} - \pi i \mathbb{Z}) \tag{1.88}$$

whose general solution are of the form

$$\boldsymbol{\Delta}_{\omega_0} f_\sharp = \underline{A}_{\omega_0} e^{-\omega_0 f^*} \qquad (\forall \omega_0 \in \pi i \mathbb{Z} - 2\pi i \mathbb{Z}) \tag{1.89}$$

$$\boldsymbol{\Delta}_{\omega_0} f_{\sharp\sharp} = \underline{\underline{A}}_{\omega_0} e^{-\omega_0 f^*} \qquad \left(\forall \omega_0 \in \frac{1}{2}\pi i \mathbb{Z} - \pi i \mathbb{Z}\right) \tag{1.90}$$

with resurgent constants $\underline{A}_{\omega_0}$ and $\underline{\underline{A}}_{\omega_0}$ unrelated to the invariants $A_\omega(f)$. In fact, $\underline{A}_{\omega_0}$ and $\underline{\underline{A}}_{\omega_0}$ are not invariant under analytic changes of z-coordinates and, unlike the invariants $A_\omega(f)$, they involve *coloured multizetas* as their transcendental ingredients, as we shall see in Section 3.6. But the mediators' alien derivatives of second (and higher) order obviously depend only on the iterator f^* and involve no new resurgent constants other than the invariants A_ω:

$$\boldsymbol{\Delta}_{\omega_1} \boldsymbol{\Delta}_{\omega_0} f_\sharp = \omega_0 \underline{A}_{\omega_0} A_{\omega_1} e^{-(\omega_0+\omega_1) f^*} \qquad (\forall \omega_1 \in 2\pi i \mathbb{Z}) \tag{1.91}$$

$$\boldsymbol{\Delta}_{\omega_1} \boldsymbol{\Delta}_{\omega_0} f_{\sharp\sharp} = \omega_0 \underline{\underline{A}}_{\omega_0} A_{\omega_1} e^{-(\omega_0+\omega_1) f^*} \qquad (\forall \omega_1 \in 2\pi i \mathbb{Z}) \tag{1.92}$$

Both systems still hold if we replace $f_\sharp(z) := F_\sharp.z$ and $f_{\sharp\sharp}(z) := F_{\sharp\sharp}.z$ by $\Phi_\sharp(z) := F_\sharp.\phi(z)$ and $\Phi_{\sharp\sharp}(z) := F_{\sharp\sharp}.\phi(z)$ for any convergent ϕ, except that the first resurgent constants $\underline{A}_{\omega_0}$ and $\underline{\underline{A}}_{\omega_0}$ now depend on ϕ (while the A_{ω_1} depend on f alone). It would thus be possible to recover the invariants of f from any such $\Phi_\sharp$ or $\Phi_{\sharp\sharp}$, barring the highly exceptional (but not impossible) case when all *initial* resurgent constants $\underline{A}_{\omega_0}$ or $\underline{\underline{A}}_{\omega_0}$ vanish.

This state of affairs is fairly typical for the general affiliates: whenever γ is meromorphic with actual poles, the affiliate $f_\diamondsuit(z) := \gamma(F-1).z$ of f verifies resurgent equations that involve, alongside the invariants A_ω of f, non-invariant constants like $\underline{A}_{\omega_0}$ and $\underline{\underline{A}}_{\omega_0}$.

1.9 Invariants, connectors, collectors

Let us survey in one table some of the main objects introduced so far or yet to come.

$$\begin{array}{ccccccccccc}
\textit{diffeo} & & \textit{collectors} & & & & \textit{connectors} & & & & \textit{invariants} \\
g_\sharp & \xrightarrow{3'_\sharp} & \mathfrak{p}_\sharp & \xrightarrow{3''_\sharp} & \mathfrak{sp}_\sharp & \xrightarrow{3'''_\sharp} & \boldsymbol{\pi}_\sharp = (\boldsymbol{\pi}_{\sharp\,\mathrm{no}}, \boldsymbol{\pi}_{\sharp\,\mathrm{so}}) & & \xrightarrow{3''''_\sharp} & & \{A^\sharp_\omega\} \\
\uparrow 2_\sharp & & \downarrow 4_\sharp & & & & \downarrow 5_{\sharp\,\mathrm{no}} \quad \downarrow 5_{\sharp\,\mathrm{so}} & & & & \downarrow 6_\sharp \\
f = l \circ g & \xrightarrow{1'} & \mathfrak{p}^\pm & \xrightarrow{1''} & \mathfrak{sp}^\pm & \xrightarrow{1'''} & \boldsymbol{\pi}^\pm = (\boldsymbol{\pi}^\pm_{\mathrm{no}}, \boldsymbol{\pi}^\pm_{\mathrm{so}}) & & \xrightarrow{1''''} & & \{A^\pm_\omega\} \\
\downarrow 2_* & & \uparrow 4_* & & & & \uparrow 5_{*\,\mathrm{no}} \quad \uparrow 5_{*\,\mathrm{so}} & & & & \uparrow 6_* \\
g_* & \xrightarrow{3'_*} & \mathfrak{p}_* & \xrightarrow{3''_*} & \mathfrak{sp}_* & \xrightarrow{3'''_*} & \boldsymbol{\pi}_* = (\boldsymbol{\pi}_{*\mathrm{no}}, \boldsymbol{\pi}_{*\mathrm{so}}) & & \xrightarrow{3''''_*} & & \{A_\omega\}
\end{array}$$

The middle row carries the objects of direct interest to us, while the upper and lower rows carry their two main affiliates (the first mediator and the infinitesimal generator), which are more in the nature of auxiliary constructs.

The first, third and fourth columns carry objects already familiar to us. The second column, however, carries novel, highly interesting objects, the *collectors*, which are very close in a sense to the *connectors*, yet should be, for the sake of conceptual cleanness, clearly held apart. The *collectors* may assume four distinct forms:

(i) formal series of multitangents, noted $\mathfrak{p}$;
(ii) formal series of monotangents, also noted $\mathfrak{p}$;
(iii) formal Laurent series of z^{-1}, noted $\mathfrak{lp}$
(iv) the singular part, noted $\mathfrak{sp}$, of these Laurent series.

One goes from (i) to (ii) by multitangent reduction as in Section 2.3 ; and from (ii) to (iv) by the change $Te^{s_1} \mapsto z^{-s_1}$.

In any of these incarnations, the collectors are but a step removed from the invariants. Yet they are not invariant themselves: they depend on the z-chart in which the diffeo f is taken. Another difference is that whereas the collectors $\pi^\pm$ are convergent Fourier series, the collectors $\mathfrak{p}^\pm$ are condemned to remain formal power series in the countably many coefficients f_n of f. But this is perfectly all right, since the function of the collectors is precisely to carry, in conveniently compact form, all the information about the f-dependence of the connector π and, ultimately, of the invariants A_ω.

One last remark is in order here: although we are basically interested in the objects of the middle row, and more specifically in getting from f to the invariants $\{A^{\pm}_{\omega}\}$, we shall see that the most advantageous route is not the straight path through the arrows 1, $1'$, $1''$, $1''''$, but any of the roundabout paths that start with 2_* or $2_\sharp$: these indirect routes are much more economical in terms of calculations and also more respectful of the underlying symmetries and parities.

1.10 The reverse problem: canonical synthesis

It can be shown that any convergent pair $\boldsymbol{\pi} = (\boldsymbol{\pi}_{\text{no}}, \boldsymbol{\pi}_{\text{so}})$ is the connector pair of some standard diffeo $f = l \circ g$. This raises the problem of *synthesis*: how to reconstitute a germ f with a prescribed set of (admissible) invariants? And how to select a canonical f among all possible choices? A semi-canonical synthesis was sketched in [5] and a fully canonical one was constructed in [9]. The latter depends on a single parameter c whose real part must be chosen large enough.[24] The construction produces a canonical $f_c := {}^*f_c \circ l \circ f_c^*$ from its iterator f_c^*, which in turn is explicitly given, in operator form, by the formula

$$F_c^* := 1 + \sum_r \sum_{\omega_i \in \Omega} (-1)^r\, \mathcal{U}e_c^{\omega_1,\omega_2,\ldots,\omega_r}(z)\ \mathbb{A}_{\omega_r} \ldots \mathbb{A}_{\omega_2}\, \mathbb{A}_{\omega_1} \tag{1.93}$$

with a careful re-arrangement of the terms[25] necessary to ensure convergence. The two ingredients in (1.93) are the invariants $\mathbb{A}_\omega$ taken in operator form (1.69), and some special resurgence monomials $\mathcal{U}e_c^{\boldsymbol{\omega}}(z)$ defined by

$$\mathcal{U}e_c^{\boldsymbol{\omega}}(z) := e^{||\boldsymbol{\omega}||z + c^2||\bar{\boldsymbol{\omega}}||z^{-1}}\, \text{SPA}\!\int_0^\infty \frac{e^{-\sum(\omega_i\, t_i + c^2 \bar{\omega}_i\, t_i^{-1})}}{(t_r - t_{r-1})\ldots(t_2 - t_1)(t_1 - z)} dt_1 \ldots dt_r \tag{1.94}$$

where *SPA* denotes a suitable average of all the 2^{r-1} possible integration multipaths that reflect the 2^{r-1} manners in which the variables t_j may circumvent each other on their way from 0 to ∞.

2 Multitangents and multizetas.

The *multitangents* and *multizetas*, being the transcendental ingredient in the analytical expression of the invariants of identity-tangent diffeos[26],

[24] Synthesis cannot be *absolute*, *i.e.* parameter-free.

[25] Known as arborification-coarborification.

[26] And of much else – they are almost coextensive with the whole field of difference equations.

deserve a short excursus. But we must begin with a brief reminder about *moulds*, which are the proper tool for handling multi-indexed objects of whatever description.

2.1 Mould operations and mould symmetries

Main mould operations. Moulds are functions of finite sequences $\boldsymbol{\omega} = (\omega_1, ..., \omega_r)$ of any length $r \geq 0$, noted as right-upper indices and rendered, as mute variables, by a plain bold dot $\bullet$. Moulds can be *multiplied* and *composed*:

$$C^\bullet = A^\bullet \times B^\bullet \Longleftrightarrow C^{\boldsymbol{\omega}} = \sum_{\boldsymbol{\omega}'\boldsymbol{\omega}''=\boldsymbol{\omega}} A^{\boldsymbol{\omega}'} B^{\boldsymbol{\omega}''} \tag{2.1}$$

$$C^\bullet = A^\bullet \circ B^\bullet \Longleftrightarrow C^{\boldsymbol{\omega}} = \sum_{\boldsymbol{\omega}^1\ldots\boldsymbol{\omega}^s=\boldsymbol{\omega}} A^{|\boldsymbol{\omega}^1|,\ldots,|\boldsymbol{\omega}^s|} B^{\boldsymbol{\omega}^s}\ldots B^{\boldsymbol{\omega}^s} \quad (\boldsymbol{\omega}^i \neq \emptyset)$$

with all the predictable relations, including

$$(A^\bullet \times B^\bullet) \circ C^\bullet = (A^\bullet \circ C^\bullet) \times (B^\bullet \circ C^\bullet)\,.$$

The units for multiplication or composition are the moulds $\mathbf{1}^\bullet$, $Id^\bullet$ respectively defined by:

$$\mathbf{1}^\emptyset := 1 \quad ; \quad \mathbf{1}^{\omega_1,\ldots,\omega_r} := 0 \quad \textit{if} \quad r \neq 0 \tag{2.2}$$

$$Id^{\omega_1} := 1 \quad ; \quad Id^{\omega_1,\ldots,\omega_r} := 0 \quad \textit{if} \quad r \neq 1 \tag{2.3}$$

There exist scores of other mould operations, unary or binary. They are far too numerous to be assigned distinct symbols. So we resort to short letter combinations instead – even, retroactively, for mould multiplication and composition, which for clarity are often noted $mu(M_1^\bullet, M_2^\bullet)$ and $ko(M_1^\bullet, M_2^\bullet)$ instead of $M_1^\bullet \times M_2^\bullet$ and $M_1^\bullet \circ M_2^\bullet$. The corresponding Lie brackets are noted $lu(M_1^\bullet, M_2^\bullet)$ and $lo(M_1^\bullet, M_2^\bullet)$.

The multiplicative inverse of a mould $M^\bullet$ is usually noted $muM^\bullet$. It exists if and only if $M^\emptyset \neq 0$.

The composition inverse of a mould $M^\bullet$ is usually noted $koM^\bullet$. It exists if and only if $M^\emptyset = 0$ and $M^{\omega_1} \neq 0\ \forall\omega_1$.

A mould $M^\bullet$ is said to be of *constant type* if $M^{\boldsymbol{\omega}}$ depends only on the length $r := r(\boldsymbol{\omega})$ of the sequence $\boldsymbol{\omega}$, *i.e.* if $M^{\boldsymbol{\omega}} := m_r$. Such moulds may conveniently be noted $m(Id^\bullet)$ with $m(t) := \sum m_r\, t^r$. Multiplying or composing constant-type moulds $M^\bullet$ reduces to multiplying or composing the underlying power series $m(t)$.

Main mould symmetries. Most moulds tend to fall into one or the other of four symmetry classes or types:

$$M^\bullet \textit{symmetral} \ (\textit{respectively alternal}) \Leftrightarrow$$
$$\Leftrightarrow \sum_{\omega \in \mathrm{sha}(\omega', \omega'')} M^{\omega} = M^{\omega'} M^{\omega''} \ (\textit{respectively } 0)$$
$$M^\bullet \textit{symmetrel} \ (\textit{respectively alternel}) \Leftrightarrow$$
$$\Leftrightarrow \sum_{\omega \in \mathrm{she}(\omega', \omega'')} M^{\omega} = M^{\omega'} M^{\omega''} \ (\textit{respectively } 0) \, .$$

Here, $sha(\boldsymbol{\omega}', \boldsymbol{\omega}'')$ (respectively $she(\boldsymbol{\omega}', \boldsymbol{\omega}'')$) denotes the set of all sequences $\boldsymbol{\omega}$ deducible from $\boldsymbol{\omega}'$ and $\boldsymbol{\omega}''$ under plain (respectively contracting[27]) shufflings. The main symmetry-types get exchanged under pre- or post-composition by special constant-type moulds. Thus

$$\textit{symmetral}^\bullet = \exp(Id^\bullet) \circ \textit{alternal}^\bullet \ , \ \textit{alternel}^\bullet = \textit{alternal} \circ \log(\mathbf{1}^\bullet + Id^\bullet)$$

$$\textit{symmetrel}^\bullet - \mathbf{1}^\bullet = \textit{elternel}^\bullet = \big(\exp(Id^\bullet) - \mathbf{1}^\bullet\big) \circ \textit{alternal}^\bullet \circ \log(\mathbf{1}^\bullet + Id^\bullet).$$

Hairsplitting though it may seem, the distinction between *symmetrel* and *elternel* should be maintained throughout: *symmetral* or *symmetrel* moulds are stable under multiplication, whereas *alternal* and *elternel* moulds are stable under composition. Likewise, *alternal* and *alternel* moulds are stable under the Lie bracket *lu*.

Pre- respectively post-composition of *alternal* moulds by $c^{-1} tanh(c\, Id^\bullet)$ respectively $c^{-1}\, arctanh(c\, Id^\bullet)$ (chiefly for $c = 1, 1/2, i, i/2$) generates new symmetry types, signalled by one or two "o" vowels in their name. Though second in importance and frequency of occurrence to the four main symmetry types, these new exotic types are of more than marginal importance, especially in this investigation. They will repeatedly occur in connection with the *mediators*, the *medial* alien operators, and the multi-tangents $To^\bullet$, $Too^\bullet$.

Moulds of *symmetral, symmetrel, or c-symmetrol*[28] type generate three multiplicative groups and their multiplicative inverses are given by simple

[27] *I.e.* allowing order-compatible, pairwise contactions $(\omega'_i, \omega''_j) \mapsto \omega'_i + \omega''_j$ of elements from the parent sequences.

[28] *I.e.* moulds of type $\textit{symmetral}^\bullet \circ \big(c^{-1}\, tanh(c\, Id^\bullet)\big)$ or $\textit{symmetrel}^\bullet \circ \big(\frac{Id^\bullet}{\mathbf{1}^\bullet - \frac{1}{2} Id^\bullet}\big)$ if $c = \frac{1}{2}$.

involution formulae:

$$\mathrm{muS}^\bullet = \mathrm{anti.S}^\bullet \circ (-Id^\bullet) \qquad \textit{if} \quad \mathrm{S}^\bullet \in \textit{symmetral} \tag{2.4}$$
$$\mathrm{muS}^\bullet = \mathrm{anti.S}^\bullet \circ \Big(-\frac{Id^\bullet}{\mathbf{1}^\bullet + Id^\bullet}\Big) \qquad \textit{if} \quad \mathrm{S}^\bullet \in \textit{symmetrel} \tag{2.5}$$
$$\mathrm{muS}^\bullet = \mathrm{anti.S}^\bullet \circ (-Id^\bullet) \qquad \textit{if} \quad \mathrm{S}^\bullet \in \textit{c-symmetrol} \tag{2.6}$$

with *anti* $S^{\omega_1,\dots,\omega_r} := S^{\omega_r,\dots,\omega_1}$.

Main moulds relevant to our investigation.

symmetrel	*symmetral*	*symmetrol*		
$\mathrm{ze}^\bullet$	$\mathrm{za}^\bullet$	$\mathrm{zo}^\bullet$	*scalar-valued*	(*multizetas*)
$\widetilde{\mathrm{Se}}^\bullet(z)$	$\widetilde{\mathrm{Sa}}^\bullet(z)$	$\widetilde{\mathrm{So}}^\bullet(z)$	*resurgent-valued*	(*resur.monomials*)
$\mathrm{Te}^\bullet(z)$	$\mathrm{Ta}^\bullet(z)$	$\mathrm{To}^\bullet(z)$	*meromorphic-va.*	(*multitangents*)
elternel	*alternal*	*olternol*		
$\mathrm{Tee}^\bullet(z)$	$\mathrm{Taa}^\bullet(z)$	$\mathrm{Too}^\bullet(z)$	*meromorphic-va.*	(*multitangents*)
$\mathrm{Tee}^\bullet_\omega$	$\mathrm{Taa}^\bullet_\omega$	$\mathrm{Too}^\bullet_\omega$	*scalar-valued*	(*multizeta sums*)

2.2 Multizetas

In this subsection, all indices s_i are in $\mathbb{N}^*$ and, to preempt divergence, we (provisionally) assume $s_1 \neq 1$ for multizetas and $s_1, s_r \neq 1$ for multitangents.

We first consider three multizeta-valued moulds, $ze^\bullet$, $za^\bullet$ and $zo^\bullet$:

$$\mathrm{ze}^{s_1,\dots,s_r} := \sum_{n_1>\dots>n_r>0} n_1^{-s_1}\dots n_r^{-s_r} \tag{2.7}$$
$$\mathrm{za}^{s_1,\dots,s_r} := \sum_{n_1\geq\dots\geq n_r>0} n_1^{-s_1}\dots n_r^{-s_r} \prod \frac{1}{r_j!} \tag{2.8}$$
$$\mathrm{zo}^{s_1,\dots,s_r} := \sum_{n_1\geq\dots\geq n_r>0} n_1^{-s_1}\dots n_r^{-s_r} \prod 2^{1-r_j} \tag{2.9}$$

If the monomial $\prod n_i^{-s_i}$ in (2.8) or (2.9) involves t clusters of $r_1, \dots, r_t$ identical integers n_i ($1 \leq t \leq r$), the multiplicity corrections have to be defined accordingly, as $\prod 1/r_j!$ or $\prod 2^{1-r_j}$. Clearly

$$\mathrm{za}^\bullet = \mathrm{ze}^\bullet \circ (\exp(Id^\bullet) - \mathbf{1}^\bullet) \tag{2.10}$$
$$\mathrm{zo}^\bullet = \mathrm{ze}^\bullet \circ \left(\frac{Id^\bullet}{\mathbf{1}^\bullet - \frac{1}{2}Id^\bullet}\right) = \mathrm{za}^\bullet \circ \left(2\,\mathrm{arctanh}(\frac{1}{2}Id^\bullet\right) \tag{2.11}$$

The moulds $ze^\bullet$ and $za^\bullet$ are obviously *symmetrel* and *symmetral*, while $zo^\bullet$ falls into a subaltern symmetry type: *symmetrol* (see Section 5.1).

Fast computation of the multizetas. Our two guiding concerns here are: replacing the sluggish rate of convergence of the series (2.7), (2.8), (2.9) by a *geometric rate* of convergence and making manifest the multitzetas' hidden *parity* properties.

Let $trunze_n^\bullet$ be the *truncated* multizetas, defined as in (2.7) but with summation over $n \geq n_1 > \dots n_r > 0$, and let $remze_n^\bullet$ be the *remainder* multizetas, defined again as in (2.7) but with summation over $+\infty \geq n_1 > \dots n_r > n$. Let $trunza_n^\bullet, trunzo_n^\bullet$ and $remza_n^\bullet, remzo_n^\bullet$ be similarly defined. The symmetry types are preserved, so too are the relations (2.10)-(2.11), and we have obvious mould factorisations

$$\mathrm{ze}^\bullet = \mathrm{remze}_n^\bullet \times \mathrm{trunze}_n^\bullet \tag{2.12}$$

$$\mathrm{za}^\bullet = \mathrm{remza}_n^\bullet \times \mathrm{trunza}_n^\bullet \tag{2.13}$$

$$\mathrm{zo}^\bullet = \mathrm{remzo}_n^\bullet \times \mathrm{trunzo}_n^\bullet\,. \tag{2.14}$$

Using the elementary difference equations (in n) verified by $remze_n^\bullet$, we find for that mould a divergent but Borel resummable (and resurgent) asymptotic expansion $asremze_n^\bullet$, in decreasing powers of n, of the form:

$$\begin{aligned} \mathrm{asremze}^{s_1,\dots,s_r} &= \frac{e^{\partial}}{1-e^{\partial}}\, n^{-s_r}\, \frac{e^{\partial}}{1-e^{\partial}}\, n^{-s_{r-1}} \dots \frac{e^{\partial}}{1-e^{\partial}}\, n^{-s_1} \\ &= \frac{1}{n^{s_1+\dots+s_r-r}} \prod_{1\leq i\leq r} \frac{1}{s_1+\dots+s_i-i} + o\Big(\frac{1}{n^{s_1+\dots+s_r-r}}\Big). \end{aligned} \tag{2.15}$$

Here $\partial := \partial_n$ and each operator $\frac{e^\partial}{1-e^\partial} = -\,\partial^{-1} - \frac{1}{2} - \frac{1}{12}\,\partial + \dots$ in (2.15) acts on everything standing to its right. The last two asymptotic series factor into:

$$\mathrm{asremza}_n^\bullet = \underline{\mathrm{asremza}}_n^\bullet \times \left(\frac{2}{\mathbf{1}^\bullet + e^{I_n^\bullet}}\right) \tag{2.16}$$

$$\mathrm{asremzo}_n^\bullet = \underline{\mathrm{asremzo}}_n^\bullet \times \left(\frac{I_n^\bullet}{\mathbf{1}^\bullet - \frac{1}{2}\, I_n^\bullet}\right) \tag{2.17}$$

with elementary right factors involving the moulds $I_n^\bullet$ and $K_n^\bullet = 2\,\big(\mathbf{1}^\bullet + e^{I_n^\bullet}\big)^{-1}$

$$I_n^{s_1,\dots,s_r} = 0 \quad \text{if} \quad r \neq 1 \quad \textit{and} \quad I_n^{s_1} := n^{-s_1} \ , \ I_n^{\emptyset} := 0 \tag{2.18}$$

$$K_n^{s_1,\dots,s_r} = \kappa_r\, n^{-(s_1+\dots+s_r)} \qquad \textit{with} \qquad \frac{2}{1+e^t} =: \sum \kappa_r\, t^r \tag{2.19}$$

and with non elementary but essentially (up to an elementary power of n) *even* left factors of the form

$$\underline{\mathrm{asremza}}_n^{s_1,\dots,s_r} \quad \textit{and} \quad \underline{\mathrm{asremzo}}_n^{s_1,\dots,s_r} \in n^{r-(s_1+\dots+s_r)}\, \mathbb{C}[[n^{-2}]]. \tag{2.20}$$

There is, however, a significant difference between the two factorisations. Whereas we can see, by post-composing (2.15) by $Id^\bullet \times (\mathbf{1}^\bullet - Id^\bullet)^{-1}$, that $\underline{asremzo}^\bullet$ is given by a simple induction:

$$\underline{\mathrm{asremzo}}^{s_1,\ldots,s_r} = H(\partial)\, n^{-s_r}\, H(\partial)\, n^{-s_{r-1}} \ldots H(\partial)\, n^{-s_1} \qquad (2.21)$$

with $H(\partial) := \frac{e^\partial}{1-e^\partial} + \frac{1}{2} = -\frac{1}{2}\mathrm{cotanh}(\frac{1}{2}\partial)$, no such induction holds for $\underline{asremza}^\bullet$. That moulds admits only indirect definitions, like:

$$\underline{\mathrm{asremza}}^\bullet = \underline{\mathrm{asremzo}}^\bullet \circ \Big(2\ \tanh(\frac{1}{2}\, Id^\bullet)\Big) \qquad (2.22)$$

or

$$\underline{\mathrm{asremza}}^{s_1,\ldots,s_r} = \Big[\mathcal{SA}^{d_1,\ldots,d_r}. \prod_{1\le i\le r} n_i^{-s_i}\Big]_{n_i=n} \qquad (2.23)$$

with

$$\mathcal{SA}^\bullet := \Big(\mathcal{SE}^\bullet \times (\mathbf{1}^\bullet + Id^\bullet)\Big) \circ \Big(\exp(Id^\bullet) - \mathbf{1}^\bullet\Big) \qquad (2.24)$$

and with the important symmetrel mould $\mathcal{SE}^\bullet$:

$$\mathcal{SE}^{d_1,\ldots,d_r} := \prod_{1\le i\le r} \frac{e^{d_1+\cdots+d_i}}{1-e^{d_1+\cdots+d_i}} \qquad (2.25)$$

The first definition (2.22) results directly from (2.11) restriced to the remainders. The second definition calls for some explanations. Here, each d_i denotes the operator ∂_{n_i} that acts on n_i alone. On the right-hand side of (2.23), we let the operator $\mathcal{SA}^{\boldsymbol{d}}$ act on the product $\prod n_i^{-s_i}$ and then set $n_i := n$. To establish (2.23), we observe that (2.15) may be written

$$\mathrm{asremze}^{s_1,\ldots,s_r} = \Big[\mathcal{SE}^{d_1,\ldots,d_r}. \prod_{1\le i\le r} n_i^{-s_i}\Big]_{n_i=n} \qquad (2.26)$$

and we then use the relation $asremza^\bullet = asremze^\bullet \circ (\exp(Id^\bullet) - \mathbf{1}^\bullet)$ that results from restricting (2.10) to the remainders. The interesting point about (2.23) is that it relates the *parity* property (2.20) of $\underline{asremza}^\bullet$ to the following *parity* property of $\mathcal{SE}^\bullet$

$$\mathrm{neg}.\mathcal{SE}^\bullet = \Big(\mathcal{SE}^\bullet \times (\mathbf{1}^\bullet + Id^\bullet)\Big) \circ \Big(-\frac{Id^\bullet}{\mathbf{1}^\bullet + Id^\bullet}\Big) \qquad (2.27)$$

and to the formula for its multiplicative inverse $mu\mathcal{SE}^\bullet$:

$$\mathrm{mu}\mathcal{SE}^\bullet = e^{|\bullet|}\ \mathrm{anti.neg}.\mathcal{SE}^\bullet \qquad (2.28)$$

with

$$|(s_1, \ldots, s_r)| = \sum s_i\ ,\ \ neg.S^{s_1,\ldots,s_r} := S^{-s_1,\ldots,-s_r}\ ,\ \ anti.S^{s_1,\ldots,s_r} := S^{s_r,\ldots,s_1}$$

Acceleration of the convergence . When we calculate $ze^\bullet$ according to formula (2.12) by taking the exact value of the truncated factor $trunze_n^\bullet$ and calculating the remainder factor $remze_n^\bullet$ from its asymptotic expansion (2.15) cut off at the least term, we get an excellent approximation, with an error that decreases roughly like $\exp(-2\pi n)$ as the truncation order n increases. The same applies to $za^\bullet$ and $zo^\bullet$: the truncated factors $trunza^\bullet$ and $trunzo^\bullet$ may have more summands than $trunze^\bullet$, but this is more than offset by the parity simplifications in the remainder factors $remza^\bullet$ and especially $remzo^\bullet$.

We may note that this method remains valid, and retains its high efficiency, for general complex values of the weights s_i, *even when the inequalities* $\Re(s_1+\ldots+s_i) > i$ *that guarantee the convergence of* (2.7)-(2.9) *no longer hold.*

Quadratic constraints. The symmetrelity of $ze^\bullet$, or the strictly equivalent symmetries of $za^\bullet$ and $zo^\bullet$, do not exhaust the set of algebraic constraints on the multizetas: there exists an another set of constraints, of 'equal strength', based on a radically different, essentially discrete[29] encoding: see Section 6.2.

2.3 Multitangents

The multizetas enter invariant analysis *indirectly*, as scalars attached to elementary periodic meromorphic functions – the so-called *multitangents*.

Here are the main multitangent-valued moulds with their symmetry types:

$$\begin{array}{ccccc} \mathrm{Te}^\bullet & \stackrel{1}{\to} & \mathrm{Ta}^\bullet & \stackrel{2}{\to} & \mathrm{To}^\bullet \\ \downarrow 3 & & \downarrow 4 & & \downarrow 5 \\ \mathrm{Tee}^\bullet & \stackrel{1}{\to} & \mathrm{Taa}^\bullet & \stackrel{2}{\to} & \mathrm{Too}^\bullet \end{array} \qquad \begin{array}{ccccc} \textit{symmetrel} & \stackrel{1}{\to} & \textit{symmetral} & \stackrel{2}{\to} & \textit{symmetrol} \\ \downarrow 3 & & \downarrow 4 & & \downarrow 5 \\ \textit{elternel} & \stackrel{1}{\to} & \textit{alternal} & \stackrel{2}{\to} & \textit{olternol} \end{array}$$

The two upper moulds are defined directly by[30]

$$\mathrm{Te}^{s_1,\ldots,s_r}(z) := \sum_{n_1>\ldots>n_r} (n_1+z)^{-s_1}\ldots(n_r+z)^{-s_r} \tag{2.29}$$

$$\mathrm{Ta}^{s_1,\ldots,s_r}(z) := \sum_{n_1\geq\ldots\geq n_r} (n_1+z)^{-s_1}\ldots(n_r+z)^{-s_r} \prod \frac{1}{r_i!} \tag{2.30}$$

$$\mathrm{To}^{s_1,\ldots,s_r}(z) := \sum_{n_1\geq\ldots\geq n_r} (n_1+z)^{-s_1}\ldots(n_r+z)^{-s_r} \prod 2^{1-r_i} \tag{2.31}$$

[29] Unlike the s_i-encoding, which of course extends to the complex field.

[30] With the same r_j in (2.30) as in (2.8).

and the two lower moulds are derived from them through a suitable pre-composition. Thus:

$$\begin{array}{lll} \mathrm{Te}^\bullet = \textit{see } (2.29) & \mathrm{Tee}^\bullet = \mathrm{Te}^\bullet - \mathbf{1}^\bullet & (Te\text{-}Tee) \\ \mathrm{Ta}^\bullet = \mathrm{Te}^\bullet \circ \left(e^{Id^\bullet} - \mathbf{1}^\bullet\right) & \mathrm{Taa}^\bullet = \log(\mathbf{1}^\bullet + Id^\bullet) \circ \mathrm{Ta}^\bullet \circ \left(e^{Id^\bullet} - \mathbf{1}^\bullet\right) & (Ta\text{-}Taa) \\ \mathrm{To}^\bullet = \mathrm{Te}^\bullet \circ \left(\frac{Id^\bullet}{\mathbf{1}^\bullet - \frac{1}{2} Id^\bullet}\right) & \mathrm{Too}^\bullet = \left(\frac{Id^\bullet}{\mathbf{1}^\bullet + \frac{1}{2} Id^\bullet}\right) \circ \mathrm{Te}^\bullet \circ \left(\frac{Id^\bullet}{\mathbf{1}^\bullet - \frac{1}{2} Id^\bullet}\right) & (To\text{-}Too) \end{array}$$

In the sequel, we shall also require the inverses of $Te^\bullet$, $Ta^\bullet$, $To^\bullet$ for mould multiplication. In view of (2.4)-(2.6), we get

$$\mathrm{muTe}^{s_1,\ldots,s_r}(z) = \sum_{n_1 \le \ldots \le n_r} (-1)^r (n_1+z)^{-s_1} \ldots (n_r+z)^{-s_r} \tag{2.32}$$

$$\mathrm{muTa}^{s_1,\ldots,s_r}(z) = \sum_{n_1 \le .. \le n_r} (-1)^r (n_1+z)^{-s_1} \ldots (n_r+z)^{-s_r} \prod \frac{1}{r_i!} \tag{2.33}$$

$$\mathrm{muTo}^{s_1,\ldots,s_r}(z) = \sum_{n_1 \le \ldots \le n_r} (-1)^r (n_1+z)^{-s_1} \ldots (n_r+z)^{-s_r} \prod n^{1-r_i} \tag{2.34}$$

with an order reversal in the summation rule, and large inequalities in place of the strict inequalities in (2.29)-(2.31).

Parity aspects. All six types of multitangents obviously verify

$$\mathrm{T}^{s_1,\ldots,s_r}(-z) \equiv (-1)^{s_1+\cdots+s_r}\, \mathrm{T}^{s_r,\ldots,s_1}(z) \quad (\forall\, \mathrm{T} \in \{\mathrm{Te}, \mathrm{Ta}, \mathrm{To}\ \textit{etc.}\}). \tag{2.35}$$

In the case of $Taa^\bullet$ and $Too^\bullet$, however, due to alternality/olternolity we have an additional relation

$$\mathrm{Taa}^{s_r,\ldots,s_1}(z) \equiv (-1)^{r-1}\, \mathrm{Taa}^{s_1,\ldots,s_r}(z) \tag{2.36}$$

$$\mathrm{Too}^{s_r,\ldots,s_1}(z) \equiv (-1)^{r-1}\, \mathrm{Too}^{s_1,\ldots,s_r}(z). \tag{2.37}$$

Combining (2.35) and (2.36)-(2.37) we get the crucial *parity separation* property, which sets $Taa^\bullet$, $Too^\bullet$ apart from $Te^\bullet \approx Tee^\bullet$:

$$\mathrm{Taa}^{s_1,\ldots,s_r}(-z) \equiv (-1)^{1+\sum d_i}\, \mathrm{Taa}^{s_1,\ldots,s_r}(z) \quad \textit{with } d_i := s_i - 1 \tag{2.38}$$

$$\mathrm{Too}^{s_1,\ldots,s_r}(-z) \equiv (-1)^{1+\sum d_i}\, \mathrm{Too}^{s_1,\ldots,s_r}(z) \quad \textit{with } d_i := s_i - 1 \tag{2.39}$$

Multitangents in terms of monotangents and multizetas. Multitangents are entirely determined by their polar parts at the entire points $z = n$. By calculating, based on the expansion (2.29), the Laurent expansion of $Te^s(z)$ at such points, and then retaining only the polar part, we find that $Te^s(z)$ can be expressed as a finite sum of elementary monotangents

$Te^{s_1}(z) = \sum_{n_1}(n_1+z)^{-s_1}$, also known as Eisenstein series. Here is the formula:[31]

$$\mathrm{Te}^{s_1,\dots,s_r}(z) = \sum_{\sigma=2}^{\sup(s_i)} \mathrm{teze}_{\sigma}^{s_1,\dots,s_r}\,\mathrm{Te}^{\sigma}(z) = \sum_{i=1}^{r}\sum_{\sigma_i=2}^{s_i} \mathrm{teze}_{i,\sigma_i}^{s_1,\dots,s_r}\,\mathrm{Te}^{\sigma_i}(z) \tag{2.40}$$

with

$$\mathrm{teze}_{i,\sigma_i}^{s_1,\dots,s_r} = \\ = \sum_{\substack{\sum\sigma_k=\sum s_k \\ \left\{\substack{\sigma_i\le s_i \\ s_j\le\sigma_j\,(j\ne i)}\right\}}} \mathrm{ze}^{\sigma_1,\dots,\sigma_{i-1}}\,\mathrm{ze}^{\sigma_r,\dots,\sigma_{i+1}} \prod_{j=1}^{i-1}(-1)^{\sigma_j} \prod_{1\le j\le r}^{j\ne i} \frac{(-1)^{s_j}(\sigma_j-1)!}{(\sigma_j-s_j)!(s_j-1)!}$$

or more symmetrically

$$\mathrm{teze}_{i,\sigma_i}^{s_1,\dots,s_r} = \\ = \sum_{\substack{\sum\sigma_k=\sum s_k \\ \left\{\substack{\sigma_i\le s_i \\ s_j\le\sigma_j\,(j\ne i)}\right\}}} \mathrm{ze}^{\sigma_1,\dots,\sigma_{i-1}}\,(-1)^{s_i-\sigma_i}\,\mathrm{vize}^{\sigma_{i+1},\dots,\sigma_r} \prod_{1\le j\le r}^{j\ne i} \frac{(\sigma_j-1)!}{(\sigma_j-s_j)!(s_j-1)!}$$

$$\mathrm{vize}^{s_1,\dots,s_r} = (-1)^{s_1+\dots s_r}\,\mathrm{ze}^{s_r,\dots,s_1}. \tag{2.41}$$

The leading monotangent $Te^1(z) = \frac{\pi}{tan(\pi z)}$ generates all others under differentiation, and admits the following northern and southern expansions:

$$\mathrm{Te}^1_{\mathrm{no}}(z) = -\pi i - 2\pi i \sum_{0<n} e^{+2\pi i\,n\,z} \qquad \text{if } \Im(z) > 0 \tag{2.42}$$

$$\mathrm{Te}^1_{\mathrm{so}}(z) = +\pi i + 2\pi i \sum_{0<n} e^{-2\pi i\,n\,z} \qquad \text{if } \Im(z) < 0. \tag{2.43}$$

Since $\mathrm{Te}^{s_1}(z) = \frac{(-1)^{s_1-1}}{(s_1-1)!}\,\partial_z^{s_1-1}\,\mathrm{Te}^1(z)$, this yields

$$\mathrm{Te}^{s_1}(z) = \sum_{\omega\in\Omega^{\mp}} \mathrm{Te}^{s_1}_{\omega}\,e^{-\omega z} \quad \text{on each half-plane} \quad \pm\Im(z) > 0 \tag{2.44}$$

with

$$\mathrm{Te}^{s_1}_{\omega} = \mathrm{sign}(\Im(\omega))\,2\pi i\,\frac{\omega^{s_1-1}}{(s_1-1)!} \quad \text{and} \quad \Omega^{\mp} = 2\pi i\mathbb{Z}^{\mp}. \tag{2.45}$$

[31] For a more compact expression, based on generating series, see Section 6.3.

All the above amounts to a simple procedure for calculating the Fourier expansions, north and south, of the four classes of multitangents. The three classes $\mathcal{T}ee^\bullet \approx \mathcal{T}e^\bullet$, $\mathcal{T}aa^\bullet$, $\mathcal{T}oo^\bullet$ shall be of direct concern to us:

$$\mathrm{Tee}^\bullet_{\mathrm{no}}(z) = \sum_{\omega\in\Omega^-} \mathrm{Tee}^\bullet_\omega \; e^{-\omega z} \quad ; \quad \mathrm{Tee}^\bullet_{\mathrm{so}}(z) = \sum_{\omega\in\Omega^+} \mathrm{Tee}^\bullet_\omega \; e^{-\omega z} \tag{2.46}$$

$$\mathrm{Taa}^\bullet_{\mathrm{no}}(z) = \sum_{\omega\in\Omega^-} \mathrm{Taa}^\bullet_\omega \; e^{-\omega z} \quad ; \quad \mathrm{Taa}^\bullet_{\mathrm{so}}(z) = \sum_{\omega\in\Omega^+} \mathrm{Taa}^\bullet_\omega \; e^{-\omega z} \tag{2.47}$$

$$\mathrm{Too}^\bullet_{\mathrm{no}}(z) = \sum_{\omega\in\Omega^-} \mathrm{Too}^\bullet_\omega \; e^{-\omega z} \quad ; \quad \mathrm{Too}^\bullet_{\mathrm{so}}(z) = \sum_{\omega\in\Omega^+} \mathrm{Too}^\bullet_\omega \; e^{-\omega z} \tag{2.48}$$

Localisation constraints. When dealing with a product of multitangents $\mathcal{T}e^{\boldsymbol{s}}$, we may perform the operations of *reduction* (of mutitangents into sums of monotangents) and *symmetrel linearisation* in either order. If we then identify the multizeta superpositions in front of each monotangent, we get to the so-called reduction constraints:

$$\begin{array}{ccc}
\mathrm{Te}^{\boldsymbol{s^1}}(z).\mathrm{Te}^{\boldsymbol{s^2}}(z) & \stackrel{\textit{reduction}}{\longrightarrow} & \left(\sum \tau^{\boldsymbol{s^1}}_{s_1}\mathrm{Te}^{s_1}(z)\right).\left(\sum \tau^{\boldsymbol{s^2}}_{s_2}\mathrm{Te}^{s_2}(z)\right) \\
\downarrow \textit{linearisation} & & \downarrow \textit{linearisation} \\
\sum \epsilon^{\boldsymbol{s^1},\boldsymbol{s^2}}_{\boldsymbol{s^3}}\mathrm{Te}^{\boldsymbol{s^3}}(z) & \stackrel{\textit{reduction}}{\longrightarrow} & \sum \epsilon^{\boldsymbol{s^1},\boldsymbol{s^2}}_{\boldsymbol{s^3}}\tau^{\boldsymbol{s^3}}_{s_3}\mathrm{Te}^{s_3}(z) = \sum \tau^{\boldsymbol{s^1}}_{s_1}\tau^{\boldsymbol{s^2}}_{s_2}\epsilon^{s_1,s_2}_{s_3}\mathrm{Te}^{s_3}(z)\,.
\end{array}$$

Here, the $\epsilon^{\boldsymbol{s^i},\boldsymbol{s^j}}_{\boldsymbol{s^k}}$ are elementary, integer-valued coefficients and the expressions $\tau^{\boldsymbol{s^i}}_{s_j}$ are finite, homogeneous sums of multizetas of total weight $\|\boldsymbol{s^i}\| - s_i - 1$.

If, instead of *reduction*, we use *localisation* (replacing each multitangent by its two-sided Laurent expansion at $z = 0$), we get the so-called localisation constraints:

$$\begin{array}{ccc}
\mathrm{Te}^{\boldsymbol{s^1}}(z).\mathrm{Te}^{\boldsymbol{s^2}}(z) & \stackrel{\textit{localisation}}{\longrightarrow} & (\sum \theta^{\boldsymbol{s^1}}_{n_1} z^{n_1}).(\sum \theta^{\boldsymbol{s^2}}_{n_2} z^{n_2}) \\
\downarrow \textit{linearisation} & & \downarrow \textit{linearisation} \\
\sum \epsilon^{\boldsymbol{s^1},\boldsymbol{s^2}}_{\boldsymbol{s^3}}\mathrm{Te}^{\boldsymbol{s^3}}(z) & \stackrel{\textit{localisation}}{\longrightarrow} & \sum \epsilon^{\boldsymbol{s^1},\boldsymbol{s^2}}_{\boldsymbol{s^3}}\theta^{\boldsymbol{s^3}}_{n_3} z^{n_3} = \sum \theta^{\boldsymbol{s^1}}_{n_1}\theta^{\boldsymbol{s^2}}_{n_2} z^{n_1+n_2}
\end{array}$$

with expressions $\theta^{\boldsymbol{s^i}}_{n_j}$ that are again finite, homogeneous sums of multizetas of total weight $\|\boldsymbol{s^i}\| + n_j$.

Though more numerous, the localisation constraints are actually equivalent to the reduction constraints, but they extend more smoothly to the ramified case, *i.e.* to the case of multitangents and multizetas that carry fractional indices s_i. In any case, the localisation constraints are *not* a consequence of the symmetrelness of $\mathcal{T}e^\bullet$.

The multitangents $Taa^\bullet$ **and** $Too^\bullet$ **in terms of** $Tee^\bullet \approx Te^\bullet$**.** Applying to $Too^\bullet$ a beautiful formula (see (5.16)-(5.17) in Section 5.4) that holds for multitangents $Te_\diamond^\bullet$ of *any* symmetry type and gives their explicit linearisation into sums of symmetrel multitangents $Te^\bullet$, we find:

$$\mathrm{Too}^{s_1,\ldots,s_r}(z) = \sum_{\sigma\in\mathfrak{S}_r}\ \sum_{2\le t\le r}\ \sum_{r_1+\cdots+r_t=r}^{(\mathcal{I}_1,\ldots,\mathcal{I}_t)\#\sigma} (-1)^{q(\sigma)}\, 2^{1-r}\ \mathrm{Te}^{s_{\sigma,r_1},\ldots,s_{\sigma,r_t}}(z) \tag{2.49}$$

$$\textit{with } s_{\sigma,j} := \sum_{k\in\mathcal{I}_j} s_\sigma(k) \ \ \textit{and} \ \ q(\sigma) := \#\left\{k\, k<r,\, \sigma^{-1}(k)>\sigma^{-1}(k+1)\right\}$$

The summation is over all permutations σ of r elements and, for each σ, over all partitions of $[1,\ldots,r]$ into intervals $\mathcal{I}_i$ of r_i elements, whereby we demand that the partition $(\mathcal{I}_1,\ldots,\mathcal{I}_r)$ be *'orthogonal'* to σ, *i.e.* such that

(i) on any given $\mathcal{I}_j$ the permutation σ assumes no two consecutive values;
(ii) σ increases on each interval $\mathcal{I}_j$.

In other words, we should have $\{k, k{+}1\}\in\mathcal{I}_j \Rightarrow \{\sigma(k+1)-\sigma(k)\ge 2\}$. The orthogonality condition proper is (i). The condition (ii) is there simply to ensure that any given summand $Te^{s_{\sigma,r_1},\ldots,s_{\sigma,r_t}}$ is counted only once. Lastly, $q(\sigma)$ measures the incompatibility of the natural order $<$ on $[1,\ldots,r]$ with the σ-induced order $\{i <_\sigma j\} \Leftrightarrow \{\sigma(i) < \sigma(j)\}$. Indeed, if j is not $<_\sigma$-maximal and j^+ denotes the $<_\sigma$-successor of j, we have $q(\sigma) = \#\{j\,;\, j > j^+\}$.

When applied to $Taa^\bullet$, the general formula (5.16)-(5.17) produces a similar expansion, but with more numerous $Te^\bullet$-summands and, in front of each of them, rational coefficients whose numerators possess no simple multiplicative structure.[32] They may be calculated, though, by applying the universal formula (5.17).

Remark. $Taa^\bullet$ **better than** $Te^\bullet$ **and** $Too^\bullet$ **better than** $Taa^\bullet$**.**
Actually, a systematic comparison would show that, of all types $Te_\diamond^\bullet$ of multitangents that possess the desirable parity property (2.38)-(2.39), $Taa^\bullet$ and especially $Too^\bullet$ are the simplest choices, not only where $Te^\bullet$-linearisation is concerned, but in most other respects.

$Taa^\bullet$ and $Too^\bullet$ even compare favourably with $Te^\bullet$, which in any case does not verify the parity property(2.38)-(2.39). $Taa^\bullet$ and $Too^\bullet$ may lack

[32] Although, for r small, they seem to be all equal to 1. This, however, is deceptive.

a simple direct definition like that of $Te^\bullet$, but after reduction to monotangents, it is $Taa^\bullet$ and especially $Too^\bullet$, not $Te^\bullet$, that give rise, by and large, to the simpler expansions[33], as shown by the Tables of Section 9.

2.4 Resurgence monomials

There exists an alternative, *resurgent* approach to multitangent reduction. In the convergent (*i.e.* $s_1, s_r \neq 1$) and non-ramified (*i.e.* $s_j \in \mathbb{N}^*$ rather than $\mathbb{Q}^*$) case, it hardly improves on the above procedure (see Section 2.3) but in the general case, especially when we go over to fractional indices s_j, the resurgent approach becomes the more flexible of the two methods and even, in a sense, the only practical one. For clarity, though, we first keep our two simplifying assumptions – no divergence[34] and no ramification[35] – to sketch this alternative method.

Multizetaic monomials in the formal model. We shall set about constructing three elementary resurgent-valued moulds [36] $\tilde{\mathrm{Se}}^\bullet(z)$, $\tilde{\mathrm{Sa}}^\bullet(z)$, $\tilde{\mathrm{So}}^\bullet(z)$, beginning with the *formal model*. We start with the symmetrel monomials $\tilde{\mathrm{Se}}^{\boldsymbol{s}}(z)$. They are defined by:

$$\tilde{\mathrm{Se}}^\bullet(z) = \frac{e^{\partial_z}}{(1-e^{\partial_z})}\left(\tilde{\mathrm{Se}}^\bullet(z) \times \mathrm{J}^\bullet(z)\right) \tag{2.50}$$

with an elementary mould $\mathbf{J}^\bullet(z)$:

$$\mathrm{J}^\emptyset(z) := 0 \quad ; \quad \mathrm{J}^{s_1}(z) := z^{-s_1} \quad ; \quad \mathrm{J}^{s_1,\ldots,s_r}(z) := 0 \quad (\forall r \geq 2). \tag{2.51}$$

Together with the conditions $\tilde{\mathrm{Se}}^\emptyset(z) = 1$, $\tilde{\mathrm{Se}}^{s_1,\ldots,s_r}(\infty) = 0$ ($\forall r \geq 1$) the induction (2.50) uniquely defines each $\tilde{\mathit{Se}}^{\boldsymbol{s}}(z)$ as a constant-free, formal power series in z^{-1}. The companions monomials $\tilde{\mathit{Sa}}^\bullet(z)$, $\tilde{\mathit{So}}^\bullet(z)$ are then defined in the usual way, by post-composition:

$$\tilde{\mathrm{Sa}}^\bullet(z) := \tilde{\mathrm{Se}}^\bullet(z) \circ \left(\exp(Id^\bullet - \mathbf{1}^\bullet)\right) \tag{2.52}$$

$$\tilde{\mathrm{So}}^\bullet(z) := \tilde{\mathrm{Se}}^\bullet(z) \circ \left(\frac{Id^\bullet}{\mathbf{1}^\bullet - \frac{1}{2}Id^\bullet}\right). \tag{2.53}$$

[33] Especially after the symmetral linearisation of the multizetas occuring as scalar coefficients in these expansions.

[34] *I.e.* $s_1 > 1$

[35] *I.e.* $s_i \in \mathbb{N}^*$

[36] They must be distinguished from the similar moulds $asremze_n^\bullet$, $asremza_n^\bullet$, $asremzo_n^\bullet$, because the emphasis here will be on the convolutive model and the associated monics.

Multizetaic monomials in the convolutive model. In the *convolutive model* the induction becomes

$$\widehat{\mathrm{Se}}^{s_1,\ldots,s_r}(\zeta) = \frac{e^{-\zeta}}{(1-e^{-\zeta})} \int_0^{\zeta} \widehat{\mathrm{Se}}^{s_1,\ldots,s_{r-1}}(\zeta-\zeta_r)\ \frac{\zeta_r^{s_r-1}}{\Gamma(s_r)}\ d\zeta_r \quad (2.54)$$

Multizetaic monomials in the sectorial model. Lastly, in the *sectorial* or *'geometric' models* $+$ and $-$ (*east* and *west*), corresponding to Laplace integration along the axes $arg(\zeta)=0$ and $arg(\zeta)=\pi$, we get

$$\mathrm{Se}_+^{s_1,\ldots,s_r}(z) = \sum_{0<n_r<\ldots<n_1} (n_1+z)^{-s_1}\ldots(n_r+z)^{-s_r} \quad (2.55)$$

$$\mathrm{Se}_-^{s_1,\ldots,s_r}(z) = \sum_{n_1\leq..\leq n_r\leq 0} (-1)^r\ (n_1+z)^{-s_1}\ldots(n_r+z)^{-s_r} \quad (2.56)$$

$$\mathrm{muSe}_+^{s_1,\ldots,s_r}(z) = \sum_{0<n_1\leq\ldots\leq n_r} (-1)^r\ (n_1+z)^{-s_1}\ldots(n_r+z)^{-s_r} \quad (2.57)$$

$$\mathrm{muSe}_-^{s_1,\ldots,s_r}(z) = \sum_{n_r<..<n_1\leq 0} (n_1+z)^{-s_1}\ldots(n_r+z)^{-s_r} \quad (2.58)$$

For $S^\bullet = Sa^\bullet$ or $So^\bullet$ and multiplicity corrections $\chi(r_i) = 1/r_i!$ or 2^{1-r_i}, these expansions become respectively

$$\mathrm{S}_+^{s_1,\ldots,s_r}(z) = \sum_{0<n_r\leq\ldots\leq n_1} (n_1+z)^{-s_1}\ldots(n_r+z)^{-s_r}\,\chi(r_i) \quad (2.59)$$

$$\mathrm{S}_-^{s_1,\ldots,s_r}(z) = \sum_{n_1\leq..\leq n_r\leq 0} (-1)^r\ (n_1+z)^{-s_1}\ldots(n_r+z)^{-s_r}\,\chi(r_i) \quad (2.60)$$

$$\mathrm{muS}_+^{s_1,\ldots,s_r}(z) = \sum_{0<n_1\leq\ldots\leq n_r} (-1)^r\ (n_1+z)^{-s_1}\ldots(n_r+z)^{-s_r}\,\chi(r_i) \quad (2.61)$$

$$\mathrm{muS}_-^{s_1,\ldots,s_r}(z) = \sum_{n_r\leq..\leq n_1\leq 0} (n_1+z)^{-s_1}\ldots(n_r+z)^{-s_r}\,\chi(r_i) \quad (2.62)$$

Multizetaic monics. From the structure of the induction (2.50), one infers directly (without calculation) that our monomials verify resurgence equations of the form[37]

$$\Delta^+_\omega \,\mathrm{Se}^\bullet(z) = \mathrm{Tee}^\bullet_\omega \times \mathrm{Se}^\bullet(z) \quad (\forall\omega \in \Omega^+ = 2\pi i\mathbb{Z}^+) \tag{2.63}$$
$$\Delta^-_\omega \,\mathrm{Se}^\bullet(z) = \mathrm{Tee}^\bullet_\omega \times \mathrm{Se}^\bullet(z) \quad (\forall\omega \in \Omega^- = 2\pi i\mathbb{Z}^-) \tag{2.64}$$

$$+2\pi i\, \Delta_\omega \,\mathrm{Sa}^\bullet(z) = \mathrm{Taa}^\bullet_\omega \times \mathrm{Sa}^\bullet(z) \quad (\forall\omega \in \Omega^+ = 2\pi i\mathbb{Z}^+) \tag{2.65}$$
$$-2\pi i\, \Delta_\omega \,\mathrm{Sa}^\bullet(z) = \mathrm{Taa}^\bullet_\omega \times \mathrm{Sa}^\bullet(z) \quad (\forall\omega \in \Omega^- = 2\pi i\mathbb{Z}^-) \tag{2.66}$$

$$+2\pi i\, \Delta^\sharp_\omega \,\mathrm{So}^\bullet(z) = \mathrm{Too}^\bullet_\omega \times \mathrm{So}^\bullet(z) \quad (\forall\omega \in \Omega^+ = 2\pi i\mathbb{Z}^+) \tag{2.67}$$
$$-2\pi i\, \Delta^\sharp_\omega \,\mathrm{So}^\bullet(z) = \mathrm{Too}^\bullet_\omega \times \mathrm{So}^\bullet(z) \quad (\forall\omega \in \Omega^- = 2\pi i\mathbb{Z}^-) \tag{2.68}$$

with scalar-valued moulds $Tee^\bullet_\omega$, $Taa^\bullet_\omega$, $Too^\bullet_\omega$, whose symmetry types follow from their construction.[38] These three moulds, for the moment, need not bear any relation to their namesakes in Section 2.3, but we shall show that they actually coincide with them.

Writing down the axis-crossing identity (1.49) with (2.12) and $\theta = +\frac{\pi}{2}$ and the reverse identity (1.50) with (2.13) and $\theta = -\frac{\pi}{2}$, and minding the fact that

$$\mathrm{Se}^\bullet_{\frac{\pi}{2}\pm\epsilon} = \mathrm{Se}^\bullet_\mp \quad (\mathit{inversion!}) \quad ; \quad \mathrm{Se}^\bullet_{-\frac{\pi}{2}\pm\epsilon} = \mathrm{Se}^\bullet_\pm \quad (\mathit{no\ inversion!})$$

we find respectively

$$\mathrm{Te}^\bullet_{\mathrm{so}}(z) \times \mathrm{Se}^\bullet_{-,\mathrm{so}}(z) = \mathrm{Se}^\bullet_{+,\mathrm{so}}(z) \;\mathit{with}\; \mathrm{Te}^\bullet_{\mathrm{so}}(z) = \sum_{\omega\in\Omega^+} \mathrm{Te}^\bullet_\omega\, e^{-\omega z} \tag{2.69}$$

$$Te^\bullet_{\mathrm{no}}(z) \times \mathrm{Se}^\bullet_{-,\mathrm{no}}(z) = \mathrm{Se}^\bullet_{+,\mathrm{no}}(z) \;\mathit{with}\; \mathrm{Te}^\bullet_{\mathrm{no}}(z) = \sum_{\omega\in\Omega^-} \mathrm{Te}^\bullet_\omega\, e^{-\omega z} \tag{2.70}$$

Thus, whether looking “north” or “south”, we arrive at the elementary identity

$$\mathrm{Te}^\bullet(z) = \mathrm{Se}^\bullet_+(z) \times \mathrm{muSe}^\bullet_-(z) \tag{2.71}$$

which of course can also be directly derived from the definitions (2.29) paired with (2.59)-(2.62). But we get an interesting extra – namely, that the moulds $Tee^\bullet_\omega$ of (2.63) and (2.64) coincide with those defined in the preceding subsection. If we now interpret the resurgence equations (2.63)-(2.68) in the convolutive model, we get an alternative expression

[37] We drop the tilde for simplicity.

[38] $Taa^\bullet_\omega$ is alternal, while $\sum Tee^\bullet_\omega\, e^{-\omega z}$ (respectively $\sum Too^\bullet_\omega\, e^{-\omega z}$) is elternel (respectively olternol).

of $Tee^{\bullet}_{\omega}$, $Taa^{\bullet}_{\omega}$, $Too^{\bullet}_{\omega}$ as finite integrals in the ζ-plane, which translate, after some work, into fast-convergent power series. This will stand us in good stead in the *divergent* and above all in the *ramified* cases. But we must first devote a short aside to the question of parity.

Parity aspects. There is something slightly incongruous about the formulae (2.65)-(2.68): they express the monics $Taa^{\bullet}_{\omega}$, $Too^{\bullet}_{\omega}$, which separate parity, in terms of monomials $Sa^{\bullet}(z)$, $So^{\bullet}(z)$, which do not. To remove this blemish, let us replace them by parity-separating monomials $\underline{Sa}^{\bullet}(z)$, $\underline{So}^{\bullet}(z)$:

$$\widetilde{\mathrm{Sa}}^{\bullet}(z) = \widetilde{\underline{\mathrm{Sa}}}^{\bullet}(z) \times 2\,(\mathbf{1}^{\bullet} + e^{J^{\bullet}(z)})^{-1} \tag{2.72}$$

$$\widetilde{\mathrm{So}}^{\bullet}(z) = \widetilde{\underline{\mathrm{So}}}^{\bullet}(z) \times (\mathbf{1}^{\bullet} - \frac{1}{2}\,J^{\bullet}(z)) \tag{2.73}$$

with $J^{s_1}(z) := z^{-s_1}$ and $J^{s_1,\dots,s_r}(z) := 0$ if $r \neq 1$.

In the case of $\widetilde{\underline{\mathrm{So}}}^{\bullet}$, we get the bonus of a simple induction

$$\widetilde{\underline{\mathrm{So}}}^{\bullet}(z) := H(\partial)\left(\widetilde{\underline{\mathrm{So}}}^{\bullet}(z) \times J^{\bullet}(z)\right) \quad \textit{with} \tag{2.74}$$

$$H(\partial) := \frac{e^{\partial}}{1-e^{\partial}} + \frac{1}{2} = \frac{1}{2}\,\frac{1+e^{\partial}}{1-e^{\partial}} = -\frac{1}{2}\,\mathrm{cotan}\left(\frac{\partial}{2}\right). \tag{2.75}$$

Since the right factors in (2.72)-(2.73) are convergent, the new monomials verify the same resurgence equations as the old ones, with the same resurgence constants:

$$\pm 2\pi i\,\Delta_{\omega}\,\underline{\mathrm{Sa}}^{\bullet}(z) = \mathrm{Taa}^{\bullet}_{\omega} \times \underline{\mathrm{Sa}}^{\bullet}(z) \quad (\forall \omega \in \Omega^{\pm} = 2\pi i\mathbb{Z}^{\pm}) \tag{2.76}$$

$$\pm 2\pi i\,\Delta_{\omega}\,\underline{\mathrm{So}}^{\bullet}(z) = \mathrm{Taa}^{\bullet}_{\omega} \times \underline{\mathrm{So}}^{\bullet}(z) \quad (\forall \omega \in \Omega^{\pm} = 2\pi i\mathbb{Z}^{\pm}). \tag{2.77}$$

Remark. Our new monomials may separate parity and generate the required monics, but they no longer belong to the clear-cut symmetry types *symmetral/symmetrol*, a fact that is reflected in the unusual form of their multiplicative inverses:

$$\mathrm{mu}\widetilde{\underline{\mathrm{Sa}}}^{\bullet}(z) = \left(\cosh(J^{\bullet}(z))\right)^{-2} \times \mathrm{anti}.\widetilde{\underline{\mathrm{Sa}}}^{\bullet}(z) \circ (-Id^{\bullet}) \tag{2.78}$$

$$\mathrm{mu}\widetilde{\underline{\mathrm{So}}}^{\bullet}(z) = \left(\mathbf{1}^{\bullet} - \frac{1}{4}\,J^{\bullet}(z) \times J^{\bullet}(z)\right) \times \mathrm{anti}.\widetilde{\underline{\mathrm{So}}}^{\bullet}(z) \circ (-Id^{\bullet}) \tag{2.79}$$

If we now ask for monomials that separate parity *and* possess the exact symmetries *and* produce the right monics, we can have that, too, by setting:

$$\begin{aligned}
&\mathrm{var}\widetilde{\mathrm{Se}}^{\bullet}(z) := \widetilde{\mathrm{Se}}^{\bullet}(z) \times \big(\mathbf{1}^{\bullet} + J^{\bullet}(z)\big)^{\frac{1}{2}} \\
&\mathrm{var}\widetilde{\mathrm{Sa}}^{\bullet}(z) := \widetilde{\mathrm{Se}}^{\bullet}(z) \times \left(2\,\tanh\left(\frac{1}{2}\,J^{\bullet}(z)\right)\right) = \underline{\widetilde{\mathrm{Sa}}}^{\bullet}(z) \times \cosh(\,J^{\bullet}(z))^{-1} \\
&\mathrm{var}\widetilde{\mathrm{So}}^{\bullet}(z) := \widetilde{\mathrm{Se}}^{\bullet}(z) \times \left(\frac{J^{\bullet}(z)}{\mathbf{1}^{\bullet} - \frac{1}{2}\,J^{\bullet}(z)}\right) = \underline{\widetilde{\mathrm{So}}}^{\bullet}(z) \times \left(\mathbf{1}^{\bullet} - \frac{1}{2} J^{\bullet}(z) \times J^{\bullet}(z)\right)^{\frac{1}{2}}.
\end{aligned}$$

These variants still verify the resurgence equations (2.76)-(2.77). Moreover:

$$\begin{aligned}
&\mathrm{varSe}_{+}^{s_1,\dots,s_r}(-z) \equiv (-1)^{s_1+\dots+s_r}\,\mathrm{varSe}_{-}^{s_r,\dots,s_1}(-z) \text{ and } \mathrm{varSe}^{\bullet} \textit{ symmetrel} \\
&\mathrm{varSa}_{+}^{s_1,\dots,s_r}(-z) \equiv (-1)^{s_1+\dots+s_r}\,\mathrm{varSa}_{-}^{s_r,\dots,s_1}(-z) \text{ and } \mathrm{varSa}^{\bullet} \textit{ symmetral} \\
&\mathrm{varSo}_{+}^{s_1,\dots,s_r}(-z) \equiv (-1)^{s_1+\dots+s_r}\,\mathrm{varSo}_{-}^{s_r,\dots,s_1}(-z) \text{ and } \mathrm{varSo}^{\bullet} \textit{ symmetrol}
\end{aligned}$$

Polylogarithmic monomials. We recall the inductive definition of the polylogarithmic monomials $\widetilde{\mathcal{V}}^{\bullet}(z)$ (symmetral) and monics $V^{\bullet}$ (alternal), whose proper province is the study of singular, resurgence-inducing ODEs:

$$-(\partial_z + \omega_1 + \dots + \omega_r)\,\widetilde{\mathcal{V}}^{\omega_1,\dots,\omega_r}(z) = \widetilde{\mathcal{V}}^{\omega_1,\dots,\omega_{r-1}}(z)\,z^{-1} \tag{2.80}$$

$$\Delta_{\omega_0}\,\widetilde{\mathcal{V}}^{\omega_1,\dots,\omega_r}(z) = \sum_{\omega_1+\dots+\omega_i=\omega_0} V^{\omega_1,\dots,\omega_i}\;\widetilde{\mathcal{V}}^{\omega_{i+1},\dots,\omega_r}(z) \tag{2.81}$$

We also require the (apparently) more general monomials $\mathcal{V}^{\bullet}_{\mathcal{H}}(z)$, defined by a similar induction:

$$-(\partial_z + \|\bullet\|)\,\widetilde{\mathcal{V}}^{\bullet}_{\mathcal{H}}(z) = \widetilde{\mathcal{V}}^{\bullet}_{\mathcal{H}}(z) \times \mathcal{H}^{\bullet}(z) \qquad \big(\mathcal{H}^{\omega}(z) \in z^{-1}\mathbb{C}\{z^{-1}\}\big) \tag{2.82}$$

relative to any *alternal* mould $\mathcal{H}^{\bullet}(z)$ with values in the ring of convergent power series of z^{-1} (without constant term). Modulo convergent series of z^{-1}, the mould $\widetilde{\mathcal{V}}^{\bullet}_{\mathcal{H}}(z)$ actually reduces to $\widetilde{\mathcal{V}}^{\bullet}(z)$, thanks to the formula:

$$\widetilde{\mathcal{V}}^{\bullet}_{\mathcal{H}}(z) = (\widetilde{\mathcal{V}}^{\bullet}(z) \circ L^{\bullet}_{\mathcal{H}}) \times \mathcal{L}^{\bullet}_{\mathcal{H}}(z) \textit{ with } L^{\omega}_{\mathcal{H}} \in \mathbb{C}\,,\ \mathcal{L}^{\omega}_{\mathcal{H}}(z) \in z^{-1}\mathbb{C}\{z^{-1}\} \tag{2.83}$$

with an alternal, scalar-valued mould $L^{\bullet}_{\mathcal{H}}$ and a symmetral, convergent-valued mould $\mathcal{L}^{\bullet}_{\mathcal{H}}(z)$. Both $L^{\bullet}_{\mathcal{H}}$ and $\mathcal{L}^{\bullet}_{\mathcal{H}}(z)$ are defined by the joint induction:

$$L^{\boldsymbol{\omega}}_{\mathcal{H}} = \sum_{\boldsymbol{\omega}^1\boldsymbol{\omega}^2=\boldsymbol{\omega}}^{\boldsymbol{\omega}^2\neq\emptyset} (\widehat{\mathcal{L}}^{\boldsymbol{\omega}^1}_{H} * \widehat{\mathcal{H}}^{\boldsymbol{\omega}^2})(|\boldsymbol{\omega}|) - \sum_{\boldsymbol{\omega}^1\boldsymbol{\omega}^2=\boldsymbol{\omega}}^{\boldsymbol{\omega}^1,\boldsymbol{\omega}^2\neq\emptyset} L^{\boldsymbol{\omega}^1}_{\mathcal{H}}.(1 * \widehat{\mathcal{L}}^{\boldsymbol{\omega}^2}_{H})(|\boldsymbol{\omega}|) \tag{2.84}$$

$$-(\partial_z + \|\bullet\|)\mathcal{L}^{\bullet}_{\mathcal{H}}(z) = \mathcal{L}^{\bullet}_{\mathcal{H}}(z) \times \mathcal{H}^{\bullet}(z) - z^{-1}L^{\bullet}_{\mathcal{H}} \times \mathcal{L}^{\bullet}_{\mathcal{H}}(z). \tag{2.85}$$

The first relation, (2.84), expresses the constant $L^{\boldsymbol{\omega}}_{\mathcal{H}}$ in terms of earlier (shorter) mould components. The second relation,(2.85), when interpreted in the convolutive model, says that $(\zeta - |\boldsymbol{\omega}|)\widehat{\mathcal{L}}^{\boldsymbol{\omega}}_{\mathcal{H}}(\zeta)$ is equal to an entire function $\widehat{\mathcal{E}}^{\boldsymbol{\omega}}(\zeta)$ which, due to (2.84), vanishes for $\zeta = |\boldsymbol{\omega}|$. So $\widehat{\mathcal{L}}^{\boldsymbol{\omega}}_{\mathcal{H}}(\zeta)$, too, is an entire function with at most exponential growth, and that makes $\mathcal{L}^{\boldsymbol{\omega}}_{\mathcal{H}}(z)$ a convergent power series of z^{-1}. The resurgence constants $V^{\bullet}_{\mathcal{H}}$ associated with $\widetilde{\mathcal{V}}^{\bullet}_{\mathcal{H}}(z)$ also reduce to the polylogarithmic monics $V^{\bullet}$, since $\widetilde{\mathcal{V}}^{\boldsymbol{\omega}}_{\mathcal{H}}(z)$, owing to (2.83), verifies the following resurgence equations:

$$\Delta_{\omega_0}\, \widetilde{\mathcal{V}}^{\omega_1,\ldots,\omega_r}_{\mathcal{H}}(z) = \sum_{\omega_1+\cdots+\omega_i=\omega_0} V^{\omega_1,\ldots,\omega_i}_{\mathcal{H}}\; \widetilde{\mathcal{V}}^{\omega_{i+1},\ldots,\omega_r}_{\mathcal{H}}(z) \qquad (2.86)$$
$$\textit{with } V^{\bullet}_{\mathcal{H}} = V^{\bullet} \circ L^{\bullet}_{\mathcal{H}}$$

Multizetaic monomials in terms of polylogarithmic monomials. From what precedes and from the decomposition

$$\frac{e^{-\zeta}}{1-e^{-\zeta}} + \frac{1}{2} = H(-\zeta) = \sum_{|\omega|\le\rho}^{\omega\in 2\pi i\mathbb{Z}} \frac{1}{\zeta+\omega} + H_{\rho}(-\zeta) \quad (\forall \rho > 0) \quad (2.87)$$

we can see that, for $|\zeta|, |\omega| < \rho$, the monomials $\widehat{\mathcal{S}e}^{s}(\zeta), \widehat{\mathcal{S}a}^{s}(\zeta), \widehat{\mathcal{S}o}^{s}(\zeta)$, and the monics Tee^{s}_{ω}, Taa^{s}_{ω}, Too^{s}_{ω} that go with them, can be expressed as finite sums of three ingredients:

(i) classical monomials $\widehat{\mathcal{V}}^{\boldsymbol{\omega}}(\zeta)$ and monics $V^{\boldsymbol{\omega}}(\zeta)$ indexed by sequences $\boldsymbol{\omega}$ that are ρ-small, *i.e.* such that $|\boldsymbol{\omega}^{1}| \le \rho, |\boldsymbol{\omega}^{2}| \le \rho$ for all factorisation $\boldsymbol{\omega} = \boldsymbol{\omega}^{1}.\boldsymbol{\omega}^{2}$;
(ii) functions of type $\widehat{\mathcal{L}}^{\boldsymbol{\omega}}_{\mathcal{H}}(\zeta)$ which, though not entire, are holomorphic on the disk $|\zeta| \le \rho$;
(iii) the companion monics $L^{\boldsymbol{\omega}}_{\mathcal{H}}$.

Altogether, this results in an effective procedure for calculating the monics Tee^{s}_{ω}, Taa^{s}_{ω}, Too^{s}_{ω}, with a guaranteed geometric rate of convergence which, moreover, can be arbitrarily improved by taking ρ large (albeit at the cost of increasing the number of summands).

2.5 The non-standard case ($\rho \neq 0$). Normalisation

If we now drop the condition that ensured convergence, namely $s_1, s_r \neq 1$, and yet insist on retaining all properties and symmetries of our moulds, we must do two things to our infinite series: *truncate* them and *correct* them. Concretely, we must set

$$\begin{aligned}
\mathrm{Te}^\bullet(z) \;&:= \lim_{k\to\infty} \mathrm{Te}^\bullet_k(z) \;&&:= \lim_{k\to\infty} \; \mathrm{mucoSe}^\bullet_k \times \mathrm{doTe}^\bullet_k(z) \times \mathrm{coSe}^\bullet_k \\
\mathrm{Se}^\bullet_\pm(z) \;&:= \lim_{k\to\infty} \mathrm{Se}^\bullet_{k,\pm}(z) \;&&:= \lim_{k\to\infty} \; \mathrm{mucoSe}^\bullet_k \times \mathrm{doSe}^\bullet_{k,\pm}(z) \\
\mathrm{muSe}^\bullet_\pm(z) \;&:= \lim_{k\to\infty} \mathrm{muSe}^\bullet_{k,\pm}(z) \;&&:= \lim_{k\to\infty} \; \mathrm{mudoSe}^\bullet_{k,\pm}(z) \times \mathrm{coSe}^\bullet_k \,.
\end{aligned}$$

Here, the symmetrel *dominant* factors $Te^\bullet$, $doSe^\bullet_{k,\pm}$, $mudoSe^\bullet_{k,\pm}$ are defined as in (2.29) and (2.55)-(2.58) but with sums truncated at $\pm k$ instead of $\pm\infty$. Thus

$$\mathrm{doTe}_k^{s_1,\dots,s_r}(z) \;:=\; \sum_{-k\le n_r<\dots<n_1\le k} (n_r+z)^{-s_r}\dots(n_1+z)^{-s_1} \quad (\forall s_i). \tag{2.88}$$

As for the symmetrel, z-constant *corrective* factors $coSe^\bullet_{k\pm}$ and $invcoSe^\bullet_{k\pm}$, their definition reduces to

$$\mathrm{coSe}_k^{s_1,\dots,s_r} \;:=\; \frac{(\,c\;+\log k\,)^r}{r!} \qquad \textit{if} \;\; (s_1,\dots,s_r)=(1,\dots,1) \tag{2.89}$$

$$\mathrm{mucoSe}_k^{s_1,\dots,s_r} \;:=\; \frac{(-c-\log k)^r}{r!} \qquad \textit{if} \;\; (s_1,\dots,s_r)=(1,\dots,1) \tag{2.90}$$

$$\mathrm{coSe}_k^{s_1,\dots,s_r} \;=\mathrm{mucoSe}_k^{s_1,\dots,s_r} \;:=\; 0 \quad \textit{if} \;\; (s_1,\dots,s_r)\neq(1,\dots,1). \tag{2.91}$$

In the formal model, the resurgent-valued moulds $\tilde{Se}^\bullet$ and $mu\tilde{Se}^\bullet$ are still uniquely defined by the induction (2.50) together with the condition

$$\tilde{\mathrm{Se}}^{\boldsymbol{s}}(z)\;,\;\; \mathrm{mu}\tilde{\mathrm{Se}}^{\boldsymbol{s}}(z) \in \mathbb{Q}[[z^{-1}]]\otimes\mathbb{Q}[(c+\log z)] \;\dot{-}\; \mathbb{Q} \quad (\forall \boldsymbol{s}\neq\emptyset). \tag{2.92}$$

The normalising condition, in other words, is that $\tilde{\mathrm{Se}}^{\boldsymbol{s}}(z)$ and $mu\tilde{Se}^{\boldsymbol{s}}(z)$, as formal series in z^{-1} and polynomials in the bloc $(c+log\, z)$, should have no constant term.

In the sectorial models, the c-normalisation implies:

$$Se_\pm^{\overbrace{1,\,\dots,\,1}^{r\text{ times}}}(0) = \frac{(\gamma-c)^r}{r!} \quad ; \quad muSe_\pm^{\overbrace{1,\,\dots,\,1}^{r\text{ times}}}(0) = \frac{(c-\gamma)^r}{r!} \tag{2.93}$$

with

$$\gamma = \lim_{k\to\infty}\left(1+\frac{1}{2}+\dots+\frac{1}{k}-\log k\right) = 0.577215\ldots = \textit{Euler constant}. \tag{2.94}$$

For multitangents, we may still formally apply the procedure (2.40)-(2.41) of Section 2.3 to reduce them into combinations of monotangents and multizetas, but this time we are liable to get formally divergent multizetas. The c-normalisation then amounts to setting $\zeta(1) = ze^1 := \gamma - c$ and to adopting for all divergent multizetas[39] the unique *symmetrel extension* compatible with that initial choice.[40]

There are two natural choices for the normalisation constant c:
(i) Either we set $c = 0$, in which case we eschew γ in the formal model but at the cost of introducing it in the convolutive and sectorial models. It also complicates the definition of the multitangents and multizetas, since it forces us to set $ze^1 = \gamma$, which however is not entirely unnatural, in view of the formula

$$\sigma\,\Gamma(\sigma) = \exp\left(-\gamma\,\sigma + \sum_{2\leq n}(-1)^n \frac{\zeta(n)}{n}\sigma^n\right) \tag{2.95}$$

(ii) Or we set $c = \gamma$, which forcibly introduces γ into the formal model but rids us of it everywhere else, including in the definition of multitangents and multizetas, since it amounts to setting $ze^1 = 0$. This shall be our preferred choice.

2.6 The ramified case ($p > 1$) and the localisation constraints

For diffeos f of tangency order $p > 1$, the prepared form (1.2) becomes a power series of $z^{-1/p}$. This inevitably leads to moulds whose indices s_i (the *weights*) are no longer in $\mathbb{N}^*$ but in $p^{-1}\mathbb{N}^*$ or even, in some instances, in $p^{-1}\mathbb{Z}^*$.

Most results, starting with the symmetry relations, carry over to that case, but with three significant changes:

(i) The *finite* reduction of multitangents into monotangents and multizetas breaks down,
(ii) The Fourier coefficients Tee^s_ω, Taa^s_ω, Too^s_ω, which are the *direct* ingredients of the invariants $A_\omega(f)$, cease to be expressible as finite sums of multizetas (even ramified ones).
(iii) The formulae (2.40)-(2.41) still make formal sense but lead to expansions which are not only infinite but also divergent. When properly re-summed, they yield the correct expressions, but from the point of view of calculational expediency, this approach is worthless. Of course, straightforward Fourier analysis in the upper and

[39] *I.e.* for all multizetas with initial index $s_1 = 1$.

[40] Thus $ze^{1,1} := -\frac{1}{2}ze^2 + \frac{1}{2}(\gamma - c)^2$, $ze^{1,2} := -ze^{2,1} - ze^3 + (\gamma - c)\,ze^2$ etc. There exist simple formulae for calculating the symmetrel extension of all multizetas relative to any given choice of ze^1.

lower halves of the z-plane would yield the coefficients Tee^s_ω, Taa^s_ω, Too^s_ω, but not in the form of nice convergent series, and again at great cost.

The resurgence approach of Section 2.4 and Section 2-5, on the other hand, survives ramification without any modification. When pursued to the end, this approach even leads to some sort of functional equation for multizetas, that is to say, to something vaguely resembling the classical relation between $\zeta(s)$ and $\zeta(1-s)$.

However, the presence of ramifications makes it advisable to rotate our multitangents and monomials, so that we may handle functions which (as far as the index symmetries permit) assume real values on the main real half-axis. Thus, instead of $Te^\bullet$, $\widetilde{Se}^\bullet$ etc, we shall consider:

$$\mathrm{Teh}^{s_1,\ldots,s_r}(z) := \left(\frac{1}{i}\right)^{s_1+\cdots+s_r} \mathrm{Te}^{s_1,\ldots,s_r}\left(\frac{z}{i}\right) \tag{2.96}$$

$$\widetilde{\mathrm{Seh}}^{s_1,\ldots,s_r}(z) := \left(\frac{1}{i}\right)^{s_1+\cdots+s_r} \widetilde{\mathrm{Se}}^{s_1,\ldots,s_r}\left(\frac{z}{i}\right). \tag{2.97}$$

No finite reduction to monotangents. If we consider the equation (2.63) for $r{=}1$ but with s_1 in $\mathbb{Q}^+$ and interpret it correctly in the Borel plane, we see that the familiar formula (2.45) for the Fourier coefficients of monotangents transposes (taking the $\pi/2$-rotation into account) to the fractional case:

$$\mathrm{Teh}^{s_1}(z) = \sum_{\omega\in 2\pi\mathbb{N}} \mathrm{Teh}^{s_1}_\omega \quad \textit{with} \quad \mathrm{Teh}^{s_1}_\omega = 2\pi \frac{\omega^{s_1-1}}{\Gamma(s_1)}. \tag{2.98}$$

So the product[41] $Teh^{s_1} Teh^{s_2} \equiv Teh^{s_1,s_2} + Teh^{s_2,s_1} + Teh^{s_1+s_2}$ has Fourier coefficients of the form

$$\mathrm{Teh}^{s_1,s_2}_\omega + \mathrm{Teh}^{s_2,s_1}_\omega + \mathrm{Teh}^{s_1+s_2}_\omega = \frac{(2\pi)^{s_1+s_2}}{\Gamma(s_1)\Gamma(s_2)} \sum_{n_1+n_2=n}^{\omega=2\pi n} n_1^{s_1-1} n_2^{s_2-1} \tag{2.99}$$

and this makes it obvious that Teh^{s_1,s_2} and Teh^{s_2,s_1} cannot simultaneously be finite sums of monotangents Teh^s.

[41] Since symmetrelity survives ramification.

SingTeh$^\bullet$ **still determines** *Teh*$^\bullet$ **but in a completely new way.** For $n \to +\infty$, the right-hand side of (2.99) can be shown to possess a divergent but n-resurgent and Borel resummable asymptotic expansion of the form $n^{s_1+s_2-1} \sum c_s\, n^{-s} \quad (s \in \mathbb{Q}^+)$.

More generally, by adapting the argument leading to (2.40), one can easily calculate the ramified Laurent series of any multitangent *Teh*$^{\boldsymbol{s}}$:

$$\mathrm{Teh}^{\boldsymbol{s}}(z) = \mathrm{SingTeh}^{\boldsymbol{s}}(z) + \mathrm{RegTeh}^{\boldsymbol{s}}(z) = \sum_{\nu\in\mathbb{N}}^{-|\boldsymbol{s}|\le\nu} \theta_\nu^{\boldsymbol{s}}\, z^\nu + \sum_{\nu\in\mathbb{Q}-\mathbb{N}}^{-|\boldsymbol{s}|\le\nu} \theta_\nu^{\boldsymbol{s}}\, z^\nu \quad (2.100)$$

with its multizetaic coefficients $\theta_\nu^{\boldsymbol{s}}$. As in the non-ramified case, *Teh*$^{\boldsymbol{s}}$ is still completely determined by its singular part *SingTeh*$^{\boldsymbol{s}}$. We may even, if we so wish, derive from the singular part of (2.100) a formal reduction of *Teh*$^{\boldsymbol{s}}$ into monotangents:

$$\mathrm{Teh}^{\boldsymbol{s}}(z) = \sum_{\sigma\in\mathbb{Q}-\mathbb{N}}^{-\infty<\sigma\le|\boldsymbol{s}|} \tau_\sigma^{\boldsymbol{s}}\, \mathrm{Teh}^{\sigma}(z) \qquad \textit{with} \qquad \tau_\sigma^{\boldsymbol{s}} := \theta_{-\sigma}^{\boldsymbol{s}} \tag{2.101}$$

but the series defined in this way will be, generally speaking, everywhere divergent, even if we take care to correctly define, as in (2.108) *infra*, the monotangents $\mathrm{Teh}^{s_1}(z)$ with index $s_1 \in (1, -\infty)$. If we now attempt to calculate the Fourier coefficient of a general multitangent:

$$\mathrm{Teh}^{s_1,\ldots,s_r}(z) =: \sum_{\omega\in2\pi\mathbb{N}^*} \mathrm{Teh}_\omega^{s_1,\ldots,s_r}\; e^{-\omega z} \tag{2.102}$$

by identifying the Fourier coefficients on both sides of (2.101) and taking (2.98) into account:

$$\begin{aligned}\mathrm{Teh}_\omega^{\boldsymbol{s}} &= \sum_{\sigma\in\mathbb{Q}-\mathbb{N}}^{-\infty<\sigma\le|\boldsymbol{s}|} \tau_\sigma^{\boldsymbol{s}}\, \mathrm{Teh}_\omega^{\sigma} \;=\; 2\pi \sum_{\sigma\in\mathbb{Q}-\mathbb{N}}^{-\infty<\sigma\le|\boldsymbol{s}|} \tau_\sigma^{\boldsymbol{s}}\, \frac{\omega^{\sigma-1}}{\Gamma(\sigma)}\\ &= -2 \sum_{-\nu\in\mathbb{Q}-\mathbb{N}}^{-|\boldsymbol{s}|<\nu<+\infty} \theta_\nu^{\boldsymbol{s}}\; \Gamma(1+\nu)\sin(\pi\nu)\; \omega^{-\nu-1}\end{aligned} \tag{2.103}$$

what we get on the right-hand side is again a divergent expansion, which is ω-resurgent and Borel resummable. But Borel resummation in the present instance amounts to calculating the following loop integral:

$$\mathrm{Teh}_\omega^{s_1,\ldots,s_r} = \frac{1}{i}\oint_{-\infty-\epsilon i}^{-\infty+\epsilon i} \mathrm{Teh}^{s_1,\ldots,s_r}(z)\; e^{\omega z}\, dz \tag{2.104}$$

$$= \frac{1}{i}\oint_{-\infty-\epsilon i}^{-\infty+\epsilon i} \mathrm{SingTeh}^{s_1,\ldots,s_r}(z)\; e^{\omega z}\, dz \tag{2.105}$$

with an integration path connecting $-\infty - \epsilon\, i$ to $-\infty + \epsilon\, i$ and having as its middle part a small half-circle$\{|z| = \epsilon, \Re z > 0\}$ centered at the origin $0_\bullet$ and located in the *main* positive half-plane. This is indeed the proper procedure for retrieving the Fourier coefficients of $Teh^{\boldsymbol{s}}(z)$ from the singular part $SingTeh^{\boldsymbol{s}}(z)$.

The ramified localisation constraints. Defining the formal multitangent-to-monotangent reduction as in (2.101), we get the *reduction constraints:*

$$\begin{array}{ccc}
\mathrm{Teh}^{\boldsymbol{s}^1}(z).\mathrm{Teh}^{\boldsymbol{s}^2}(z) & \overset{reduction}{\longrightarrow} & (\sum \tau^{\boldsymbol{s}^1}_{s_1}\mathrm{Teh}^{s_1}(z)).(\sum \tau^{\boldsymbol{s}^2}_{s_2}\mathrm{Teh}^{s_2}(z)) \\
\downarrow linearisation & & \downarrow linearisation \\
\sum \epsilon^{\boldsymbol{s}^1,\boldsymbol{s}^2}_{\boldsymbol{s}^3}\mathrm{Teh}^{\boldsymbol{s}^3}(z) & \overset{reduction}{\longrightarrow} & \sum \epsilon^{\boldsymbol{s}^1,\boldsymbol{s}^2}_{\boldsymbol{s}^3}\tau^{\boldsymbol{s}^3}_{s_3}\mathrm{Teh}^{s_3}(z) = \sum \tau^{\boldsymbol{s}^1}_{s_1}\tau^{\boldsymbol{s}^2}_{s_2}\epsilon^{s_1,s_2}_{s_3}\mathrm{Teh}^{s_3}(z)
\end{array}$$

with elementary, integer-valued coefficients $\epsilon^{\boldsymbol{s}^i,\boldsymbol{s}^j}_{\boldsymbol{s}^k}$ and coefficients $\tau^{\boldsymbol{s}^i}_{s_j}$ that are finite, homogeneous sums of multizetas of total weight $\|\boldsymbol{s}^i\| - s_i - 1$. Although the multitangent expansions diverge, by equating (in the right-lower corner) the coefficients in front of each $Teh^{s_3}(z)$ we get a system of finite relations between multizetas.

Using instead the (locally convergent) expansions at $z = 0$, we get the *localisation constraints*, which are only seemingly more general than the reduction constraints:

$$\begin{array}{ccc}
\mathrm{Teh}^{\boldsymbol{s}^1}(z).\mathrm{Teh}^{\boldsymbol{s}^2}(z) & \overset{localisation}{\longrightarrow} & (\sum \theta^{\boldsymbol{s}^1}_{\nu_1} z^{\nu_1}).(\sum \theta^{\boldsymbol{s}^2}_{\nu_2} z^{\nu_2}) \\
\downarrow linearisation & & \downarrow linearisation \\
\sum \epsilon^{\boldsymbol{s}^1,\boldsymbol{s}^2}_{\boldsymbol{s}^3}\mathrm{Teh}^{\boldsymbol{s}^3}(z) & \overset{localisation}{\longrightarrow} & \sum \epsilon^{\boldsymbol{s}^1,\boldsymbol{s}^2}_{\boldsymbol{s}^3}\theta^{\boldsymbol{s}^3}_{\nu_3} z^{\nu_3} = \sum \theta^{\boldsymbol{s}^1}_{\nu_1}\theta^{\boldsymbol{s}^2}_{\nu_2} z^{\nu_1+\nu_2}
\end{array}$$

Here, the coefficients $\theta^{\boldsymbol{s}^i}_{n_j}$ are finite, homogeneous sums of multizetas of total weight $\|\boldsymbol{s}^i\| + n_j$.

Lastly, for the Fourier coefficients $Teh^\bullet_\omega$ (these monics, we recall, are the direct ingredients of the holomorphic invariants $A_\omega(f)$) we get the following system of constraints:

$$\begin{array}{ccc}
\mathrm{Teh}^{\boldsymbol{s}^1}(z).\mathrm{Teh}^{\boldsymbol{s}^2}(z) \overset{Fourier}{\longrightarrow} & & (\sum \mathrm{Teh}^{\boldsymbol{s}^1}_{\omega_1} e^{-\omega_1 z}).(\sum \mathrm{Teh}^{\boldsymbol{s}^2}_{\omega_2} e^{-\omega_2 z}) \\
\downarrow linearisation & & \downarrow linearisation \\
\sum \epsilon^{\boldsymbol{s}^1,\boldsymbol{s}^2}_{\boldsymbol{s}^3}\mathrm{Teh}^{\boldsymbol{s}^3}(z) \overset{Fourier}{\longrightarrow} & \sum \epsilon^{\boldsymbol{s}^1,\boldsymbol{s}^2}_{\boldsymbol{s}^3}\mathrm{Teh}^{\boldsymbol{s}^3}_{\omega_3} e^{-\omega_3 z} & = \sum \mathrm{Teh}^{\boldsymbol{s}^1}_{\omega_1}\mathrm{Teh}^{\boldsymbol{s}^2}_{\omega_2} e^{-(\omega_1+\omega_2)z}.
\end{array}$$

2.7 Meromorphic $\boldsymbol{s}$-continuation of $Seh^{\boldsymbol{s}}$ and $Teh^{\boldsymbol{s}}$ etc.

The whole subject of $\boldsymbol{s}$-continuation, being simply incidental to our investigation, shall receive only a sketchy treatment.

Meromorphic $\boldsymbol{s}$-continuation of the multizetas $ze^{\boldsymbol{s}}$. There exist various ways of proving the existence of a meromorphic continuation of $ze^{s_1,\ldots,s_r}$ to the whole of $\mathbb{C}^r$, with a singularity locus confined to the hyperplanes $\cup_{i,n}\{s_1+\cdots+s_i \in i-n\}$ $(n \in \mathbb{N})$. One of them relies on the convergent expansions

$$\begin{aligned} \mathrm{ze}^{s_1,\ldots,s_i,\ldots,s_r} &= -\sum_{k_i\geq 1} \frac{\Gamma(k_i+s_i)}{(k_i+1)!\,\Gamma(s_i)}\ \mathrm{ze}^{s_1,\ldots,s_i+k_i,\ldots,s_r} \\ &\quad + \frac{1}{s_i-1}\ \mathrm{ze}^{s_1,\ldots,s_i+s_{i+1}-1,\ldots,s_r} \\ &\quad - \sum_{k_i\geq -1} \frac{\Gamma(k_i+s_i)}{(k_i+1)!\,\Gamma(s_i)}\ \mathrm{ze}^{s_1,\ldots,s_{i-1}+s_i+k_i,\ldots,s_r} \end{aligned} \tag{2.106}$$

valid for $1<i<r$, and with slight modifications for $i=1$ or $i=r$ as well. The expansion (2.107) in turn results from plugging the identity

$$n_i^{-s_i} = \sum_{k_i\geq 0} \frac{\Gamma(k_i+s_i)}{k_i!\,\Gamma(s_i)} (1+n_i)^{-s_i-k_i}$$

into the definition of $ze^{s_1,\ldots,s_i,\ldots,s_r}$ or rather $ze^{s_1,\ldots,s_i-1,\ldots,s_r}$.

Similar expansions hold for $za^{\boldsymbol{s}}$ and $zo^{\boldsymbol{s}}$, of course, but here the parity properties have the effect of 'halving' the number of hyperplanes in the singularity locus.

The multiresidues at singular points $\boldsymbol{s} \in \mathbb{Z}^r$ are simple combinations of convergent multizetas with indices $\boldsymbol{s}' \in \mathbb{N}^{r'}$. The more negative components s_i in $\boldsymbol{s}$, the smaller the depths r' of the convergent multizetas contributing to the multiresidues.

Meromorphic $\boldsymbol{s}$-continuation of the multitangents $Teh^{\boldsymbol{s}}(z)$. The $\boldsymbol{s}$-continuation of multitangents proceeds on the same lines as that of multizetas. The main difference is the persistence, for multitangents, of convergent 'polar' expansions that rely on convergence-restoring corrections $[\ldots]_K^{-s}$. For any integer K we set:

$$\begin{aligned} \big[z \pm i\,n\big]_K^{-s} &= \sum_{0\leq k\leq K} (\pm i)^k\, e^{\mp\frac{1}{2}\pi i s}\ \frac{\Gamma(k+s)}{k!\,\Gamma(s)}\ n^{-s-k}\, z^k \quad (0<n,\, 0<\Re z) \\ \big[z\big]_K^{-s} &= \sum_{0\leq k\leq K} 2\,(\pm i)^k \cos\left(\frac{1}{2}\pi i s\right) \frac{\Gamma(k+s)}{k!\,\Gamma(s)}\ \zeta(s+k)\, z^k\,. \end{aligned} \tag{2.107}$$

For $s \in \mathbb{C}-\mathbb{Z}^-$, the monotangents admit 'polar' expansions of the form

$$\mathrm{Teh}^s(z) = \sum_{n\in\mathbb{Z}} \left((z+i\,n)^{-s} - \big[z\pm i\,n\big]_K^{-s}\right) \quad (\Re(s)+K>2) \tag{2.108}$$

There exist exact analogues for the multitangents.

Meromorphic $\boldsymbol{s}$-continuation of the monomials $\mathit{Seh}^{\boldsymbol{s}}(z)$**.** In the convolutive model (hence in the other models as well), the $\boldsymbol{s}$-continuation of the monomials $\mathit{Seh}^{\boldsymbol{s}}(z)$ presents no difficulty, and provides an alternative approach to the $\boldsymbol{s}$-continuation of the multizetas and multitangents, since the latter can be derived from the monomials $\mathit{Seh}^{\boldsymbol{s}}(z)$.

The closest thing to a reflection equation for multizetas. Let us start for orientation with depth one, *i.e.* with ordinary zetas. Calculating the Laurent expansion of $\mathit{Teh}^{s}(z)$ at $z=0$, and assuming $\Re(s)>1$, we find:

$$\mathrm{Teh}^{s}(z) := z^{-s} + 2\,\zeta(s)\,\cos\left(\frac{\pi}{2}s\right) + o(1)\,. \tag{2.109}$$

Due to (2.108), this also extends to all regular values of s, with the only difference that when $\Re(s)<0$ the term z^{-s} is absorbed by $o(1)$. On the other hand, starting from the Fourier expansion of $\mathit{Teh}^{s}(z)$ and assuming $\Re(s)<0,\ s\notin -\mathbb{N}$, we find

$$\mathrm{Teh}^{s}(z) := 2\pi \sum_{0<n} \frac{(2\pi n)^{s-1}}{\Gamma(s)}\, e^{-2\pi n z} = (2\pi)^{s}\,\frac{\zeta(1-s)}{\Gamma(s)} + o(1). \tag{2.110}$$

Comparing (2.109) and (2.110) for $\Re(s)<0$, we recover the classical reflection equation for the Riemann zeta function:

$$\begin{aligned} 2\,\zeta(s)\cos\left(\frac{\pi}{2}s\right) &= (2\pi)^{s}\,\frac{\zeta(1-s)}{\Gamma(s)} \\ &\Longleftrightarrow \zeta(s) = 2^{s}\,\pi^{s-1}\,\sin\left(\frac{\pi}{2}s\right)\,\Gamma(1-s)\,\zeta(1-s). \end{aligned}$$

To find out if something of that reflection equation survives at depth $r\geq 2$, let us fix a sequence $\boldsymbol{s}=(s_1,\ldots,s_r)$ with $\Re(s_i)<0$ and all partial sums $s_1+\cdots+s_i,\, s_i+\cdots+s_r$ not in $\mathbb{Z}$, and let us exploit the commutative diagram:

$$\begin{array}{ccc} \mathrm{Teh}^{\boldsymbol{s}}(z) & \stackrel{\textit{reduction}}{\longrightarrow} & \mathrm{singTeh}^{\boldsymbol{s}}(z) \\ & \searrow & \downarrow \\ & & \mathrm{regTeh}^{\boldsymbol{s}}(z). \end{array}$$

The leading term of the Laurent expansion of $\mathit{Teh}^{\boldsymbol{s}}(z)$ at $z=0$ is:

$$\mathrm{Teh}^{\boldsymbol{s}}(z) = \sum_{\boldsymbol{s}'\,\boldsymbol{s}''=\boldsymbol{s}} e^{\frac{\pi i}{2}(|\boldsymbol{s}''|-|\boldsymbol{s}'|)}\ \mathrm{ze}^{\boldsymbol{s}'}\ \mathrm{vize}^{\boldsymbol{s}''} + o(1) \tag{2.111}$$

with $vize^{s_1,\dots,s_r} := ze^{s_r,\dots,s_1}$. As for the purely singular part $\sum c_s z^{-s}$ of that same Laurent exp[ansion, it yields the formal, infinite, monotangential expansion $\sum c_s Teh^s(z)$ of $Teh^s(z)$:

$$\mathrm{Teh}^{s}(z) \overset{formally}{=} \sum_{\underline{s}^i s_i \overline{s}^i = s}^{0 \leq n_i \in \mathbb{N}} \mathrm{Teh}^{s_i - n_i}(z)\ \ \mathrm{Ze}^{\underline{s}^i, n_i, \overline{s}^i}. \tag{2.112}$$

The scalars $\mathrm{Ze}^{\underline{s}^i, n_i, \overline{s}^i}$ are here finite, homogeneous superposition of multizetas of total weight $n_i - |\underline{s}^i| - |\overline{s}^i| = n_i + s_i - |s|$. All monotangents $Teh^{s_i - n_i}(z)$ having indices of negative real part, they tend to known constants as z goes to 0:

$$\mathrm{Teh}^{s}(z) \overset{formally}{=} \sum_{\underline{s}^i s_i \overline{s}^i = s}^{0 \leq n_i \in \mathbb{N}} (2\pi)^{s_i - n_i} \ \frac{\zeta(1+n_i - s_i)}{\Gamma(s_i - n_i)}\ \mathrm{Ze}^{\underline{s}^i, n_i, \overline{s}^i} + o(1). \tag{2.113}$$

Finally, formally equating (2.111) and (2.113), we get:

$$\sum_{s' s'' = s} e^{\frac{\pi i}{2}(|s''| - |s'|)} \mathrm{ze}^{s'} \mathrm{vize}^{s''} \approx \sum_{\underline{s}^i s_i \overline{s}^i = s}^{0 \leq n_i \in \mathbb{N}} (2\pi)^{s_i - n_i} \frac{\zeta(1+n_i - s_i)}{\Gamma(s_i - n_i)}\, \mathrm{Ze}^{\underline{s}^i, n_i, \overline{s}^i}. \tag{2.114}$$

The finitely many multizetas on the left-hand side all carry indices with *negative* real parts, and two of them (ze^s and $vise^s$) are exactly of depth r. On the right-hand side, all but a finite number of multizetas carry indices with *positive* real parts, and all are of depth $< r$.

This, sadly, is the closest thing we can get, with this approach, to a reflection identity for multizetas. Note that the expansion on the right-hand side of (2.114) is divergent, but Borel resummable when viewed as a series in negative powers of the 'variable' $t := 2\pi$.

Ultimately, the obstruction to finding a satisfactory reflection formula is the non-existence of a multivariate, symmetrel Poisson formula. The fact is that the Fourier transform of the symmetrel Poisson distribution $De^{\bullet}$

$$\mathrm{De}^{x_1,\dots,x_r} := \sum_{-\infty < n_1 < \dots < n_r < +\infty} \delta(x_1 - n_1) \dots \delta(x_r - n_r) \ \ (\delta = Dirac) \tag{2.115}$$

not only differs from $De^{\bullet}$, but is not even an atomic distribution.

3 Collectors and connectors in terms of f

3.1 Operator relations

We begin with identity-tangent germs f in the standard class $(p, \rho) = (1, 0)$, *i.e.* of the form $f = l \circ g$, with the unit shift $l(z) = z{+}1$ and a germ

$g(z) = z+\underline{g}(z) = z+\mathcal{O}(z^{-2})$ which may be viewed as a perturbation. This is an invitation to expand everything (collectors, connectors, invariants) in series with a 1-linear, 2-linear, etc, part in $\underline{g}$ or, more conveniently, in the corresponding operator $\underline{G} := G-1$.

The iterator f^* is characterised by the germ identities $f^* = l^{-1} \circ f^* \circ f \equiv l^{-1} \circ f^* \circ l \circ g$ which in order-reversing operator notation[42] read:

$$F^* \; = \; G \; F^*_{:1} \qquad \textit{with} \quad F^*_{:1} := L \, F^* L^{-1}. \tag{3.1}$$

To solve (3.1) while respecting the symmetry between f, g and f^{-1}, g^{-1}, we take as our basic 'infinitesimals' the following operators

$$\underline{G}^+_{:n} := L^n.(G \; - \; 1).L^{-n} \qquad (n_i \in \mathbb{Z}) \tag{3.2}$$

$$\underline{G}^-_{:n} := L^n.(G^{-1} - 1).L^{-n} \qquad (n_i \in \mathbb{Z}). \tag{3.3}$$

With the notations of Section 1.2, this leads straightaway to simple formal expansions for the iterators

$$F^*_+ = 1 + \sum_{1 \le r} \; \sum_{0 \le n_r < \ldots < n_1} \underline{G}^+_{:n_r} \ldots \underline{G}^+_{:n_1} \qquad (n_i \in \mathbb{Z}) \tag{3.4}$$

$$F^*_- = 1 + \sum_{1 \le r} \; \sum_{n_1 < \ldots < n_r < 0} \underline{G}^-_{:n_r} \ldots \underline{G}^-_{:n_1} \qquad (n_i \in \mathbb{Z}) \tag{3.5}$$

$${}^*F_+ = 1 + \sum_{1 \le r} \; \sum_{0 \le n_1 < \ldots < n_r} \underline{G}^-_{:n_r} \ldots \underline{G}^-_{:n_1} \qquad (n_i \in \mathbb{Z}) \tag{3.6}$$

$${}^*F_- = 1 + \sum_{1 \le r} \; \sum_{n_r < \ldots < n_1 < 0} \underline{G}^+_{:n_r} \ldots \underline{G}^+_{:n_1} \qquad (n_i \in \mathbb{Z}). \tag{3.7}$$

These formulae, in turn, combine to produce new expansions which, depending on how we analyse them (- whether in terms of multitangents or Fourier series -) shall yield the collectors $\mathfrak{P}$ or the connectors Π in operator form:

$$\mathfrak{P}^+ \approx \mathbf{\Pi}^+ := {}^*F_-.F^*_+ = 1 + \sum_{1 \le r} \; \sum_{n_r < \ldots < n_1} \underline{G}^+_{:n_r} \ldots \underline{G}^+_{:n_1} \quad (n_i \in \mathbb{Z}) \tag{3.8}$$

$$\mathfrak{P}^- \approx \mathbf{\Pi}^- := {}^*F_+.F^*_- = 1 + \sum_{1 \le r} \; \sum_{n_1 < \ldots < n_r} \underline{G}^-_{:n_r} \ldots \underline{G}^-_{:n_1} \quad (n_i \in \mathbb{Z}). \tag{3.9}$$

For standard diffeos f, the above expansions for $F^*, {}^*F$ (respectively $\Pi^{\pm 1}$) are easily shown to *converge* when they are made to act on test

[42] To diffeos $f, g\ldots$ we associate the operators $F, G\ldots$ of postcomposition by $f, g\ldots$

functions that are defined on suitably extended U-shaped domains (respectively on suitably distant half-planes $|\Im(z)| \gg 1$). See Section 7.2. *But at this stage we do not have to worry about convergence: we shall provisionnaly (up to Section* 6 *inclusively) regard our connectors and collectors as* generating functions *that carry, in conveniently compact form, the various k-linear contributions*[43]. *Each k-linear contribution unproblematically converges, and for the moment this is all we require.*

The real challenge is to extract from these expansions (- first in the standard, then in the general case -) *theoretically appealing*, *analytically transparent*, and *computationally manageable* expressions for (in that order) the *collectors*, *connectors*, and *invariants*.

3.2 The direct scheme: from g to $\mathfrak{p}$

To break down the expansions (3.8)-(3.9) into sums of multitangents, we require scalar coefficients $\Gamma_{\pm}^{\boldsymbol{n}}$ that can be collectively defined by the generating function:

$$\left[\underline{G}^{\pm}_{:c_r^{-1}} \dots \underline{G}^{\pm}_{:c_1^{-1}} . z\right]_{z=0} =: \sum \Gamma_{\pm}^{n_1,\dots,n_r}\, c_1^{n_1} \dots c_r^{n_r} \tag{3.10}$$

with
$$\underline{G}^{\pm}_{:c^{-1}} = \sum_{1 \le k} \frac{1}{k!} \left(g^{\pm 1}(z+c^{-1}) - (z+c^{-1})\right)^k \partial_z^k . \tag{3.11}$$

The collectors then read:

$$\mathfrak{p}^{+}(z) = z + \sum_{1 \le r} \sum_{n_i} \Gamma_{+}^{n_1,\dots,n_r}\, \mathrm{Te}^{n_1,\dots,n_r}(z) \tag{3.12}$$

$$\mathfrak{p}^{-}(z) = z + \sum_{1 \le r} \sum_{n_i} \Gamma_{-}^{n_r,\dots,n_1}\, \mathrm{Te}^{n_1,\dots,n_r}(z) \tag{3.13}$$

with an order reversal between (3.10) and (3.12) that reflects the order reversal between (3.8) and (3.9).

Let us give an alternative, more analytical expansion. We first set

$$\frac{1}{n!}\left(g(z)-z\right)^n =: \sum_{2n \le s} g^{+}_{n,s}\, z^{-s+1} \quad , \quad \frac{1}{n!}\left(g^{-1}(z)-z\right)^n =: \sum_{2n \le s} g^{-}_{n,s}\, z^{-s+1}$$

Next, to account for the action of the *derivation* operators ∂_z implicit in the definition of the *substitution* operators $\underline{G}^{\pm}_{:n}$, we require integers $\delta^{\bullet}_{\bullet}$

[43] k-linear, that is, in the 'perturbation' $\underline{g}$ or its coefficients g_n

defined by[44]

$$\sum_{\sum(n_i-l_i)=1} \delta^{l_1,\dots,l_r}_{n_1,\dots,n_r}\; x_1^{l_1}\dots x_r^{l_r} \equiv x_1^{n_2}\,(x_1+x_2)^{n_3}\dots(x_1+\dots+x_{r-1})^{n_r} \quad (3.14)$$

Letting the operators on both sides of (3.8) respectively (3.9) act on the test function z, and collecting all r-linear summands, we find the sought-after expansions for the collectors $\mathfrak{p}^{\pm}$:

$$\mathfrak{p}^+(z) = z + \sum_{1\le r}\sum_{\substack{0\le l_i\\ 1\le n_i}}^{n_i+l_i\le s_i} (-1)^{n-1}\delta^{l_1,\dots,l_r}_{n_1,\dots,n_r}\,\mathrm{Te}^{s_1,\dots,s_r}(z)\prod_{1\le i\le r}\frac{(s_i-1)!\,g^+_{n_i,s_i-l_i+1}}{(s_i-l_i-1)!} \quad (3.15)$$

$$\mathfrak{p}^-(z) = z + \sum_{1\le r}\sum_{\substack{0\le l_i\\ 1\le n_i}}^{n_i+l_i\le s_i} (-1)^{n-1}\delta^{l_1,\dots,l_r}_{n_1,\dots,n_r}\,\mathrm{Te}^{s_r,\dots,s_1}(z)\prod_{1\le i\le r}\frac{(s_i-1)!\,g^-_{n_i,s_i-l_i+1}}{(s_i-l_i-1)!} \quad (3.16)$$

with $n := n_1 + ...n_r$.

3.3 The affiliate-based scheme: from $g_\diamondsuit$ to $\mathfrak{p}_\diamondsuit$

We shall now express the general affiliate $\mathfrak{p}_\diamondsuit$ of $\mathfrak{p}$ in terms of the corresponding affiliate $g_\diamondsuit$ of g – not so much for the sake of $\mathfrak{p}_\diamondsuit$, but to prepare for the specialisations g_* (generator) and $g_\sharp$, $g_{\sharp\sharp}$ (mediators), and to show what is so special about these three cases.

The first step is to take our stand on the trivial affiliate - $\mathfrak{p}$ itself - and to observe that after re-indexation, (3.8) may be re-written as

$$\underline{\mathbf{\Pi}}^+ = \sum_{1\le r}\sum_{n_i\in\mathbb{Z}} \mathfrak{O}^{n_1,\dots,n_r}\,\underline{G}^+_{:n_1}\dots\underline{G}^+_{:n_r} \quad (3.17)$$

with $\underline{\mathbf{\Pi}}^+ := \mathbf{\Pi}^+ - 1$, $\underline{G}^+ := G^+ - 1$, $\underline{G}^+_{:n} := L^n\,\underline{G}^+\,L^{-n}$ and with an elementary *'ordering mould'* $\mathfrak{O}^\bullet$, clearly of symmetrel type:

$$\mathfrak{O}^{n_1} := 1\ ,\ \mathfrak{O}^{n_1,\dots,n_r} := 1 \quad \textit{if}\ n_1 < \dots < n_r \quad \textit{resp.} := 0\ \textit{otherwise}. \quad (3.18)$$

Let us show that for any $\gamma(t) = t + \sum \gamma_r\, t^{r+1}$, an expansion exactly analogous to (3.17) holds for the corresponding affiliates

$$\mathbf{\Pi}_\diamondsuit = \sum_{1\le r}\sum_{n_i\in\mathbb{Z}} \mathfrak{O}^{n_1,\dots,n_r}_\diamondsuit\, G_{\diamondsuit:n_1}\dots G_{\diamondsuit:n_r} \quad (3.19)$$

[44] For $r = 1$, one should of course take $\delta^0_1 := 1$ and $\delta^{l_1}_{n_1} := 0$ if $\binom{l_1}{n_1} \neq \binom{0}{1}$. The presence of n_1, x_r on the left-hand side and their absence on the right-hand side is no oversight. It simply implies that $\delta^{l_1,\dots,l_r}_{n_1,\dots,n_r} = 0$ when $n_1 \neq 1$ or $l_r \neq 0$. If one finds (3.14) confusing, one should think of it as $\sum \delta^{l_1,l_2,\dots,l_{r-1},0}_{1,n_2,\dots,n_{r-1},n_r}\, x_1^{l_1}\dots x_{r-1}^{l_{r-1}} \equiv x_1^{n_2}\,(x_1+x_2)^{n_3}\dots(x_1+\dots x_{r-1})^{n_r}$.

with

$$\mathbf{\Pi}_{\Diamond} := \gamma(\underline{\mathbf{\Pi}}) = \gamma(\mathbf{\Pi} - 1) \,,$$
$$G_{\Diamond} := \gamma(\underline{G}) = \gamma(G - 1) \,,$$
$$G_{\Diamond:n} := L^n.G_{\Diamond}.L^{-n}$$

and with a suitable variant $\mathfrak{O}^{\bullet}_{\Diamond}$ of the ordering mould $\mathfrak{O}^{\bullet}$:

$$\mathfrak{O}^{\bullet}_{\Diamond} := \gamma(Id^{\bullet}) \circ \mathfrak{O}^{\bullet} \ddot{\circ} \gamma^{-1}(Id^{\bullet})$$

$\mathfrak{O}^{\bullet}_{\Diamond}$ is derived from $\mathfrak{O}^{\bullet}$ by ordidary pre-composition by $\gamma(Id^{\bullet})$ and *modified post-composition* by $\gamma^{-1}(Id^{\bullet})$. See (3.20) below. The order in which these two operations are performed does not matter. The formula for $\ddot{\circ}$-composition is patterned on the formula (2.1) for $\circ$-composition:

$$C^{\bullet} = A^{\bullet} \ddot{\circ} B^{\bullet} \Longleftrightarrow C^{\boldsymbol{\omega}} = \sum_{\boldsymbol{\omega}^1 \ldots \boldsymbol{\omega}^s = \boldsymbol{\omega}}^{\boldsymbol{\omega}^i \text{ monoindicial}} A^{\langle \boldsymbol{\omega}^1 \rangle, \ldots, \langle \boldsymbol{\omega}^s \rangle} B^{\boldsymbol{\omega}^1} \ldots B^{\boldsymbol{\omega}^s} \quad (3.20)$$

except that the sum on the right-hand side of (3.20) extends only to those factorisations of $\boldsymbol{\omega}$ that involve *mono-indicial* factor sequences $\boldsymbol{\omega}^i$, *i.e.* factor sequences consisting each of *one* index ω_i repeated r_i times. And $\langle \boldsymbol{\omega}^i \rangle := (\omega_i)$ denotes that same factor sequence collapsed to its one index. Thus we get:

$$C^{3,3,3,5} = A^{3,3,3,5} B^3 B^3 B^3 B^5 + A^{3,3,5} B^{3,3} B^3 B^5 + A^{3,3,5} B^3 B^{3,3} B^5 + A^{3,5} B^{3,3,3} B^5.$$

The last missing items are the multitangents $Tee^{\bullet}_{\Diamond}$ and the corresponding structure coefficients. The former are defined by:

$$\mathrm{Tee}^{\bullet}_{\Diamond} = \gamma(Id^{\bullet}) \circ \mathrm{Tee}^{\bullet} \circ \delta(Id^{\bullet}) \qquad (\gamma \circ \delta = id) \qquad (3.21)$$

The latter are given by the generating series:

$$\left[G_{\Diamond,\, c_r^{-1}} \ldots G_{\Diamond,\, c_1^{-1}} \cdot z \right]_{z=0} =: \sum \Gamma_{\Diamond}^{n_1,\ldots,n_r} c_1^{n_1} \ldots c_r^{n_r} \qquad (3.22)$$

where $G_{\Diamond,\, c^{-1}}$ denotes the translated γ-affiliate of G:

$$G_{\Diamond,\, c^{-1}} := \sum_{1 \le r} \sum_{1 \le n_i} \Diamond^{n_1,\ldots,n_r} g_{\Diamond}^{n_1}(z + c^{-1}) \frac{\partial^{n_1}}{n_1!} \ldots g_{\Diamond}^{n_r}(z + c^{-1}) \frac{\partial^{n_r}}{n_r!}. \quad (3.23)$$

See Section 1.3 and Section 3.2 and recall that $\Diamond^1 = 1$ and $\Diamond^{n_1,\ldots,n_r} = 0$ if $1 < r$ and $n_r = 1$. We are now in a position to expand $p_{\Diamond}$ in series of multitangents $Tee_{\Diamond}$:

$$p_{\Diamond}(z) = z + \sum_{1 \le r} \sum_{n_i} \Gamma_{\Diamond}^{n_1,\ldots,n_r} \mathrm{Tee}_{\Diamond}^{n_1,\ldots,n_r}(z) \qquad (3.24)$$

Short proof: One should compare step by step the derivation of (3.24) with that of the expansion (3.8) for $\mathfrak{p}^+$. The key point here is that changing from operators to multitangents changes $\ddot{\circ}$ to $\circ$. Indeed, in a sum of the form

$$\sum_{n_i \in \mathbb{Z}} C^{n_1,\dots,n_r} (z+n_1)^{-\sigma_1} \dots (z+n_r)^{-\sigma_r} \quad \textit{with} \quad C^\bullet := A^\bullet \ddot{\circ} B^\bullet \tag{3.25}$$

any contribution to $C^{\boldsymbol{n}}$ of the form $A^{\langle \boldsymbol{n}^1 \rangle,\dots,\langle \boldsymbol{n}^t \rangle} B^{\boldsymbol{n}^1} \dots B^{\boldsymbol{n}^t}$, with monoindicial factor sequences $\boldsymbol{n}^k$ consisting of identical indices n_k, will contract to

$$\prod_{1 \le k \le t} \prod_{n_i \in \boldsymbol{n}^k} (z+n_i)^{-s_i} = \prod_{1 \le k \le t} (z+n_k)^{-\sum_{n_i \in \boldsymbol{n}^k} s_i} \tag{3.26}$$

3.4 Parity separation and affiliate selection

The relative complexity of $g_\diamondsuit$ counts for nothing. What matters is

(i) to get $Tee^\bullet_\diamondsuit$ and the corresponding expansions for $\mathfrak{p}$ as simple as possible;
(ii) to pick parity-respecting affiliates: $(g^{-1})_\diamondsuit \equiv -g_\diamondsuit$, $(\mathfrak{p}^{-1})_\diamondsuit \equiv -\mathfrak{p}_\diamondsuit$.

We already know three parity-respecting affiliates:

$$\gamma_0(t) = \log(1+t) \quad (\textit{infinitesimal generator}), \tag{3.27}$$

$$\gamma_1(t) = \frac{t}{1+\frac{1}{2}t}, \quad (\textit{first mediator}) \tag{3.28}$$

$$\gamma_2(t) = \frac{(1+t)^2-1}{(1+t)^2+1} \quad (\textit{second mediator}) \tag{3.29}$$

and the general parity-respecting affiliate obviously corresponds to functions of the form $\gamma = h_i \circ \gamma_i$ $(0 \le i \le 2)$ with h_i odd. So the task now is to select one of those γ so as to optimise $Tee^\bullet_\diamondsuit$ and in particular to make the formulae for their symmetrel $Te^\bullet$-linearisation as simple as possible. But we have already suggested in Section 2.3 and we shall show more conclusiveely in Section 5.4 that there exist no simpler choices than $\gamma_0, \gamma_1, \gamma_2$, with γ_1 topping the list, and γ_0 coming second. So we shall focus here on these three choices.

3.5 The generator-based scheme: from g_* to $\mathfrak{p}_*$

Here, the structure coefficients $\Gamma^{\boldsymbol{n}}_*$ are given by the series:

$$\Big[g_*(z+c_r^{-1})\,\partial \dots g_*(z+c_1^{-1})\,\partial\,.\,z \Big]_{z=0} =: \sum \Gamma_*^{n_1,\dots,n_r}\, c_1^{n_1} \dots c_r^{n_r} \tag{3.30}$$

The corresponding expansion for $\mathfrak{p}_*$ reads:

$$\mathfrak{p}_*(z) = \sum_{1\le r} \sum_{n_i} \Gamma_*^{n_1,\dots,n_r} \,\mathrm{Taa}^{n_1,\dots,n_r}(z) \tag{3.31}$$

Like with $\Gamma_\pm^\bullet$, one may prefer more analytical variants. These rely on integers $\delta^\bullet$ and $\delta_1^\bullet$ much simpler than the $\delta_\bullet^\bullet$ of Section 3.2

$$\sum_{l_i\ge 0\,,\,\sum l_i = r-1} \delta^{l_1,\dots,l_r}\, x_1^{l_1}\dots x_r^{l_r} \equiv x_1.(x_1+x_2)\dots(x_1+\dots+x_{r-1}) \tag{3.32}$$

$$\sum_{l_i\ge 0\,,\,\sum l_i = r} \delta_1^{l_1,\dots,l_r}\, x_1^{l_1}\dots x_r^{l_r} \equiv x_1.(x_1+x_2)\dots(x_1+\dots+x_r) \tag{3.33}$$

and of course on the coefficients g_{*s} of g_*: $g_*(z) = \sum_{2\le s} g_{*s}\, z^{1-s}$.

The corresponding expansion for $\mathfrak{p}_*$ and $\mathfrak{p}'_*$ read:

$$\mathfrak{p}_*(z) = \sum_{1\le r}(-1)^{r-1} \sum_{0\le l_i<s_i} \delta^{l_1,\dots,l_r}\,\mathrm{Taa}^{s_1,\dots,s_r}(z) \prod_{1\le i\le r} \frac{(s_i-1)!\, g_{*s_i-l_i+1}}{(s_i-l_i-1)!} \tag{3.34}$$

$$\mathfrak{p}'_*(z) = \sum_{1\le r}(-1)^{r} \sum_{0\le l_i<s_i} \delta_1^{l_1,\dots,l_r}\,\mathrm{Taa}^{s_1,\dots,s_r}(z) \prod_{1\le i\le r} \frac{(s_i-1)!\, g_{*s_i-l_i+1}}{(s_i-l_i-1)!} \tag{3.35}$$

The second expansion is formally more appealing in that its multitangents *Taa*$^\bullet$ have exactly the same total weight $\sum s_j$ as the accompanying coefficient clusters. We may note that while it would be possible (though rather pointless) to produce similar expansions for all derivatives $\mathfrak{p}_*^{(n)}$, nothing analogous exists for the indefinite integrals ${}^{`}\mathfrak{p}_*, {}^{``}\mathfrak{p}_* \dots$

3.6 The mediator-based scheme: from $g_\sharp$, $g_{\sharp\sharp}$ to $\mathfrak{p}_\sharp$, $\mathfrak{p}_{\sharp\sharp}$

The relevant structure coefficients $\Gamma_\sharp$ are defined in the usual way

$$\Big[G_{\sharp,\,c_r^{-1}}\dots G_{\sharp,\,c_1^{-1}}\,.\,z\Big]_{z=0} =: \sum \Gamma_\sharp^{n_1,\dots,n_r}\, c_1^{n_1}\dots c_r^{n_r} \tag{3.36}$$

using the translates of the mediator in operator form:

$$G_{\sharp,\,c^{-1}} := 2\left(\sum_{1\le n\ odd} \frac{(g_\sharp(z+c^{-1}))^n}{2^n\, n!}\partial^n\right)\left(\sum_{0\le n\ even} \frac{(g_\sharp(z+c^{-1}))^n}{2^n\, n!}\partial^n\right)^{-1}. \tag{3.37}$$

The corresponding expansion for the collector involves *Too*$^\bullet$ and reads:

$$p_\sharp(z) = \sum_{1\le r} \sum_{n_i} \Gamma_\sharp^{n_1,\dots,n_r}\,\mathrm{Too}^{n_1,\dots,n_r}(z) \tag{3.38}$$

Appearance of coloured multitangents and multizetas. Although, as pointed out in Section 1.8, the resurgence properties of the mediators $f_\sharp$ and $g_\sharp$ are completely unrelated (both have distinct critical times and distinct resurgence constants) and have no bearing on the object of interest to us, namely $\mathfrak{p}_\sharp$, a few complements about the very specific resurgence regimen of mediators, quite different from that of infinitesimal generators but fairly typical for the behaviour of general affiliates, may not be superfluous. The actual resurgence equations were obtained in Section 1.8. Here, we shall focus on the nature of their resurgence constants $\underline{A}_\omega$ and $\underline{\underline{A}}_\omega$.

The definition of the (first) mediator leads formally to an expansion

$$F_\sharp = 2 - 4\,(1 + L + \underline{G}\,L)^{-1} \tag{3.39}$$

$$= 2 - 4\,(1{+}L)^{-1} - 4\,(1{+}L)^{-1} \sum_{1\le r} (-1)^r \left(\underline{G}L(1{+}L)^{-1}\right)^r \tag{3.40}$$

valid in the formal model and, after the proper transpositions, in the convolutive model. In the right sectorial model this becomes:

$$F_{\sharp,+} = 2 - 4 \sum_{0\le n_0} L^{n_0} \;\; - 4 \sum_{0\le n_r<\ldots<n_1<n_0}^{0\le r} (-1)^{r+n_0}\, \underline{G}_{:n_r} \ldots \underline{G}_{:n_1}\, L^{n_0}. \tag{3.41}$$

Note that, due to the rightmost factor L^{n_0}, this expansion is only superficially similar to the expansion (3.4) of F^*_+. However, applying both sides of (3.41) to z and using

$$L\,(1+L)^{-1} . z = \frac{1}{2}\,z + \frac{1}{4} \qquad , \qquad \underline{G}_{:n_1}\, L\,(1+L)^{-1} . z = \frac{1}{2}\,\underline{G}_{:n_1} . z$$

we get for $f_{\sharp,+}$an expansion much closer in outward shape to that of $f^*_+(z)$:

$$f_{\sharp,+}(z) = 1 - 2 \sum_{0\le n_r<\ldots<n_1}^{1\le r} (-1)^{r+n_1}\, \underline{G}_{:n_r} \ldots \underline{G}_{:n_1} . z. \tag{3.42}$$

Mind the change $(-1)^{r+n_0} \to (-1)^{r+n_1}$ from (3.41) to (3.42), which is correct. If we now consider the limit $\Lambda_\sharp(z) := \lim_{n\to+\infty} f_{\sharp,+}(z-n)$, we obtain for $\Lambda_\sharp(z)$ a formal expansion

$$\Lambda_\sharp(z) = -2 \sum_{-\infty\le n_r<\ldots<n_1<+\infty}^{1\le r} (-1)^{r+n_1}\, \underline{G}_{:n_r} \ldots \underline{G}_{:n_1} . z \tag{3.43}$$

which, like the expansion (3.8) of $\Pi^+(z)$ and for much the same reasons, is going to converge in the half-planes $|\Im z| > y$ for y large enough, and whose Fourier coefficient are going to give the resurgence constants of $f_\sharp$. (See Section 1.8). That said, the main difference with (3.8) is not so much the presence of a factor $(-1)^r$ in (3.43), but of the factor $(-1)^{n_1}$, which will be responsible for introducing *bi-coloured* multitangents and *bi-coloured* multizetas: see (6.2) and take $\epsilon_j \in \frac{1}{2}\mathbb{Z}/\mathbb{Z}$.

The picture for the second mediator $f_{\sharp\sharp}$ would be quite similar, leading to $\Lambda_{\sharp\sharp}(z) := \lim_{n\to+\infty} f_{\sharp\sharp,+}(z-n)$ and a periodic expansion

$$\Lambda_{\sharp\sharp}(z) = -\sum_{-\infty\le n_r<\ldots<n_1<+\infty}^{1\le r} (-1)^{r+n_1}\, \underline{GG}_{:n_r} \ldots \underline{GG}_{:n_1} \,.\, z \tag{3.44}$$

with $\underline{GG}_{:n} := L^n \,.\, (G.G-1) \,.\, L^{-n}$ In any case, we see that while $\Lambda_\sharp$ and $\Lambda_{\sharp\sharp}$ bear some resemblance to Π^+, they are completely unrelated to $p_\sharp$ and $p_{\sharp\sharp}$.

3.7 From collectors to connectors

The dichotomy collector/connector. The various objects $\mathfrak{p}_\diamondsuit$ constructed so far in this section have to be simultaneously examined under the viewpoint of their f- and z-dependence.

They depend on a germ $f = l \circ g$ that moves freely within the formal class $(p,\rho) = (1,0)$. As such, they are to begin with nothing more than formal power series in the coefficients g_s of g or, equivalently, the coefficients $g_{\diamondsuit,s}$ of its affiliates $g_\diamondsuit$:

$$\mathfrak{p}_\diamondsuit(z) = \sum_{1\le r} \sum_{s_i,n_i}^{s_i<s_{i+1}} \prod_{1\le i\le r} (g_{\diamondsuit,s_i})^{n_i}\, T_\diamondsuit^{\binom{n_1\ ,\ldots,\ n_r}{s_1\ ,\ldots,\ s_r}}(z). \tag{3.45}$$

As functions of z, however, our objects may be viewed

(i) either as *collectors* (and noted $\mathfrak{p}_\diamondsuit$), *i.e.* as global meromorphic functions defined on the whole of $\mathbb{C}$ with all their poles on $\mathbb{Z}$ and with well defined expansions as finite sums of multitangents or, after reduction, as sums of monotangents with multizeta coefficients;
(ii) or as *connectors* (and noted $\pi_\diamondsuit$), *i.e.* as pairs of 1-periodic functions defined in the upper or lower half-plane and possessing their own distinct Fourier expansions there.

So far, the distinction between *collectors* and *connectors* may appear tenuous, but it acquires all its significance when, ceasing to regard the f-dependence as formal, we focus on individual, convergent germs $f = l \circ g$ and try to associate with them global z-functions (impossible) or pairs of periodic z-germs (possible).

To that end, let us consider the s-truncations $trunc_s.\mathfrak{p}_\diamondsuit(z)$ obtained by retaining in (3.45) the sole terms of global weight[45] $\sum n_i s_i \leq s$. Notice that weight-truncation is intrinsical, in the sense that, in any given z-chart[46], it stays the same whether we choose the natural coefficient system $\{g_s, s \geq 3\}$ or any affiliate-based system $\{g_{\diamondsuit,s}, s \geq 3\}$.

Divergence of the collectors. When $s \to +\infty$, $trunc_s.\mathfrak{p}_\diamondsuit(z)$ does not tend to a global function, irrespective of the choice of affiliation $\diamondsuit$. Moreover, even after finite reduction to monotangents, $trunc_s.\mathfrak{p}_\diamondsuit(z)$ does not converge to an infinite sum (even a formal one) of monotangents. This may seem surprising, because:

(*) reducing $trunc_s.\mathfrak{p}_\diamondsuit(z)$ to a series of montangents $\sum_{0<\sigma} a^{\diamondsuit}_{s,\sigma}\, Te^{\sigma}(z)$ is the same as taking the negative part $\sum_{0<\sigma} a^{\diamondsuit}_{s,\sigma}\, z^{-\sigma}$ of the Laurent expansion at $z = 0$ of $trunc_s.\mathfrak{p}_\diamondsuit(z)$;

(**) the Borel transform $\sum_{0<\sigma} a^{\diamondsuit}_{s,\sigma}\, \zeta^{\sigma-1}/(\sigma-1)!$ of that negative part, when evaluated at the points $\zeta = 2\pi i n$, yields precisely the Fourier coefficients of the truncated connectors $trunc_s.\pi_\diamondsuit(z)$ — and these Fourier coefficients, as we shall see in a moment, *do converge* when $s \to \infty$.

We shall have more to say about this apparent paradox and the reasons behind it in Section 7, but for the moment let us observe that the only meaning that can be attached to the limit $\lim_{s\to\infty} trunc_s.\mathfrak{p}_\diamondsuit(z)$ is the formal series (3.45) with its individual clusters $\prod_i g^{n_i}_{\diamondsuit,s_i}\; T^{\binom{n}{s}}(z)$ kept separate.

Convergence of the connectors.
(*) From $\mathfrak{p}$ to $\boldsymbol{\pi} = (\boldsymbol{\pi}_{\rm no}, \boldsymbol{\pi}_{\rm so})$:
As s goes to ∞ and for $K_\pm$ large enough, $trunc_s.\pi(z) - z$ tends uniformly to a 1-periodic limit $\pi_{\rm no}(z) - z$ (respectively $\pi_{\rm so}(z) - z$) on the upper or 'northern' half-plane $\Im z > K_+$ (respectively on the lower or 'southern' half-plane $\Im z < -K_-$) .

(**) From $\mathfrak{p}_\diamondsuit$ to $\boldsymbol{\pi}_\diamondsuit = (\boldsymbol{\pi}_{\diamondsuit,\rm no}, \boldsymbol{\pi}_{\diamondsuit,\rm so})$:
The affiliate $\pi_\diamondsuit(z)$ of π being of the form $\gamma(\Pi-1).z$, the $n^{\rm th}$ Fourier coefficient of its northern or southern component is a polynomial in the first n

[45] The 'weight' in question is that of the coefficient clusters. But the weight of the accompanying multitangents (or, after reduction, of the multizeta-monotangent combinations) differs from the first only by one unit.

[46] But the weight truncation is of course dependent on the choice of z-chart.

Fourier coefficients of π_{no} or π_{so}. So, as $s \to \infty$, the (convergent) Fourier series $trunc_s.\pi_{\diamondsuit,\text{no}}$ and $trunc_s.\pi_{\diamondsuit,\text{so}}$ converge (coefficient-wise)[47], to two formal Fourier series $\pi_{\diamondsuit,\text{no}}$ and $\pi_{\diamondsuit,\text{so}}$. These are generally divergent, but usually (and definitely so in the case of the generators π_* or mediators $\pi_\sharp$, $\pi_{\sharp\sharp}$) resurgent and Borel-resummable, with respect to some critical time of the form $z' := \exp(\pm n\pi i z)$. In any case their Fourier coefficients are well-defined, and this is all that matters to us at the moment.

More on the dichotomy collector/connector. Despite being very close to the *connectors*, the *collectors* differ from them in two fundamental respects: they are *not invariant* and they are *of one piece*.

The *non-invariance* is fairly obvious when $\mathfrak{p}$ is taken in its *natural* multitangent expansion, but even after monotangent reduction (when at all it exists), $\mathfrak{p}$ still remains non-invariant. Indeed, even when a formal limit $\sum_{s\in\mathbb{N}} Te^s(z)$ exists (it sometimes does, though very exceptionally) as the truncation goes to infinity, the 'Borel transform' $\sum_{s\in\mathbb{N}} \zeta^{s-1}/(s-1)!$ assumes invariant values only when restricted to the set $2\pi\mathbb{Z}^*$.

As for being *of one piece*, this is a property not so much of the collectors as of their constituent multitangents or monotangents, which are meromorphic over the whole of $\mathbb{C}$, in complete contrast to the connectors, whose northern and southern components are usually completely unrelated: each one may a priori be *anything*.

3.8 The ramified case ($p > 1$)

Everything carries over to the general case, when f ranges though a general formal class (p, ρ). But when $p > 1$, we must take f to a prepared form $f = f_{norm} \circ g$ (see (1.2)) with a ramified perturbation $g(z) = z + \sum g_s z^{1-s}$ and with fractional indexation: $s \in p^{-1}\mathbb{N}^*$.

The *connectors* are of course still invariant, but even more 'fragmented' than usual: there are now $2\,p$ of them – p northern and p southern ones. Each of these $2\,p$ periodic germs is unrelated to the others and may *a priori* be anything.

As for the *collectors*, as formal objects they are still *of one piece*, but things get more tangled when we regard the truncations $trunk_s.\mathfrak{p}_\diamondsuit(z)$ or the individual clusters $\widetilde{T^{\binom{n}{s}}(z)}$ in the ramified equivalent of (3.45) as *global functions* on $(\mathbb{C} - 2\pi i\mathbb{Z})_p$ (the p-ramified covering of $\mathbb{C} - 2\pi i\mathbb{Z}$). The thing is that we can no longer go from one upper-plane determination to the two neighbouring lower-plane determination by simply cross-

[47] Recall that s-truncation is independent of $\diamondsuit$.

ing the real axis between two consecutive singularities n and $n+1$: by so doing, one would get a wrong determination, dependent on n, and not even periodic.

3.9 Reflexive and unitary diffeomorphisms

In this section, we find it convenient to switch from the s- or *weight*-indexation $g(z) = z + \sum g_s\, z^{1-s}$ to the d- or *degree*-indexation $g(z) = z + \sum g_{1+d}\, z^{-d}$.

In Section 3.4 we observed that in the expansion (3.34) of $\mathfrak{p}_*$, coefficient clusters $\prod g_{*1+d_i}$ of even (respectively odd) total degree $\sum d_i$ accompany multitangents $\mathcal{T}aa^\bullet$ that are even functions with real Fourier coefficients (respectively odd functions with purely imaginary Fourier coefficients). As a consequence, there is no simple condition on the coefficients g_{*1+d_i} of g_* capable of ensuring that $\mathfrak{p}_*$ be *odd*, whereas three elementary conditions may ensure that it be *even*, namely:

(i) all coefficients g_{*1+d_i} of odd degree d_i vanish and those of even degree are real;
(ii) all coefficients g_{*1+d_i} of even degree d_i vanish and those of odd degree are purely imaginary;
(iii) all coefficients g_{*1+d_i} of even degree d_i are real and those of odd degree are purely imaginary.

No special significance attaches to case (ii), but the cases (i) and (iii) present interesting stability properties, with collectors and connectors inheriting the nature of f. This is an incentive for singling out the following three types of diffeos f whose inverses f^{-1} either coincide with, or are analytically conjugate to, the image of f under an elementary involution:

$$\begin{array}{rl} \textit{reflexive}:\ \check{f} = f^{-1} \ || & \textit{weakly reflexive}:\ \check{f} \overset{\text{an. cj.}}{\sim} f^{-1} \\ \textit{unitary}:\ \bar{f} = f^{-1} \ || & \textit{weakly unitary}:\ \bar{f} \overset{\text{an. cj.}}{\sim} f^{-1} \\ \textit{counitary}:\ \check{\bar{f}} = f^{-1} \ || & \textit{weakly counitary}:\ \check{\bar{f}} \overset{\text{an. cj.}}{\sim} f^{-1}. \end{array}$$

Here, $\bar{f}$ denotes the complex conjugate of f, and $\check{f} := \sigma \circ f \circ \sigma$ with $\sigma(z) \equiv -z$. Conjugation by τ, with $\tau(z) \equiv i\, z$, clearly exchanges *unitary* and *counitary*, so that *weakly unitary* is equivalent to *weakly counitary*. Though *unitariness* seems a more natural notion, we shall work here with *counitariness*, which is better adapted to the correspondance $f \mapsto \boldsymbol{\pi}$ and enables us to take f in standard form $f = l \circ g$.

$\mathbf{P_1}$: f is reflexive iff the power series f_* respectively f^* are *even* respectively *odd*, in which case $f_{*\pm}(-z) \equiv f_{*\mp}(z)$ and $f^*_{\pm}(-z) \equiv -f^*_{\mp}(z)$.

Likewise, f is counitary iff the power series f_* respectively f^* are of the form $f_{*\mathrm{re}} \circ \tau$ respectively $\tau^{-1} \circ f^*_{\mathrm{re}} \circ \tau$ with real $f_{*\mathrm{re}}$, f^*_{re}, in which case $\bar{f}_{*\pm}(-z) \equiv f_{*\mp}(z)$ and $\bar{f}^*_{\pm}(-z) \equiv -f^*_{\mp}(z)$.

P$_2$: If a standard f is reflexive respectively counitary, then its conjugate $l^{+\frac{1}{2}} \circ f \circ l^{-\frac{1}{2}}$ is of the standard form $f = l \circ g$ with reflexive respectively counintary factors l and $g := l^{-\frac{1}{2}} \circ f \circ l^{-\frac{1}{2}}$.

P$_3$: If f is (weakly or strictly) reflexive respectively counitary, then its connector $\boldsymbol{\pi}$ is (strictly) reflexive respectively counitary. This is geometrically obvious, from the relations **P$_1$** injected into the definition (1.6), but the remarkable fact is that the analytical procedure (3.34) also respects this conservation of reflexivity or counitariness at every single step. Thus, if we apply it to the decomposition $f = l \circ g$ (as in **P$_2$**) of a reflexive f, we have to do with an *even* infinitesimal generator g_* that carries only coefficients g_{*1+d} of *even* degree d, and (3.34) automatically produces an *even* $\boldsymbol{p}_*$. The diffeo g itself is of mixed parity, but its coefficients of g_{*1+d} of *odd* degree are fully determined by the earlier coefficients of *even* degree, and can thus be used in place of the g_{*1+d}. Either way, for reflexive diffeos the calculation of the invariants is a much more pleasant affair than for general diffeos, due to the drastic reduction in the mass of coefficients and (provided f be real) to the realness of $\mathfrak{p}_*$ and $\boldsymbol{\pi}_*$.

P$_4$: Conversely, any reflexive respectively counitary $\boldsymbol{\pi}$ is the invariant of some reflexive respectively counitary f. This follows from the *canonical synthesis* (see Section 1.4) which, for c real and large enough, automatically produces diffeos f_c of the required type.[48]

P$_5$: (Reinhard Schäfke). The product or quotient of two reflexive (respectively unitary) diffeomorphisms is obviously conjugate to a reflexive (respectively unitary) diffeomorphisms, but the converse is also true: any weakly reflexive (respectively unitary) f can, for any consecutive integers n_j, be represented as a quotient of two strictly reflexive (respectively unitary) diffeos f_j:

$$f := f_1 \circ f_2^{-1} \quad \textit{with}$$
$$f(z) := z + 1 + o(1), \;\; f_j(z) := z + n_j + o(1), \;\; n_1 - n_2 = 1$$

48 As pointed out to us by Reinhard Schäfke, this can also be deduced from the bifactorisation of f in **P$_5$** below, provided we admit the existence of a pre-image f for any given $\boldsymbol{\pi}$, which fact again follows from the canonical synthesis, but may also be established more directly.

and that too with explicit factors f_j:

f weakly reflexive		f weakly counitary	
$f_j := ({}^*f) \circ l^{n_j} \circ \sigma \circ (f^*) \circ \sigma$	||	$f_j := ({}^*f) \circ l^{n_j} \circ \sigma \circ (\bar{f}^*) \circ \sigma$	(3.46)
$= f^{n_j} \circ ({}^*f) \circ \sigma \circ (f^*) \circ \sigma$	||	$= f^{n_j} \circ ({}^*f) \circ \sigma \circ (\bar{f}^*) \circ \sigma$	(3.47)
$= f^{n_j} \circ h^{-1} \circ \sigma \circ h \circ \sigma$	||	$= f^{n_j} \circ h^{-1} \circ \sigma \circ \bar{h} \circ \sigma$	(3.48)

Indeed, the equivalent definitions (3.46), (3.47), (3.48) make it clear, respectively:
– that f_1, f_2 are reflexive (respectively counitary);
– that $f = f_1 \circ f_2^{-1}$;
– that f_1, f_2 are analytic.[49]

P$_6$: Piecing together all the above, we see that the commutative, non-associative[50] operation *mix$_c$*:

$$\mathrm{mix_c} \;:\; (\boldsymbol{\pi_1}, \boldsymbol{\pi_2}) \mapsto \boldsymbol{\pi} := \boldsymbol{\pi}_{f_{1,c} \circ f_{2,c}} = \boldsymbol{\pi}_{f_{2,c} \circ f_{1,c}} \tag{3.49}$$

(where $f_{j,c}$ stands for the c-canonical pre-image of $\boldsymbol{\pi}_j$) respects reflexivity and counitariness.

4 Scalar invariants in terms of f

4.1 The invariants A_ω as entire functions of f

Let $\pi_\omega^\pm$ and $\pi_{\Diamond,\omega}$ be the Fourier coefficients of the *connectors*, as defined in Section 3.5 by weight-wise truncation of the *collectors* and passage to the limit:

$$\textit{If } +\Im(z) \gg 1 : \pi^{\pm 1}(z) = z + \sum_{\omega \in \Omega^-} \pi_\omega^\pm \, e^{-\omega z} \;;\; \pi_\Diamond(z) = \sum_{\omega \in \Omega^-} \pi_{\Diamond,\omega} \, e^{-\omega z} \tag{4.1}$$

$$\textit{If } -\Im(z) \gg 1 : \pi^{\pm 1}(z) = z + \sum_{\omega \in \Omega^+} \pi_\omega^\pm \, e^{-\omega z} \;;\; \pi_\Diamond(z) = \sum_{\omega \in \Omega^+} \pi_{\Diamond,\omega} \, e^{-\omega z} \tag{4.2}$$

The Fourier series for $\pi^\pm(z) - z$ are convergent, whereas those for $\pi_\Diamond(z)$, π_*, $\pi_\sharp$ etc are (usually) merely formal. But this makes no difference to

[49] The analytic h in (3.48) conjugates the weakly reflexive/counitary f with a strictly reflexive/counitary f_0, *i.e.* $h \circ f = f_0 \circ h$. By definition, such a pair h, f_0 exists. We may note in passing that the factorisation $f = f_1 \circ f_2^{-1}$ would still hold for complex (in the reflexive case) or real (in the unitary case) values of n_j, but in that case the above formulae break down (f_1, f_2 are no longer analytic) and we must take recourse to another, more involved construction.

[50] $mix_c(\boldsymbol{\pi_1}, \boldsymbol{\pi_2})$ is doubly germinal: for a given $(\boldsymbol{\pi_1}, \boldsymbol{\pi_2})$, it is defined for c large enough, and for a given c , it is defined for $(\boldsymbol{\pi_1}, \boldsymbol{\pi_2})$ close enough to (id, id).

the Fourier coefficients, which are always given by *convergent* series:

$$\pi^{\pm}_{\omega} = z + \sum_{1\le r}\sum_{n_i} \Gamma^{n_1,\dots,n_r}_{\pm}\, \mathrm{Tee}^{n_1,\dots,n_r}_{\omega} \tag{4.3}$$

$$\pi_{*\,\omega} = \sum_{1\le r}\sum_{n_i} \Gamma^{n_1,\dots,n_r}_{*}\, \mathrm{Taa}^{n_1,\dots,n_r}_{\omega} \tag{4.4}$$

$$\pi_{\sharp\,\omega} = \sum_{1\le r}\sum_{n_i} \Gamma^{n_1,\dots,n_r}_{\sharp}\, \mathrm{Too}^{n_1,\dots,n_r}_{\omega} \tag{4.5}$$

with the g-dependendence implicit in the coefficients $\Gamma_{\pm}, \Gamma_{*}, \Gamma_{\sharp}$ as defined in (3.10), (3.30), (3.36), or explicit in the definitions (3.14), (3.32).

However, the need to define the alien operators $\Delta^{\pm}_{\omega}$ and Δ_{ω} in uniform manner for all ω clashes with the need to associate within one and the same pair $(\pi_{\mathrm{no}}, \pi_{\mathrm{so}})$ respectively $(\pi^{-1}_{\mathrm{no}}, \pi^{-1}_{\mathrm{so}})$ northern and southern components originating from the same collector $\mathfrak{p}$ or $\mathfrak{p}^{-1}$. This clash leads to a regrettable but unavoidable disharmony in the correspondance between the invariants $A^{\pm}_{\omega}$ and A_{ω}, as defined from the resurgence equations, and the Fourier coefficients of the connectors, as derived from the collectors. This correspondance takes the form:

$$\forall\omega\in\Omega^{-}\ :\ A^{+}_{\omega} = \pi^{+}_{\omega}\ ;\ A^{-}_{\omega} = \pi^{-}_{\omega}\ ;\ +2\pi i\, A_{\omega} = \pi_{*\omega}$$

$$\forall\omega\in\Omega^{+}\ :\ A^{-}_{\omega} = \pi^{+}_{\omega}\ ;\ A^{+}_{\omega} = \pi^{-}_{\omega}\ ;\ -2\pi i\, A_{\omega} = \pi_{*\omega}$$

Remark. Nature of the convergence

(i) The convergence in (4.3) is completely unproblematic – absolute with respect to the contributions attached to individual clusters $\prod_i (g_{s_i})^{n_i}$

(ii) We also have absolute, cluster-wise convergence in (4.4) and (4.5) provided we take the precaution of switching from the coefficient systems $\{g_{*,s}\}$ or $\{g_{\sharp,s}\}$ back to the natural system $\{g_s\}$.

(iii) But we can also dispense with that change if we take the precaution of collecting in (4.4) or (4.5) all terms (in finite number) of total weight s, and then of summing all s-contributions. But summing separately the contributions attached to the clusters $\prod_i (g_{*,s_i})^{n_i}$ or $\prod_i (g_{\sharp,s_i})^{n_i}$ would not do.

4.2 The case $\rho(f) \neq 0$. Normalisation

For diffeos of the form $f(z) = z + 1 - \rho z^{-1} + \mathcal{O}(z^{-2})$ with a non-vanishing 'iterative residue' ρ, the defining relation (1.5) for the right and left iterators must be changed to

$$f^{*}_{\pm}(z) = \lim_{k\to\pm\infty} f^{k}(z) - k \pm \rho\,(c + \log|k|) \tag{4.6}$$

with the normalisation constant c as in Section 2.5. In the formal model, this leads to

$$\tilde{f}^*(z) = z + \rho\,(c + \log z) + o(z^{-1}). \tag{4.7}$$

That apart, nothing changes and all the previous results and formulae still hold, including the explicit expansions (3.12)-(3.13) and (4.3), provided we set $ze^1 := \gamma - c$ and normalise all multizetas and multitangents accordingly. As mentioned in Section 2.6, the recommended choice is $c = \gamma$, since it amounts to setting $ze^1 := 0$.

4.3 The case $p \neq 1$. Ramification

Here again, the transition is straightforward. The 'prepared' form (1.2) for the diffeo now carries fractional exponents $s \in p^{-1}\,\mathbb{N}^*$. As a consequence, the multiplicative z-plane and the convolutive ζ-plane are now p-ramified, and so is the index set Ω, which is embedded in the ζ-plane. We still have one single collector $\mathfrak{p}$ respectively $\mathfrak{p}_*$, $\mathfrak{p}_\sharp$ etc, *ramified* yet *of one piece*, but p distinct pairs of connectors, $\pi = (\pi_{\text{no}}, \pi_{\text{so}})$ respectively $\pi_* = (\pi_{*\text{no}}, \pi_{*\text{so}})$ or $\pi_\sharp = (\pi_{\sharp\text{no}}, \pi_{\sharp\text{so}})$ etc, separately *unramified* and mutually *unrelated*. The invariants $\pi_\omega^{\pm}$ respectively $\pi_{*\omega}$, $\pi_{\sharp\omega}$ are still given by the familiar formulae (4.3), (4.4), (4.5) but with Fourier coefficients Tee_ω^s respectively Taa_ω^s, Too_ω^s etc that are best calculated by resurgent analysis, as in Section 2.7, and are no longer finite sums of multizetas, even of ramified ones.

The transition to the most general case, with (ρ, p) any element of $(\mathbb{C}, \mathbb{N}^*)$, follows on exactly the same lines, and merely combines the partial adjustments of the present and preceding subsections.

4.4 Growth properties of the invariants

Growth in ω for a given analytic f: For a diffeo f in prepared form (1.2), any majorisation of its coefficients easily translates into a majorisation of its invariants:

$$\left\{\,|f_{[s]}| \leq c_0\, c_1^s\,\right\} \Longrightarrow \left\{\,|A_\omega^{\pm}| \leq C_0\, C_1^{|\omega|}\,\right\}. \tag{4.8}$$

Rough estimates of (C_0, C_1) in terms of (c_0, c_1) were given in [5] and sharper ones in [1]. These results can be derived from a geometric analysis in the z-plane or from a resurgent analysis in the ζ-plane. Things change, though, when we go over to the Gevrey case.

Growth in ω for a given f of Gevrey class: Formal diffeos f (in prepared form) of Gevrey class τ are easily shown to be stable under formal conjugations (also in prepared form) of the same Gevrey class. For $0 < \tau$, the Gevrey class is non-analytic, and Gevrey conjugacy turns out to be strictly stronger than formal conjugacy if and only if $\tau < 1$. This implies, for $0 < \tau < 1$, the existence of Gevrey conjugation invariants. These, however, can no longer be defined in the z-plane, since f is purely formal and has no geometric realisation there. In the ζ-plane, though, the Borel tranforms of the iterators ${}^*f$ and f^* still exist (again, assuming $\tau < 1$); still extend to uniform analytic functions on $\widetilde{\mathbb{C} - 2\pi i\, \mathbb{Z}}$; still verify the familiar resurgence equations (1.66)-(1.67); and still unambigously define invariants $A_\omega^\pm$ and A_ω, which are still given by the explicit expansions (4.3)-(4.4). The only difference lies in the faster than exponential growth of $\hat{f}^*(\zeta)$ and ${}^*\hat{f}(\zeta)$ as $|\zeta| \to \infty$, and in the faster than exponential growth of $A_\omega^\pm$ as $|\omega| \to \infty$. More precisely, for $0 < \tau < 1$, the earlier implication (4.8) becomes[51]:

$$\{\ |f_{[s]}| \leq c_0\, c_1^s\, s^{\tau s}\ \} \Longrightarrow \{\ |A_\omega^\pm| \leq C_0\, C_1^{|\omega|}\ \exp(C_2\, |\omega|^{\frac{1}{1-\tau}})\} \tag{4.9}$$

Growth in f for a given ω. We may now fix ω and ask how $A_\omega^+(f)$, $A_\omega^-(f)$, $A_\omega(f)$ behave as functions of f or, to simplify, as entire functions of any given coefficient $f_{[s]}$ $(s \geq 2)$ relative to a prepared form (1.2). Unlike with the ω-growth, there is little difference here between $A_\omega^\pm$ and A_ω.

(i) If $s > 2$, all three entire functions $A_\omega^+(f_{[s]})$, $A_\omega^-(f_{[s]})$, $A_\omega(f_{[s]})$ have at most exponential growth in $|f_{[s]}|^{\frac{1}{s-1}}$.

(ii) If $s = 2$, the corresponding coefficient coincides up to sign with the iterative residue (*i.e.* $f_{[2]} = -\rho$), and the entire functions $A_\omega^+(\rho)$, $A_\omega^-(\rho)$, $A_\omega(\rho)$ have at most exponential growth in $|\rho \log \rho|$. The result appears to be sharp.[52]

These results are almost "special cases" of the following statement: at any given point ζ_0 on $\widetilde{\mathbb{C} - \Omega}$, the Borel transform of the direct iterator assumes a value $\hat{f}^*(\zeta_0)$ which, as an entire function of $f_{[s]}$, is exactly of exponential type in $|f_{[s]}|^{\frac{1}{s-1}}$. This applies even for $s = 2$. The difference between the cases $s \neq 2$ and $s = 2$ makes itself felt only when we move ζ_0 to some point ω_0 located over Ω, to investigate the leading singularity there and infer from it the value of the invariants. When $\rho = 0$, the leading singularity in question is a simple pole $a_{\omega_0}(\zeta - \omega_0)^{-1}$, but when

[51] For details, see [5, page 424]

[52] See the argument in [2, Section 8].

$\rho \neq 0$ it is of the form $a_{\omega_0}(\zeta - \omega_0)^{\rho\,\omega_0 - 1}/\,\Gamma(\rho\,\omega_0)$ and can be quite violent if ρ has an imaginary part.

We shall take up these growth and convergence issues more systematically in Section 7.

4.5 Alternative computational strategies

Direct Fourier analysis in the multiplicative plane. The methods amounts to calculating the limit:[53]

$$A_{\omega}^{\mp\epsilon(\omega)} = \boldsymbol{\pi}_{\omega}^{\pm} = \lim_{k \to \pm\infty} \int_{z_0}^{1+z_0} \left(l^{-k} \circ f^{2k} \circ l^{-k}(z) - z \right) e^{\omega\, z}\, dz \quad (4.10)$$

with $\epsilon(\omega) := sign(\Im(\omega))$. Although the parenthesised part of the integrand converges to $\boldsymbol{\pi}^{\pm}(z) - z$ for $|\Im(z)|$ large enough, the above scheme, even after optimisation in the choice of z_0, is computationally costly (integral instead of series) and inefficient (arithmetical convergence) as well as theoretically opaque (it sheds no light on the internal structure of the invariants as functions of f). But it has the merit of being almost insensitive to the choice of ω, unlike the next method.

(ii) Asymptotic coefficient analysis in the formal model. The method starts with the inductive calculation of the first N coefficients of the direct iterator $f^*(z)$ from its functional equation (1.11). One then switches to the Borel transform $\hat{f}^*(\zeta)$ and uses the method of *coefficient asymptotics*[54] to derive the form of the two singularities[55] closest to the origin (they are located over $\pm 2\pi i$). When applied to a parameter-free diffeo f with proper optimising precautions, the method is superbly efficient for computing $A_{\pm 2\pi i}$, even for diffeos f that are 'large', *i.e.* distant from the identity. Thus, with N taken in the region of 200 or 300, one typically gets $A_{\pm 2\pi i}$ with 100 exact digits or more, in less than half an hour of Maple time.

The method works less well, however, for $\omega_0 = 2\pi i n$ with $n > 1$. One must then start with a conformal mapping $\zeta \mapsto \zeta' = h(\zeta)$ of $\mathcal{R} = \widetilde{\mathbb{C} - 2\pi i \mathbb{Z}}$ that keeps $0_\bullet$ fixed and takes the points $+\omega_0^{\text{main}}$ and $-\omega_0^{\text{main}}$

[53] If $\rho(f) \neq 0$, the shift l^{-k} should of course be replaced by $l^{-k+(c+\log k)\rho}$, with $c = \gamma$ as recommended choice for the normalisation constant c. See Section 2.6.

[54] For a brief exposition of the method, see for ex. the section Section 2.3 of *Power Series with sum-product Taylor coefficients and their resurgence algebra*, J. Ecalle and S. Sharma, Ed. Scuola Normale Superiore, Pisa, 2011.

[55] Or of the 2 p closest singularities when $p(f) \neq 0$.

closer to the origin than all other points $\pm\omega^{\text{main}}$, with ω^{main} denoting *the* ramification point of $\mathcal{R}$ over ω that abuts the *main* real half-plane. One can then apply the method of coefficient asymptotics in the ζ'-plane, with the Taylor series $\widehat{f}^*(h^{-1}(\zeta'))$ in place of the series $\widehat{f}^*(\zeta)$, to calculate $A^+_{\omega_0}$ and $A^-_{-\omega_0}$.

(iii) Resurgent analysis in the Poincaré plane. That method also is based on the resurgence equation (1.67) verified by the direct iterator f^*. But instead of interpreting that resurgence equation, as usual, in the highly ramified ζ-plane, one performs a conformal transform $\zeta \to \xi$ derived from the classical modular function λ :

$$\zeta = q(\xi) := -\log(1-\lambda(\xi)) = -\log\lambda\left(-\frac{1}{\xi}\right) = 16\sum_{n\,\text{odd}} q_n\, e^{2\pi i \xi} \tag{4.11}$$

$$q_n := \sum_{d|n}\frac{1}{d} = \frac{1}{n}\sum_{d|n} d \tag{4.12}$$

That comformal transform does three things:
(*) it maps the Riemann surface $\mathcal{R} := \widetilde{\mathbb{C} - 2\pi i\,\mathbb{Z}}$ of the ζ variable uniformly onto the Poincaré half-plane $\Im(\xi) > 0$;
(**) it changes the power series $\widehat{f}^*(\zeta)$ with finite radius of convergence into a Fourier series $\widehat{f}^*(q(\xi))$ that converges on the entire Poincaré half-plane.
(***) it turns the alien operators into finite superpositions of post-composition operators – more precisely, post-composition by simple homographies $h^{\pm}_{\omega,j}$ or $h^{\pm}_{\omega,j}$ with entire coefficients:

$$\Delta^{\pm}_{\omega}\widehat{\varphi}(\xi) := \widehat{\varphi}\circ h^{\pm}_{\omega,1}(\xi) - \widehat{\varphi}\circ h^{\pm}_{\omega,2}(\xi) \tag{4.13}$$

$$\Delta_{\omega}\widehat{\varphi}(\xi) := \sum_{1\le j\le 2^r} m_{\omega,j}\,\widehat{\varphi}\circ h_{\omega,j}(\xi) \quad \left(r := |\frac{\omega}{2\pi i}|,\, m_{\omega,j}\in\mathbb{Q}\right) \tag{4.14}$$

The method is efficient enough for small values of ω, but as $r := |\frac{\omega}{2\pi i}|$ increases, the distances

$$H^{\pm}(\omega) := \max_{\Im(\xi)>0} \inf\{\Im(\xi)\,,\ \Im(h^{\mp}_{\omega,1}(\xi))\,,\ \Im(h^{\mp}_{\omega,2}(\xi))\} \tag{4.15}$$

$$H(\omega) := \max_{\Im(\xi)>0} \inf\{\Im(\xi)\,,\ \Im(h_{\omega,1}(\xi))\,,\ldots,\ \Im(h_{\omega,2^r}(\xi))\} \tag{4.16}$$

rapidly decrease to zero, making it necessary to evaluate our Fourier series for $\widehat{f}^*(q(\xi))$ close to the boundary of their domain of convergence, *i.e.* the real axis, which of course is computationally costly.

(iv) Explicit multizetaic expansions. This method, to which the present paper is devoted, has the advantage of explicitness and theoretical transparency, expressing as it does the invariants in terms of universal transcendental constants (the multizetas) and of the diffeo's Taylor coefficients. It has the further advantage of handling large values of ω almost as efficiently as small ones. But the method's chief drawback would seem to be this: it involves expansions which converge very fast (faster than geometrically) once they reach 'cruising speed', but which often take a damn long time to reach that speed. This is the case, not so much for ω large, but for f large, *i.e.* for diffeos too distant from *id*.

4.6 Concluding remarks

(i) The invariants as autark functions.
Local, analytic, resonant vector fields X ranging through a *fixed* formal conjugacy class, possess holomorphic invariants A_ω which are *autark* functions of X, that is to say, of any given *free*[56] Taylor coefficient of X. Autark functions, very informally, are entire functions whose asymptotic behaviour in every sector of exponential increase or decrease admits a complete description, with dominant exponential terms accompanied by divergent-resurgent power series, which in turn verify a *closed* system of resurgence equations. Whether the invariants A_ω of diffeos are autark, too, seems likely but is yet unproved. Be that as it may, one would like to fully understand the asymptotic behaviour of A_ω as f grows, or as any given coefficient or parameter in f grows, since for very 'large' diffeos f the *direct* computation of the invariants would in any case be very costly.

(ii) Formal multizetas: dynamical vs arithmetical variants.
There exist several distinct but most probably equivalent notions of *arithmetical formal multizetas*, like the multizeta symbols subject to the two systems of so-called *quadratic multizeta relations*, or again to the *pentagonal, hexagonal* and *digonal relations*. But there also exists a demonstrably distinct and *weaker* notion of *dynamical formal multizetas* (and *multitangents*), by which we mean any system $\mathbb{S}$ of scalar-valued multizeta symbols (respectively function-valued multitangent symbols) that, when inserted into the expansions (4.3) (respectively (3.15)) guarantees, first, the convergence of these expansions, and, second, the invariance of the A_ω (respectively $\boldsymbol{\pi}$) so produced. This immediately suggests a pro-

[56] *I.e.* of each coefficient that may freely vary without causing X to leave its formal conjugacy class.

gramme: to repeat for the dynamical multizetas what has been successfully done for their arithmetical counterparts, in particular to construct *explicit, complete and canonical systems of irreducibles*.

(iii) Abstract invariants.
Let $\{{}^{\mathbb{S}}\!A_\omega,\ \omega \in \Omega\}$ be the system of 'abstract' invariants induced by a system $\mathbb{S}$ of dynamical multizetas as above. Since the system of natural invariants $\{A_\omega,\ \omega \in \Omega\}$ is complete, there necessarily exist conversion formulae of the form:

$$ {}^{\mathbb{S}}\!A_{\omega_0} = \sum_{1\leq r} \sum_{\omega_1+\ldots\omega_r=\omega_0} H_{\mathbb{S}}^{\omega_1,\ldots,\omega_r} A_{\omega_1} \ldots A_{\omega_r} \tag{4.17} $$

that respect the basic ω-gradation and carry interesting 'universal' structure constants $H_{\mathbb{S}}^{\bullet}$. These constants ought to be of particular significance in the case of the system $\mathbb{S}_0$ of 'rational' dynamical multizetas which is analogous, on the dynamical side, to the canonical system of 'rational'[57] multizetas on the arithmetical side.

5 Complement: twisted symmetries and multitangents.

The aim of this section is twofold:

(i) to review in a systematic and orderly fashion the combinatorial lemmas relevant to this investigation
(ii) to examine the most general *symmetry types* and the structure coefficients attached to them — less for their own sake than for showing how exceptional and deserving of attention the dozen or so *special symmetry types* are.

5.1 Twisted alien operators

Let $\gamma(t) = \sum_{0\leq r} \gamma_r\, t^{r+1}$ and consider the alien operator

$$ \boldsymbol{\mathcal{D}}^{\diamondsuit} := \gamma(\boldsymbol{\mathcal{D}}^{+} - 1) = \gamma(e^{2\pi i \boldsymbol{\mathcal{D}}} - 1) \tag{5.1} $$

The ω-components of $\boldsymbol{\mathcal{D}}^{\diamondsuit}$ are of the form:

$$ \boldsymbol{\mathcal{D}}^{\diamondsuit} = \sum_{\arg(\omega)=0} \boldsymbol{\Delta}_{\omega}^{\diamondsuit} = \sum_{\arg(\omega)=0} e^{-\omega.z}\, \Delta_{\omega}^{\diamondsuit} \tag{5.2} $$

$$ (\widehat{\Delta}_{\omega}^{\diamondsuit}\hat{\varphi})(\zeta) := \sum_{\epsilon_1,\ldots,\epsilon_r} \frac{\epsilon_r}{2\pi i}\, \lambda^{\diamondsuit}_{\epsilon_1,\ldots,\epsilon_{r-1}}\, \hat{\varphi}^{\left(\begin{smallmatrix}\epsilon_1 & ,\ldots, & \epsilon_r\\ \omega_1 & ,\ldots, & \epsilon_r\end{smallmatrix}\right)}(\omega+\zeta). \tag{5.3} $$

[57] They become rational, of course, only after a homogeneous rescaling that amounts to setting $\pi := 1$.

Like with the λ-coefficients of the already familiar operators Δ_ω, $\Delta_\omega^\pm$, $\Delta_\omega^\sharp$, $\Delta_\omega^{\sharp\sharp}$, the coefficients $\lambda^{\diamondsuit}_{\epsilon_1,\dots,\epsilon_{r-1}}$ that describe the action of $\Delta_\omega^{\diamondsuit}$ depend only on the *crossing pattern*, *i.e.* on the number p, q of plus and minus signs in the sequence $\{\epsilon_i\}$. But in this case they are given by:

$$\lambda^{\diamondsuit}_{\epsilon_1,\dots,\epsilon_{r-1}} = \lambda^{[p,q]}_{\diamondsuit} = (-1)^q \sum_{0\le k\le p} \frac{p!}{(p-k)!\,k!}\,\gamma_{q+k}. \tag{5.4}$$

For $\gamma(t) = \frac{t}{1+t/2}$ or $\gamma(t) = \frac{(1+t)^2-1}{(1+t)^2+1}$, we recover the structure coefficients $\lambda_\sharp^{[p,q]}, \lambda_{\sharp\sharp}^{[p,q]}$ for the alien operators $\Delta_\omega^\sharp$ and $\Delta_\omega^{\sharp\sharp}$ introduced in Section 1.6.

$$\lambda_\sharp^{[p,q]} = 2^{-p-q} \quad , \quad \lambda_{\sharp\sharp}^{[p,q]} = \varrho(p-q)\,2^{-int(\frac{p+q+1}{2})}$$

where ϱ is *the* even function from $\mathbb{Z}/8\,\mathbb{Z}$ to $\mathbb{Z}$ verifying $\varrho(k+4) = -\varrho(k)$ and $\varrho(0) = \varrho(\pm 1) = 1$. Since $\varrho(2) = -\varrho(2+4) = -\varrho(-2) = -\varrho(2)$, it follows that $\varrho(\pm 2) = 0$.

Short proof: After checking that the λ-coefficients of $\mathcal{D}^{\diamondsuit}$ inherit from those of $\mathcal{D}^+$ the crucial property of depending solely on the crossing pattern (p, q), we are left with the simple task of considering the case of p initial right-crossings followed by q final left crossings. As in Section 1.6 we begin with the situation when all singularities are located over $\mathbb{N}$. Next we define the non-commuting elementary shifts σ, τ as in Section 1.6, then use the expansion

$$\begin{aligned}\mathcal{D}^+ - 1 &= (1-\tau)(1-\sigma)^{-1} - 1 = (\sigma-\tau)(1-\sigma)^{-1}\\ &= \Big(\sigma - \tau)(1 + \sum_{1\le p} \sigma^p\Big)\end{aligned}$$

and in each power $(\mathcal{D}^+ - 1)^r$ collect the terms that contribute to $(\sigma - \tau)\tau^q\sigma^p$.

5.2 Twisted mould symmetries

Given any two power series without constant term

$$\alpha(t) = \sum_{0\le r} \alpha_r\, t^{1+r} \quad , \quad \beta(t) = \sum_{0\le r} \beta_r\, t^{1+r} \qquad (\alpha_0 \ne 0, \beta_0 \ne 0)$$

we denote by $\alpha(Id^\bullet)$, $\beta(Id^\bullet)$, or simply $\alpha^\bullet$, $\beta^\bullet$ the moulds whose length-0 components vanish and whose length-r components are equal to

$$\alpha^{\omega_1} \equiv \alpha_0 \ , \ \alpha^{\omega_1,\dots,\omega_r} \equiv \alpha_{r-1} \quad , \quad \beta^{\omega_1} \equiv \beta_0 \ , \ \beta^{\omega_1,\dots,\omega_r} \equiv \beta_{r-1}$$

irrespective of the actual values of ω_i. We then define coefficients $\alpha^{p,q}$ and $\beta_{p,q}$ by setting

$$\sum \alpha^{p,q}\, t_1^p\, t_2^q \;:=\; \alpha\big(\alpha^{-1}(t_1) + \alpha^{-1}(t_2)\big) \tag{5.5}$$

$$\sum \beta_{p,q}\, t_1^p\, t_2^q \;:=\; \beta^{-1}\big(\beta(t_1) + \beta(t_2)\big). \tag{5.6}$$

If $M^\bullet \in \alpha^\bullet \circ \textit{alternal}^\bullet$, then for any two sequences $\boldsymbol{\omega}', \boldsymbol{\omega}'' \neq \boldsymbol{\emptyset}$:

$$\sum_{\substack{\boldsymbol{\omega}'^1..\boldsymbol{\omega}'^p=\boldsymbol{\omega}' \\ \boldsymbol{\omega}''^1..\boldsymbol{\omega}''^q=\boldsymbol{\omega}''}}^{1\le p,1\le q} \alpha^{p,q}\; M^{\boldsymbol{\omega}'^1}\ldots M^{\boldsymbol{\omega}'^p} M^{\boldsymbol{\omega}''^1}\ldots M^{\boldsymbol{\omega}''^q} \equiv \sum_{\boldsymbol{\omega}\in \mathrm{sha}(\boldsymbol{\omega}',\boldsymbol{\omega}'')} M^{\boldsymbol{\omega}}. \tag{5.7}$$

If $M^\bullet \in \textit{alternal}^\bullet \circ \beta^\bullet$, then for any two sequences $\boldsymbol{\omega}', \boldsymbol{\omega}'' \neq \boldsymbol{\emptyset}$:

$$0 \;\equiv\; \sum_{\boldsymbol{\omega}\in \mathrm{sha}_{p,q}(\boldsymbol{\omega}',\boldsymbol{\omega}'')}^{1\le p,1\le q} \beta_{p,q}\, M^{\boldsymbol{\omega}}. \tag{5.8}$$

If $M^\bullet \in \alpha^\bullet \circ \textit{alternal}^\bullet \circ \beta^\bullet$, then for any two sequences $\boldsymbol{\omega}', \boldsymbol{\omega}'' \neq \boldsymbol{\emptyset}$:

$$\sum_{\substack{\boldsymbol{\omega}'^1..\boldsymbol{\omega}'^p=\boldsymbol{\omega}' \\ \boldsymbol{\omega}''^1..\boldsymbol{\omega}''^q=\boldsymbol{\omega}''}}^{1\le p,1\le q} \alpha^{p,q}\; M^{\boldsymbol{\omega}'^1}\ldots M^{\boldsymbol{\omega}'^p} M^{\boldsymbol{\omega}''^1}\ldots M^{\boldsymbol{\omega}''^q} \equiv \sum_{\boldsymbol{\omega}\in \mathrm{sha}_{p,q}(\boldsymbol{\omega}',\boldsymbol{\omega}'')}^{1\le p,1\le q} \beta_{p,q}\; M^{\boldsymbol{\omega}}. \tag{5.9}$$

An important sub-case is when α, β are reciprocal, for it corresponds to a symmetry type $\alpha^\bullet \circ \textit{alternal}^\bullet \circ \beta^\bullet$ stable under mould-composition and leads to identical coefficients $\alpha^{p,q} = \beta_{p,q}$ on both sides of (5.9).

It is often preferable to take *elternel* rather than *alternal* as a standard of reference. Since

$$\textit{elternel}^\bullet = (\exp(\textit{Id}^\bullet) - 1^\bullet) \circ \textit{alternal}^\bullet \circ \log(1^\bullet + \textit{Id}^\bullet) \tag{5.10}$$

we see at once that moulds respectively of type

$$\textit{elternel}^\bullet \circ \delta^\bullet \quad , \quad \gamma^\bullet \circ \textit{elternel}^\bullet \quad , \quad \gamma^\bullet \circ \textit{elternel}^\bullet \circ \delta^\bullet$$

still verify identities of the form (5.7), (5.8), (5.9), but with new coefficients $\gamma^{[p,q]}$, $\delta_{[p,q]}$, defined by

$$\sum \gamma^{[p,q]}\, t_1^p\, t_2^q \;:=\; \gamma\big(\gamma^{-1}(t_1) + \gamma^{-1}(t_2) + \gamma^{-1}(t_1)\,\gamma^{-1}(t_2)\big) \tag{5.11}$$

$$\sum \delta_{[p,q]}\, t_1^p\, t_2^q \;:=\; \delta^{-1}\big(\delta(t_1) + \delta(t_2) + \delta(t_1)\,\delta(t_2)\big) \tag{5.12}$$

in place of $\alpha_{p,q}$, $\beta^{p,q}$. Indeed, in view of (5.10), (5.11)-(5.12) results from (5.5)-(5.6) under the change $\alpha(t) = \gamma(e^t - 1)$, $\beta(t) = \log(1 + \delta(t))$

5.3 Twisted co-products

As useful as the statements of Section 5.2 are the dual statements:

(i) If $\theta_\Diamond = \alpha(\theta_*)$ with $cop(\theta_*) = 1 \oplus \theta_* + \theta_* \oplus 1$, then

$$cop(\theta_\Diamond) = 1 \oplus \theta_\Diamond + \theta_\Diamond \oplus 1 + \sum_{1 \le p,q} \alpha^{p,q}\,(\theta_\Diamond)^p \oplus (\theta_\Diamond)^q. \tag{5.13}$$

(ii) If $\theta_\Diamond = \gamma(\theta)$ with $cop(\theta) = 1 \oplus \theta + \theta \oplus 1 + \theta \oplus \theta$, then

$$cop(\theta_\Diamond) = 1 \oplus \theta_\Diamond + \theta_\Diamond \oplus 1 + \sum_{1 \le p,q} \gamma^{[p,q]}\,(\theta_\Diamond)^p \oplus (\theta_\Diamond)^q. \tag{5.14}$$

5.4 Twisted multitangents

Let $\gamma(t) = \sum_{0 \le r} \gamma_r\, t^{r+1}$ and $\delta(t) = \sum_{0 \le r} \delta_r\, t^{r+1}$ as usual[58] and let

$$\mathrm{Te}^\bullet_{\gamma,\delta} := \gamma(Id^\bullet) \circ (\mathrm{Te}^\bullet - 1^\bullet) \circ \delta(Id^\bullet) = \gamma(Id^\bullet) \circ \mathrm{Tee}^\bullet \circ \delta(Id^\bullet). \tag{5.15}$$

Linearisation lemma: *The twisted multitangents* $\mathrm{Te}^\bullet_{\gamma,\delta}(z)$ *can be uniquely expanded into sums of symmetrel multitangents* $\mathrm{Te}^\bullet(z)$

$$\mathrm{Te}^{n_1,\ldots,n_r}_{\gamma,\delta}(z) = \sum_{1 \le s \le r} \sum_{1 \le r_i}^{r_1+\cdots+r_s=r} \sum_{\sigma \in \mathfrak{S}_{r_1,\ldots,r_s}} H_\sigma^{r_1,\ldots,r_s}\ \mathrm{Te}^{n_{\sigma,1},\ldots,n_{\sigma,s}}(z) \tag{5.16}$$

with universal coefficients $\mathrm{H}^{\boldsymbol{r}}_\sigma = \mathrm{H}^{\boldsymbol{r^*}}_{[p,q]}$ *defined as follows*

$$\begin{aligned} H^{r_1,\ldots,r_s}(\sigma) &= H^{r_1^*,\ldots,r_{s^*}^*}_{[p,q]} \\ &= \sum_{k=0}^{r-s^*} \left[\sum_{l=0}^{p} \gamma_{k+q+l} \frac{p!}{(p-l)!\,l!} \right] \left[\frac{\nabla^k}{k!} \left(\delta_{r_1^*-1} \ldots \delta_{r_{s^*}^*-1} \right) \right]. \end{aligned} \tag{5.17}$$

(i) The sum (5.16) ranges over all ordered sequences $(r_1, \ldots, r_s)$ and all permutations σ in $\mathfrak{S}_{r_1,\ldots,r_s}$, *i.e.* all σ that *increase* on each of the intervals I_{r_k} of the partition

$$\mathcal{I}_{r_1} \sqcup \cdots \sqcup \mathcal{I}_{r_s} = [1, \ldots, r] \in \mathbb{Z} \qquad (\mathrm{card}(\mathcal{I}_{r_i}) = r_i).$$

(ii) The indices of $Te^\bullet(z)$ on the right-hand side of (5.16) are given by

$$n_{\sigma,i} = \sum_{j \in \mathcal{I}_{r_i}} n_{\sigma(j)} \qquad \forall\ i \in [1, s].$$

[58] For the moment, we assume neither $\gamma \circ \delta = id$ nor $\gamma_0 \neq 0$, $\delta_0 \neq 0$.

(iii) $\mathcal{I}_{r_1^*} \sqcup \cdots \sqcup \mathcal{I}_{r_{s^*}^*}$ denotes the *minimal* sub-partition of $\mathcal{I}_{r_1} \sqcup \cdots \sqcup \mathcal{I}_{r_s}$ such that σ increases *without gaps* on each $\mathcal{I}_{r_k^*}$, *i.e.* such that

$$\sigma(j) - \sigma(i) \equiv j - i \qquad \forall i, j \in \mathcal{I}_{r_k^*} \quad , \quad \forall k \in [1, s^*].$$

(iv) There exist two full orders $<$ and $<_\sigma$ on the set $\{\mathcal{I}_{r_1^*}, \ldots, \mathcal{I}_{r_{s^*}^*}\}$:

$$\begin{aligned} \{\mathcal{I}_{r_k^*} < \mathcal{I}_{r_l^*}\} \;&\Leftrightarrow\; \quad i < j \qquad \forall (i, j) \in (\mathcal{I}_{r_k^*}, \mathcal{I}_{r_l^*}) \;\Leftrightarrow\; k < l \\ \{\mathcal{I}_{r_k^*} <_\sigma \mathcal{I}_{r_l^*}\} \;&\Leftrightarrow\; \sigma(i) < \sigma(j) \quad \forall (i, j) \in (\mathcal{I}_{r_k^*}, \mathcal{I}_{r_l^*}). \end{aligned}$$

For each $k \leq s^*$ the immediate $<_\sigma$-successor of $\mathcal{I}_{r_k^*}$ is noted $\mathcal{I}_{r_{k+}^*}$ (when it exists, *i.e.* when $\mathcal{I}_{r_{k+}^*}$ is not $<_\sigma$-maximal). The integer p (respectively q) so defined

$$p := \sum_{k<k^+} 1 \quad , \quad q := \sum_{k>k^+} 1 \qquad (p + q \equiv s^* - 1)$$

measures the compatibility (respectively incompatibility) of $<$ and $<_\sigma$.

(v) ∇ denotes the *derivation* on $\mathbb{Q}[\delta_0, \delta_1, \delta_2 \ldots]$ characterised by

$$\nabla\delta_0 := 0, \quad \nabla\delta_1 := (\delta_0)^2, \quad \nabla\delta_2 := 2\,\delta_0\,\delta_1, \ldots, \quad \nabla\delta_r := \sum_{r'=0}^{r-1} \delta_{r'}\,\delta_{r-1-r'}.$$

It readily follows that

$$\frac{\nabla^r}{r!}\,\delta_r \equiv (\delta_0)^{r+1} \quad , \quad \frac{\nabla^l}{l!}\,\delta_r \equiv 0 \quad \textit{iff } r < l.$$

Remark 1. When k takes either of its extreme values 0 or $r - s^*$, the formula (5.17) gives for $H^{\boldsymbol{r^*}}_{[p,q]}$ two γ-dependent parts respectively of the form

$$\begin{aligned} (*) &\quad \gamma_q + \cdots + \gamma_{p+q} \\ (**) &\quad \gamma_{q+r-s^*} + \cdots + \gamma_{p+q+r-s^*} = \gamma_{r-1-p} + \cdots + \gamma_{r-1} \end{aligned}$$

while the δ-dependent parts reduce to

$$\begin{aligned} (*) &\quad \frac{\nabla^0}{0!} \prod_i \delta_{r_i^*-1} = \prod_i \delta_{r_i^*-1} \\ (**) &\quad \frac{\nabla^{r-s^*}}{(r-s^*)!} \prod_i \delta_{r_i^*-1} = \prod_i \left(\frac{\nabla^{r_i^*-1}}{(r_i^*-1)!} \delta_{r_i^*-1} \right) = \prod_i (\delta_0)^{r_i^*} = (\delta_0)^r \end{aligned}$$

As a consequece of (∗∗), $H^{\boldsymbol{r^*}}_{[p,q]}$ always contains the term $\gamma_{r-1}\,(\delta_0)^r$ among its summands.

Remark 2. Exchanging two adjacent intervals $\mathcal{I}_{r_i^*}$ and $\mathcal{I}_{r_{i+1}^*}$ with *non-adjacent* images[59] $\sigma(\mathcal{I}_{r_i^*})$ and $\sigma(\mathcal{I}_{r_{i+1}^*})$ leaves the pair (p,q) unchanged. On the other hand, once (p,q) has been determined in function of σ and the ordered sequence $\boldsymbol{r^*}$, the order in $\boldsymbol{r^*}$ no longer counts for the determination of $H^{\boldsymbol{r^*}}_{\gamma,\delta}(p,q)$. For a given depth r, therefore, the maximum number of distinct values assumed by $H^{\boldsymbol{r^*}}_{\gamma,\delta}(p,q)$ cannot exceed $\sum_{k=1}^{r} k\,p(r,k)$, with $p(r,k)$ denoting the number of k-multiple partitions of r.

Example. Let us calculate the coefficients of $Te^{n_1,n_3+n_4,n_2+n_6+n_7}$ in the expansion (5.16) of $Te^{n_1,\ldots,n_6}_{\gamma,\delta}$. Starting from a partition $\boldsymbol{r}=(1,2,3)$ with $s=3$ we arrive at the refined partition $\boldsymbol{r^*}=(1,2,1,2)$ with $s^*=4$. Applying (5.17) and the rules for handling ∇, we successively find:

$$\begin{aligned}
H^{1,2,1,2}_{[2,1]} &= \sum_{k=0}^{2}(\gamma_{1+k}+2\,\gamma_{2+k}+\gamma_{3+k})\,\frac{\nabla^k}{k!}(\delta_0\delta_1\delta_0\delta_1)\\
&= +(\gamma_1+2\,\gamma_2+\gamma_3)\,(\delta_0^2\,\delta_1^2)\\
&\quad\; +(\gamma_2+2\,\gamma_3+\gamma_4)\,(2\,\delta_0^4\,\delta_1)\\
&\quad\; +(\gamma_3+2\,\gamma_4+\gamma_5)\,(\delta_0^6).
\end{aligned}$$

We would find exactly the same coefficient for $Te^{n_1,n_3+n_4,n_2,n_6+n_7}$ and for $Te^{n_1,n_3+n_4,n_6+n_7,n_2}$, in agreement with the observation of Remark 2 above.

Special cases. If we now assume that $\gamma\circ\delta=id$, we find few noteworthy simplications, apart from the automatic vanishing of the coefficient $H^{r}_{[r-1,0]}$ that stands in front of the lone 'monotangent' $Te^{|\boldsymbol{n}|}$ in the $Te^{\bullet}$-expansion (5.16) of $Te^{\boldsymbol{n}}$. For real simplications, we must turn to the multitangents $Te^{\bullet}_{\natural c}=Te^{\bullet}_{\gamma_c,\delta_c}$ with homographic driving series $\gamma_c(t)=\frac{t}{1+c\,t}$ and $\delta_c(t)=\frac{t}{1-c\,t}$. In that case, a simple calculation shows that in the expansion (5.16) of $Te^{\boldsymbol{n}}_{\gamma_c,\delta_c}$ the only surviving terms $Te^{n_{\sigma,1},\ldots,n_{\sigma,s}}$ are those whose indices $n_{\sigma,k}$ carry no sums n_i+n_{i+1} of consecutive terms. This implies that the only non-zero coefficients $H^{\boldsymbol{r^*}}_{\gamma_c,\delta_c}(p,q)$ correspond to reduced sequences $\boldsymbol{r^*}$ with all multiplicities $r_i^*\equiv 1$, so that $s=r$. Moreover, even these surviving $H^{\boldsymbol{r^*}}_{[p,q]}$ turn out to be extremely simple:

$$H^{1,\ldots,1}_{[p,q]}=(1-c)^p\,(-c)^q. \tag{5.18}$$

59 This of course is possible only if $\mathcal{I}_{r_i^*}$ and $\mathcal{I}_{r_{i+1}^*}$ do not stem from one and the same $\mathcal{I}_k$.

When $c = 1/2$, we recover the formula (2.49) for the $Te^\bullet$-expansion of the olternol multitangents $Too^\bullet$.

The family

$$\gamma(t) := \frac{1}{c}\frac{(1+t)^{2c}-1}{(1+t)^{2c}+1} \quad , \quad \delta(t) := \left(\frac{1+c\,t}{1-c\,t}\right)^{\frac{1}{2c}} - 1 \tag{5.19}$$

does not lead to simple results, except of course in the case $c = 1/2$, where it coincides with (5.18), and in the case $c = 1$, where all coefficients $H^{\boldsymbol{r^*}}_{\gamma_c,\delta_c}(p,q)$ turn out to be simple products of Catalan numbers times a negative power of 2 and an appropriate sign in front. Here is the precise statement:

8-periodicity of $H^{\boldsymbol{r^*}}_{[p,q]}$. For γ, δ of the form

$$\gamma(t) := \frac{t+\frac{1}{2}t^2}{1+t+\frac{1}{2}t^2} \quad , \quad \delta(t) := \left(\frac{1+t}{1-t}\right)^{\frac{1}{2}} - 1 \tag{5.20}$$

we have

$$H^{r_1^*,\dots,r_s^*}_{[p,q]} = \rho_*(s_u - s_e + 2\,p) \;\; 2^{\mathrm{int}(s/2)} \prod_{1\le i\le s} \kappa(r_i^*) \tag{5.21}$$

$$= \rho(2\,s_u + p - q) \;\; 2^{\mathrm{int}(s/2)} \prod_{1\le i\le s} \kappa(r_i^*) \tag{5.22}$$

with

$$s_u := \sum_{r_i^*=1} 1 \,,\, s_e := \sum_{r_i^*\,even\ge 2} 1 \,,\, s_o := \sum_{r_i^*\,odd\ge 3} 1 \qquad (1+p+q \equiv s_u+s_o+s_e)$$

$$int(s) = integer\,part\,of\,s \tag{5.23}$$

$$\rho_*(m) : \mathbb{Z}/8\mathbb{Z} \to \mathbb{Z},\, [0,1,2,3,4,5,6,7] \mapsto [-1,2,-1,0,1,-2,1,0] \tag{5.24}$$

$$\rho(m) : \mathbb{Z}/8\mathbb{Z} \to \mathbb{Z},\, [0,1,2,3,4,5,6,7] \mapsto [0,-1,2,-1,0,1,-2,1] \tag{5.25}$$

$$\kappa(1) := 1/2 \,,\, \kappa(2n) := \frac{1}{2^{2n}}\frac{(2n-2)!}{n!\,(n-1)!} \,,\, \kappa(2n+1) := 0 \;\forall n > 1. \tag{5.26}$$

Due to (5.26), $H^{\boldsymbol{r^*}}_{[p,q]}$ vanishes unless none of the indices r_i^* is odd ≥ 3. Moreover, when all r_i^* are either 1 or even, after division by elementary factors (- powers of 2 and Catalan numbers -) we get an expression h:

$$h(p,q,s_u,s_e) := H^{r_1^*,\dots,r_s^*}_{[p,q]} \;\; 2^{-int(s/2)} \prod_i (1/\kappa(r_i^*)) \tag{5.27}$$

$$= \rho(2\,s_u + p - q) \tag{5.28}$$

$$= \rho_*(s_u - s_e + 2\,p) = \rho_*(3\,s_u + s_e - 2\,q - 2) \tag{5.29}$$

which turns out, quite unexpectedly, to be 8-periodic in the order-compatibility coefficients p, q and the multiplicities s_u, s_e.

5.5 Affliates : from function to operator

We have the choice between relating an affiliate $F_\Diamond$ to F itself or to its infinitesimal generator F_*:

$$F_\Diamond := \gamma(F-1) = \alpha(F_*) \qquad \big(F_* := \log(F)\big). \tag{5.30}$$

This implies handling two distinct systems of coefficients:

$$\alpha(t)=t+\sum_{1\le r}\alpha_r t^{r+1},\ \gamma(t)=t+\sum_{1\le r}\gamma_r t^{r+1},\ \big(\gamma(t)=\alpha(\log(1+t))\big). \tag{5.31}$$

The choice impacts the analytic expression of the correspondence $f_\Diamond \mapsto F_\Diamond$:

$$F_\Diamond \mapsto f_\Diamond = F_\Diamond . z \tag{5.32}$$

$$f_\Diamond \mapsto F_\Diamond = \sum_{1\le r}\sum_{1\le n_i} \Diamond^{n_1,\dots,n_r}\left(f_\Diamond^{n_1}\frac{\partial_z^{n_1}}{n_1!}\right)\dots\left(f_\Diamond^{n_r}\frac{\partial_z^{n_r}}{n_r!}\right)\ (n_r>1\,if\,r>1). \tag{5.33}$$

Although $F_\Diamond$ is usually derived from F rather than F_*, the structure coefficients $\Diamond_{n_1,\dots,n_r}$ are simpler to express in terms of the coefficients α_n than in terms of γ_n: in the former case, the sums involve fewer terms $\prod \alpha_{m_j}$ due to the homogeneity constraints $\sum n_i = \sum m_j$. The simplest way to ensure (5.32) is to set $\Diamond_1 = 1$ and to impose that all other coefficients $\Diamond_{n_1,\dots,n_r}$ ending with $n_r = 1$ should vanish. This, however, is not enough to enforce the uniqueness of the expansion (5.33), due to the existence, for n large enough, of universal identities of the form

$$0 \equiv \sum_{n_1+\dots+n_r=n} c_{n_1,\dots,n_r}\left(f_\Diamond^{n_1}\frac{\partial_z^{n_1}}{n_1!}\right)\dots\left(f_\Diamond^{n_r}\frac{\partial_z^{n_r}}{n_r!}\right) \quad (c_{n_1,\dots,n_r}\in\mathbb{Z}). \tag{5.34}$$

The latitude in the choice of the structure coefficients being $2^{r-2}-par(r)$ for $r > 1$ (par = partition number), it is clear that even imposing a natural condition[60] like

$$\left\{\alpha_n = \frac{1}{(n+1)!}\ \ \forall n\right\} \Longrightarrow \left\{\Diamond^{n_1}=1\ \ \forall n_1\ ,\ \ \Diamond^{n_1,\dots,n_r}=0\ \ \forall r\ge 2\right\} \tag{5.35}$$

is not enough to restore uniqueness. In fact, we know of no *simple* condition that does. In any case, here is a natural choice for the first structure

[60] Natural indeed, since this choice of α leads to the fonction $f_\Diamond(z) = f(z) - z$ and to the operator $F_\Diamond = F - 1 = \sum_{1\le n} f_\Diamond^n \frac{\partial_z^n}{n!}$.

coefficients:

$$\begin{array}{lcl}
\Diamond^{1} &=& 1\\
\Diamond^{2} &=& 2\,\alpha_1\\[4pt]
\Diamond^{3} &=& -3\,\alpha_2 \quad +6\,\alpha_1^2\\
\Diamond^{1,2} &=& 3\,\alpha_2 \quad -2\,\alpha_1^2\\[4pt]
\Diamond^{4} &=& 4\,\alpha_3\ -20\,\alpha_1\,\alpha_2\ +20\,\alpha_1^3\\
\Diamond^{1,3} &=& -7\,\alpha_3\ +20\,\alpha_1\,\alpha_2\ -11\,\alpha_1^3\\
\Diamond^{2,2} &=& 2\,\alpha_3\ +2\,\alpha_1\,\alpha_2\ -2\,\alpha_1^3\\
\Diamond^{1,1,2} &=& 3\,\alpha_3\ -6\,\alpha_1\,\alpha_2\ +3\,\alpha_1^3
\end{array}$$

$$\begin{array}{lcl}
\Diamond^{5} &=& -5\,\alpha_4\ +30\,\alpha_1\,\alpha_3\ +15\,\alpha_2^2\ -105\,\alpha_1^2\,\alpha_2\ +70\,\alpha_1^4\\
\Diamond^{1,4} &=& 21\,\alpha_4\ -\frac{366}{5}\,\alpha_1\,\alpha_3\ -\frac{171}{5}\,\alpha_2^2\ +\frac{789}{5}\,\alpha_1^2\,\alpha_2\ -\frac{342}{5}\,\alpha_1^4\\
\Diamond^{2,3} &=& -28\,\alpha_4\ +\frac{348}{5}\,\alpha_1\,\alpha_3\ +\frac{168}{5}\,\alpha_2^2\ -\frac{552}{5}\,\alpha_1^2\,\alpha_2\ +\frac{196}{5}\,\alpha_1^4\\
\Diamond^{3,2} &=& 9\,\alpha_4\ -\frac{114}{5}\,\alpha_1\,\alpha_3\ -\frac{99}{5}\,\alpha_2^2\ +\frac{321}{5}\,\alpha_1^2\,\alpha_2\ -\frac{138}{5}\,\alpha_1^4\\
\Diamond^{1,1,3} &=& 0\\
\Diamond^{1,2,2} &=& -\alpha_4\ +\frac{86}{5}\,\alpha_1\,\alpha_3\ +\frac{51}{5}\,\alpha_2^2\ -\frac{229}{5}\,\alpha_1^2\,\alpha_2\ +\frac{102}{5}\,\alpha_1^4\\
\Diamond^{2,1,2} &=& +4\,\alpha_1\,\alpha_3\ -8\,\alpha_1^2\,\alpha_2\ +4\,\alpha_1^4\\
\Diamond^{1,1,1,2} &=& 4\,\alpha_4\ -\frac{64}{5}\,\alpha_1\,\alpha_3\ -\frac{24}{5}\,\alpha_2^2\ +\frac{116}{5}\,\alpha_1^2\,\alpha_2\ -\frac{48}{5}\,\alpha_1^4\,.
\end{array}$$

Remarkably enough, for index sums $|\boldsymbol{n}| \geq 5$, a fair number of structure coefficients $\Diamond^{\boldsymbol{n}}$ are always $= 0$, irrespective of α and despite having a last index $n_r \neq 1$. Here are the first unconditionally vanishing coefficients:

$$\begin{array}{rl}
|\boldsymbol{n}|=5\ : & \Diamond^{1,1,3}\\
|\boldsymbol{n}|=6\ : & \Diamond^{2,4},\Diamond^{3,1,2},\Diamond^{1,1,1,3},\Diamond^{1,1,1,1,2}\\
|\boldsymbol{n}|=7\ : & \Diamond^{2,5},\Diamond^{1,3,3},\Diamond^{1,1,1,4},\Diamond^{1,1,2,3},\Diamond^{1,2,1,3},\Diamond^{2,1,1,3},\Diamond^{2,1,2,2},\Diamond^{1,1,1,1,3},\\
& \Diamond^{1,1,1,2,2},\Diamond^{1,1,2,1,2},\Diamond^{1,2,1,1,2},\Diamond^{2,1,1,1,2},\Diamond^{1,1,1,1,1,2}.
\end{array}$$

Here again, the case

$$\alpha(t) = \frac{1}{c}\tanh(c\,t) \quad , \quad \gamma(t) = \frac{1}{c}\frac{(1+t)^{2c}-1}{(1+t)^{2c}+1} \tag{5.36}$$

stands out for simplicity. It makes it possible to choose a system of structure coefficients which are all $\equiv 0$ except those of the form:

$$\Diamond^{2m_1-1,2m_2,2m_3,\ldots,2m_r} = (-1)^{r-1}\,c^{-2+2\sum m_i} \quad (\forall r\ ,\ \forall m_i \geq 1). \tag{5.37}$$

When $c = \frac{1}{2}$ we recover the earlier formula (1.26) for the mediator.

5.6 Main and secondary symmetry types

Let us stand back and take stock. Alongside the four ubiquitous symmetry types:

$$
\begin{aligned}
\boldsymbol{alternal}^{\bullet} &= \boldsymbol{basic\ symmetry\ type}\\
\boldsymbol{symmetral}^{\bullet} &= (\exp Id^{\bullet}) \circ \boldsymbol{alternal}^{\bullet}\\
\boldsymbol{alternel}^{\bullet} &= \boldsymbol{alternal}^{\bullet} \circ \log(1^{\bullet} + Id^{\bullet})\\
\boldsymbol{symmetrel}^{\bullet} &= (\exp Id^{\bullet}) \circ \boldsymbol{alternal}^{\bullet} \circ \log(1^{\bullet} + Id^{\bullet})
\end{aligned}
$$

we have a number of special symmetry types, of secondary but non-negligible importance:

$$
\begin{aligned}
\boldsymbol{olternal}^{\bullet} &= \alpha(Id^{\bullet}) \circ \boldsymbol{alternal}^{\bullet}\\
&= \gamma(Id^{\bullet}) \circ (\boldsymbol{symmetral}^{\bullet} - \mathbf{1}^{\bullet})\\
\boldsymbol{alternol}^{\bullet} &= \boldsymbol{alternal}^{\bullet} \circ \beta(Id^{\bullet})\\
&= \boldsymbol{alternel}^{\bullet} \circ \delta(Id^{\bullet})\\
\boldsymbol{olternol}^{\bullet} &= \alpha(Id^{\bullet}) \circ \boldsymbol{alternal}^{\bullet} \circ \beta(Id^{\bullet})\\
&= \gamma(Id^{\bullet}) \circ (\boldsymbol{symmetrel}^{\bullet} - \mathbf{1}^{\bullet}) \circ \delta(Id^{\bullet})
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{symmetrol}^{\bullet} &= \boldsymbol{symmetral}^{\bullet} \circ \beta(Id^{\bullet})\\
&= \boldsymbol{symmetrel}^{\bullet} \circ \delta(Id^{\bullet}).
\end{aligned}
$$

Choice 1. The most common choice for the quartet $(\alpha, \beta, \gamma, \delta)$ is

$$\alpha(t) := 2\tanh\left(\frac{1}{2}\,t\right) \quad , \quad \beta(t) := 2\,\mathrm{arctanh}\left(\frac{1}{2}\,t\right) \tag{5.38}$$

$$\gamma(t) := \frac{t}{1+\frac{1}{2}\,t} \quad , \quad \delta(t) := \frac{t}{1-\frac{1}{2}\,t}\,. \tag{5.39}$$

The corresponding structure constants are:

$$
\begin{aligned}
&\lambda^{[p,q]} = 2^{-p-q}\\
&\gamma^{[p,q]} = \left(-\tfrac{1}{4}\right)^{\inf(p,q)}\ \textit{if}\ |p-q|=1 && (\textit{resp.}\ 0\ \textit{otherwise})\\
&\Diamond^{n_1,\ldots,n_r} = (-1)^{r-1}\,2^{1-\sum n_i}\ \textit{if}\ r, n_1\ \textit{odd}, n_2,\ldots,n_r\ \textit{even} && (\textit{resp.}\ 0\ \textit{otherwise})\\
&H^{r_1^*,\ldots,r_s^*}_{[p,q]} = (-1)^q \left(\tfrac{1}{4}\right)^{s-1}\ \textit{if}\ r_1^* = \ldots = r_s^* = 1 && (\textit{resp.}\ 0\ \textit{otherwise}).
\end{aligned}
$$

Choice 2. More rarely we take

$$\alpha(t) := \tanh(t) \quad , \quad \beta(t) := \operatorname{arctanh}(t) \tag{5.40}$$

$$\gamma(t) := \frac{t + \frac{1}{2}t^2}{1 + t + \frac{1}{2}t^2} \quad , \quad \delta(t) := \left(\frac{1+t}{1-t}\right)^{\frac{1}{2}} - 1. \tag{5.41}$$

This choice leads to marginally less simple structure coefficients:

$$\begin{aligned}
\lambda^{[p,q]} &= \varrho(p-q)\ 2^{-int(\frac{p+q+1}{2})} \\
\gamma^{[p,q]} &= (-1)^{\inf(p,q)}\ \textit{if}\ |p-q| = 1 \qquad (\textit{resp.}\ 0\ \textit{otherwise}) \\
\lozenge^{n_1,\dots,n_r} &= (-1)^{r-1}\ \textit{if}\ r, n_1\ \textit{odd}, n_2, \dots, n_r\ \textit{even} \quad (\textit{resp.}\ 0\ \textit{otherwise}) \\
H_{[p,q]}^{r_1^*,\dots,r_s^*} &= \rho(2\,s_u + p - q)\ \ 2^{\mathrm{int}(s/2)} \prod_{1 \le i \le s} \kappa(r_i^*) \quad (\textit{resp.}\ 0\ \textit{if}\ s_0 \neq 0)
\end{aligned}$$

with $s_u, s_o, s_e, \rho, \varrho, \kappa$ as in (5.24)-(5.26). In particular:

$$\begin{aligned}
\rho &: \mathbb{Z}/8\mathbb{Z} \to \mathbb{Z} \ , \ [0,1,2,3,4,5,6,7] \mapsto [0,-1,2,-1,0,1,-2,1] \\
\varrho &: \mathbb{Z}/8\mathbb{Z} \to \mathbb{Z} \ , \ [0,1,2,3,4,5,6,7] \mapsto [1,1,0,-1,-1,-1,0,1]
\end{aligned}$$

ρ is odd and ϱ even but both change signs under 4-shifts

$$\rho(k+4) \equiv -\rho(k) \quad , \quad \varrho(k+4) \equiv -\varrho(k).$$

Choice 3. If $c \notin \{\pm 1, \pm i, \pm\frac{1}{2}, \pm\frac{i}{2}\}$ the one-parameter family

$$\alpha_c(t) := \frac{1}{c}\tanh(c\,t) \quad , \quad \beta_c(t) := \frac{1}{c}\operatorname{arctanh}(c\,t) \tag{5.42}$$

$$\gamma_c(t) := \frac{1}{c}\frac{(1+t)^{2c} - 1}{(1+t)^{2c} + 1} \quad , \quad \delta_c(t) := \left(\frac{1+c\,t}{1-c\,t}\right)^{\frac{1}{2c}} - 1\,. \tag{5.43}$$

makes only $\gamma^{[p,q]}$ and $\lozenge^{\bullet}$ simple:

$$\begin{aligned}
\lambda^{[p,q]} &= \textit{no simple multiplicative structure} \\
\gamma^{[p,q]} &= (-c^2)^{\inf(p,q)} \quad \textit{if}\ |p-q| = 1\,, \qquad (\textit{else} = 0) \\
\lozenge^{n_1,\dots,n_r} &= (-1)^{r-1}\, c^{-1+\sum n_i}\ \textit{if}\ r, n_1\ \textit{odd}, n_2, \dots, n_r\ \textit{even} \quad (\textit{else} = 0) \\
H_{[p,q]}^{r_1^*,\dots,r_s^*} &= \textit{no simple multiplicative structure.}
\end{aligned}$$

Choice 4. The homographic quartet:

$$\underline{\alpha}_c(t) := \frac{(e^t - 1)}{1 + c\,(e^t - 1)} \quad , \quad \underline{\beta}_c(t) := \frac{t}{1 - c\,t} \tag{5.44}$$

$$\underline{\gamma}_c(t) := \frac{t}{1 + c\,t} \quad , \quad \underline{\delta}_c(t) := \frac{t}{1 - c\,t} \tag{5.45}$$

predictably leads to simpler structure coefficients:

$$\begin{aligned}
\lambda^{[p,q]} &= c^q\ (1-c)^p \\
\gamma^{[p,q]} &= 0 \quad \textit{if} \quad |p-q| \geq 2 \\
\gamma^{[p,p]} &= (1-2\,c)\,c^{\,p-1}\,(c-1)^{p-1} \\
\gamma^{[p,p+1]} &= \gamma_c^{[p+1,p]} \;=\; c^{\,p}\,(c-1)^p \\
\Diamond^{n_1,\ldots,n_r} &= \textit{ no simple multiplicative structure} \\
H_{[p,q]}^{r_1^*,\ldots,r_s^*} &= (-c)^q\,(1-c)^p \ \textit{if}\ r_1^* = \ldots = r_s^* = 1\ \ (\textit{resp.}\,0\ \textit{otherwise}).
\end{aligned}$$

General case. Lasty, for arbitrary but mutually reciprocal (γ,δ), the formulae read

$$\begin{aligned}
\lambda^{[p,q]} &= (-1)^q \sum_{0\leq k\leq p} \frac{p!}{(p-k)!\,k!}\,\gamma_{q+k} \\
\gamma^{[p,q]} &: \textit{ generated by } \gamma\big(\delta(t_1)+\delta(t_2)+\delta(t_1)\,\delta(t_2)\big) = \sum \gamma^{[p,q]}\,t_1^p\,t_2^q \\
\Diamond^{n_1,\ldots,n_r} &: \textit{multiple competing expressions.}
\end{aligned}$$

$$H_{[p,q]}^{r_1^*,\ldots,r_s^*} = \sum_{k=0}^{r-s^*}\left[\sum_{l=0}^{p}\gamma_{k+q+l}\frac{p!}{(p-l)!\,l!}\right]\left[\frac{\nabla^k}{k!}\big(\delta_{r_1^*-1}\ldots\delta_{r_{s^*}^*-1}\big)\right]$$

In conclusion, of all secondary symmetry types, the simplest (and most frequently occuring in practice) is the one at the intersection of the two one-parameter families: $\gamma = \gamma_{\frac{1}{2}} = \underline{\gamma}_{\frac{1}{2}} \quad , \quad \delta = \delta_{\frac{1}{2}} = \underline{\delta}_{\frac{1}{2}}$.

Remark 1. Consider the $\mathbb{N}$-indexed mould *har*$^\bullet$ defined by the induction

$$|\bullet|\ \mathrm{har}^\bullet = \mathrm{har}^\bullet \times Id^\bullet \times \mathrm{har}^\bullet \quad (\textit{resp.} = 0) \quad \textit{if} \quad r(\bullet) \quad \textit{odd} \quad (\textit{resp.} \quad \textit{even})$$

or more explicitely

$$\mathrm{har}^{n_1} = \frac{1}{n_1} \tag{5.46}$$

$$\mathrm{har}^{n_1,\ldots,n_r} := 0 \qquad \forall r \textit{ even} \qquad (\textit{in particular}\,\mathrm{har}^{\emptyset} := 0\,) \tag{5.47}$$

$$\mathrm{har}^{n_1,\ldots,n_r} := \frac{1}{n_1+\cdots+n_r}\sum_{1<i<r}\mathrm{har}^{n_1,\ldots,n_{i-1}}\mathrm{har}^{n_{i+1},\ldots,n_r}\ (\forall r\ \textit{odd} \geq 3) \tag{5.48}$$

har$^\bullet$ is the simplest example of a i-olternal mould. It occurs naturally in the study of some special trigonometric flexion algebras.[61] Its inverse *kohar*$^\bullet$ under mould composition is even more elementary:

$$\mathrm{kohar}^{n1,\ldots,n_{2r}} \equiv 0 \quad , \quad \mathrm{kohar}^{n1,\ldots,n_{2r+1}} \equiv (-1)^r\,n_r \tag{5.49}$$

kohar$^\bullet$ is the simplest instance of a i-alternol mould.

[61] *Cf.* [10, page 177].

Remark 2. There is an important operator $\mathfrak{H}$, also acting on a trigonometric flexion algebra[62], that happens to verify a co-symmetrol co-product.[63]

Remark 3. There seems to exist no simple notion of *bracket* (anti-commutative and rational in its two arguments) for mediators and consequently no proper equivalent of the Campbell-Hausdorff formula for expressing $(F.G)_\sharp$ in terms of $F_\sharp$ and $G_\sharp$, other than the obvious expansion that relies on the coefficients $\gamma^{[\bullet]}$ defined by the series in the non-commutative variables t_1, t_2:

$$\sum \gamma^{[[p_1,q_1,\ldots,p_r,q_r]]} t_1^{p_1} t_2^{q_1} \ldots t_1^{p_r} t_2^{q_r} := \gamma\left(\gamma^{-1}(t_1)+\gamma^{-1}(t_2)+\gamma^{-1}(t_1)\gamma^{-1}(t_2)\right)$$

with $p_1, q_r \geq 0$ and all other $p_i, q_i \geq 1$.

6 Complement: arithmetical vs dynamical monics

6.1 Distinguishing Stokes constants from holomorphic invariants

The scalars $A_\omega(f)$ may be viewed

(i) as Stokes constants;
(ii) as holomorphic invariants.

In their first capacity, they govern the Stokes transitions and are rigidly determined. So too are the (presumably transcendental) monics — the multizetas — which enter their expansions. We speak accordingly of *rigid* or *arithmetical monics*.

There is more latitude, however, when we look upon the saclars $A_\omega(f)$ as holomorphic invariants and retain only those multizeta properties which are directly responsible for their invariance. We speak in that case of *dynamical monics*.

Both types of monics verify various types of relations, some infinite, some finite-algebraic. When viewed as subject only to their various systems of algebraic relations over $\mathbb{Q}$, our monics (whether rigid-arithmetical or dynamical) become *formal monics*. As such, they possess their own system of independent generators, the so-called *irreducibles*. Being subject to laxer constraints, the *dynamical irreducibles* should be expected to be, and in fact are, more 'numerous' than the *rigid-arithmetical irreducibles*.[64]

[62] *Cf.* [10, (11.42)-(11-43)].

[63] *Cf.* [10, (11.47)].

[64] Though of course any complete system of irreducibles, of either sort, has to be countably infinite.

6.2 Arithmetical multizetas

The two classical systems of algebraic (quadratic) constraints. Either system of constraints is best expressed as a specific multiplication rule relative to a specific encoding.

In the *first* or $\boldsymbol{\alpha}$*-encoding*, the multizetas are given by polylogarithmic integrals:

$$\mathrm{wa}_*^{\alpha_1,\dots,\alpha_l} := (-1)^{l_0} \int_0^1 \frac{dt_l}{(\alpha_l - t_l)} \dots \int_0^{t_3} \frac{dt_2}{(\alpha_2 - t_2)} \int_0^{t_2} \frac{dt_1}{(\alpha_1 - t_1)} \tag{6.1}$$

with indices α_j that are either 0 or unit roots, and $l_0 := \sum_{\alpha_i=0} 1$.

In the *second* or $\binom{\epsilon}{s}$*-encoding*, the multizetas are expressed as "harmonic sums":

$$\mathrm{ze}_*^{\binom{\epsilon_1\ ,\dots,\ \epsilon_r}{s_1\ ,\dots,\ s_r}} := \sum_{n_1>\dots>n_r>0} n_1^{-s_1}\dots n_r^{-s_r}\ e_1^{-n_1}\dots e_r^{-n_r} \tag{6.2}$$

with $s_j \in \mathbb{N}^*$ and unit roots $e_j := \exp(2\pi i \epsilon_j)$ of 'logarithms' $\epsilon_j \in \mathbb{Q}/\mathbb{Z}$.

The stars $*$ means that the integrals or sums are provisionally assumed to be convergent or semi-convergent: for $wa_*^{\boldsymbol{\alpha}}$ this means that $\alpha_1 \neq 0$ and $\alpha_l \neq 1$, and for $ze_*^{\binom{\boldsymbol{\epsilon}}{\mathbf{s}}}$ this means that $\binom{\epsilon_1}{s_1} \neq \binom{0}{1}$ *i.e.* $\binom{e_1}{s_1} \neq \binom{1}{1}$.

The corresponding moulds $wa_*^{\bullet}$ and $ze_*^{\bullet}$ turn out to be respectively *symmetral* and *symmetrel*:[65]

$$\mathrm{wa}_*^{\boldsymbol{\alpha}^1}\ \mathrm{wa}_*^{\boldsymbol{\alpha}^2} = \sum_{\boldsymbol{\alpha}\,\in\,\mathrm{sha}(\boldsymbol{\alpha}^1,\boldsymbol{\alpha}^2)} \mathrm{wa}_*^{\boldsymbol{\alpha}} \qquad \forall \boldsymbol{\alpha}^1, \forall \boldsymbol{\alpha}^2 \tag{6.3}$$

$$\mathrm{ze}_*^{\binom{\boldsymbol{\epsilon}^1}{\mathbf{s}^1}}\ \mathrm{ze}_*^{\binom{\boldsymbol{\epsilon}^2}{\mathbf{s}^2}} = \sum_{\binom{\boldsymbol{\epsilon}}{\mathbf{s}}\,\in\,\mathrm{she}\left(\binom{\boldsymbol{\epsilon}^1}{\mathbf{s}^1},\binom{\boldsymbol{\epsilon}^2}{\mathbf{s}^2}\right)} \mathrm{ze}_*^{\binom{\boldsymbol{\epsilon}}{\mathbf{s}}} \qquad \forall \binom{\boldsymbol{\epsilon}^1}{\mathbf{s}^1}, \forall \binom{\boldsymbol{\epsilon}^2}{\mathbf{s}^2}. \tag{6.4}$$

These are the so-called *quadratic relations*, which express *multizeta dimorphy*. As for the conversion rule, it reads:[66]

$$\mathrm{wa}_*{}^{e_1,0^{[s_1-1]},\dots,e_r,0^{[s_r-1]}} := \mathrm{ze}_*^{\binom{\epsilon_r\ ,\ \epsilon_{r-1:r}\ ,\dots,\ \epsilon_{1:2}}{s_r\ ,\ s_{r-1}\ ,\dots,\ s_1}} \tag{6.5}$$

$$\mathrm{ze}_*^{\binom{\epsilon_1\ ,\ \epsilon_2\ ,\dots,\ \epsilon_r}{s_1\ ,\ s_2\ ,\dots,\ s_r}} =: \mathrm{wa}_*{}^{e_1\dots e_r,0^{[s_r-1]},\dots,e_1e_2,0^{[s_2-1]},e_1,0^{[s_1-1]}} \tag{6.6}$$

with $0^{[k]}$ denoting a subsequence of k zeros.

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[65] As usual, $sha(\boldsymbol{\omega}', \boldsymbol{\omega}'')$ denotes the set of all simple shufflings of the sequences $\boldsymbol{\omega}', \boldsymbol{\omega}''$, whereas in $she(\boldsymbol{\omega}', \boldsymbol{\omega}'')$ we allow (any number of) order-compatible contractions $\omega_i' + \omega_j''$.

[66] With the usual shorthand for differences: $\epsilon_{i:j} := \epsilon_i - \epsilon_j$.

There happen to be unique extensions $wa^\bullet_* \to wa^\bullet$ and $ze^\bullet_* \to ze^\bullet$ that cover the divergent cases and keep our moulds symmetral or symmetrel while conforming to the 'initial conditions' $wa^0 = wa^1 = 0$ and $ze^{\binom{0}{1}} = 0$. As we shall see in a moment, however, the divergent case calls for a slight modification of the conversion rules (6.5)-(6.6).

Arithmetical multizeta irreducibles. The $\mathbb{Q}$-ring $\mathbb{ZE}$ of *formal multizetas*, *i.e.* the $\mathbb{Q}$-ring generated by the symbols $wa^{\boldsymbol{\alpha}}$ and $ze^{\binom{\epsilon}{s}}$ subject only to the conversion rule (6.5)-(6.6) and the quadratic relations[67] (6.3)-(6.4), is known to be a polynomial ring, freely generated by a countable number of so-called *irreducibles*.

Generating series. As borne out by past experience, it is advisable, for most intents and purposes, to switch from the scalar multizetas $wa^\bullet$ and $ze^\bullet$ to the generating series $Zag^\bullet$ and $Zig^\bullet$:

$$\mathrm{Zag}^{\binom{u_1\ ,\dots,\ u_r}{\epsilon_1\ ,\dots,\ \epsilon_r}} := \sum_{1\leq s_j} \mathrm{wa}^{e_1,0^{[s_1-1]},\dots,e_r,0^{[s_r-1]}}\ u_1^{s_1-1}u_{1,2}^{s_2-1}\dots u_{1\dots r}^{s_r-1} \quad (6.7)$$

$$\mathrm{Zig}^{\binom{\epsilon_1\ ,\dots,\ \epsilon_r}{v_1\ ,\dots,\ v_r}} := \sum_{1\leq s_j} \mathrm{ze}^{\binom{\epsilon_1\ ,\dots,\ \epsilon_r}{s_1\ ,\dots,\ s_r}}\ v_1^{s_1-1}\dots v_r^{s_r-1} \quad (6.8)$$

The bimould[68] $Zag^\bullet$ is *symmetral*, just as $wa^\bullet$ was, while the bimould $Zig^\bullet$ has its own symmetry type: *symmetril*. The *symmetrility* relations are patterned on the *symmetrelity* relations, but with the additive contractions $w_i + w_j$ replaced by 'polar' contractions $\widehat{w_i, w_j}$, according to the rules:

$$\begin{aligned} S^{\binom{\dots,\ \widehat{u_i\ ,\ u_j}\ ,\dots}{\dots,\ v_i\ ,\ v_j\ ,\dots}} &= S^{\binom{\dots,\ u_i+u_j\ ,\dots}{\dots,\ v_i\ ,\dots}}\ P(v_i-v_j) \\ &\quad + S^{\binom{\dots,\ u_i+u_j\ ,\dots}{\dots,\ v_j\ ,\dots}}\ P(v_j-v_i). \end{aligned} \quad (6.9)$$

67 Though yet unproven, it is generally assumed (and backed by massive numerical evidence) that the two systems of quadratic relations imply all other (known or yet to be discovered) algebraic relations between multizetas.

68 What turns $Zag^\bullet$, $Zig^\bullet$ into *bimoulds* is not so much their two-tier indexation $w_i = \binom{u_i}{v_i}$ but rather the fact that the u_i's and v_i's interact in a very special way, through so-called *flexions*, which allow only the addition of (several consecutive) u_i's and the subtraction of (two not necessarily consecutive) v_i's with conservation of $\sum u_i v_i$.

Here $P(t) := 1/t$. In (6.9) the dots may themselves contain any number of additional contractions $\widehat{w_k, w_l}$. Thus:

$$S^{\left(\begin{smallmatrix} \dots, & \widehat{u_i\ ,\ u_j} & ,\dots\ \dots, & \widehat{u_k\ ,\ u_l} & ,\dots \\ \dots, & v_i\ ,\ v_j & ,\dots\ \dots, & v_k\ ,\ v_l & ,\dots \end{smallmatrix}\right)} = +S^{\left(\begin{smallmatrix} \dots, & u_i+u_j & ,\dots\ \dots, & u_k+u_l & ,\dots \\ \dots, & v_i & ,\dots\ \dots, & v_k & ,\dots \end{smallmatrix}\right)} P(v_i-v_j)\,P(v_k-v_l)$$
$$+S^{\left(\begin{smallmatrix} \dots, & u_i+u_j & ,\dots\ \dots, & u_k+u_l & ,\dots \\ \dots, & v_j & ,\dots\ \dots, & v_k & ,\dots \end{smallmatrix}\right)} P(v_j-v_i)\,P(v_k-v_l)$$
$$+S^{\left(\begin{smallmatrix} \dots, & u_i+u_j & ,\dots\ \dots, & u_k+u_l & ,\dots \\ \dots, & v_i & ,\dots\ \dots, & v_l & ,\dots \end{smallmatrix}\right)} P(v_i-v_j)\,P(v_l-v_k)$$
$$+S^{\left(\begin{smallmatrix} \dots, & u_i+u_j & ,\dots\ \dots, & u_k+u_l & ,\dots \\ \dots, & v_j & ,\dots\ \dots, & v_l & ,\dots \end{smallmatrix}\right)} P(v_j-v_i)\,P(v_l-v_k)$$

A typical symmetrility relation reads:

$$\begin{aligned} S^{w_1,w_2}\,S^{w_3,w_4} = {} & +S^{w_1,w_2,w_3,w_4}+S^{w_1,w_3,w_2,w_4}+S^{w_3,w_1,w_2,w_4}+S^{w_1,w_3,w_4,w_2} \\ & +S^{w_3,w_1,w_4,w_2}+S^{w_3,w_4,w_1,w_2}+S^{\widehat{w_1,w_3},w_2,w_4}+S^{\widehat{w_1,w_3},w_4,w_2} \\ & +S^{w_1,\widehat{w_2,w_3},w_4}+S^{w_3,\widehat{w_1,w_4},w_2}+S^{w_1,w_3,\widehat{w_2,w_4}}+S^{w_1,w_3,\widehat{w_2,w_4}} \\ & +S^{\widehat{w_1,w_3},\widehat{w_2,w_4}}. \end{aligned}$$

Summing up, not only do we have an exact equivalence between the old and new symmetries:

$$\{\mathrm{wa}^\bullet\ \ symmetral\} \Longleftrightarrow \{\mathrm{Zag}^\bullet\ \ symmetral\} \tag{6.10}$$
$$\{\mathrm{ze}^\bullet\ \ symmetrel\} \Longleftrightarrow \{\mathrm{Zig}^\bullet\ \ symmetril\} \tag{6.11}$$

but the old conversion rule for scalar multizetas [69] becomes:

$$\mathrm{Zig}^\bullet = \mathrm{Mini}^\bullet \times \mathrm{swap(Zag)}^\bullet \tag{6.12}$$
$$\big(\Longleftrightarrow \mathrm{swap(Zig^\bullet)} = \mathrm{Zag}^\bullet \times \mathrm{Mana}^\bullet\big). \tag{6.13}$$

Here, *swap* is the basic involution of the flexion structure:

$$(\mathrm{swap}.S)^{\left(\begin{smallmatrix} u_1 & ,\dots, & u_r \\ v_1 & ,\dots, & v_r \end{smallmatrix}\right)} := S^{\left(\begin{smallmatrix} v'_r & ,\dots, & v'_1 \\ u'_r & ,\dots, & u'_1 \end{smallmatrix}\right)} \tag{6.14}$$

with $u'_i := u_1+\dots+u_i$ and $v'_i := v_i - v_{i+1}$ if $i < r$ respectively $v'_r := v_r$.

As for *Mana*$^\bullet$ and *Mini*$^\bullet$:= *swap.Mana*$^\bullet$, they are elementary bimoulds whose only non-vanishing components are those carrying only zeros in the lower (respectively upper) index row:

$$\mathrm{Mana}^{\left(\begin{smallmatrix} u_1 & ,\dots, & u_r \\ 0 & ,\dots, & 0 \end{smallmatrix}\right)} \equiv \mathrm{Mini}^{\left(\begin{smallmatrix} 0 & ,\dots, & 0 \\ v_1 & ,\dots, & v_r \end{smallmatrix}\right)} \equiv \mathrm{mono}_r\,. \tag{6.15}$$

[69] Namely the rules (6.5)-(6.6) suitably modified to cover the *divergent* case.

They can be expressed in terms of monozetas:

$$1+\sum_{r\geq 2} \text{mono}_r \; t^r := \exp\left(\sum_{s\geq 2}(-1)^{s-1}\zeta(s)\,\frac{t^s}{s}\right) \tag{6.16}$$

Even-odd separation. The natural environment of $Zag^\bullet$ is the group $GARI$, central to flexion theory. Its complicated product *gari* is highly non-linear in its second factor. Nonetheless $Zag^\bullet$ admits remarkable factorisations in $GARI$:

$$\text{Zag}^\bullet := \text{gari}\big(\text{Zag}^\bullet_{\text{I}}, \text{Zag}^\bullet_{\text{II}}, \text{Zag}^\bullet_{\text{III}}\big) = \text{gari}\big(\text{Zag}_{\text{ev}}, \text{Zag}_{\text{odd}}\big) \tag{6.17}$$

$$\text{Zag}^\bullet_{\text{ev}} := \text{gari}\big(\text{Zag}_{\text{I}}, \text{Zag}^\bullet_{\text{II}}\big) \tag{6.18}$$

$$\text{Zag}^\bullet_{\text{odd}} := \text{Zag}^\bullet_{\text{III}} \tag{6.19}$$

where the various factors, like $Zag^\bullet$ itself, possess a double symmetry: $Zag^\bullet_{ev}, Zag^\bullet_{odd}$ etc are *symmetral*, while the swappees $Zig^\bullet_{ev}, Zig^\bullet_{odd}$ etc are *symmetril*. The 'even' and 'odd' factors $Zag^\bullet_{ev}$ and $Zag^\bullet_{odd}$ are characterized by their behaviour under the involutions $neg, pari$:

$$(\text{neg}S)^{\binom{u_1\ ,\dots,\ u_r}{v_1\ ,\dots,\ v_r}} := S^{\binom{-u_1\ ,\dots,\ -u_r}{-v_1\ ,\dots,\ -v_r}}\,;\,(\text{pari}S)^{\binom{u_1\ ,\dots,\ u_r}{v_1\ ,\dots,\ v_r}} := (-1)^r\, S^{\binom{u_1\ ,\dots,\ u_r}{v_1\ ,\dots,\ v_r}} \tag{6.20}$$

and under *invgari*, *i.e.* the taking of the *gari*-inverse:

$$\text{neg.pari.Zag}^\bullet_{\text{ev}} = \text{Zag}^\bullet_{\text{ev}} \tag{6.21}$$

$$\text{neg.pari.Zag}^\bullet_{\text{odd}} = \text{invgari.Zag}^\bullet_{\text{odd}} \tag{6.22}$$

$$\text{gari}(\text{Zag}^\bullet_{\text{odd}}, \text{Zag}^\bullet_{\text{odd}}) = \text{gari}(\text{neg.pari.invgari.Zag}^\bullet, \text{Zag}^\bullet). \tag{6.23}$$

Since all elements of $GARI$ have one well-defined square-root,[70] the last identity (6.23) readily yields $Zag^\bullet_{odd}$. Separating the last factor from the first two is thus an easy matter (assuming the flexion machinery). Separating $Zag^\bullet_{\text{I}}$ from $Zag^\bullet_{\text{II}}$ is easy too, unless we insist on doing this in a 'canonical' way.

Here is the significance of these $Zag^\bullet$-factors in terms of multizeta irreducibles.[71] For simplicity, we consider only the case of ordinary or 'colourless' multizetas:

(i) The factor $Zag^\bullet_{\text{I}}$ carries only powers of the special irreducibe $\zeta(2) = \pi^2/6$, of weight 2.

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[70] Apply $expari.\frac{1}{2}.logari$.

[71] Recall that the weight s, length (or depth) r , and degree d are related by $s = r+d$.

(ii) The factor $Zag^{\bullet}_{\mathrm{II}}$ carries only irreducibles of even weight $s \geq 4$ and even *depth*, along with their products.
(iii) The factor $Zag^{\bullet}_{\mathrm{III}}$ carries only irreducibles of odd weight $s \geq 3$ and odd *depth*, along with their products.

The even-multizeta / odd-multizeta irreducibles. The even/odd factorisation (6.17) of $Zag^{\bullet}$ leads to a canonical decomposition $\mathbb{ZE} = \mathbb{ZE}_{\mathrm{ev}} \oplus \mathbb{ZE}_{\mathrm{odd}}$ of the $\mathbb{Q}$-ring of multizetas into a direct sum of two sub-rings, each with its own irreducibles. These *even-irreducibles* and *odd-irreducibles* will lead in Section 9 to simpler expansions for the holomorphic invariants $A_{\omega}(f)$. Mark in passing the importance of the hyphenation: a system of, say, odd-irreducibles is not simply a system of irreducibles with odd weight and odd depth; it must also consist of elements in $\mathbb{ZE}_{\mathrm{odd}}$, *i.e.* of elements generated by the scalar coefficients of $Zag^{\bullet}_{odd}$.

The even-multitangents $Te^{\bullet}_{ev}(z)$. For any multitangent $Te^{s}(z)$ of monotangential expansion $Te^{s}(z)=\sum ze^{s}_{\sigma}\, Te^{\sigma}(z)$ we set $Te^{s}_{ev}(z)=\sum ev(ze^{s}_{\sigma})\, Te^{\sigma}(z)$, with ev the natural projection of $\mathbb{ZE}$ onto $\mathbb{ZE}_{\mathrm{ev}}$. Since the multiplication of monotangents involves only rational powers of π^2, *i.e.* elements of $\mathbb{ZE}_{\mathrm{ev}}$, the *even-multitangents* $Te^{s}_{ev}(z)$ are stable under multiplication, and their multiplication stays commutative.

6.3 Dynamical multizetas

If we review those multizeta properties on which our expansions of the invariants $A_{\omega}(f)$ effectively relied, we find three systems of 'dynamical constraints':

(i) the *symmetrelness constraints:* $ze^{s'}\, ze^{s''}=\sum_{s\in \mathrm{she}(s',s'')} ze^{s}$, which are none other than the *second quadratic relations* (6.4).
(ii) the *localisation constraints* (see Section 2.3) which take into account the commutation of two operations on multitangents – multiplication and localisation[72] – and derive from this fact finite multizetas relations much weaker than the *first quadratic relations*.
(iii) the *shift constraints* (non-algebraic, see Section 2.7) which, for any $i \leq r$, expand $ze^{s_1,\ldots,s_i,\ldots,s_r}$ as a convergent series of:

(*) all s_i-translates $ze^{s_1,\ldots,s_i+k_i,\ldots,s_r}$ of depth r and shift $k_i \geq 1$;
(**) some multizetas of depth $< r$.

[72] *I.e.* taking the Laurent expansion of a multitangent at $z = 0$.

Although the shift constraints (iii) are the ones most directly responsible for the invariance of the $A_\omega(f)$, they are not finite. So we shall concentrate on the algebraic constraints (i)-(ii).

Algebraic dynamical constraints. We begin by introducing the *coloured* symmetrel multitangent mould $Te^\bullet(z)$ and the bimould $Tig^\bullet(z)$ formed from the generating series of multitangents. The definitions are transparently patterned on those of $ze^\bullet$ and $Zig^\bullet$:

$$\mathrm{Te}^{\binom{\epsilon_1\ ,\dots,\ \epsilon_r}{s_1\ ,\dots,\ s_r}}(z) := \sum_{+\infty>n_1>\dots>n_r>-\infty} \prod_{i=1}^{i=r} \left(e_i^{-n_i}\,(n_i+z)^{-s_1} \right) \tag{6.24}$$

$$\mathrm{Tig}^{\binom{\epsilon_1\ ,\dots,\ \epsilon_r}{v_1\ ,\dots,\ v_r}}(z) := \sum_{s_i\geq 1} Te^{\binom{\epsilon_1\ ,\dots,\ \epsilon_r}{s_1\ ,\dots,\ s_r}}(z)\, v_1^{s_1-1}\dots v_r^{s_r-1}. \tag{6.25}$$

Clearly $\{Tig^\bullet$ *symmetril*$\} \Leftrightarrow \{Te^\bullet$ *symmetrel*$\} \Rightarrow \{ze^\bullet$ *symmetrel*$\}$.

To see now how the localisation constraints compare with the *first quadratic relations* (6.3), we must express the multitangents in terms of multizetas, in two distinct ways that reflect (at the level of the generating series $Tig^\bullet(z)$ and $Zig^\bullet$) the two paths in the corresponding commutative diagram of Section 2.3. We find:

$$\mathrm{Tig}^{\boldsymbol{w}}(z) = \sum_{\boldsymbol{w}=\boldsymbol{w}^+\boldsymbol{w}^-} \mathrm{Zig}^{\boldsymbol{w}^+}(z)\ \mathrm{viZig}^{\boldsymbol{w}^-}(z) - \sum_{\boldsymbol{w}=\boldsymbol{w}^+w_0\boldsymbol{w}^-} \mathrm{Zig}^{\boldsymbol{w}^+}(z)\ \mathrm{Pi}^{w_0}(z)\ \mathrm{viZig}^{\boldsymbol{w}^-}(z)$$

$$\mathrm{Tig}^{\boldsymbol{w}}(z) = \mathrm{Rig}^{\boldsymbol{w}} - \sum_{\boldsymbol{w}=\boldsymbol{w}^+w_0\boldsymbol{w}^-} \mathrm{Zig}^{\boldsymbol{w}^+\rfloor}\ \mathrm{Qii}^{\lceil w_0\rceil}(z)\ \mathrm{viZig}^{\lfloor\boldsymbol{w}^-}. \tag{6.26}$$

The ingredient *Pi*, *Qii*, $Rig^\bullet$ in the above formulae are defined as follows:

$$\mathrm{Pi}^{\binom{\epsilon_1}{v_1}} := \frac{1}{v_1},\quad \mathrm{Qii}^{\binom{\epsilon_1}{v_1}} := \sum_{n_1\in\mathbb{Z}} \frac{e^{-2\pi i n_1\epsilon_1}}{n_1+v_1} \quad \forall\epsilon_1 \tag{6.27}$$

$$\mathrm{Pi}^{\binom{\epsilon_1\ \dots\ \epsilon_r}{v_1\ \dots\ v_r}} := 0 \quad , \quad \mathrm{Qii}^{\binom{\epsilon_1\ \dots\ \epsilon_r}{v_1\ \dots\ v_r}} := 0 \quad \forall r\neq 1 \tag{6.28}$$

$$\mathrm{Rig}^{w_1,\dots,w_r} := 0 \quad \textit{for}\ \ r=0\ \ \textit{or}\ \ r\ \ \textit{odd} \tag{6.29}$$

$$\mathrm{Rig}^{w_1,\dots,w_r} := \frac{(\pi i)^r}{r!}\ \delta(\epsilon_1)\dots\delta(\epsilon_r) \quad \textit{for}\ \ r\ \ \textit{even} > 0 \tag{6.30}$$

with δ denoting as usual the discrete dirac[73] and $viZig^\bullet := neg.pari.anti.Zig^\bullet$. Lastly, the bimoulds $Pi^\bullet(z)$, $Qii^\bullet(z)$, $Zig^\bullet(z)$, $viZig^\bullet(z)$ are derived from $Pi^\bullet$, $Qii^\bullet$, $Zig^\bullet$, $viZig^\bullet$ by changing v_i into $v_i - z$ $(\forall i)$.

73 $\delta(0) := 1$ and $\delta(t) := 0$ *for* $t \neq 0$.

Dynamical multizeta irreducibles. Finding a system of irreducibles relative to the sole symmetrelness contraints on multizetas (*'second quadratic relations'*) is very easy.[74] So let us examine instead the full (algebraic) dynamical constraints (i.e *symmetrelness* plus *'localisation'*) and show that we can derive from them a simple algorithm for *expressing every (colourless) multizeta of odd degree and depth* ≥ 2 *as a finite sum, with rational coefficients, of multizetas of even degree*.[75] By equating our *uninflected* and *inflected* expressions of $Tig^{\boldsymbol{w}}(z)$ and then setting $z = 0$, we get the remarkable identity:

$$\sum_{\boldsymbol{w}=\boldsymbol{w}^{+}\boldsymbol{w}^{-}} \mathrm{Zig}^{\boldsymbol{w}^{+}}\,\mathrm{viZig}^{\boldsymbol{w}^{-}} - \sum_{\boldsymbol{w}=\boldsymbol{w}^{+}w_0\boldsymbol{w}^{-}} \mathrm{Zig}^{\boldsymbol{w}^{+}}\,\mathrm{Pi}^{w_0}\,\mathrm{viZig}^{\boldsymbol{w}^{-}}$$
$$= \mathrm{Rig}^{\boldsymbol{w}} - \sum_{\boldsymbol{w}=\boldsymbol{w}^{+}w_0\boldsymbol{w}^{-}} \mathrm{Zig}^{\boldsymbol{w}^{+}\rfloor}\,\mathrm{Qii}^{\lceil w_0\rceil}\,\mathrm{viZig}^{\lfloor\boldsymbol{w}^{-}} \qquad (\forall \boldsymbol{w}) \qquad (6.31)$$

with factor sequences $\boldsymbol{w}^{\pm}$ that can be $\boldsymbol{\emptyset}$, and with the usual flexion conventions.[76] As a consequence, (6.31) is of the form:

$$\mathrm{Zig}^{w_1,\ldots,w_r} + (-1)^r\,\mathrm{Zig}^{-w_r,\ldots,-w_1} = \text{“\textit{shorter terms}”}. \qquad (6.32)$$

But *Zig*$^{\bullet}$ is symmetril and therefore *mantir*-invariant[77], which again yields an identity of the form:

$$\mathrm{Zig}^{-w_1,\ldots,-w_r} + (-1)^r\,\mathrm{Zig}^{-w_r,\ldots,-w_1} = \text{“\textit{shorter terms}”}. \qquad (6.33)$$

If we now take 'colourless' indices w_i, *i.e.* indices $w_i := (\begin{smallmatrix}0\\ v_i\end{smallmatrix})$, then subtract (6.32) from (6.33), and calculate therein the coefficient of $\prod v_i^{s_i-1}$, we find:

$$(1-(-1)^d)\,\mathrm{ze}^{\left(\begin{smallmatrix}0 & ,\ldots, & 0\\ s_1 & ,\ldots, & s_r\end{smallmatrix}\right)} = \text{“\textit{shorter terms}”} \quad \left(d := -r + \sum s_i\right) \quad (6.34)$$

with quite explicit 'shorter terms'.

---

[74] For the uncoloured multizetas, it amounts to constructing a basis (the Lyndon basis will do, or any other) on the Lie algebra freely generated by the symbols $\mathfrak{e}_s$ with $s \in \mathbb{N}^*$.

[75] Recall that the degree $d := s - r$ of a multizeta is defined as its total weight s minus its length (or depth) r.

[76] One goes from w_0 to $\lceil w_0 \rceil$ by changing the upper index ϵ_0 to $|\boldsymbol{\epsilon}^{+}| + \epsilon_0 + |\boldsymbol{\epsilon}^{-}|$, and from $\boldsymbol{w}^{+}$ (respectively $\boldsymbol{w}^{-}$) to $\boldsymbol{w}^{+}\rfloor$ (respectively $\lfloor\boldsymbol{w}^{-}$) by changing the lower indices v_i to $v_i - v_0$.

[77] *Mantir* is a non-linear involution on bimoulds, whose definition is given in [10, pages 67-69]. But all we need to know here is that *mantir*.$S^{\bullet}$ = $-$*pari.anti*.$S^{\bullet}$ + *shorter terms*.

The dynamical constraints on multizetas thus provide us with a very effective algorithm for the *reduction* (to simpler multizetas) of all *uncoloured* multizetas $\zeta(s_1, \ldots, s_r)$ of depth $r \geq 2$ and *odd* degree $d := \sum_i (s_i - 1)$. We may note that, at depth $r = 1$, the monozetas of odd degree are precisely the $\zeta(s)$ of even weight s. These are of course commensurate with $\pi^s = (6\zeta(2))^{s/2}$, but this is a consequence of the *rigid-arithmetical* constraints, *not* of the *dynamical* ones!

6.4 The ramified case (tangency order $p > 1$)

Another striking difference between the (algebraic) dynamical constraints and the (algebraic) arithmetical ones makes itself felt when we go over to the ramified situation, for diffeos f of tangency order $p \geq 2$ and multizetas with indices $s_i \in p^{-1}\mathbb{N}^*$.

The *dynamical constraints* on the multizetas[78] carry over almost unchanged: the symmetrelness of $ze^\bullet$ survives, of course, and so do the finite localisation constraints (although the finite reduction of multitangents into monotangents breaks down), as shown in Section 2.3.

On the other hand, it is not only the symmetralness of $wa^\bullet$ — the first leg of the *arithmetical constraints* — that cannot survive ramification: the very definition of the mould $wa^\bullet$ and the conversion rules (6.5)-(6.6) cease to make sense, since these rules would equate the *entire* lengths of 0-sequences in $\boldsymbol{\alpha}$ with the *fractional* weights s_i in $\boldsymbol{s}$.

7 Complement: convergence issues and phantom dynamics

7.1 The scalar invariants

Although convergence issues are by no means central to this investigation — the analytical expressions of the invariants $A_\omega(f)$ in terms of f is — there seems to be a lot of muddled thinking about these questions, with some authors insisting on seeing difficulties where there are none. So a short section entirely devoted to the subject may not be superfluous, even if it entails some repetitions and leads us, now and then, to state the obvious.

Scalar invariant attached to convergent diffeos f. There are two ways of establishing the existence of the scalar invariants as entire functions of

[78] Recall, though, that in the ramified case the monics $Te^{\boldsymbol{s}}_\omega$ take the place of the multizetas as direct transcendental ingredient of the invariants $A_\omega(f)$, and these $Te^{\boldsymbol{s}}_\omega$ are no longer finite superpositions of multizetas.

f (*i.e.* of $\{f_n\}$) when f ranges through a formal class $\mathbb{G}^{p,\rho}$ of identity-tangent diffeomorphisms. Briefly restated in the terminology of this paper, they are:

(i) The quite old and very elementary *geometric approach*. It constructs the iterators $f^*_\pm$ and ${}^*f_\pm$ in the z-plane; derives from them the connectors $\pi^\pm$; then subjects the 1-periodic germs $\pi^\pm(z) - z$ to Fourier analysis; and arrives directly at the invariants $A^\pm_\omega(f)$.

(ii) The more informative *resurgent approach*, less ancient but already four decades old. It focuses on the formal iterator $\widetilde{f}^*(z)$; forms its Borel transform $\widehat{f}^*(\zeta)$; readily finds its resurgence locus $2\pi i\mathbb{Z}$; then, based solely on the functional equation $f^* \circ f = 1 + f^*$, it immediately infers the form of the resurgence equations. Lastly, depending on which alien operators it applies to $\widehat{f}^*(\zeta)$, it directly reaches all systems of invariants, whether $\{A^\pm_\omega(f)\}$ or $\{A_\omega(f)\}$ or $\{A^\sharp_\omega(f)\}$ etc, plus a wealth of information about them.

Having once establish the existence of the invariants $A_\omega(f)$ as entire functions of f, the only task left is to find their Taylor expansion in the countably many coefficients f_n – or rather g_n if $f = l \circ g$:

$$A_\omega(f) = \sum_r \sum_{n_i, s_i} H_\omega^{\binom{n_1\ ,\dots,\ n_r}{s_1\ ,\dots,\ s_r}} \prod_i (g_{s_i})^{n_i} \tag{7.1}$$

Series like (4.3) do just that, since their mode of derivation exactly mimics the parallel constructions of the invariants according to the geometric and resurgent methods. *And the shape of the expansion* (7.1) *once found, its convergence is guaranteed beforehand by the mere fact of $A_\omega(f)$ being an entire function of f. We do not have to bother about majorising the coefficients $H_\omega^{\binom{n}{s}}$ to prove the convergence of* (7.1). *It is exactly the other way round: it is by directly establishing bounds on the growth of $A_\omega(f)$ as a function of f or $\{g_n\}$ (as in the next subsection) that we can most easily derive bounds on the coefficients $H^{\binom{n}{s}}$.*

f-growth of the scalar invariants. This is yet another context where the d-indexation (degree-based) is preferable to the s-indexation (weight-based), for reasons spelled out in *Remark 3* at the end of this paragraph. So let us consider a diffeo $f = l \circ g$ in the standard class $(p, \rho) = (1, 0)$, with $\underline{g}(z) := g(z) - z = \sum_{2 \le d} g_{1+d}\, z^{-d}$. The iterator $\widetilde{f}^*$, or rather its essential part $\underline{\widetilde{f}}(z) := \widetilde{f}(z) - z$, is given in the formal model by

$$\underline{\widetilde{f}}^*(z) = \sum_{1\le r}\left[\frac{e^{\partial}}{1-e^{\partial}}.\sum_{1\le k_r}(\underline{g}(z))^{k_r}\frac{\partial^{k_r}}{k_r!}\right]\dots\left[\frac{e^{\partial}}{1-e^{\partial}}.\sum_{1\le k_r}(\underline{g}(z))^{k_1}\frac{\partial^{k_1}}{k_1!}\right].z \tag{7.2}$$
$$= \underline{g}(z) + \sum_{2\le r}\left[\frac{e^{\partial}}{1-e^{\partial}}.\sum_{1\le k_r}(\underline{g}(z))^{k_r}\frac{\partial^{k_r}}{k_r!}\right]\dots\left[\frac{e^{\partial}}{1-e^{\partial}}.\sum_{1\le k_r}(\underline{g}(z))^{k_2}\frac{\partial^{k_2}}{k_2!}\right].\underline{g}(z).$$

In the convolution model, this translates to an everywhere[79] convergent series

$$\widehat{\underline{f}^*}(\zeta) = \widehat{\underline{g}}(\zeta) + \sum_{1\le n}\widehat{W}^n\,\widehat{\underline{g}}(\zeta) \tag{7.3}$$

with the mixed (multiplication-convolution) operators $\widehat{K}$ acting thus:

$$\left(\widehat{W}\,\widehat{\varphi}\right)(\zeta) := \frac{e^{-\zeta}}{1-e^{-\zeta}}\cdot\sum_{1\le k}\left[(\widehat{\underline{g}})^{*k}(\zeta)\right] *_{\zeta}\left[\frac{(-\zeta)^k}{k!}\widehat{\varphi}(\zeta)\right]. \tag{7.4}$$

A product of two consecutive operators $\widehat{W}$ involves a series of middle terms of the form

$$\widehat{W}.\widehat{W} = \big(\dots\big).\left(\sum_{1\le k}\frac{(-\zeta)^k}{k!}\,\frac{e^{-\zeta}}{1-e^{-\zeta}}\right).\big(\dots\big) \tag{7.5}$$

with bounds

$$\left|\frac{(-\zeta)^k}{k!}\,\frac{e^{-\zeta}}{1-e^{-\zeta}}\right| \le c_\epsilon\,\frac{|\zeta|^{k-1}}{(k-1)!}\,(1+|\zeta|)\quad(\forall\zeta\in K_\epsilon\ ,\ c_\epsilon^{\pm}>0) \tag{7.6}$$

uniformly valid on the K_ϵ

$$K_\epsilon := \{\zeta\in\mathbb{C}, \mathrm{dist}(\zeta, 2\pi i\mathbb{Z}^*)\ge\epsilon\}. \tag{7.7}$$
$$\mathcal{K}_\epsilon := \{\zeta\in\mathcal{R}, \mathrm{dist}(\zeta,\mathcal{R}_{ram}-0_\bullet)\ge\epsilon\}\quad \textit{with}\ \ \mathcal{R} = \widetilde{\mathbb{C}-2\pi i\mathbb{Z}}. \tag{7.8}$$

Note that K_ϵ (respectively $\mathcal{K}_\epsilon$) contains a neighbourhood of the origin 0 (resp $0_\bullet$). Using the expansion (7.4)-(7.5), the bounds (7.6), and the estimates

$$|(\widehat{\underline{g}})^{*k}(\zeta)| < \gamma_0\,\exp(\gamma_1|\zeta|)\,|\zeta|^{2k-1}/(2k-1)! \tag{7.9}$$

[79] *I.e.* at all points ζ not located *over* the singularity locus $2\pi i\mathbb{Z}$.

tedious but elementary calculations[80] lead to optimal[81] estimates of type:

$$
\begin{aligned}
&|\underline{\widehat{f}}(\zeta)| \\
&< c_{0,d}(\zeta)\ \exp\left(c_d(\zeta)\,|g_{1+d}|^{\frac{1}{d}}\right) \qquad (2 \le d) \qquad (7.10)\\
&< c_{0,D}(\zeta)\ \exp\left(\sum_{d\in D} c_{d,D}(\zeta)\,|g_{1+d}|^{\frac{1}{d}}\right) \qquad (D \textit{ finite} \subset \{2,3,\ldots\}) \ (7.11)\\
&< c_{0,\infty}(\zeta)\ \exp\left(c_\infty(\zeta)\ \sup_d |g_{1+d}|^{\frac{1}{d}}\right) \qquad (7.12)
\end{aligned}
$$

for any ζ on the convolution domain $\mathcal{R} := \widetilde{\mathbb{C}-2\pi i\mathbb{Z}}$. The main point to observe is that all the terms $\widehat{W}^n\,\underline{\widehat{g}}(\zeta)$ in (7.3) can be calculated inductively as convolution integrals of the form

$$\frac{e^{-\zeta}}{1-e^{-\zeta}}\int_{0_\bullet}^{\zeta} (\underline{\widehat{g}})^{*k}(\zeta-\zeta_1)\,\widehat{\varphi}_{n,k}(\zeta_1)\,d\zeta_1 \qquad (7.13)$$

with a first convolution factor $(\underline{\widehat{g}})^{*k}(\zeta-\zeta_1)$ that is uniform on $\mathbb{C}$ with the bounds (7.9) and a second factor that is uniform on $\mathcal{R}$ and easily bounded (by induction) on any $\mathcal{K}_\epsilon$. To continue the induction, it is enough to calculate the integral on a ζ_1-path confined within the largest $\mathcal{K}_\epsilon$ that contains ζ, without worrying about $\zeta-\zeta_1$.

To derive from the estimates (7.10)-(7.12) analogous estimates for the invariants A^+_ω , we write the resurgence equations $\Delta^\pm_\omega \underline{\widehat{f}}^*(z) = -A^\pm_\omega \exp(-\omega\,\underline{\widehat{f}}^*(z))$. In the Borel plane this becomes[82]

$$
\begin{aligned}
&\underline{\widehat{f}}^*(\zeta'_\pm) - \underline{\widehat{f}}^*(\zeta''_\pm) = A^+_\omega\,.\,\underline{\widehat{f}}^*_\omega(\zeta) \textit{ with}\\
&\qquad \underline{\widetilde{f}}^*_\omega(z) = e^{-\omega\underline{\widetilde{f}}(z)} - 1 \sim -\omega\, g_{s_0}.z^{1-s_0}
\end{aligned} \qquad (7.14)
$$

with ζ close to $0_\bullet$ on the main Riemann sheet and $\zeta'_\pm$, $\zeta''_\pm$ both over $\dot{\zeta}+\omega$ but on two consecutive Riemann sheets. Since $\underline{\widehat{f}}^*_\omega(\zeta) \sim -\omega g_{s_0}\zeta^{s_0-2}/(s_0-2)!$ for ζ close to $0_\bullet$, there exists for each value of the variable coefficient g_{1+d} at least one point $\zeta = \zeta(g_{1+d})$ on the circle $|\zeta| = 1$ where

80 Even if one were to retain only the part of the operators $\widehat{W}$ that correspond to $k = 1$, the (much simpler) calculations would already show that the estimates (7.10)-(7.12) cannot be improved upon. Taking all k-parts into account does not alter the shape of the estimates, due to the bounds (7.9).

81 *Optimal* as long as we consider the absolute values $|g_{1+d}|^{1/d}$. But one might improve on (7.10) by finding the indicatrix of exponential growth in $|g_{1+d}|^{1/d}$.

82 Since the first term "1" in $\exp(-\omega\,\underline{\widehat{f}}^*(z)) = 1 + \ldots$ contributes nothing to the minors.

$|\widehat{\underline{f}}^*_\omega(\zeta)| = |\omega\, g_{s_0}/(s_0{-}1)!|$. Considering the identity (7.14) for this particular ζ and its images $\zeta'_\pm$ and $\zeta''_\pm$ and using (7.10), we get (7.15) for A^+_ω, as well as (7.16) and (7.17) by a similar argument. The analogous estimates for A_ω, $A^\sharp_\omega$, $A^{\sharp\sharp}_\omega$ etc. follow in view of the bipolynomial correspondance between any two systems of invariants.

$$\begin{aligned} &|A^\pm_\omega|,\, |A_\omega|,\, |A^\sharp_\omega|,\, |A^{\sharp\sharp}_\omega|\; etc. \\ &< c_{0,d}(\omega)\;\; \exp\left(c_d(\omega)\, |g_{1+d}|^{\frac{1}{d}}\right) \qquad (\forall\omega, d \geq 2) && (7.15) \\ &< c_{0,D}(\omega)\;\; \exp\Big(\sum_{d\in D} c_{d,D}(\omega)\, |g_{1+d}|^{\frac{1}{d}}\Big)\;\; (D\; finite) && (7.16) \\ &< c_{0,\infty}(\omega)\;\; \exp\left(c_\infty(\omega)\; \sup_d |g_{1+d}|^{\frac{1}{d}}\right) && (7.17) \end{aligned}$$

Remark 1. The case of the iteration residue ρ. If we now let $f = l \circ g$ range through all classes $(1, \rho)$ by taking $g(z) = -\rho z^{-1} + \mathcal{O}(z^{-1})$, and ask about the asymptotics in ρ, we would get the wrong result by simply setting $g_2 = -\rho$ in the estimate (7.15). The correct estimate is rather:

$$|A^\pm_\omega|,\, |A_\omega|,\, |A^\sharp_\omega|,\, |A^{\sharp\sharp}_\omega|\; etc. < c_{0,1}(\omega) \exp\left(c_1(\omega)|\rho \log|\rho||\right) \quad (\forall\omega). \quad (7.18)$$

The reason is *not* the change from (7.9) to the weaker estimates:

$$|\widehat{\underline{g}}^{*k}(\zeta)| < \gamma_0\; \exp(\gamma_1|\zeta|)\, |\zeta|^{k-1}/(k-1)! \qquad (7.19)$$

The real reason is that we now have $\widehat{\underline{f}}^*(z) = \rho\; \log z + \widehat{\underline{\underline{f}}}^*(z)$ and

$$\widetilde{\underline{f}}^*_\omega(z) = z^{-\omega\rho}\;\; \exp(-\omega \widetilde{\underline{\underline{f}}}^*(z)) = z^{-\omega\rho}\;\; \widetilde{\underline{\underline{f}}}^*_\omega(z) \qquad (7.20)$$

so that (7.14) presently becomes[83]

$$\widehat{\underline{f}}^*(\zeta') - \widehat{\underline{f}}^*(\zeta'') = A^+_\omega \cdot \frac{\zeta^{\omega\rho-1}}{\Gamma(\omega\rho)}\; *_\zeta\; \widehat{\underline{\underline{f}}}^*_\omega(\zeta) \qquad (7.21)$$

Remark 2. 'Uniformisation'. Due to the 'uniformisation' formula (1.54) or (1.55), we see that for any $\zeta \in \mathcal{R}$ (but not above the imaginary axis), $\widehat{\underline{f}}^*(\zeta)$ reduces to a finite sum

$$\widehat{\underline{f}}^*(\zeta) = a_0\; \widehat{\underline{f}}^*(\dot\zeta) + \sum_{\omega\in 2\pi i\mathbb{Z}^*} a_\omega\; \widehat{\underline{f}}^*_\omega(\dot\zeta - \omega) \qquad (7.22)$$

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[83] At least when $-\omega\rho \notin \mathbb{N}$. When $-\omega\rho \in \mathbb{N}$, the positive z-powers in $z^{-\omega\rho}\;\; \widetilde{\underline{\underline{f}}}^*_\omega(z)$ should be neglected, as contributing nothing to the minors in the Borel plane.

(i) with $\underline{\widehat{f}}^*_\omega$ as in (7.20)
(ii) with coefficients a_0, a_ω polynomial in the A_ω
(iii) with $\dot{\zeta}$ the projection of $\zeta \in \mathcal{R}$ onto the main Riemann sheet.

Remark 3. Weight-based vs degree-based indexation. While the s-indexation $f(z) = z + \sum f_s\, z^{1-s}$ is well-adapted to germ composition, the d-indexation $\sum f_{\{d\}}\, z^{-d}$ is better suited to germ conjugation and, consequently, to studying the asymptotics of $A_\omega(f)$. Indeed, take a diffeo f in the standard class and fix $2 \le d \le d'$. There clearly exists a unique diffeo h of the form $h(z) := z + \sum_{d-1 \le n \le d'-1} h_{\{n\}}\, z^{-n}$ that conjugates f to ${}^{var}f$ so as to remove the coefficient $f_{\{d\}}$ while keeping all other coefficients between d and d' unchanged:

$$f(z) := z + 1 + \sum_{2 \le d} f_{\{n\}}\, z^{-n} \;\to\; {}^{var}f(z) := (h \circ f \circ h^{-1})(z)$$
$$= z + 1 + \sum_{2 \le d} {}^{var}f_{\{n\}}\, z^{-n}$$

On top of the defining condition (i), the h-conjugation verifies (ii)-(iii):

(i) ${}^{var}f_{\{d\}} = 0$ if $n \le d'$, and ${}^{var}f_{\{n\}} = f_{\{n\}}$ with $n \ne d$;
(ii) if $d' < n$, ${}^{var}f_{\{n\}}$ is a polynomial in $f_{\{2\}}, f_{\{3\}}, \ldots, f_{\{n\}}$ involving only 'subhomogeneous' monomials of form $\prod_i (f_{\{n_i\}})^{m_i}$ with $n_1 m_1 + \cdots + n_r m_r \le n$;
(iii) if $d|n$ and $d' < n$, the monomial $(f_{\{d\}})^{n/d}$ is effectively present, with a nonzero rational coefficient, in the expression of ${}^{var}f_{\{n\}}$.

Since $A_\omega(f) = A_\omega({}^{var}f)$, we see that the additional properties (ii)-(iii) are perfectly coherent with the asymptotic estimates (7.10)-(7.15).

ω-growth of the scalar invariants. Fixing $f = l \circ g$ and $\epsilon_0 < \pi$, using the relations (7.3), and calculating the successive integrals in (7.4) on ζ_1-paths contained in $\mathcal{K}_{\epsilon_0}$, one easily arrives at exponential estimates

$$|A^{\pm}_\omega| < \gamma_0^{\pm}\ \exp(\gamma_1^{\pm}\,|\omega|) \qquad (\forall \omega \in 2\pi\mathbb{Z}^*\ ,\ \gamma_0^{\pm}, \gamma_1^{\pm} > 0) \tag{7.23}$$

with constants $\gamma_0^{\pm}, \gamma_1^{\pm}$ that depend only on the growth of $\underline{\widehat{g}}(\zeta)$ in the vertical stripes $|\Re(\zeta)| < \epsilon$. This, however, does not apply to the other systems of invariants, like A_ω, $A^{\sharp}_\omega$, $A^{\sharp\sharp}_\omega$ etc, which, being the coefficients of generically divergent but resummable Fourier series (see below), generically possess exponential growth in $|\omega|.\log|\omega|$ rather than $|\omega|$.

7.2 The connectors

For $f = l \circ g$ fixed and convergent, only the connectors $\pi^{\pm}(z)$ with Fourier coefficients $A^{\pm}_{\omega}$ have guaranteed convergence is some bi-domain $|\Im(z)| > y$. But as shown in Section 1.7, and 1.8, most other connectors $\pi_{\diamondsuit}(z)$ are merely resurgent and Borel resummable, each with a definite critical time $z_0 := \exp(\mp 2\pi i z)$, where n_0 is the index of the first non-vanishing invariant. This is definitely the case with the connectors $\pi_*(z), \pi_{\sharp}(z), \pi_{\sharp\sharp}(z)$.

7.3 The collectors

As already pointed out, collectors can be classified unter two viewpoints:

(i) *type*: there is $\mathfrak{p}(z)$ itself and its various affiliates $\mathfrak{p}_{\diamondsuit}(z)$ — generators, mediators etc,

(ii) *nature*: we can consider their natural multitangent expansions; or their reduced monotangent expansions; or their local Laurent expansions at $z = 0$.

Now, as long as the collectors are viewed as generating series in the coefficients g_n, as in Section 3, the question of their convergence does not arise — the coefficients of each bloc is always convergent, and this is all that matters from the perspective of this paper. But we may also ask, gratuitously so to speak: given a fixed convergent germ f, which impersonations of the collectors do converge, and in what sense?

From what we already know about the connectors, the question makes sense only for $\mathfrak{p}(z)$ itself, not for its affiliates. And $\mathfrak{p}(z)$, as we shall see, *convergences only in its natural multitangent presentation*.[84]

Convergence of the multitangential collectors $\mathfrak{p}(z)$. The convergence of the connectors π as scalar germs can be established in any number of ways (*e.g.* from the estimates (7.23)) and it implies the convergence of the associated substitution operators Π. However, in order to ease the transition to the collectors $\mathfrak{p}$ and $\mathfrak{P}$, we need to look more closely at these operators Π and *their constituent parts*.

Set $\Pi := \Pi^+$, $G := G^+$, $G_{:n} := L^n \,.\, G \,.\, L^{-n}$ consider the (for the moment, formal) operator Π as given by (3.8) and replace its bifactorisation $\Pi = {}^*F_- \,.\, F^*_+$ by the trifactorisation

$$\Pi = \Pi_{L,n} \,.\, \Pi_{M,n} \,.\, \Pi_{R,n} \,. \qquad (n \textit{ large}) \tag{7.24}$$

[84] Natural means that we take the $Te^{\bullet}$-expansions as they naturally result from the series (3.12) in Section 3 and resort, at most, to symmetrel linearisation.

with L, M, R standing for *left, middle, right* and with the truncated expansions

$$\mathbf{\Pi}_{R,n} := 1 + \sum_{1\le r} \sum_{n\le n_r<\ldots<n_1} \underline{G}^{+}_{:n_r} \ldots \underline{G}^{+}_{:n_1} = L^n \,.\, F^*_+ \,.\, L^{-n} \tag{7.25}$$

$$\mathbf{\Pi}_{M,n} := 1 + \sum_{1\le 2n} \sum_{-n\le n_r<\ldots<n_1<n} \underline{G}^{+}_{:n_r} \ldots \underline{G}^{+}_{:n_1} = G_{:(-n)} \ldots G_{:(n-1)} \tag{7.26}$$

$$\mathbf{\Pi}_{L,n} := 1 + \sum_{1\le r} \sum_{n_r<\ldots<n_1<-n} \underline{G}^{+}_{:n_r} \ldots \underline{G}^{+}_{:n_1} = L^{-n} \,.\, {}^*F_- \,.\, L^n . \tag{7.27}$$

For any two open sets $\mathcal{D}_1, \mathcal{D}_2$ of $\mathbb{C}$, bounded or not, connected or not, but with $\overline{\mathcal{D}_2} \subset \mathcal{D}_1$, and any operator H, we set

$$\|H\|_{\mathcal{D}_1,\mathcal{D}_2} := \sup_{\|\varphi\|_{\mathcal{D}_1}\le 1} \|H\,\varphi\|_{\mathcal{D}_2} \quad \textit{and} \quad \|H\|_{\mathcal{D}} := \|H\|_{\mathcal{D},\mathcal{D}^*} \tag{7.28}$$

where $\mathcal{D}^*$ denotes the set of all points in $\mathcal{D}$ whose distance from the boundary of $\mathcal{D}$ is more than 1.

For any ϵ we can find $n \in \mathbb{N}$ and $y \in \mathbb{R}^+$ large enough to ensure

$$\|\Pi_{R,n} - 1\|_{\mathcal{D}_R} \le \epsilon \quad \forall \mathcal{D}_R \subset \{z, \Re z \ge -6\} \tag{7.29}$$

$$\|\Pi_{M,n} - 1\|_{\mathcal{D}_M} \le \epsilon \quad \forall \mathcal{D}_M \subset \{z, |\Im z| \ge y\} \tag{7.30}$$

$$\|\Pi_{L,n} - 1\|_{\mathcal{D}_L} \le \epsilon \quad \forall \mathcal{D}_L \subset \{z, \Re z \le +6\} \tag{7.31}$$

and therefore

$$\|\Pi - 1\|_{\mathcal{D}} \le 4\,\epsilon \quad \forall \mathcal{D} \subset \{z, |\Re z| \le 3, |\Im z| \ge y+3\}. \tag{7.32}$$

Moreover, one can show that the statement would still hold (for a slightly larger choice of n, y) if, instead of considering the norm $\|\Pi - 1\|_{\mathcal{D}}$, we were to consider the larger norms:

$$\|\Pi - 1\|^{\mathcal{S}}_{\mathcal{D}} = \sum \|H^{\binom{n}{s}}\|_{\mathcal{D}} \prod |g_{s_i}|^{n_i} \quad \textit{with} \quad \Pi - 1 = \sum H^{\binom{n}{s}} \prod (g_{s_i})^{n_i}$$

relative to any *natural* expansion $\mathcal{S}$ of $\Pi - 1$ as a series of monomials $\prod(g_{s_i})^{n_i}$. But expanding Π in this way is tantamount to viewing it as the *collector* $\mathfrak{P}$ with its natural multitangent expansion (relative to the system $Te^\bullet$). Of course, the multitangential $\mathfrak{P}$ and $\mathfrak{p}$ converge *separately* on the two half-planes $|\Im(z)| > y$, but in that sense, *qua* convergent objects, already cease to be of one piece.

Divergence of the monotangential collectors $\mathfrak{p}(z)$. By multiplying the Laurent expansions of $Te^{s_1}(z)$ and $Te^{s_2}(z)$ at $z = 0$ and then retaining only the z-negative powers in the product, we get the multiplication rule for (integer-indexed) monotangents:

$$\mathrm{Te}^{s_1}(z)\mathrm{Te}^{s_1}(z) = \mathrm{Te}^{s_1+s_2}(z) + \sum_{2 \leq s_3 < \max(s_1,s_2)} \mathrm{te}_{s_3}^{s_1,s_2}\, \mathrm{Te}^{s_3}(z) \;\; (s_1, s_2 \in N^*) \quad (7.33)$$

with

$$\begin{aligned} \mathrm{te}_{s_3}^{s_1,s_2} &= \left[1 + (-1)^{s_1+s_2-s_3}\right] \zeta(s_1+s_2-s_3) \\ &\times \left[\frac{(-1)_+^{s_1-s_3}(s_1+s_2-s_3)!}{(s_1-s_3)!(s_2-1)!} + \frac{(-1)_+^{s_2-s_3}(s_1+s_2-s_3)!}{(s_2-s_3)!(s_1-1)!} \right] \end{aligned} \quad (7.34)$$

and $(-1)_+^s := (-1)^s$ if $s > 0$ respectively $(-1)_+^s := 0$ if $s \leq 0$. Now, if the monotangential expansions for $\mathfrak{p}^+$ and $\mathfrak{p}^-$ always existed, since $\mathfrak{p}^+ \circ \mathfrak{p}^- = id$, going from the one to the other would involve mutiplying many infinite sums of the form

$$\left(\sum_{s_1\ even} a_{s_1}\, \mathrm{Te}^{s_1}(z)\right)\left(\sum_{s_2\ even} b_{s_2}\, \mathrm{Te}^{s_2}(z)\right) \mapsto \left(\sum_{s_3\ even} c_{s_3}\, \mathrm{Te}^{s_3}(z)\right) \quad (7.35)$$

with series $\sum a_{s_1} z^{-s_1}$ and $\sum b_{s_2} z^{-s_2}$ whose convergence radii might be small, since the convergence radius of the underlying series $g(z)$ may be anything. But the coefficient c_{s_3} on the right-hand side of (7.35) are given by

$$c_{s_3} = \sum_{s_3 = s_1+s_2} a_{s_1}\, b_{s_2} + \sum_{s_3 < \max(s_1,s_2)} a_{s_1}\, b_{s_2}\, \mathrm{te}_{s_3}^{s_1,s_2} \quad (7.36)$$

with a second sum that diverges if, for instance, all a_{s_1} and b_{s_2} are positive with $\lim |a_{s_1}|^{\frac{1}{s_1}} = a > 0$, $\lim |b_{s_s}|^{\frac{1}{s_2}} = b > 0$ and $2\,a\,b > 1$. In that case, the coefficients c_{s_3} are not even defined.

So it would be more accurate to say that the monotangential collectors, rather than diverging, *generally do not even exist*: they cannot be defined, not even as formal series. What exists but fails to converge as $s \to +\infty$ is the *weight-truncated, monotangential collectors*[85] $trunc_{s_0}\, \mathfrak{p}^{\pm}(z)$ (see Section 3.7).

[85] They exist unproblematically as *finite* sums, whether in multi- or monotangential form.

7.4 Groups of invariant-carrying formal diffeos

One of the many advantages of the resurgent approach to the study of holomorphic invariants is that it extends effortlessly to many subgroups $\mathbb{G}_\chi$ of the group $\mathbb{G}$ of all *formal* identity-tangent diffeos. Typically, these groups $\mathbb{G}_\chi$ are defined by a growth condition on the coefficients f_s of their elements that is

(i) stable under composition and reciprocation[86];
(ii) stringent enough to ensure that formal conjugacy (in $\mathbb{G}$) does not imply actual conjugacy (in $\mathbb{G}_\chi$).

This implies the existence on these groups $\mathbb{G}_\chi$ of non-formal invariants, and immediately raises the question of their description/calculation.

If we put aside a few pathological instances[87], all such groups $\mathbb{G}_\chi$ consist of elements $\widetilde{f}$ whose Borel transforms $\widehat{f}(\zeta)$ extend to well-defined entire functions (albeit with supra-exponential growth), with iterators $\widetilde{f}^*$, ${}^*\widetilde{f}$ that verify the familiar resurgence equations and produce complete systems of holomorphic invariants $A_\omega(\widetilde{f})$, exactly as on the analytic group $\mathbb{G}_0$.

Before taking a closer look at some examples of 'invariant-carrying' groups $\mathbb{G}_\chi$, let us state a few useful lemmas.

Given a system $\{a_n, n \in \mathbb{C}\}$ with a geometric or slightly faster-than-geometric rate of growth, and a number $\omega_0 \in \mathbb{C}^*$, we set $b_m := \sum_n \frac{|\omega_0\, m|^n}{n!}\, a_n$. Using the rough estimates $\log^+ |b_m| \sim \sup_n \log^+ |\frac{|\omega_0\, m|^n}{n!}\, a_n|$, we easily infer the growth rate of $\log |b_m|$ from that of $\log |a_m|$ in these four important cases:

$$\left\{\log^+|a_n| = \mathcal{O}(n)\right\} \Longrightarrow \left\{\log^+|b_m| = \mathcal{O}(m)\right\} \tag{7.37}$$

$$\left\{\log^+|a_n| = \mathcal{O}(n \log_k n)\right\} \Longrightarrow \left\{\log^+|b_m| = \mathcal{O}(m \log_{k-1} m)\right\} \tag{7.38}$$

$$\left\{\log^+|a_n| = \mathcal{O}\left(n \frac{\log n}{\log_k n}\right)\right\} \Longrightarrow \left\{\log^+|b_m| = \mathcal{O}\left(m \exp\left(\frac{\log m}{\log_k m}\right)\right)\right\} \tag{7.39}$$

$$\left\{\limsup \frac{\log^+|a_n|}{n \log n} \le \tau < 1\right\} \Longrightarrow \left\{\limsup \frac{\log^+|b_m|}{m^{1/(1-\tau)}} \le 1\right\}. \tag{7.40}$$

Here, $\log^+ x := \log x$ if $1 < x$ (respectively $:= 0$ if $0 \le x \le 1$). As we can see, the actual value of ω_0 is immaterial.

[86] *I.e*. the taking of the composition inverse.

[87] Corresponding to wildly irregular ('oscillating' in some sense) growth conditions χ.

Moreover, if we set

$$b(w) = w + \sum b_m \, e^{-m\omega_0 w} \tag{7.41}$$

$$c(z) = z + \sum c_m z^{1-m} = \exp\left(-\omega_0 \, b\left(\frac{1}{\omega_0}\right) \log\left(\frac{1}{z}\right)\right) \tag{7.42}$$

the Taylor coefficients c_m are, in all four instances (7.37)-(7.40), subject to exactly the same growth constraints as the Fourier coefficients b_m.

Lastly, it is an easy matter to check that each of the growth conditions listed in (7.37)-(7.40) is stable under composition and reciprocation, and thus defines a group $\mathbb{G}_\chi$.

The analytic subgroup $\mathbb{G}_0$. There is no need to return to the group $\mathbb{G}_0$ and its invariants, except to emphasise a remarkable feature: any germ $f \neq id$ in $\mathbb{G}_0$ has $2p$ connectors which, after a rescaling of type (7.42), produce $2p$ new germs $f_{(i_1)}$ still in $\mathbb{G}_0$. Each one of these $f_{(i_1)}$ produces $2p_{i_1}$ new germs $f_{(i_1,i_2)}$, each of which in turn produces $2p_{i_1,i_2}$ germs $f_{(i_1,i_2,i_3)}$, and so on indefinitely[88], without ever leaving the group $\mathbb{G}_0$. This infinite self-replication property of $\mathbb{G}_0$ is more than a curiosity: it has practical implications.[89] It also raises the question: is self-replication an exclusive feature of $\mathbb{G}_0$, or does it extend to other invariant-carrying groups $\mathbb{G}_\chi$? It does, as we shall see, provided the growth condition χ is *extremely* close to geometric growth (which ensures analyticity).

The near-analytic, self-replicating subgroup $\mathbb{G}_{0^+}$. The implication (7.38) being optimal, on the group $\mathbb{G}_{[k]}$ consisting of all f (let us drop the clumsy tilda) whose coefficients verify

$$\lim_{n\to+\infty} \frac{\log^+ |f_n|}{n \, \log_k n} = 0 \tag{7.43}$$

the mapping[90] $f \mapsto resc.\pi$ is from $\mathbb{G}_{[k]}$ to $\mathbb{G}_{[k-1]} \subset \mathbb{G}_{[k]}$. So it is only the limit or intersection

$$\mathbb{G}_{0^+} := \lim_k \mathbb{G}_{[k]} = \bigcap_k \mathbb{G}_{[k]} \tag{7.44}$$

88 For the process to stop, at a certain stage all $f_{(i_1,\ldots,i_r)}$ would have to be *id*, which of course almost never happens.

89 *E.g.*, in fractal analysis (see [12]) and in resummation theory: it played a part in the original proof of Dulac's conjecture about the non-accumulation of limit-cycles, prior to the introduction of *well-behaved* convolution averages (see [7]).

90 $resc.\pi$ is the connector π *rescaled* so as to become an element of $\mathbb{G}$.

that possess the property of self-replication. To realise how close $\mathbb{G}_{0^+}$ is to $\mathbb{G}_0$, we may note that verifying (7.43) for any k is a far more severe condition than verifying the Denjoy quasi-analyticity conditions. Expressed in terms of Taylor coefficients, these read:

$$|g_n|^{\frac{1}{n}} \leq \mathcal{O}(\log_1 n \log_2 n \ldots \log_{k-1} n) \tag{7.45}$$

for some given k. That merely implies

$$\log^+ |f_n| \leq n \left(\log_2 n + \cdots + \log_k n + o(\log_k n) \right) \tag{7.46}$$

which is much weaker than (7.43), let alone (7.44). This is not to say, of course, that $\mathbb{G}_{0^+}$ consists only of quasi-analytic germs, since a smooth function f must verify a Denjoy condition on a whole interval to qualify as quasi-analytic.[91]

The maximal subgroup $\mathbb{G}_{0^{++}}$. Consider the Gevrey subgroups of $\mathbb{G}$ defined by the growth conditions

$$\mathbb{G}_{[[\tau]]} := \left\{ f \ ; \ \limsup_{n\to+\infty} \frac{\log^+ |f_n|}{n \log n} \leq \tau \right\}. \tag{7.47}$$

For all elements f in $\mathbb{G}_{[[\tau]]}$ of tangency order $p = 1$ to have everywhere convergent Borel transforms, τ has to be < 1, in which case these f possess invariants whose growth pattern is bounded by the b_m-estimates of (7.40). Elements f of tangency order $p > 1$, however, must first be brought to a prepared form $(f(z^{1/p})^p$, which belongs to $\mathbb{G}_{[[p\tau]]}$, or rather to the ramified equivalent of $\mathbb{G}_{[[p\tau]]}$. So the largest group whose elements all possess holomorphic invariants is the intersection $\mathbb{G}_{0^{++}}$ of all these Gevrey goups:

$$\mathbb{G}_{0^{++}} := \left\{ f \ ; \ \lim_{n\to+\infty} \frac{\log^+ |f_n|}{n \log n} = 0 \right\} \tag{7.48}$$

Elements of $\mathbb{G}_{0^{++}}$ have connectors which are usually not in $\mathbb{G}_{0^{++}}$. since their coefficients are subject only to the very weak growth constraints

$$\log^+ \log^+ |c_r| = o(r \log r) \tag{7.49}$$

[91] Growth conditions at *one* point never suffice to ensure the existence of a quasi-analytic 'continuation' on a neighbourhood of that point. In fact, when the coefficients are all > 0 and with faster than geometric growth, the 'continuation' never exists.

This results from the optimal implication (7.39) or rather from its – still valid – extension to the case where log_k is replaced on both sides by any regular[92] germ $\mathcal{L}$ with ultra-slow growth.

7.5 A glimpse of phantom holomorphic dynamics

Let us for definiteness consider the "near-analytic" group $\mathbb{G}_{0^+}$. It has much more in common with its analytic prototype $\mathbb{G}_0$ than the existence of non-trivial (*i.e.* non-formal) conjugacy classes characterisable by holomorphic invariants $A_\omega(f)$. The notion of *polarised sectorial model* too has its equivalent, but with *acceleration operators* taking the place of Laplace integration. Indeed, for any slow acceleration $z \to z_\dagger$ with

$$\frac{z_\dagger}{z} \to +\infty \quad but \quad \frac{\log z_\dagger}{\log z} \to 1 \quad e.g. \quad z = \mathfrak{F}(z_\dagger) := \frac{z_\dagger}{\log z_\dagger} \tag{7.50}$$

the acceleration integrals $\zeta \to \zeta_\dagger$

$$\widehat{f}^{*}_{\dagger,\pm}(\zeta_\dagger) = \int_0^{(1\pm\epsilon)\, i\, \infty} C_{\mathfrak{F}}(\zeta_\dagger, \zeta)\ \widehat{f}^{*}(\zeta) \tag{7.51}$$

$${}^{*}\widehat{f}_{\dagger,\pm}(\zeta_\dagger) = \int_0^{(1\pm\epsilon)\, i\, \infty} C_{\mathfrak{F}}(\zeta_\dagger, \zeta)\ {}^{*}\widehat{f}(\zeta) \tag{7.52}$$

turns the non-polarised iterators $\widehat{f}^{*}, {}^{*}\widehat{f}$ into polarised iterators $\widehat{f}^{*}_{\dagger,\pm}, {}^{*}\widehat{f}_{\dagger,\pm}$ defined and regular in sectors $\mathcal{S}_{\dagger,\pm}$ of the $\zeta_\dagger$-plane. Moreover, on the intersection $\mathcal{S}_{\dagger,+} \cap \mathcal{S}_{\dagger,-}$, which contains a southern half-plane $\{\Im\, \zeta_\dagger < -y\}$, these polarised iterators can be subjected to the operation $\hat{\circ}$ (which transposes the ordinary composition $\circ$ to the Borel planes[93]) to produce an object $\widehat{\pi}_{\dagger,\mathrm{so}}(\zeta_\dagger)$ that will be the exact counterpart of a connector's southern component $\pi_{\mathrm{so}}(z)$ for an ordinary analytic germs f in $\mathbb{G}_0$.

One may even perform Fourier analysis on $\widehat{\pi}_{\dagger,\mathrm{so}}(\zeta_\dagger)$ and $\widehat{\pi}_{\dagger,\mathrm{no}}(\zeta_\dagger)$ in the $\zeta_\dagger$-plane[94] to calculate the invariants $A_\omega(f)$. This procedure (inefficient but perfectly workable) would essentially differ from the (efficient) resurgent analysis in the ζ-plane. It would exactly mirror the (moderately efficient – see Section 4.5) Fourier analysis performed on ordinary connectors $\pi_{\mathrm{so}}(z), \pi_{\mathrm{no}}(z)$ in the multiplicative z-plane.

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92 "Regular" in the sense of verifying the *universal asymptotics of slow-growing germs*. See *e.g.* [7, 8]. For instance, we may take $\mathcal{L}$ to be any transfinite exponential of *log*, again in the sense of [7,8].

93 $(\widehat{\varphi}\,\hat{\circ}\,\widehat{f})(\zeta) := \widehat{\varphi}(\zeta) + \sum_{1\le k} \frac{1}{k!}\,(\widehat{\underline{f}})^{*k}(\zeta) *_\zeta \Big((-\zeta)^k\, \widehat{\varphi}(\zeta)\Big)$ with $\underline{f}(z) = f(z) - z$.

94 There is no contradiction here: the exponentials $e^{\pm\omega z}$ have no image in the ζ-plane, but they have one in the $\zeta_\dagger$-plane, since $e^{\pm\omega z} = e^{\pm\omega \mathfrak{F}(z_\dagger)}$ is strictly sub-exponential in $z_\dagger$.

For any f in $\mathbb{G}_{0^+}$, the mapping $\widehat{\varphi} \mapsto \widehat{\varphi} \,\hat{\circ}\, \widehat{f}$ is an algebra isomorphism (relative to the convolution product), just as the substitution operators are (relative to ordinary multiplication). Another aspect of "phantom holomorphic dynamics" (in non-polarised and polarised Borel planes) is the notion of invariant subspaces or *fuzzy orbits*, which in a sense fill the role of *orbits* in the (here non-existent) multiplicative plane. But the subject is still in its infancy, and we had better stop here.

8 Conclusion

8.1 Some historical background

(i) Identity-tangent diffeos in holomorphic dynamics.
The iteration of one-dimensional analytic mappings – whether local or global; identity-tangent or not – has a long history going back a century or more. Fatou, for one, knew about the analytic classes of identity-tangent diffeos and had formed a clear, geometry-based idea of their invariants. The subject then when into something of a hibernation until the advent of high-power computation, which brought about an explosive revival of holomorphic dynamics, one- and many-dimensional. For the specific subject of analytic invariants, however, the main impetus for renewal came from an unexpected quarter: resurgent analysis.
(ii) Identity-tangent diffeos and resurgent analysis.
The fact is that identity-tangent diffeos possess generically divergent but always resurgent iterators and fractional iterates, with an interesting, nonlinear pattern of resurgence or self-reproduction at the singular points in the Borel plane, and it was in the process of sorting out these phenomena that resurgence theory was born, and later applied to general local objects and much else. In a sense, this involved a retreat from dynamics proper, since it meant focusing on the Borel plane, where the key dynamic notions of trajectory, fixed point etc admit no simple interpretation. For the invariants A_ω, however, the shift in focus brought a definite advantage, since in the Borel plane these invariants are automatically *localised* and *isolated* (they appear as coefficients of the leading singularities over the point ω) whereas in the multiplicative plane they are *diffuse* and *intertwined* (they make themselves felt only collectively and indirectly, via Stokes phenomena and the like, and the only way to isolate them is by Fourier analysis of type (4.10), which is but a half-hearted way of doing what Borel analysis does neatly and efficiently). This applies not just to identity-tangent diffeos, but to a huge range of local objects and equations. It also works in both directions: in that of *"analysis"*, *i.e.* calculating and investigating the invariants of a given object; and in that of *"synthesis"*, *i.e.* prescribing an admissible system of 'invariants' and

then constructing an object of which they are the actual invariants. And it has to be said that in both directions resurgence theory performs rather better than geometry. It leads in particular to a privileged or "canonical" synthesis, a notion which eludes geometry.

8.2 Multitangents and multizetas

(iii) Identity-tangent diffeos and the resuscitation of multizetas. Multizetas (of depth 2, to be precise) were first considered by Euler as an isolated curiosity, and later fell into a protracted oblivion for want of applications. They resurfaced only in the late 1970s and early 1980s in [3–5], precisely in the context of holomorphic dynamics and identity-tangent diffeos, as *the* transcendental ingredient in the make-up of their invariants. Ten years later, the multizetas started cropping up in half a dozen, largely unconnected contexts: braid groups and knot theory; Feynman diagrams; Galois theory; mixed Tate motives; arithmetical dimorphy; ARI/GARI and the flexion structure, etc. At the moment, all these strands are in the process of merging or at least cross-fertilising, and constitute a vibrant field of research.

(iv) Identity-tangent diffeos and the actual computation of their invariants.
The sections of [5] devoted to the invariants of identity-tangent diffeos were written with no computational applications in mind, and no attempt was made to optimise the calculational procedures. On the contrary, the PhD thesis [1], which revisits the subject 30 years on, lays its main emphasis on these neglected aspects and provides effective Maple programmes for the computations of the invariants; it also offers copious asides on the algebraic aspects of multitangents, which largely, but not exactly, mirror those of multizetas.

8.3 Remark about the general composition equation

The equations verified by the iterators and iteration roots of identity-tangent diffeos are extremely special cases of the general composition equation:

$$f^{\circ m_r} \circ g_r \circ \ldots f^{\circ m_2} \circ g_2 \circ f^{\circ m_1} \circ g_1 = id \tag{8.1}$$

with f unknown, $m_i \in \mathbb{Z}$ and $g_i(z) = z + \tau_i + \mathcal{O}(z^{-1})$. The general solution[95] of (8.1) is also generally divergent but always resurgent

[95] It is *unique* under the genericity assumption $\sum m_i \neq 0$.

and resummable.[96] The subject is investigated in Section 11 and 12 of a preprint accessible on the author's homepage.[97]

The critical set Ω (containing the indices ω of all *active* alien derivations Δ_ω) is often huge: it usually consists of all finite combinations $-\lambda_{j_0} + \sum n_j \lambda_i$ $(n_j > 0)$ spanned by the (countably many) roots of some exponential polynomial constructed from the data m_i and τ_i. We may adjust these data m_i, τ_i so as to ensure $\Omega = 2\pi i \mathbb{Z}$, for example by considering composition equations of the form

$$f \circ g_r \circ \ldots f \circ g_2 \circ f \circ g_1 = id \tag{8.2}$$

with $g_1(z) = z+1+\mathcal{O}(z^{-2})$, $g_i(z) = z+\mathcal{O}(z^{-2})$ $(i \geq 2)$. But even then the complete formal solution remains extremely complex, and still depends non-linearly on a countable infinity of parameters u_j:

$$\widetilde{f}(z,u) = \widetilde{f}(z) + \sum \boldsymbol{u}^{\boldsymbol{n}}\, e^{\omega z}\, \widetilde{f}_{\boldsymbol{n}}(z) \qquad \Big(\boldsymbol{u}^{\boldsymbol{n}} = \prod u_j^{n_j}\Big). \tag{8.3}$$

The bridge equation reads $\boldsymbol{\Delta}_\omega \widetilde{f}(z,u) = \mathbb{A}_\omega \widetilde{f}(z,u)$ with operators $\mathbb{A}_\omega$ that are hardly less complex:

$$\mathbb{A}_\omega = \sum_{\langle \boldsymbol{n},\, \boldsymbol{j}\rangle - j = k} u_{j_1}^{n_{j_r}} \ldots u_{j_1}^{n_{j_r}}\; A^{j}_{\omega,\boldsymbol{n}}\, \partial_{u_j} \quad \big(\dot{\omega} = 2\pi i\, k, k \in \mathbb{Z} - r\,\mathbb{Z}\big). \tag{8.4}$$

However, a drastic simplification occurs in the case $r = 2$:

$$\mathbb{A}_\omega = 2\pi i\; A_\omega \sum_{k \in \mathbb{Z}^*} (j+k)\, u_{j+k}\, \partial_{u_j} \qquad \big(\dot{\omega} = 2\pi i\, k, k \in \mathbb{Z} - 2\,\mathbb{Z}\big). \tag{8.5}$$

Instead of depending on a huge set of unrelated resurgence constants $A^j_{\omega,\boldsymbol{n}}$, with $\omega \in 2\pi i \mathbb{Z}^*$ but an index $\boldsymbol{n}$ running through all finite parts of $\mathbb{Z}$, the operators $\mathbb{A}_\omega$ now depend on an incomparably smaller set of resurgence constants A_ω, with $\omega \in 2\pi i \mathbb{Z}^*$.

The reason is of course that in the case $r = 2$, the composition equation reduces to an iteration equation - to the taking of a 'square root':

$$f \circ g_2 \circ f \circ g_1 = id \Longleftrightarrow (f \circ g_2) \circ (f \circ g_2) = g_1^{-1} \circ g_2. \tag{8.6}$$

This huge complexity gap between the case $r \geq 3$ and $r = 2$ is reminiscent of the equally dramatic simplification that takes place with first order singular ODE's of 'Euler type' :

$$\partial_z Y = Y + \sum_{-1 \leq n \leq n_0} b_n(z)\, Y^{1+n} \qquad \big(b_n(z) \in z^{-1}\mathbb{C}\{z^{-1}\}\big). \tag{8.7}$$

[96] The critical time too is unique under the same genericity assumption $\sum m_i \neq 0$.

[97] *The Natural Growth Scale*.

In the general case ($2 \leq n_0 \leq \infty$), we get a resurgent formal solution $\widetilde{f}(z,u)$ in $\mathbb{C}[[z^{-1}, u\, z^{\tau}\, e^{z}]]$, a critical set $\Omega = \{-1\} \cup \mathbb{N}^*$, and an infinite series of independent invariants $\mathbb{A}_n = A_\omega\, u^{n+1} \partial_u$ with indices $n \in \{-1, 1, 2, 3 \dots\}$, whereas in the case $n_0 = 1$, the equation (8.7) becomes an ODE of Riccati type; the critical set Ω reduces to $\{-1, 1\}$; and we are left with just two independent invariants $\mathbb{A}_{-1}, \mathbb{A}_1$.

9 Tables

9.1 Multitangents: symmetrel, alternal, olternol

We express $Taa^\bullet$ and $Too^\bullet$ in terms of $Te^\bullet \approx Tee^\bullet$ according to the linearisation lemma of Section 5.4, using throughout the shorthand $n_{i,j,\dots}$ for $n_i + n_j + \dots$.

Table 1 : Comparing $Te^\bullet \sim Tee^\bullet, Taa^\bullet, Too^\bullet$.

$$\mathrm{Taa}^{n_1} = \mathrm{Too}^{n_1} = \mathrm{Te}^{n_1} \quad , \quad \mathrm{Taa}^{n_1,n_2} = \mathrm{Too}^{n_1,n_2} = \frac{1}{2}\mathrm{Te}^{n_1,n_2} - \frac{1}{2}\mathrm{Te}^{n_2,n_1}$$

$$\begin{aligned} 6\,\mathrm{Taa}^{n_1,n_2,n_3} = {} & 2\mathrm{Te}^{n_1,n_2,n_3} - \mathrm{Te}^{n_1,n_3,n_2} - \mathrm{Te}^{n_2,n_1,n_3} - \mathrm{Te}^{n_2,n_3,n_1} - \mathrm{Te}^{n_3,n_1,n_2} \\ & + 2\,\mathrm{Te}^{n_3,n_2,n_1} - \mathrm{Te}^{n_1+n_3,n_2} + \frac{1}{2}\mathrm{Te}^{n_1,n_{2,3}} + \frac{1}{2}\mathrm{Te}^{n_{1,2},n_3} + \frac{1}{2}\mathrm{Te}^{n_3,n_{1,2}} \\ & + \frac{1}{2}\mathrm{Te}^{n_{2,3},n_1} - \mathrm{Te}^{n_2,n_{1,3}} \end{aligned}$$

$$\begin{aligned} 4\,\mathrm{Too}^{n_1,n_2,n_3} = {} & \mathrm{Te}^{n_1,n_2,n_3} - \mathrm{Te}^{n_1,n_3,n_2} - \mathrm{Te}^{n_2,n_1,n_3} - \mathrm{Te}^{n_2,n_3,n_1} - \mathrm{Te}^{n_3,n_1,n_2} \\ & + \mathrm{Te}^{n_3,n_2,n_1} - \mathrm{Te}^{n_{1,3},n_2} - \mathrm{Te}^{n_2,n_{1,3}} \end{aligned}$$

$$\begin{aligned} & 12\,\mathrm{Taa}^{n_1,n_2,n_3,n_4} = \\ & 3\,\mathrm{Te}^{n_1,n_2,n_3,n_4} - \mathrm{Te}^{n_1,n_2,n_4,n_3} - \mathrm{Te}^{n_1,n_3,n_2,n_4} - \mathrm{Te}^{n_1,n_3,n_4,n_2} - \mathrm{Te}^{n_1,n_4,n_2,n_3} \\ & + \mathrm{Te}^{n_1,n_4,n_3,n_2} - \mathrm{Te}^{n_2,n_1,n_3,n_4} + \mathrm{Te}^{n_2,n_1,n_4,n_3} - \mathrm{Te}^{n_2,n_3,n_1,n_4} - \mathrm{Te}^{n_2,n_3,n_4,n_1} \\ & + \mathrm{Te}^{n_2,n_4,n_1,n_3} + \mathrm{Te}^{n_2,n_4,n_3,n_1} - \mathrm{Te}^{n_3,n_1,n_2,n_4} - \mathrm{Te}^{n_3,n_1,n_4,n_2} + \mathrm{Te}^{n_3,n_2,n_1,n_4} \\ & + \mathrm{Te}^{n_3,n_2,n_4,n_1} - \mathrm{Te}^{n_4,n_1,n_2,n_3} + \mathrm{Te}^{n_4,n_1,n_3,n_2} + \mathrm{Te}^{n_4,n_2,n_1,n_3} + \mathrm{Te}^{n_4,n_2,n_3,n_1} \\ & + \mathrm{Te}^{n_4,n_3,n_1,n_2} - 3\,\mathrm{Te}^{n_4,n_3,n_2,n_1} + \mathrm{Te}^{n_1,n_2,n_{3,4}} - \mathrm{Te}^{n_1,n_3,n_{2,4}} - \mathrm{Te}^{n_2,n_3,n_{1,4}} \\ & + \mathrm{Te}^{n_2,n_4,n_{1,3}} - \mathrm{Te}^{n_3,n_1,n_{2,4}} + \mathrm{Te}^{n_3,n_2,n_{1,4}} + \mathrm{Te}^{n_4,n_2,n_{1,3}} - \mathrm{Te}^{n_4,n_3,n_{1,2}} \\ & + \mathrm{Te}^{n_1,n_{2,3},n_4} - \mathrm{Te}^{n_1,n_{2,4},n_3} - \mathrm{Te}^{n_2,n_{1,3},n_4} + \mathrm{Te}^{n_2,n_{1,4},n_3} - \mathrm{Te}^{n_3,n_{1,4},n_2} \\ & + \mathrm{Te}^{n_3,n_{2,4},n_1} + \mathrm{Te}^{n_4,n_{1,3},n_2} - \mathrm{Te}^{n_4,n_{2,3},n_1} + \mathrm{Te}^{n_{1,2},n_3,n_4} - \mathrm{Te}^{n_{1,3},n_2,n_4} \\ & - \mathrm{Te}^{n_{1,3},n_4,n_2} + \mathrm{Te}^{n_{2,4},n_1,n_3} + \mathrm{Te}^{n_{2,4},n_3,n_1} - \mathrm{Te}^{n_{1,4},n_2,n_3} + \mathrm{Te}^{n_{1,4},n_3,n_2} \\ & - \mathrm{Te}^{n_{3,4},n_2,n_1} + \frac{1}{2}\,\mathrm{Te}^{n_{1,2},n_{3,4}} - \mathrm{Te}^{n_{1,3},n_{2,4}} + \mathrm{Te}^{n_{2,4},n_{1,3}} - \frac{1}{2}\,\mathrm{Te}^{n_{3,4},n_{1,2}} \end{aligned}$$

$$
\begin{aligned}
&8\ \mathrm{Too}^{n_1,n_2,n_3,n_4} = \\
&+\mathrm{Te}^{n_1,n_2,n_3,n_4} - \mathrm{Te}^{n_1,n_2,n_4,n_3} - \mathrm{Te}^{n_1,n_3,n_2,n_4} - \mathrm{Te}^{n_1,n_3,n_4,n_2} - \mathrm{Te}^{n_1,n_4,n_2,n_3} \\
&+\mathrm{Te}^{n_1,n_4,n_3,n_2} - \mathrm{Te}^{n_2,n_1,n_3,n_4} + \mathrm{Te}^{n_2,n_1,n_4,n_3} - \mathrm{Te}^{n_2,n_3,n_1,n_4} - \mathrm{Te}^{n_2,n_3,n_4,n_1} \\
&+\mathrm{Te}^{n_2,n_4,n_1,n_3} + \mathrm{Te}^{n_2,n_4,n_3,n_1} - \mathrm{Te}^{n_3,n_1,n_2,n_4} - \mathrm{Te}^{n_3,n_1,n_4,n_2} + \mathrm{Te}^{n_3,n_2,n_1,n_4} \\
&+\mathrm{Te}^{n_3,n_2,n_4,n_1} - \mathrm{Te}^{n_3,n_4,n_1,n_2} + \mathrm{Te}^{n_3,n_4,n_2,n_1} - \mathrm{Te}^{n_4,n_1,n_2,n_3} + \mathrm{Te}^{n_4,n_1,n_3,n_2} \\
&+\mathrm{Te}^{n_4,n_2,n_1,n_3} + \mathrm{Te}^{n_4,n_2,n_3,n_1} + \mathrm{Te}^{n_4,n_3,n_1,n_2} - \mathrm{Te}^{n_4,n_3,n_2,n_1} - \mathrm{Te}^{n_{1,3},n_2,n_4} \\
&-\mathrm{Te}^{n_{1,3},n_4,n_2} - \mathrm{Te}^{n_2,n_{1,3},n_4} + \mathrm{Te}^{n_4,n_{1,3},n_2} + \mathrm{Te}^{n_2,n_4,n_{1,3}} + \mathrm{Te}^{n_4,n_2,n_{1,3}} \\
&+\mathrm{Te}^{n_{2,4},n_1,n_3} + \mathrm{Te}^{n_{2,4},n_3,n_1} - \mathrm{Te}^{n_1,n_{2,4},n_3} + \mathrm{Te}^{n_3,n_{2,4},n_1} - \mathrm{Te}^{n_1,n_3,n_{2,4}} \\
&-\mathrm{Te}^{n_3,n_1,n_{2,4}} - \mathrm{Te}^{n_{1,4},n_2,n_3} + \mathrm{Te}^{n_{1,4},n_3,n_2} + \mathrm{Te}^{n_2,n_{1,4},n_3} - \mathrm{Te}^{n_3,n_{1,4},n_2} \\
&-\mathrm{Te}^{n_2,n_3,n_{1,4}} + \mathrm{Te}^{n_3,n_2,n_{1,4}} + \mathrm{Te}^{n_{2,4},n_{1,3}} - \mathrm{Te}^{n_{1,3},n_{2,4}}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Taa}^{n_1,\ldots,n_5} &= 540\ \mathrm{Te}^\bullet\text{-}\textit{summands}\ , & \mathrm{Too}^{n_1,\ldots,n_5} &= 308\ \mathrm{Te}^\bullet\text{-}\textit{summands} \\
\mathrm{Taa}^{n_1,\ldots,n_6} &= 3\,688\ \mathrm{Te}^\bullet\text{-}\textit{summands}\ , & \mathrm{Too}^{n_1,\ldots,n_6} &= 2\,612\ \mathrm{Te}^\bullet\text{-}\textit{summands} \\
\mathrm{Taa}^{n_1,\ldots,n_7} &= 47\,292\ \mathrm{Te}^\bullet\text{-}\textit{summands}\ , & \mathrm{Too}^{n_1,\ldots,n_7} &= 25\,988\ \mathrm{Te}^\bullet\text{-}\textit{summands}
\end{aligned}
$$

9.2 Parity properties of alternal and olternol multitangents

We begin by comparing the number of summands in the monotangent reductions $red_1(Te^\bullet)$ and $red_1(Taa^\bullet)$ (respectively $red_2(Te^\bullet)$ and $red_2(Taa^\bullet)$) of $Te^\bullet$ and $Taa^\bullet$ *before* (respectively *after*) symmetrel linearisation of the resulting multizetas. N.B. A further reduction $red_3(Te^\bullet)$ and $red_3(Taa^\bullet)$, corresponding to a complete decomposition of the multizeta into *arithmetical irreducibles*, would yield even fewer summands.

The triplets $[\boldsymbol{N_1}, N_2, \boldsymbol{N_3}]$ of Table 2 are defined as follows. $\boldsymbol{N_1}$ is the number of summands after reduction into a sum of monotangents Te^{n_i} and symmetrel multizeta coefficients $ze^\bullet$. N_2 and $\boldsymbol{N_3}$ represent the number of summands left after taking multizeta dimorphy into account and expressing everything in terms of *multizeta irreducibles* – either plain irreducibles from $Zig^\bullet$ or even-odd irreducibles from $Zig^\bullet_{ev}$, $Zig^\bullet_{odd}$. See Section 6.2, Section 6.3. Note that N_2 is about the same as $\boldsymbol{N_1}$, but that $\boldsymbol{N_3}$ is much smaller.[98]

[98] Of course, unlike $\boldsymbol{N_1}$, which has absolute significance, N_2 and $\boldsymbol{N_3}$ depend on the particular system of irreducibles chosen for the reduction. There exist privileged systems, but we cannot go into that here. But whatever system we choose, the *average values* $\boldsymbol{N_3}$ will always be much smaller than that N_2.

Table 2.

$(n_1, \ldots, n_r)$	$\#(\mathrm{Te}^\bullet)$	$\#(\mathrm{Taa}^\bullet)$	$\#(\mathrm{Too}^\bullet)$
(2, 7, 4)	**47**, 45, **17**	**28**, 26, **8**	**15**, 15, **5**
(5, 2, 2, 4)	**40**, 39, **21**	**37**, 37, **13**	**30**, 30, **11**
(5, 3, 3, 4, 2)	**210**, 209, **69**	**294**, 289, **38**	**212**, 207, **32**
(3, 1, 2, 3, 4, 2)	**455**, 455, **33**	**491**, 488, **30**	**382**, 382, **26**
(2, 1, 2, 1, 2, 2, 3)	**220**, 203, **15**	**659**, 578, **15**	**631**, 567, **12**

. . . .

Table 2 bis: Here are the even-irreducibles and odd-irreducibles to appear in the sequel, with their expression in terms of ordinary irreducibles.

$$\begin{aligned}
\zeta^{\mathrm{ev}}_{6,2} &= \zeta_{6,2} - 3\zeta_5\zeta_3\\
\zeta^{\mathrm{ev}}_{8,2} &= \zeta_{8,2} - 4\zeta_7\zeta_3 - 2\zeta_5^2\\
\zeta^{\mathrm{ev}}_{10,2} &= \zeta_{10,2} - 5\zeta_3\zeta_9 - 5\zeta_7\zeta_5\\
\zeta^{\mathrm{odd}}_{8,1,2} &= \zeta_{8,1,2} + \zeta_{6,2}\zeta_3 - 3\zeta_5\zeta_3^2 - \frac{27}{2}\zeta_9\zeta_2 - \frac{13}{10}\zeta_7\zeta_2^2 - \frac{44}{105}\zeta_2^3\zeta_5 + \frac{72}{175}\zeta_3\zeta_2^4\\
\zeta^{\mathrm{odd}}_{9,3,1} &= \zeta_{9,3,1} + 82\zeta_{11}\zeta_2 + \frac{193}{10}\zeta_9\zeta_2^2 + \frac{8}{55}\zeta_3\zeta_2^5 + \frac{226}{35}\zeta_7\zeta_2^3 + \frac{288}{175}\zeta_5\zeta_2^4\\
\zeta^{\mathrm{odd}}_{10,2,1} &= \zeta_{10,2,1} - 28\zeta_{11}\zeta_2 - \frac{41}{5}\zeta_9\zeta_2^2 - \frac{36}{25}\zeta_5\zeta_2^4 - \frac{124}{35}\zeta_7\zeta_2^3 - \frac{208}{385}\zeta_3\zeta_2^5
\end{aligned}$$

. . . .

The following twelve examples of multitangent reduction (of type red_2) are meant to cover all situations. They illustrate the phenomenon of *parity separation* in $Taa^\bullet$ and $Too^\bullet$, and its absence in $Te^\bullet \approx Tee^\bullet$. The last examples involve irreducibles of depth 2 and 3.

Table 3 : $Te^{2,7,3}(z)$ has no definite parity in z.

$$\mathrm{Te}^{2,7,3}(z) = \sum_{2\le m\le 7} \mathrm{teze}_m^{2,7,3}\,\mathrm{Te}^m(z)$$

$$\mathrm{teze}_1^{2,7,3} = 10\mathrm{ze}^{5,6}+10\mathrm{ze}^{6,5}+35\mathrm{ze}^{8,3}+56\mathrm{ze}^{3,8}-10\mathrm{ze}^{11}-21\mathrm{ze}^{4,7} -27\mathrm{ze}^{7,4}-28\mathrm{ze}^{9,2}=0$$

$$\mathrm{teze}_2^{2,7,3} = 35\mathrm{ze}^{3,7}+36\mathrm{ze}^{7,3}+48\mathrm{ze}^{5,5}-6\mathrm{ze}^{10}-21\mathrm{ze}^{8,2}-28\mathrm{ze}^{2,8} -45\mathrm{ze}^{4,6}-45\mathrm{ze}^{6,4}=\frac{7}{2}\zeta^{\mathrm{ev}}_{8,2}+56\zeta_7\zeta_3+35\zeta_5^2-\frac{2296}{275}\zeta_2^5$$

$$\mathrm{teze}_3^{2,7,3} = 15\mathrm{ze}^{3,6}+15\mathrm{ze}^{6,3}-6\mathrm{ze}^{9}-6\mathrm{ze}^{4,5}-6\mathrm{ze}^{5,4}-14\mathrm{ze}^{2,7} -15\mathrm{ze}^{7,2}=\frac{35}{2}\zeta_9+\frac{104}{35}\zeta_3\zeta_2^3-21\zeta_7\zeta_2-4\zeta_5\zeta_2^2$$

$$\mathrm{teze}_4^{2,7,3} = 16\mathrm{ze}^{3,5}+16\mathrm{ze}^{5,3}-3\mathrm{ze}^{8}-10\mathrm{ze}^{2,6}-10\mathrm{ze}^{6,2}-18\mathrm{ze}^{4,4} = 16\zeta_5\zeta_3-\frac{652}{175}\zeta_2^4$$

$$\mathrm{teze}_5^{2,7,3} = 3\mathrm{ze}^{3,4}+3\mathrm{ze}^{4,3}-3\mathrm{ze}^{7}-6\mathrm{ze}^{2,5}-6\mathrm{ze}^{5,2}=\frac{6}{5}\zeta_3\zeta_2^2-6\zeta_5\zeta_2$$

$$\mathrm{teze}_6^{2,7,3} = 4\mathrm{ze}^{3,3}-\mathrm{ze}^{6}-3\mathrm{ze}^{2,4}-3\mathrm{ze}^{4,2}=2\zeta_3^2-\frac{6}{5}\zeta_2^3$$

$$\mathrm{teze}_7^{2,7,3} = -\mathrm{ze}^{5}-\mathrm{ze}^{2,3}-\mathrm{ze}^{3,2}=-\zeta_3\zeta_2$$

....

Table 4 : $Taa^{2,7,3}(z)$ is even in z since 2+7+3-3 is odd.

$$\mathrm{Taa}^{2,7,3}(z) = \sum_{2\le m\,\mathrm{even}\le 10} \mathrm{taaze}_m^{2,7,3}\,\mathrm{Te}^m(z)$$

$$\mathrm{taaze}_2^{2,7,3} = 35\mathrm{ze}^{3,7}+36\mathrm{ze}^{7,3}+48\mathrm{ze}^{5,5}+\frac{373}{6}\mathrm{ze}^{10}-\frac{28}{3}\mathrm{ze}^{2,8}-\frac{7}{3}\mathrm{ze}^{8,2} -15\mathrm{ze}^{4,6}-15\mathrm{ze}^{6,4}=35\zeta_5^2+56\zeta_7\zeta_3+\frac{7}{2}\zeta^{\mathrm{ev}}_{8,2}-\frac{392}{275}\zeta_2^5$$

$$\mathrm{taaze}_4^{2,7,3} = 16\mathrm{ze}^{3,5}+16\mathrm{ze}^{5,3}+\frac{29}{3}\mathrm{ze}^{8}-\frac{10}{3}\mathrm{ze}^{2,6}-6\mathrm{ze}^{4,4}-\frac{10}{3}\mathrm{ze}^{6,2} = 16\zeta_5\zeta_3-\frac{652}{525}\zeta_2^4$$

$$\mathrm{taaze}_6^{2,7,3} = 4\mathrm{ze}^{3,3}+\frac{1}{6}\mathrm{ze}^{6}-\mathrm{ze}^{2,4}-\mathrm{ze}^{4,2}=2\zeta_3^2-\frac{62}{105}\zeta_2^3$$

$$\mathrm{taaze}_8^{2,7,3} = 0$$

$$\mathrm{taaze}_{10}^{2,7,3} = \frac{1}{6}\mathrm{ze}^{2}=\frac{1}{6}\zeta_2$$

....

Table 5 : $Too^{2,7,3}(z)$ is even in z since 2+7+3-3 is odd.

$$\mathrm{Too}^{2,7,3}(z) = \sum_{2\leq m\,\mathrm{even}\leq 6} \mathrm{tooze}_m^{2,7,3}\,\mathrm{Te}^m(z)$$

$$\mathrm{tooze}_2^{2,7,3} = 7\mathrm{ze}^{8,2}+35\mathrm{ze}^{3,7}+36\mathrm{ze}^{7,3}+48\mathrm{ze}^{5,5}+105\mathrm{ze}^{10}$$

$$= 35\zeta_5^2+56\zeta_7\zeta_3+7/2\zeta_{8,2}^{\mathrm{ev}}+\frac{152}{55}\zeta_2^5$$

$$\mathrm{tooze}_4^{2,7,3} = 16\mathrm{ze}^{3,5}+16\mathrm{ze}^{5,3}+\frac{39}{2}\mathrm{ze}^8=+16\zeta_5\zeta_3]+\frac{12}{25}\zeta_2^4$$

$$\mathrm{tooze}_6^{2,7,3} = 2\mathrm{ze}^6+4\mathrm{ze}^{3,3}=2\zeta_3^2$$

. . . .

Table 6 : $Te^{2,7,4}(z)$ has no definite parity in z.

$$\mathrm{Te}^{2,7,4}(z) = \sum_{2\leq m\leq 7} \mathrm{teze}_m^{2,7,4}\,\mathrm{Te}^m(z)$$

$$\mathrm{teze}_1^{2,7,4} = 30\mathrm{ze}^{12}+84\mathrm{ze}^{4,8}+84\mathrm{ze}^{10,2}+100\mathrm{ze}^{6,6}+112\mathrm{ze}^{8,4}-104\mathrm{ze}^{7,5}$$

$$-112\mathrm{ze}^{5,7}-112\mathrm{ze}^{9,3}-168\mathrm{ze}^{3,9}=0$$

$$\mathrm{teze}_3^{2,7,4} = 14\mathrm{ze}^{10}+28\mathrm{ze}^{2,8}+35\mathrm{ze}^{8,2}+35\mathrm{ze}^{4,6}+35\mathrm{ze}^{6,4}-32\mathrm{ze}^{5,5}$$

$$-40\mathrm{ze}^{7,3}-42\mathrm{ze}^{3,7}=\frac{992}{175}\zeta_2^5-8\zeta_5^2-28\zeta_7\zeta_3$$

$$\mathrm{teze}_4^{2,7,4} = 8\mathrm{ze}^9+8\mathrm{ze}^{4,5}+8\mathrm{ze}^{5,4}+20\mathrm{ze}^{7,2}+21\mathrm{ze}^{2,7}-20\mathrm{ze}^{3,6}-20\mathrm{ze}^{6,3}$$

$$= 14\zeta_7\zeta_2+\frac{8}{5}\zeta_5\zeta_2^2+\frac{35}{2}\zeta_9-\frac{176}{35}\zeta_3\zeta_2^3$$

$$\mathrm{teze}_5^{2,7,4} = 5\mathrm{ze}^8+6\mathrm{ze}^{4,4}+10\mathrm{ze}^{2,6}+10\mathrm{ze}^{6,2}-8\mathrm{ze}^{3,5}-8\mathrm{ze}^{5,3}$$

$$= \frac{484}{175}\zeta_2^4-8\zeta_5\zeta_3$$

$$\mathrm{teze}_6^{2,7,4} = 2\mathrm{ze}^7+4\mathrm{ze}^{2,5}+4\mathrm{ze}^{5,2}-2\mathrm{ze}^{3,4}-2\mathrm{ze}^{4,3}=4\zeta_5\zeta_2-\frac{4}{5}\zeta_3\zeta_2^2$$

$$\mathrm{teze}_7^{2,7,4} = \mathrm{ze}^6+\mathrm{ze}^{2,4}+\mathrm{ze}^{4,2}=\frac{2}{5}\zeta_2^3$$

. . . .

Table 7 : $Taa^{2,7,4}(z)$ is odd in z since 2+7+4-3 is even.

$$\mathrm{Taa}^{2,7,4}(z) = \sum_{2\leq m\,\mathrm{odd}\leq 11} \mathrm{taaze}_m^{2,7,4}\mathrm{Te}^m(z)$$

$$\mathrm{taaze}_1^{2,7,4} = 28\mathrm{ze}^{4,8}+36\mathrm{ze}^{12}+56\mathrm{ze}^{8,4}+84\mathrm{ze}^{10,2}+\frac{100}{3}\mathrm{ze}^{6,6}-104\mathrm{ze}^{7,5} -112\mathrm{ze}^{5,7}-112\mathrm{ze}^{9,3}-168\mathrm{ze}^{3,9}=0$$

$$\mathrm{taaze}_3^{2,7,4} = 11\mathrm{ze}^{10}+\frac{28}{3}\mathrm{ze}^{2,8}+\frac{35}{3}\mathrm{ze}^{4,6}+\frac{35}{3}\mathrm{ze}^{6,4}+\frac{49}{3}\mathrm{ze}^{8,2}-32\mathrm{ze}^{5,5} -40\mathrm{ze}^{7,3}-42\mathrm{ze}^{3,7}=\frac{24352}{5775}\zeta_2^5-8\zeta_5^2-28\zeta_7\zeta_3$$

$$\mathrm{taaze}_5^{2,7,4} = 2\mathrm{ze}^{4,4}+\frac{10}{3}\mathrm{ze}^{2,6}+\frac{10}{3}\mathrm{ze}^{6,2}+\frac{17}{3}\mathrm{ze}^{8}-8\mathrm{ze}^{3,5}-8\mathrm{ze}^{5,3} =\frac{1156}{525}\zeta_2^4-8\zeta_5\zeta_3$$

$$\mathrm{taaze}_7^{2,7,4} = \frac{1}{3}\mathrm{ze}^{2,4}+\frac{1}{3}\mathrm{ze}^{4,2}+\frac{17}{6}\mathrm{ze}^{6}=\frac{74}{105}\zeta_2^3$$

$$\mathrm{taaze}_9^{2,7,4} = \frac{2}{3}\mathrm{ze}^{4}=\frac{4}{15}\zeta_2^2$$

$$\mathrm{taaze}_{11}^{2,7,4} = \frac{1}{6}\mathrm{ze}^{2}=\frac{1}{6}\zeta_2$$

. . . .

Table 8 : $Too^{2,7,4}(z)$ is odd in z since 2+7+4-3 is even.

$$\mathrm{Too}^{2,7,4}(z) = \sum_{3\leq m\,\mathrm{odd}\leq 5} \mathrm{tooze}_m^{2,7,4}\mathrm{Te}^m(z)$$

$$\mathrm{tooze}_1^{2,7,4} = 39\mathrm{ze}^{12}+28\mathrm{ze}^{8,4}+84\mathrm{ze}^{10,2}-104\mathrm{ze}^{7,5}-112\mathrm{ze}^{5,7}-112\mathrm{ze}^{9,3} -168\mathrm{ze}^{3,9}=0$$

$$\mathrm{tooze}_3^{2,7,4} = 7\mathrm{ze}^{8,2}-\frac{23}{2}\mathrm{ze}^{10}-32\mathrm{ze}^{5,5}-40\mathrm{ze}^{7,3}-42\mathrm{ze}^{3,7} =\frac{96}{55}\zeta_2^5-8\zeta_5^2-28\zeta_7\zeta_3$$

$$\mathrm{tooze}_5^{2,7,4} = -8\mathrm{ze}^{3,5}-8\mathrm{ze}^{5,3}-\frac{9}{2}\mathrm{ze}^{8}=\frac{12}{25}\zeta_2^4-8\zeta_5\zeta_3$$

. . . .

Table 9 : $Te^{5,3,3,4}(z)$ has no definite parity in z.

$$\mathrm{Te}^{5,3,3,4}(z) = \sum_{2\le m\le 5} \mathrm{teze}_m^{5,3,3,4}\,\mathrm{Te}^{m}(z)$$

$$\begin{aligned}
\mathrm{teze}_1^{5,3,3,4} &= 6\mathrm{ze}^{10,4}+12\mathrm{ze}^{5,9}+15\mathrm{ze}^{7,7}+12\mathrm{ze}^{5,5,4}+15\mathrm{ze}^{7,4,3}+15\mathrm{ze}^{4,7,3}\\
&\quad +30\mathrm{ze}^{6,5,3}+30\mathrm{ze}^{5,6,3}+30\mathrm{ze}^{5,4,5}+30\mathrm{ze}^{4,5,5}+30\mathrm{ze}^{7,3,4}\\
&\quad +60\mathrm{ze}^{4,6,4}+60\mathrm{ze}^{5,3,6}+45\mathrm{ze}^{4,4,6}+90\mathrm{ze}^{6,4,4}-15\mathrm{ze}^{6,8}\\
&= -6\mathrm{ze}^{4,10}=0\\
\mathrm{teze}_2^{5,3,3,4} &= 2\mathrm{ze}^{4,9}+10\mathrm{ze}^{4,6,3}+12\mathrm{ze}^{4,5,4}+15\mathrm{ze}^{4,4,5}+15\mathrm{ze}^{4,3,6}+30\mathrm{ze}^{5,3,5}\\
&\quad +30\mathrm{ze}^{5,5,3}+35\mathrm{ze}^{7,3,3}+36\mathrm{ze}^{5,4,4}+40\mathrm{ze}^{6,3,4}+45\mathrm{ze}^{6,4,3}-3\mathrm{ze}^{5,8}\\
&\quad -5\mathrm{ze}^{6,7}-6\mathrm{ze}^{9,4}=\frac{240}{7}\zeta_7\zeta_2^3-72\zeta_9\zeta_2^2-175\zeta_{6,2}^{\mathrm{ev}}\zeta_5-775\zeta_5^2\zeta_3\\
&\quad -600\zeta_7\zeta_3^2-200\zeta_{9,3,1}^{\mathrm{odd}}-700\zeta_{10,2,1}^{\mathrm{odd}}-\frac{71900}{3}\zeta_{13}-\frac{3198}{35}\zeta_5\zeta_2^4\\
\mathrm{teze}_3^{5,3,3,4} &= \mathrm{ze}^{5,7}+5\mathrm{ze}^{6,3,3}+5\mathrm{ze}^{4,3,5}+6\mathrm{ze}^{4,5,3}+9\mathrm{ze}^{5,4,3}+10\mathrm{ze}^{5,3,4}\\
&\quad +9\mathrm{ze}^{4,4,4}-\mathrm{ze}^{4,8}=14\zeta_{6,2}^{\mathrm{ev}}\zeta_2^2+14\zeta_5\zeta_3\zeta_2^2+15\zeta_{10,2}^{\mathrm{ev}}+45\zeta_9\zeta_3\\
&\quad +55\zeta_7\zeta_5+\frac{10576684}{875875}\zeta_2^6-50\zeta_5^2\zeta_2\\
\mathrm{teze}_4^{5,3,3,4} &= 3\mathrm{ze}^{4,3,4}+3\mathrm{ze}^{4,4,3}+5\mathrm{ze}^{5,3,3}=\frac{35}{2}\zeta_5\zeta_3^2+\frac{35}{4}\zeta_{8,1,2}^{\mathrm{odd}}+\frac{72}{5}\zeta_7\zeta_2^2\\
&\quad +\frac{29893}{96}\zeta_{11}-45\zeta_9\zeta_2-\frac{80}{7}\zeta_5\zeta_2^3\\
\mathrm{teze}_5^{5,3,3,4} &= \mathrm{ze}^{4,3,3}=10\zeta_5\zeta_3\zeta_2+7\zeta_7\zeta_3+\frac{12932}{1925}\zeta_2^5+\frac{7}{2}\zeta_{8,2}^{\mathrm{ev}}+10\zeta_{6,2}^{\mathrm{ev}}\zeta_2-\frac{45}{2}\zeta_5^2
\end{aligned}$$

. . . .

Table 10 : $Taa^{5,3,3,4}(z)$ is even in z since 5+3+3+4-4 is odd.

$$\mathrm{Taa}^{5,3,3,4}(z) = \sum_{2\leq m\,\mathrm{even}\leq 8} \mathrm{taaze}_m^{5,3,3,4}\mathrm{Te}^m(z)$$

$$\begin{aligned}
\mathrm{taaze}_2^{5,3,3,4} &= 5\mathrm{ze}^{4,3,6}+22\mathrm{ze}^{5,8}+30\mathrm{ze}^{5,5,3}+30\mathrm{ze}^{5,3,5}+35\mathrm{ze}^{7,3,3}+\frac{25}{3}\mathrm{ze}^{6,4,3}\\
&\quad+\frac{40}{3}\mathrm{ze}^{6,3,4}+\frac{184}{3}\mathrm{ze}^{9,4}+\frac{295}{3}\mathrm{ze}^{6,7}+\frac{260}{3}\mathrm{ze}^{7,6}+\frac{323}{3}\mathrm{ze}^{4,9}\\
&\quad+\frac{291}{2}\mathrm{ze}^{13}-16\mathrm{ze}^{5,4,4}-24\mathrm{ze}^{4,5,4}-5\mathrm{ze}^{4,4,5}-40\mathrm{ze}^{8,5}\\
&\quad-\frac{35}{3}\mathrm{ze}^{10,3}-\frac{80}{3}\mathrm{ze}^{4,6,3}=-175\zeta_{6,2}^{\mathrm{ev}}\zeta_5-200\zeta_{9,3,1}^{\mathrm{odd}}-700\zeta_{10,2,1}^{\mathrm{odd}}\\
&\quad-600\zeta_7\zeta_3^2-775\zeta_5^2\zeta_3-\frac{3102}{35}\zeta_5\zeta_2^4-\frac{71614}{3}\zeta_{13}\\
\mathrm{taaze}_4^{5,3,3,4} &= \mathrm{ze}^{4,3,4}+35\mathrm{ze}^{4,7}+\frac{41}{2}\mathrm{ze}^{7,4}+\frac{55}{6}\mathrm{ze}^{5,6}+\frac{155}{6}\mathrm{ze}^{11}-\frac{29}{3}\mathrm{ze}^{8,3}\\
&\quad+5\mathrm{ze}^{5,3,3}-\mathrm{ze}^{4,4,3}=\frac{35}{4}\zeta_{8,1,2}^{\mathrm{odd}}+\frac{35}{2}\zeta_5\zeta_3^2+\frac{8967}{32}\zeta_{11}-\frac{124}{21}\zeta_5\zeta_2^3\\
\mathrm{taaze}_6^{5,3,3,4} &= \frac{8}{3}\mathrm{ze}^{5,4}+\frac{25}{6}\mathrm{ze}^{4,5}+\frac{13}{6}\mathrm{ze}^{9}-\frac{5}{2}\mathrm{ze}^{6,3}=\frac{14}{3}\zeta_5\zeta_2^2-\frac{21}{2}\zeta_9\\
\mathrm{taaze}_8^{5,3,3,4} &= -\frac{1}{6}\mathrm{ze}^{4,3}-\frac{1}{12}\mathrm{ze}^{7}=\frac{5}{3}\zeta_2\zeta_5-\frac{35}{12}\zeta_7
\end{aligned}$$

. . . .

Table 11 : $Too^{5,3,3,4}(z)$ is even in z since 5+3+3+4-4 is odd.

$$\mathrm{Too}^{5,3,3,4}(z) = \sum_{2\leq m\,even\leq 6} \mathrm{tooze}_m^{5,3,3,4}\mathrm{Te}^m(z)$$

$$\begin{aligned}
\mathrm{tooze}_2^{5,3,3,4} &= 5\mathrm{ze}^{10,3}+30\mathrm{ze}^{5,3,5}+30\mathrm{ze}^{5,5,3}+35\mathrm{ze}^{7,3,3}+60\mathrm{ze}^{8,5}+138\mathrm{ze}^{4,9}\\
&\quad+147\mathrm{ze}^{9,4}+170\mathrm{ze}^{6,7}+\frac{123}{2}\mathrm{ze}^{5,8}+\frac{385}{2}\mathrm{ze}^{7,6}+\frac{861}{2}\mathrm{ze}^{13}\\
&\quad-42\mathrm{ze}^{5,4,4}-42\mathrm{ze}^{4,5,4}-15\mathrm{ze}^{4,4,5}-10\mathrm{ze}^{6,4,3}-45\mathrm{ze}^{4,6,3}\\
&= -775\zeta_3\zeta_5^2-200\zeta_{9,3,1}^{\mathrm{odd}}-700\zeta_{10,2,1}^{\mathrm{odd}}-175\zeta_{6,2}^{\mathrm{ev}}\zeta_5-\frac{306}{5}\zeta_5\zeta_2^4\\
&\quad-600\zeta_7\zeta_3^2-\frac{285455}{12}\zeta_{13}\\
\mathrm{tooze}_4^{5,3,3,4} &= \mathrm{ze}^{8,3}+25\mathrm{ze}^{6,5}+51\mathrm{ze}^{4,7}+\frac{55}{2}\mathrm{ze}^{5,6}+\frac{105}{2}\mathrm{ze}^{7,4}+\frac{315}{4}\mathrm{ze}^{11}\\
&\quad+5\mathrm{ze}^{5,3,3}-3\mathrm{ze}^{4,4,3}=\frac{35}{4}\zeta_{8,1,2}^{\mathrm{odd}}+\frac{29629}{96}\zeta_{11}+\frac{35}{2}\zeta_5\zeta_3^2\\
\mathrm{tooze}_6^{5,3,3,4} &= \frac{15}{2}\mathrm{ze}^{5,4}+\frac{15}{2}\mathrm{ze}^{4,5}+\frac{15}{2}\mathrm{ze}^{9}=3\zeta_5\zeta_2^2
\end{aligned}$$

. . . .

Table 12 : $Te^{5,2,3,4}$(z) has no definite parity in z.

$$\mathrm{Te}^{5,2,3,4}(z) = \sum_{2\le m\le 5} \mathrm{teze}_m^{5,2,3,4}\mathrm{Te}^m(z)$$

$$\begin{aligned}
\mathrm{teze}_1^{5,2,3,4} &= \mathrm{ze}^{5,8}+2\mathrm{ze}^{4,9}+3\mathrm{ze}^{10,3}+5\mathrm{ze}^{11,2}+15\mathrm{ze}^{7,6}-10\mathrm{ze}^{6,7}-5\mathrm{ze}^{4,6,3}\\
&\quad -15\mathrm{ze}^{4,4,5}-35\mathrm{ze}^{8,3,2}-35\mathrm{ze}^{8,2,3}-42\mathrm{ze}^{5,4,4}-18\mathrm{ze}^{4,5,4}\\
&\quad -30\mathrm{ze}^{7,2,4}-40\mathrm{ze}^{5,3,5}-45\mathrm{ze}^{7,4,2}-50\mathrm{ze}^{6,3,4}-50\mathrm{ze}^{5,6,2}\\
&\quad -50\mathrm{ze}^{6,2,5}-60\mathrm{ze}^{6,4,3}-70\mathrm{ze}^{7,3,3}-70\mathrm{ze}^{6,5,2}-76\mathrm{ze}^{5,5,3}=0\\
\mathrm{teze}_2^{5,2,3,4} &= 2\mathrm{ze}^{5,7}+10\mathrm{ze}^{6,2,4}+10\mathrm{ze}^{5,2,5}+15\mathrm{ze}^{7,2,3}-\mathrm{ze}^{4,8}-\mathrm{ze}^{10,2}-5\mathrm{ze}^{6,6}\\
&\quad -3\mathrm{ze}^{9,3}-5\mathrm{ze}^{4,3,5}-8\mathrm{ze}^{5,3,4}-9\mathrm{ze}^{4,4,4}-10\mathrm{ze}^{6,3,3}-15\mathrm{ze}^{4,5,3}\\
&\quad -15\mathrm{ze}^{4,6,2}-16\mathrm{ze}^{5,5,2}-20\mathrm{ze}^{7,3,2}-24\mathrm{ze}^{5,4,3}-35\mathrm{ze}^{6,4,2}\\
&= 16\zeta^{\mathrm{ev}}_{6,2}\zeta_2^2+35\zeta_7\zeta_5+100\zeta_5^2\zeta_2+105\zeta_9\zeta_3-35\zeta^{\mathrm{ev}}_{10,2}-16\zeta_3^2\zeta_2^3\\
&\quad -\frac{12462448}{525525}\zeta_2^6\\
\mathrm{teze}_3^{5,2,3,4} &= \mathrm{ze}^{5,6}+\mathrm{ze}^{9,2}-2\mathrm{ze}^{5,2,4}-3\mathrm{ze}^{4,3,4}-5\mathrm{ze}^{6,2,3}-5\mathrm{ze}^{4,5,2}-6\mathrm{ze}^{4,4,3}\\
&\quad -10\mathrm{ze}^{6,3,2}-10\mathrm{ze}^{5,3,3}-11\mathrm{ze}^{5,4,2}=8\zeta_5\zeta_2^3+60\zeta^{\mathrm{ev}}_{6,2}\zeta_3\\
&\quad +\frac{4136}{175}\zeta_3\zeta_2^4-30\zeta_5\zeta_3^2-40\zeta^{\mathrm{odd}}_{8,1,2}-\frac{112}{5}\zeta_7\zeta_2^2-\frac{3040}{3}\zeta_{11}\\
\mathrm{teze}_4^{5,2,3,4} &= \mathrm{ze}^{5,2,3}-2\mathrm{ze}^{4,3,3}-3\mathrm{ze}^{4,4,2}-4\mathrm{ze}^{5,3,2}=10\zeta^{\mathrm{ev}}_{6,2}\zeta_2+\frac{21}{2}\zeta_7\zeta_3\\
&\quad +\frac{105}{4}\zeta_5^2-4\zeta_3^2\zeta_2^2-\frac{63}{4}\zeta^{\mathrm{ev}}_{8,2}-\frac{1696}{275}\zeta_2^5\\
\mathrm{teze}_5^{5,2,3,4} &= -Ze^{4,3,2}=7\zeta_5\zeta_2^2+\frac{53}{36}\zeta_9+\frac{64}{105}\zeta_3\zeta_2^3-14\zeta_7\zeta_2-\frac{2}{3}\zeta_3^3
\end{aligned}$$

....

Table 13 : $Taa^{5,2,3,4}(z)$ is odd in z since 5+2+3+4-4 is even.

$$\mathrm{Taa}^{5,2,3,4}(z) = \sum_{3\le m\,\mathrm{odd}\le 9} \mathrm{taaze}_m^{5,2,3,4}\mathrm{Te}^m(z)$$

$$\begin{aligned}
\mathrm{taaze}_1^{5,2,3,4} &= 5\mathrm{ze}^{4,4,5}+10\mathrm{ze}^{4,3,6}+18\mathrm{ze}^{5,4,4}+22\mathrm{ze}^{4,5,4}+30\mathrm{ze}^{7,2,4}\\
&\quad +15\mathrm{ze}^{7,4,2}+20\mathrm{ze}^{2,5,6}+30\mathrm{ze}^{2,7,4}+30\mathrm{ze}^{4,7,2}+40\mathrm{ze}^{5,2,6}\\
&\quad +\frac{70}{3}\mathrm{ze}^{2,8,3}+\frac{100}{3}\mathrm{ze}^{2,6,5}+\frac{145}{3}\mathrm{ze}^{4,6,3}+\frac{8}{3}\mathrm{ze}^{10,3}+\frac{80}{3}\mathrm{ze}^{7,6}\\
&\quad +\frac{176}{3}\mathrm{ze}^{4,9}+\frac{238}{3}\mathrm{ze}^{9,4}-10\mathrm{ze}^{5,6,2}-\frac{5}{3}\mathrm{ze}^{11,2}-\frac{5}{3}\mathrm{ze}^{2,11}\\
&\quad -\frac{11}{6}\mathrm{ze}^{13}-\frac{20}{3}\mathrm{ze}^{6,4,3}-\frac{20}{3}\mathrm{ze}^{6,3,4}-40\mathrm{ze}^{5,3,5}-70\mathrm{ze}^{7,3,3}\\
&\quad -76\mathrm{ze}^{5,5,3}-\frac{35}{3}\mathrm{ze}^{8,3,2}-\frac{35}{3}\mathrm{ze}^{8,2,3}-\frac{50}{3}\mathrm{ze}^{6,5,2}-\frac{50}{3}\mathrm{ze}^{6,2,5}\\
&\quad -\frac{70}{3}\mathrm{ze}^{6,7}-\frac{115}{3}\mathrm{ze}^{8,5}-\frac{200}{3}\mathrm{ze}^{5,8}=0\\
\mathrm{taaze}_3^{5,2,3,4} &= Ze^{4,3,4}+2\mathrm{ze}^{4,4,3}+4\mathrm{ze}^{2,5,4}+\frac{43}{3}\mathrm{ze}^{8,3}+\frac{1}{3}\mathrm{ze}^{4,5,2}+\frac{10}{3}\mathrm{ze}^{2,6,3}\\
&\quad +\frac{22}{3}\mathrm{ze}^{5,2,4}+\frac{7}{6}\mathrm{ze}^{7,4}-26\mathrm{ze}^{2,9}-10\mathrm{ze}^{5,3,3}-\frac{5}{3}\mathrm{ze}^{5,4,2}-\frac{5}{3}\mathrm{ze}^{6,2,3}\\
&\quad -\frac{10}{3}\mathrm{ze}^{6,3,2}-\frac{10}{3}\mathrm{ze}^{2,3,6}-\frac{28}{3}\mathrm{ze}^{9,2}-\frac{37}{3}\mathrm{ze}^{4,7}-\frac{44}{3}\mathrm{ze}^{5,6}\\
&\quad -\frac{65}{6}\mathrm{ze}^{6,5}-\frac{169}{6}\mathrm{ze}^{11}\\
&= 60\zeta^{\mathrm{ev}}_{6,2}\zeta_3+\frac{15112}{525}\zeta_3\zeta_2^4-\frac{16}{3}\zeta_5\zeta_2^3-40\zeta^{\mathrm{odd}}_{8,1,2}-30\zeta_5\zeta_3^2-1105\zeta_{11}\\
\mathrm{taaze}_5^{5,2,3,4} &= 5\mathrm{ze}^{6,3}-6\mathrm{ze}^{4,5}-12\mathrm{ze}^{2,7}-\frac{1}{3}\mathrm{ze}^{4,3,2}-\frac{2}{3}\mathrm{ze}^{2,3,4}-\frac{13}{3}\mathrm{ze}^{7,2}\\
&\quad -\frac{14}{3}\mathrm{ze}^{5,4}-\frac{32}{3}\mathrm{ze}^{9}=\frac{152}{35}\zeta_3\zeta_2^3-\frac{1}{3}\zeta_5\zeta_2^2-\frac{2}{3}\zeta_3^3-\frac{1447}{36}\zeta_9\\
\mathrm{taaze}_7^{5,2,3,4} &= \frac{5}{6}\mathrm{ze}^{4,3}-\frac{10}{3}\mathrm{ze}^{2,5}-\frac{11}{6}\mathrm{ze}^{7}-\frac{7}{6}\mathrm{ze}^{5,2}=\frac{26}{15}\zeta_3\zeta_2^2-\frac{5}{6}\zeta_5\zeta_2-\frac{49}{6}\zeta_7\\
\mathrm{taaze}_9^{5,2,3,4} &= -\frac{1}{6}\mathrm{ze}^{5}-\frac{1}{3}\mathrm{ze}^{2,3}=\frac{2}{3}\zeta_3\zeta_2-\frac{5}{3}\zeta_5
\end{aligned}$$

. . . .

Table 14 : *Too*$^{5,2,3,4}(z)$ is odd in z since $5+2+3+4-4$ is even.

$$
\begin{aligned}
\mathrm{Too}^{5,2,3,4}(z) &= \sum_{3\leq m \text{ odd} \leq 9} \mathrm{tooze}_m^{5,2,3,4}\mathrm{Te}^m(z)\\
\mathrm{tooze}_1^{5,2,3,4} &= 10\mathrm{ze}^{6,5,2}+10\mathrm{ze}^{5,6,2}+15\mathrm{ze}^{4,4,5}+15\mathrm{ze}^{4,3,6}+15\mathrm{ze}^{6,3,4}\\
&\quad +20\mathrm{ze}^{6,4,3}+30\mathrm{ze}^{2,5,6}+35\mathrm{ze}^{2,8,3}+42\mathrm{ze}^{4,5,4}+45\mathrm{ze}^{4,7,2}\\
&\quad +45\mathrm{ze}^{7,4,2}+45\mathrm{ze}^{2,7,4}+48\mathrm{ze}^{5,4,4}+50\mathrm{ze}^{2,6,5}+60\mathrm{ze}^{7,2,4}\\
&\quad +60\mathrm{ze}^{5,2,6}+75\mathrm{ze}^{4,6,3}+30\mathrm{ze}^{8,5}+40\mathrm{ze}^{6,7}+87\mathrm{ze}^{4,9}+95\mathrm{ze}^{7,6}\\
&\quad +126\mathrm{ze}^{9,4}+\frac{47}{2}\mathrm{ze}^{10,3}-\frac{5}{2}\mathrm{ze}^{5,8}-5\mathrm{ze}^{11,2}-\frac{5}{2}\mathrm{ze}^{2,11}-\frac{17}{2}\mathrm{ze}^{13}\\
&\quad -40\mathrm{ze}^{5,3,5}-70\mathrm{ze}^{7,3,3}-76\mathrm{ze}^{5,5,3}=0\\
\mathrm{tooze}_3^{5,2,3,4} &= 3\mathrm{ze}^{7,4}+18\mathrm{ze}^{8,3}+3\mathrm{ze}^{4,3,4}+3\mathrm{ze}^{5,4,2}+3\mathrm{ze}^{4,5,2}+5\mathrm{ze}^{2,6,3}\\
&\quad +6\mathrm{ze}^{2,5,4}+6\mathrm{ze}^{4,4,3}+12\mathrm{ze}^{5,2,4}-5\mathrm{ze}^{2,3,6}-10\mathrm{ze}^{5,3,3}-5\mathrm{ze}^{5,6}\\
&\quad -15\mathrm{ze}^{4,7}-25\mathrm{ze}^{9,2}-39\mathrm{ze}^{2,9}-\frac{175}{4}\mathrm{ze}^{11}\\
&= 60\zeta^{\mathrm{ev}}_{6,2}\zeta_3+\frac{712}{25}\zeta_3\zeta_2^4-30\zeta_5\zeta_3^2-40\zeta^{\mathrm{odd}}_{8,1,2}-\frac{104}{7}\zeta_5\zeta_2^3-\frac{12985}{12}\zeta_{11}\\
\mathrm{tooze}_5^{5,2,3,4} &= \frac{15}{2}\mathrm{ze}^{6,3}-\mathrm{ze}^{2,3,4}-5\mathrm{ze}^{5,4}-18\mathrm{ze}^{2,7}-10\mathrm{ze}^{7,2}-\frac{15}{2}\mathrm{ze}^{4,5}-\frac{65}{4}\mathrm{ze}^{9}\\
&= 484/105\zeta_3\zeta_2^3-8\zeta_5\zeta_2^2-\frac{2}{3}\zeta_3^3-\frac{268}{9}\zeta_9\\
\mathrm{tooze}_7^{5,2,3,4} &= \frac{3}{2}\mathrm{ze}^{4,3}-2\mathrm{ze}^{5,2}-5\mathrm{ze}^{2,5}-\frac{11}{4}\mathrm{ze}^{7}=-5\zeta_5\zeta_2-\frac{21}{4}\zeta_7+\frac{12}{5}\zeta_3\zeta_2^2\\
\mathrm{tooze}_9^{5,2,3,4} &= -\frac{1}{4}\mathrm{ze}^{5}-\frac{1}{2}\mathrm{ze}^{2,3}=\zeta_3\zeta_2-\frac{5}{2}\zeta_5
\end{aligned}
$$

9.3 The invariants as entire functions of f: the general case

We write down, up to weight 10 inclusively, the expansions of the collectors $\mathfrak{p}$, $\mathfrak{p}_*$, $\mathfrak{p}_\sharp$ in terms of the g, g_*, $g_\sharp$. We assume $p(f)=1$ but impose no restriction on $\rho(f)\equiv -g_2$. In these and all further examples, we order the terms according to their total weight and, within a given total weight, we start with the lowest monotangents.

Table 15: $\mathfrak{p} = id + \sum \mathfrak{P}_s$ up to weight 10 with $f = l \circ g$, $g(z) = z + \sum_{2 \le s} g_s\, z^{1-s}$.

$$
\begin{aligned}
\mathfrak{P}_2 &= \mathbf{Te}^1 g_2, \mathfrak{P}_3 = \mathbf{Te}^2 g_3, \mathfrak{P}_4 = \mathbf{Te}^3 g_4, \mathfrak{P}_5 = \mathbf{Te}^4 g_5,\\
\mathfrak{P}_6 &= \mathbf{Te}^2 \big[3\zeta_3 g_2^3 + 6\zeta_3 g_4 g_2 - 6\zeta_3 g_3^2\big] + \mathbf{Te}^3 \big[2\zeta_2 g_4 g_2 - 2\zeta_2 g_3^2\big] + \mathbf{Te}^5 g_6,\\
\mathfrak{P}_7 &= \mathbf{Te}^3 \big[6\zeta_3 g_2^2 g_3 + 6\zeta_3 g_5 g_2 - 6\zeta_3 g_4 g_3\big] + \mathbf{Te}^4 \big[3\zeta_2 g_5 g_2 - 3\zeta_2 g_4 g_3\big] + \mathbf{Te}^6 g_7,\\
\mathfrak{P}_8 &= \mathbf{Te}^2 \big[10\zeta_5 g_2^4 + 10\zeta_5 g_6 g_2 + 30\zeta_5 g_4^2 - 40\zeta_5 g_5 g_3 + 50\zeta_5 g_4 g_2^2 - 50\zeta_5 g_2 g_3^2\big[\\
&\quad + \mathbf{Te}^3 \big[\frac{4}{5}\zeta_2^2 g_6 g_2 + \frac{12}{5}\zeta_2^2 g_4^2 + \frac{16}{5}\zeta_2^2 g_4 g_2^2 - \frac{2}{5}\zeta_2^2 g_2^4 - \frac{16}{5}\zeta_2^2 g_2 g_3^2\\
&\quad - \frac{16}{5}\zeta_2^2 g_5 g_3\big] + \mathbf{Te}^4 \big[\zeta_3 g_2 g_3^2 + 3\zeta_3 g_4^2 + 7\zeta_3 g_6 g_2 + 8\zeta_3 g_4 g_2^2 - 10\zeta_3 g_5 g_3\big]\\
&\quad + \mathbf{Te}^5 \big[4\zeta_2 g_6 g_2 - 4\zeta_2 g_5 g_3\big] + \mathbf{Te}^7 g_8,\\
\mathfrak{P}_9 &= \mathbf{Te}^2 \big[\frac{16}{7}\zeta_2^3 g_5 g_2^2 + \frac{32}{7}\zeta_2^3 g_3^3 - \frac{48}{7}\zeta_2^3 g_4 g_3 g_2 + 18\zeta_3^2 g_5 g_2^2 + 36\zeta_3^2 g_3^3\\
&\quad - 54\zeta_3^2 g_4, g_3 g_2\big] + \mathbf{Te}^3 \big[10\zeta_5 g_2 g_7 + 20\zeta_5 g_2^3 g_3 + 20\zeta_5 g_5 g_4 + 35\zeta_5 g_5 g_2^2\\
&\quad - 5\zeta_5 g_4 g_3 g_2 - 30\zeta_5 g_3^3 - 30\zeta_5 g_6 g_3 + 12\zeta_3\zeta_2 g_5 g_2^2 + 24\zeta_3\zeta_2 g_3^3\\
&\quad - 36\zeta_3\zeta_2 g_4 g_2 g_3\big] + \mathbf{Te}^4 \big[\frac{6}{5}\zeta_2^2 g_2 g_7 + \frac{12}{5}\zeta_2^2 g_5 g_4 + \frac{21}{5}\zeta_2^2 g_3^3 + \frac{69}{10}\zeta_2^2 g_5 g_2^2\\
&\quad - \frac{6}{5}\zeta_2^2 g_3 g_2^3 - \frac{18}{5}\zeta_2^2 g_6 g_3 - \frac{111}{10}\zeta_2^2 g_4 g_3 g_2\big] + \mathbf{Te}^5 \big[2\zeta_3 g_4 g_3 g_2 + 4\zeta_3 g_5 g_4\\
&\quad + 8\zeta_3 g_7 g_2 + 10\zeta_3 g_5 g_2^2 - 12\zeta_3 g_6 g_3\big] + \mathbf{Te}^6 \big[5\zeta(2) g_2 g_7 - 5\zeta(2) g_6 g_3\big]\\
&\quad + \mathbf{Te}^8 g_9,\\
\mathfrak{P}_{10} &= \mathbf{Te}^2 \big[14\zeta_7 g_2 g_8 + \frac{77}{2}\zeta_7 g_2^5 + 147\zeta_7 g_6 g_2^2 + 210\zeta_7 g_6 g_4 + 322\zeta_7 g_4 g_2^3\\
&\quad + 441\zeta_7 g_4^2 g_2 - 84\zeta_7 g_3 g_7 - 140\zeta_7 g_5^2 - 322\zeta_7 g_3^2 g_2^2 - 588\zeta_7 g_5 g_3 g_2\big]\\
&\quad + \mathbf{Te}^3 \big[9\zeta_3^2 g_2^5 + 21\zeta_3^2 g_6 g_2^2 + 33\zeta_3^2 g_4 g_2^3 + 36\zeta_3^2 g_4 g_3^2 - 9\zeta_3^2 g_4^2 g_2 - 33\zeta_3^2 g_3^2 g_2^2\\
&\quad - 48\zeta_3^2 g_5 g_3 g_2 + \frac{16}{35}\zeta_2^3 g_8 g_2 + \frac{32}{7}\zeta_2^3 g_4 g_3^2 + \frac{48}{7}\zeta_2^3 g_6 g_4 + \frac{32}{105}\zeta_2^3 g_4 g_2^3\\
&\quad + \frac{248}{35}\zeta_2^3 g_4^2 g_2 + \frac{568}{105}\zeta_2^3 g_6 g_2^2 - \frac{32}{7}\zeta_2^3 g_5^2 - \frac{32}{105}\zeta_2^3 g_3^2 g_2^2 - \frac{244}{105}\zeta_2^3 g_2^5\\
&\quad - \frac{256}{15}\zeta_2^3 g_5 g_3 g_2 - \frac{96}{35}\zeta_2^3 g_3 g_7\big] + \mathbf{Te}^4 \big[\zeta_5 g_2^5 + \frac{1}{2}\zeta_5 g_3^2 g_2^2 + 11\zeta_5 g_8 g_2\\
&\quad + 45\zeta_5 g_6 g_4 + \frac{59}{2}\zeta_5 g_4 g_2^3 + \frac{81}{2}\zeta_5 g_6 g_2^2 + \frac{123}{2}\zeta_5 g_4^2 g_2 - 20\zeta_5 g_5^2 - 36\zeta_5 g_7 g_3\\
&\quad - 45\zeta_5 g_4 g_3^2 - 57\zeta_5 g_5 g_3 g_2 + 15\zeta_3\zeta_2 g_4 g_2^3 + 21\zeta_3\zeta_2 g_6 g_2^2 + 36\zeta_3\zeta_2 g_4 g_3^2\\
&\quad - 9\zeta_3\zeta_2 g_4^2 g_2 - 15\zeta_3\zeta_2 g_3^2 g_2^2 - 48\zeta_3\zeta_2 g_5 g_3 g_2\big] + \mathbf{Te}^5 \big[\frac{8}{5}\zeta_2^2 g_8 g_2 + \frac{24}{5}\zeta_2^2 g_6 g_4\\
&\quad + \frac{42}{5}\zeta_2^2 g_4 g_3^2 + \frac{58}{5}\zeta_2^2 g_6 g_2^2 - \frac{6}{5}\zeta_2^2 g_4 g_2^3 - \frac{6}{5}\zeta_2^2 g_3^2 g_2^2 - \frac{6}{5}\zeta_2^2 g_4^2 g_2 - \frac{8}{5}\zeta_2^2 g_5^2\\
&\quad - \frac{24}{5}\zeta_2^2 g_7 g_3 - \frac{94}{5}\zeta_2^2 g_5 g_3 g_2\big] + \mathbf{Te}^6 \big[\zeta_3 g_4^2 g_2 + 2\zeta_3 g_5 g_3 g_2 + 5\zeta_3 g_6 g_4\\
&\quad + 9\zeta_3 g_8 g_2 + 12\zeta_3 g_6 g_2^2 - 14\zeta_3 g_7 g_3\big] + \mathbf{Te}^7 \big[6\zeta_2 g_8 g_2 - 6\zeta_2 g_7 g_3\big] + \mathbf{Te}^9 g_{10}
\end{aligned}
$$

Table 16: $\mathfrak{p}_* = \sum \mathfrak{P}_{*s}$ up to weight 10 with $f = l \circ g$, $g_*(z) = \sum_{2\leq s} g_{*s}\, z^{1-s}$.

$$
\begin{aligned}
\mathfrak{P}_{*2} &= \mathbf{Te}^1 g_{*2}, \mathfrak{P}_{*3} = \mathbf{Te}^2 g_{*3}, \mathfrak{P}_{*4} = \mathbf{Te}^3 g_{*4}, \mathfrak{P}_{*5} = \mathbf{Te}^4 g_{*5},\\
\mathfrak{P}_{*6} &= \mathbf{Te}^2\big[6\zeta_3 g_{*2} g_{*4} - 6\zeta_3 g_{*3}^2\big] + \mathbf{Te}^5\big[g_{*6}\big]\\
\mathfrak{P}_{*7} &= \mathbf{Te}^3\big[6\zeta_3 g_{*2} g_{*5} - 6\zeta_3 g_{*3} g_{*4}\big] + \mathbf{Te}^6\big[g_{*7}\big]\\
\mathfrak{P}_{*8} &= \mathbf{Te}^2\big[30\zeta_5 g_{*4}^2 - \frac{5}{2}\zeta_5 g_{*2}^4 + 10\zeta_5 g_{*2} g_{*6} - 40\zeta_5 g_{*3} g_{*5}\big]\\
&\quad \mathbf{Te}^3\big[\frac{4}{3}\zeta_2^2 g_{*2} g_{*3}^2 - \frac{4}{3}\zeta_2^2 g_{*2}^2 g_{*4}\big] + \mathbf{Te}^4\big[3\zeta_3 g_{*4}^2 + \frac{1}{4}\zeta_3 g_{*2}^4 - 10\zeta_3 g_{*3} g_{*5}\\
&\quad +7\zeta_3 g_{*2} g_{*6}\big] + \mathbf{Te}^5\big[-\frac{2}{3}\zeta_2 g_{*2} g_{*3}^2 + \frac{2}{3}\zeta_2 g_{*2}^2 g_{*4}\big] + \mathbf{Te}^7\big[g_{*8}\big]\\
\mathfrak{P}_{*9} &= \mathbf{Te}^2\big[36\zeta(3)^2 g_{*3}^3 - \frac{32}{5}\zeta_2^3 g_{*3}^3 + 18\zeta_3^2 g_{*5} g_{*2}^2 + \frac{48}{5}\zeta_2^3 g_{*2} g_{*3} g_{*4}\\
&\quad -54\zeta_3^2 g_{*2} g_{*3} g_{*4} - \frac{16}{5}\zeta_2^3 g_{*5} g_{*2}^2\big] + \mathbf{Te}^3\big[20\zeta_5 g_{*4} g_{*5} + 10\zeta_5 g_{*2} g_{*7}\\
&\quad -30\zeta_5 g_{*3} g_{*6} - 5\zeta_5 g_{*2}^3 g_{*3}\big] + \mathbf{Te}^4\big[-\frac{1}{5}\zeta_2^2 g_{*3}^3 - \frac{21}{10}\zeta_2^2 g_{*2}^2 g_{*5}\\
&\quad +\frac{23}{10}\zeta_2^2 g_{*2} g_{*3} g_{*4}\big] + \mathbf{Te}^5\big[8\zeta_3 g_{*2} g_{*7} - 12\zeta_3 g_{*3} g_{*6} + 4\zeta_3 g_{*4} g_{*5}\\
&\quad +\zeta_3 g_{*2}^3 g_{*3}\big] + \mathbf{Te}^6\big[\frac{3}{2}\zeta_2 g_{*2}^2 g_{*5} - \frac{1}{3}\zeta_2 g_{*3}^3 - \frac{7}{6}\zeta_2 g_{*2} g_{*3} g_{*4}\big] + \mathbf{Te}^8\big[g_{*9}\big]\\
\mathfrak{P}_{*10} &= \mathbf{Te}^2\big[210\zeta_7 g_{*4} g_{*6} - 140\zeta_7 g_{*5}^2 - 84\zeta_7 g_{*3} g_{*7} + 14\zeta_7 g_{*2} g_{*8}\big]\\
&\quad -\frac{133}{3}\zeta_7 g_{*2}^3 g_{*4} + \frac{133}{3}\zeta_7 g_{*2}^2 g_{*3}^2\big] + \mathbf{Te}^3\big[36\zeta_3^2 g_{*3}^2 g_{*4} - 9\zeta_3^2 g_{*2} g_{*4}^2\\
&\quad +21\zeta_3^2 g_{*2}^2 g_{*6} + \frac{3}{4}\zeta_3^2 g_{*2}^5 - \frac{32}{5}\zeta_2^3 g_{*3}^2 g_{*4} - \frac{64}{15}\zeta_2^3 g_{*2}^2 g_{*6}\\
&\quad +\frac{32}{3}\zeta_2^3 g_{*2} g_{*3} g_{*5} - 48\zeta_3^2 g_{*2} g_{*3} g_{*5}\big] + \mathbf{Te}^4\big[45\zeta_5 g_{*4} g_{*6} - 20\zeta_5 g_{*5}^2\\
&\quad -36\zeta_5 g_{*3} g_{*7} + 11\zeta_5 g_{*2} g_{*8} - \frac{10}{3}\zeta_5 g_{*2}^3 g_{*4} - \frac{25}{6}\zeta_5 g_{*2}^2 g_{*3}^2\big]\\
&\quad +\mathbf{Te}^5\big[\frac{10}{3}\zeta_2^2 g_{*2} g_{*3} g_{*5} - \frac{2}{5}\zeta_2^2 g_{*3}^2 g_{*4} - \frac{44}{15}\zeta_2^2 g_{*2}^2 g_{*6}\big] + \mathbf{Te}^6\big[9\zeta_3 g_{*2} g_{*8}\\
&\quad -14\zeta_3 g_{*3} g_{*7} + 5\zeta_3 g_{*4} g_{*6} + \frac{1}{2}\zeta(3) g_{*2}^2 g_{*3}^2 + 2\zeta_3 g_{*2}^3 g_{*4}\big]\\
&\quad +\mathbf{Te}^7\big[\frac{8}{3}\zeta_2 g_{*2}^2 g_{*6} - \frac{5}{3}\zeta_2 g_{*2} g_{*3} g_{*5} - \zeta_2 g_{*3}^2 g_{*4}\big] + \mathbf{Te}^9\big[g_{*10}\big]
\end{aligned}
$$

Table 17: $\mathfrak{p}_\sharp = \sum \mathfrak{P}_{\sharp s}$ up to weight 10 with $f = l \circ g\,,\; g_\sharp(z) = \sum_{2\le s} g_{\sharp s}\, z^{1-s}$.

$$\mathfrak{P}_{\sharp 2} = \mathbf{Te}^1 g_{\sharp 2}, \mathfrak{P}_{\sharp 3} = \mathbf{Te}^2 g_{\sharp 3}, \mathfrak{P}_{\sharp 4} = \mathbf{Te}^3 g_{\sharp 4}, \mathfrak{P}_{\sharp 5} = \mathbf{Te}^4 g_{\sharp 5},$$

$$\mathfrak{P}_{\sharp 6} = \mathbf{Te}^2\big[\mathbf{Te}^5 g_{\sharp 6} + 6\zeta_3 g_{\sharp 4} g_{\sharp 2} - 6\zeta_3 g_{\sharp 3}^2\big] + \mathbf{Te}^3 \zeta_2 g_{\sharp 2}^3$$

$$\mathfrak{P}_{\sharp 7} = \mathbf{Te}^3\big[6\zeta_3 g_{\sharp 5} g_{\sharp 2} - 6\zeta_3 g_\sharp 4 g_{\sharp 3}\big] + \mathbf{Te}^4 \zeta_2 \frac{3}{2} g_{\sharp 3} g_{\sharp 2}^2 + \mathbf{Te}^6 g_{\sharp 7}$$

$$\begin{aligned}
\mathfrak{P}_{\sharp 8} = {} & \mathbf{Te}^2\big[10\zeta_5 g_{\sharp 6} g_{\sharp 2} + 30\zeta_5 g_{\sharp 4}^2 - 40\zeta_5 g_{\sharp 5} g_{\sharp 3}\big] \\
& + \mathbf{Te}^3\big[\frac{8}{5}\zeta_2^2 g_{\sharp 3}^2 g_{\sharp 2} - \frac{8}{5}\zeta_2^2 g_{\sharp 4} g_{\sharp 2}^2\big] \\
& + \mathbf{Te}^4\big[3\zeta_3 g_{\sharp 4}^2 + 2\zeta_3 g_{\sharp 2}^4 - 10\zeta_3 g_{\sharp 5} g_{\sharp 3} + 7\zeta_3 g_{\sharp 6} g_{\sharp 2}\big] \\
& + \mathbf{Te}^5\big[5\zeta_2 g_{\sharp 4} g_{\sharp 2}^2 - 2\zeta_2 g_{\sharp 3}^2 g_{\sharp 2}\big] + \mathbf{Te}^7 g_{\sharp 8}
\end{aligned}$$

$$\begin{aligned}
\mathfrak{P}_{\sharp 9} = {} & \mathbf{Te}^2\big[18\zeta_3^2 g_{\sharp 5} g_{\sharp 2}^2 + 36\zeta_3^2 g_{\sharp 3}^3 - 54\zeta_3^2 g_{\sharp 4} g_{\sharp 3} g_{\sharp 2} + \frac{624}{35}\zeta_2^3 g_{\sharp 4} g_{\sharp 3} g_{\sharp 2} \\
& - \frac{208}{35}\zeta_2^3 g_{\sharp 5} g_{\sharp 2}^2 - \frac{416}{35}\zeta_2^3 g_{\sharp 3}^3\big] + \mathbf{Te}^3\big[10\zeta_5 g_{\sharp 7} g_{\sharp 2} + 20\zeta_5 g_{\sharp 5} g_{\sharp 4} \\
& - 30\zeta_5 g_{\sharp 6} g_{\sharp 3}\big] + \mathbf{Te}^4\big[\frac{87}{10}\zeta_2^2 g_{\sharp 4} g_{\sharp 3} g_{\sharp 2} - \frac{9}{2}\zeta_2^2 g_{\sharp 5} g_{\sharp 2}^2 - \frac{21}{5}\zeta_2^2 g_{\sharp 3}^3\big] \\
& + \mathbf{Te}^5\big[8\zeta_3 g_{\sharp 3} g_{\sharp 2}^3 + 8\zeta_3 g_{\sharp 7} g_{\sharp 2} + 4\zeta_3 g_{\sharp 5} g_{\sharp 4} - 12\zeta_3 g_{\sharp 6} g_{\sharp 3}\big] \\
& + \mathbf{Te}^6\big[\frac{17}{2}\zeta_2 g_{\sharp 5} g_{\sharp 2}^2 - \frac{1}{2}\zeta_2 g_{\sharp 4} g_{\sharp 3} g_{\sharp 2} - 3\zeta_2 g_{\sharp 3}^3\big] + \mathbf{Te}^8 g_{\sharp 9}
\end{aligned}$$

$$\begin{aligned}
\mathfrak{P}_{\sharp 10} = {} & \mathbf{Te}^2\big[14\zeta_7 g_{\sharp 8} g_{\sharp 2} + 35\zeta_7 g_{\sharp 2}^3 g_{\sharp 4} + 210\zeta_7 g_{\sharp 6} g_{\sharp 4} - 84\zeta_7 g_{\sharp 7} g_{\sharp 3} \\
& - 35\zeta_7 g_{\sharp 3}^2 g_{\sharp 2}^2 - 140\zeta_7 g_{\sharp 5}^2\big] + \mathbf{Te}^3\big[\frac{224}{15}\zeta_2^3 g_{\sharp 5} g_{\sharp 3} g_{\sharp 2} + \frac{128}{35}\zeta_2^3 g_{\sharp 4}^2 g_{\sharp 2} \\
& - \frac{704}{105}\zeta_2^3 g_{\sharp 6} g_{\sharp 2}^2 - \frac{416}{35}\zeta_2^3 g_{\sharp 4} g_{\sharp 3}^2 - \frac{176}{105}\zeta_2^3 g_{\sharp 2}^5 + 6\zeta_3^2 g_{\sharp 2}^5 + 21\zeta_3^2 g_{\sharp 6} g_{\sharp 2}^2 \\
& + 36\zeta_3^2 g_{\sharp 4} g_{\sharp 3}^2 - 9\zeta_3^2 g_{\sharp 4}^2 g_{\sharp 2} - 48\zeta_3^2 g_{\sharp 5} g_{\sharp 3} g_{\sharp 2}\big] + \mathbf{Te}^4\big[11\zeta_5 g_{\sharp 8} g_{\sharp 2} \\
& + 29\zeta_5 g_{\sharp 4} g_{\sharp 2}^3 + 45\zeta_5 g_{\sharp 6} g_{\sharp 4} - 36\zeta_5 g_{\sharp 7} g_\sharp[3] - 29\zeta_5 g_{\sharp 3}^2 g_{\sharp 2}^2 - 20\zeta_5 g_{\sharp 5}^2 \\
& + 9\zeta_2\zeta_3 g_{\sharp 4} g_{\sharp 2}^3 - 9\zeta_2\zeta_3 g_{\sharp 3}^2 g_{\sharp 2}^2\big] + \mathbf{Te}^5\big[\frac{9}{4}\zeta_2^2 g_{\sharp 2}^5 + \frac{42}{5}\zeta_2^2 g_{\sharp 5} g_{\sharp 3} g_{\sharp 2} \\
& + \frac{33}{5}\zeta_2^2 g_{\sharp 4}^2 g_{\sharp 2} - \frac{33}{5}\zeta_2^2 g_{\sharp 6} g_{\sharp 2}^2 - \frac{42}{5}\zeta_2^2 g_{\sharp 4} g_{\sharp 3}^2\big] + \mathbf{Te}^6\big[+5\zeta_3 g_{\sharp 6} g_{\sharp 4} \\
& + 7\zeta_3 g_{\sharp 3}^2 g_{\sharp 2}^2 + 9\zeta_3 g_{\sharp 8} g_{\sharp 2} + 13\zeta_3 g_{\sharp 4} g_{\sharp 2}^3 - 14\zeta_3 g_{\sharp 7} g_{\sharp 3}\big] \\
& + \mathbf{Te}^7\big[13\zeta_2 g_{\sharp 6} g_{\sharp 2}^2 + \frac{9}{2}\zeta_2 g_{\sharp 4}^2 g_{\sharp 2} - \frac{1}{2}\zeta_2 g_{\sharp 2}^5 - 9\zeta_2 g_{\sharp 4} g_{\sharp 3}^2 - \zeta_2 g_{\sharp 5} g_{\sharp 3} g_{\sharp 2}\big] \\
& + \mathbf{Te}^9 g_{\sharp 10}
\end{aligned}$$

9.4 The invariants as entire functions of f: the reflexive case

As in Table 16, we write down the expansion of the collector $\mathfrak{p}_*$ in terms of g_*, but this time for a reflexive f. Recall that a standard f is reflexive iff $f(-f(-z)) \equiv z$, in which case its conjugate $l^{1/2} \circ f \circ f^{-1/2}$ is of the form $l \circ g$ with g also reflexive. See Section 3.9. Reflexivity automatically implies $\rho(f) \equiv -g_{*2} \equiv 0$. There being fewer coefficients g_{*s}, we reach weight 13.

Example 18: $\mathfrak{p}_*$ up to weight 13 for $f = l \circ g$ with $g_*(z) = \sum_{1 \le d} g_{*1+2d} z^{-2d}$.

$$
\begin{aligned}
\mathfrak{P}_{*3} &= \mathbf{Te}^2 g_{*3}, \mathfrak{P}_{*5} = \mathbf{Te}^4 g_{*5}, \mathfrak{P}_{*6} = \mathbf{Te}^2\big[-6\zeta_3 g_{*3}^2\big], \mathfrak{P}_{*7} = \mathbf{Te}^6 g_{*7},\\
\mathfrak{P}_{*8} &= \mathbf{Te}^2\big[-40\zeta_5 g_{*5} g_{*3}\big] + \mathbf{Te}^4\big[-10\zeta_3 g_{*5} g_{*3}\big],\\
\mathfrak{P}_{*9} &= \mathbf{Te}^2\big[36\zeta_3^2 g_{*3}^3 - \frac{32}{5}\zeta_2^3 g_{*3}^3\big] + \mathbf{Te}^4\big[-\frac{1}{5}\zeta_2^2 g_{*3}^3\big] + \mathbf{Te}^6\big[-\frac{1}{3}\zeta_2 g_{*3}^3\big]\\
&\quad + \mathbf{Te}^8\big[g_{*9}\big],\\
\mathfrak{P}_{*10} &= \mathbf{Te}^2\big[-84\zeta_7 g_{*7} g_{*3} - 140\zeta_7 g_{*5}^2\big] + \mathbf{Te}^4\big[-36\zeta_5 g_{*7} g_{*3} - 20\zeta_5 g_{*5}^2\big]\\
&\quad + \mathbf{Te}^6\big[-14\zeta_3 g_{*7} g_{*3}\big]\\
\mathfrak{P}_{*11} &= \mathbf{Te}^2\big[560\zeta_5\zeta_3 g_{*5} g_{*3}^2 - \frac{15648}{175}\zeta_2^4 g_{*5} g_{*3}^2 - 80\zeta_{6,2}^{\mathrm{ev}} g_{*5} g_{*3}^2\big]\\
&\quad + \mathbf{Te}^4\big[80\zeta_3^2 g_{*5} g_{*3}^2 - \frac{272}{21}\zeta_2^3 g_{*5} g_{*3}^2\big] + \mathbf{Te}^6\big[-\frac{34}{15}\zeta_2^2 g_{*5} g_{*3}^2\big]\\
&\quad + \mathbf{Te}^8\big[-\frac{5}{3}\zeta_2 g_{*5} g_{*3}^2\big] + \mathbf{Te}^1 0 g_{*11}\big],\\
\mathfrak{P}_{*12} &= \mathbf{Te}^2\big[\frac{576}{5}\zeta_3\zeta_2^3 g_{*3}^4 - 216\zeta_3^3 g_{*3}^4 - 144\zeta_9 g_{*9} g_{*3} - 210\zeta_9 g_{*3}^4 - 1008\zeta_9 g_{*7} g_{*5}\big]\\
&\quad + \mathbf{Te}^4\big[\frac{18}{5}\zeta_3\zeta_2^2 g_{*3}^4 + 14\zeta_7 g_{*3}^4 - 210\zeta_7 g_{*7} g_{*5} - 78\zeta_7 g_{*3} g_{*9}\big]\\
&\quad + \mathbf{Te}^6\big[6\zeta_3\zeta_2 g_{*3}^4 - \frac{10}{3}\zeta_5 g_{*3}^4 - 28\zeta_5 g_{*7} g_{*5} - 44\zeta_5 g_{*9} g_{*3}\big]\\
&\quad + \mathbf{Te}^8\big[-18\zeta_3 g_{*9} g_{*3}\big],\\
\mathfrak{P}_{*13} &= \mathbf{Te}^2\big[720\zeta_5^2 g_{*7} g_{*3}^2 + 1200\zeta_5^2 g_{*5}^2 g_{*3} + 1344\zeta_7\zeta_3 g_{*7} g_{*3}^2\\
&\quad + 2240\zeta_7\zeta_3 g_{*5}^2 g_{*3} - 168\zeta_{8,2}^{\mathrm{ev}} g_{*7} g_{*3}^2 - 280\zeta_{8,2}^{\mathrm{ev}} g_{*5}^2 g_{*3} - \frac{125056}{385}\zeta_2^5 g_{*5}^2 g_{*3}\\
&\quad - \frac{375168}{1925}\zeta_2^5 g_{*7} g_{*3}^2\big] + \mathbf{Te}^4\big[100\zeta_{6,2}^{\mathrm{ev}} g_{*5}^2 g_{*3} + 500\zeta_5\zeta_3 g_{*5}^2 g_{*3}\\
&\quad + 540\zeta_5\zeta_3 g_{*7} g_{*3}^2 + \frac{6544}{525}\zeta_2^4 g_{*5}^2 g_{*3} - \frac{23824}{175}\zeta_2^4 g_{*7} g_{*3}^2 - 180\zeta_{6,2}^{\mathrm{ev}} g_{*7} g_{*3}^2\big]\\
&\quad + \mathbf{Te}^6\big[140\zeta_3^2 g_{*7} g_{*3}^2 + \frac{88}{21}\zeta_2^3 g_{*5}^2 g_{*3} - \frac{3064}{105}\zeta_2^3 g_{*7} g_{*3}^2\big] + \mathbf{Te}^8\big[\frac{8}{15}\zeta_2^2 g_{*5}^2 g_{*3}\\
&\quad - \frac{39}{5}\zeta_2^2 g_{*7} g_{*3}^2\big] + \mathbf{Te}^{10}\big[-4\zeta_2 g_{*7} g_{*3}^2 - \frac{2}{3}\zeta_2 g_{*5}^2 g_{*3}\big] + \mathbf{Te}^{12} g_{*13}
\end{aligned}
$$

9.5 The invariants as entire functions of f: one-parameter cases

Table 19: $\mathfrak{p}_*$ up to weight 12 for $f = l \circ g$ with $g(z) = z + g_2\, z^{-1}$.

$$
\begin{aligned}
\mathfrak{P}_2 &= g_2\mathbf{Te}^2, \mathfrak{P}_4{=}0, \mathfrak{P}_6{=}g_2^3\mathbf{Te}^2\big[3\zeta_3\big],\\
\mathfrak{P}_8 &= g_2^4\Big(\mathbf{Te}^2[10\zeta_5]+\mathbf{Te}^3\Big[-\frac{2}{5}\zeta_2^2\Big]\Big),\\
\mathfrak{P}_{10} &= g_2^5\Big(\mathbf{Te}^2\Big[\frac{77}{2}\zeta_7\Big]+\mathbf{Te}^3\Big[9\zeta_3^2-\frac{244}{105}\zeta_2^3\Big]+\mathbf{Te}^4\zeta_5\Big),\\
\mathfrak{P}_{12} &= g_2^6\Big(\mathbf{Te}^2[151\zeta_9]+\mathbf{Te}^3\Big[3\zeta_{6,2}^{\mathrm{ev}}+63\zeta_3\zeta_5-\frac{878}{105}\zeta_2^4\Big]\\
&\qquad +\mathbf{Te}^4\Big[10\zeta_7+3\zeta_2\zeta_5-\frac{18}{5}\zeta_2^2\zeta_3\Big]+\mathbf{Te}^5\Big[-\frac{8}{35}\zeta_2^3\Big]\Big),\\
\mathfrak{P}_{14} &= g_2^7\Big(\mathbf{Te}^2\Big[\frac{16}{7}\zeta_2^3\zeta_5+18\zeta_3^2\zeta_5+9\zeta_{8,1,2}^{\mathrm{odd}}+\frac{19343}{24}\zeta_{11}\Big]\\
&\qquad +\mathbf{Te}^3\Big[15\zeta_{8,2}^{\mathrm{ev}}+6\zeta_{6,2}^{\mathrm{ev}}\zeta_2+261\zeta_7\zeta_3-\frac{5972}{231}\zeta_2^5+\frac{235}{2}\zeta_5^2+6\zeta_5\zeta_3\zeta_2\Big]\\
&\qquad +\mathbf{Te}^4\Big[+27\zeta_3^3+\frac{5027}{72}\zeta_9+30\zeta_7\zeta_2-\frac{51}{10}\zeta_2^2\zeta_5-\frac{732}{35}\zeta_3\zeta_2^3\Big]\\
&\qquad +\mathbf{Te}^5\Big[11\zeta_3\zeta_5-\zeta_{6,2}^{\mathrm{ev}}-\frac{508}{175}\zeta_2^4\Big]+\mathbf{Te}^6\big[\zeta_7\big]\Big)
\end{aligned}
$$

Table 20: $\mathfrak{p}_*$ up to weight 12 for $f = l \circ g$ with $g(z) = z\left[1+2\, g_{*2}\, z^{-2}\right]^{\frac{1}{2}}$.

$$
\begin{aligned}
\mathfrak{P}_{*2} &= g_{*2}\mathbf{Te}^1, \mathfrak{P}_{*4}{=}0, \mathfrak{P}_{*6}{=}0,\\
\mathfrak{P}_{*8} &= g_{*2}^4\Big(\mathbf{Te}^2\Big[-\frac{5}{2}\zeta_5\Big]+\mathbf{Te}^4\Big[\frac{1}{4}\zeta_3\Big]\Big)\\
\mathfrak{P}_{*10} &= g_{*2}^5\mathbf{Te}^{\mathbf{3}}\Big[\frac{3}{4}*\zeta_3^2\Big]\\
\mathfrak{P}_{*12} &= g_{*2}^6\Big(\mathbf{Te}^2\Big[\frac{3}{2}\zeta_3^3+\frac{47}{6}\zeta[9]-\frac{4}{5}\zeta_3\zeta_2^3\Big]+\mathbf{Te}^4\Big[-\frac{21}{40}\zeta_3\zeta_2^2-\frac{63}{64}\zeta_7\Big]\\
&\qquad +\mathbf{Te}^6\Big[\frac{3}{8}\zeta_3\zeta_2+\frac{1}{16}\zeta_5\Big]+\mathbf{Te}^8\Big[-\frac{1}{16}\zeta_3\Big]\Big)\\
\mathfrak{P}_{*14} &= g_{*2}^7\Big(\mathbf{Te}^3\Big[\frac{105}{16}\zeta_5^2-\zeta_3^2\zeta_2^2-\frac{189}{32}\zeta_7\zeta_3\Big]\\
&\qquad +\mathbf{Te}^5\Big[\frac{1}{2}\zeta_3^2\zeta_2-2\zeta_5\zeta_3\Big]+\mathbf{Te}^7\Big[\frac{1}{8}\zeta_3^2\Big]\Big)
\end{aligned}
$$

Table 21: $\mathfrak{p}_*$ up to weight 15 for $f = l \circ g$ with $g(z) = z\left[1+3\,g_{*3}\,z^{-3}\right]^{\frac{1}{3}}$.

$$
\begin{aligned}
\mathfrak{P}_{*3} &= g_{*3}\mathbf{Te}\\
\mathfrak{P}_{*6} &= g_{*3}^2\Big(\mathbf{Te}^2[-6\zeta_3]\Big)\\
\mathfrak{P}_{*9} &= g_{*3}^3\left(\mathbf{Te}^2\left[36\zeta_3^2-\frac{32}{5}\zeta_2^3\right]+\mathbf{Te}^4\left[-\frac{1}{5}\zeta_2^2\right]+\mathbf{Te}^6\left[-\frac{1}{3}\zeta_2\right]\right)\\
\mathfrak{P}_{*12} &= g_{*3}^4\left(\mathbf{Te}^2\left[\frac{576}{5}\zeta_3\zeta_2^3-216\zeta_3^3-210\zeta_9\right]\right.\\
&\quad\left.+\mathbf{Te}^4\left[\frac{18}{5}\zeta_3\zeta_2^2+14\zeta_7\right]+\mathbf{Te}^6\left[6\zeta_3\zeta_2-\frac{10}{3}\zeta_5\right]\right)\\
\mathfrak{P}_{*15} &= g_{*3}^5\left(\mathbf{Te}^2\left[1296\zeta_3^4+3780\zeta_9\zeta_3-140\zeta_7\zeta_5-\frac{23054144}{125125}\zeta_2^6-\frac{6912}{5}\zeta_3^2\zeta_2^3\right.\right.\\
&\quad\left.-420\zeta_{10,2}^{\mathrm{ev}}\right]+\mathbf{Te}^4\left[\frac{1332224}{28875}\zeta_2^5-\frac{216}{5}\zeta_3^2\zeta_2^2+60\zeta_5^2-238\zeta_7\zeta_3+49\zeta_{8,2}^{\mathrm{ev}}\right]\\
&\quad+\mathbf{Te}^6\left[\frac{1007}{1575}\zeta_2^4-72\zeta_3^2\zeta_2+\frac{190}{3}\zeta_5\zeta_3-\frac{50}{3}\zeta_{6,2}^{\mathrm{ev}}\right]+\mathbf{Te}^8\left[\frac{193}{75}\zeta_2^3\right]\\
&\quad\left.+\mathbf{Te}^{10}\left[\frac{16}{15}\zeta_2^2\right]+\mathbf{Te}^{12}\left[\frac{7}{45}\zeta_2\right]\right)
\end{aligned}
$$

10 Synopsis

10.1 Diffeos, collectors, connectors, invariants

Given a general local identity-tangent mapping f of $\mathbb{C}_{,\infty} \mapsto \mathbb{C}_{,\infty}$, whether of tangency order 1 (*i.e.* $f(z) - z \sim Cst$) or of order $p > 1$ (*i.e.* $f(z) - z \sim Cst\, z^{1-p}$), what can be said of its **analytic invariants**? What are the most natural, complete systems $\{A_\omega,\ \omega \in \Omega\}$ of invariants? What methods are there for computing these A_ω, singly or collectively? How do these methods compare as to efficiency? Above all, on the more theoretical side: which are the most explicit and/or economical formulae for expanding the A_ω into convergent series of f-dependent inputs (such as the Taylor coefficients of f) and f-independent, universal constants?

Practically all natural, complete systems $\{A_\omega,\ \omega \in \Omega\}$ of invariants consist of the Fourier coefficients of the so-called **connectors** $\pi(z)$ – *i.e.* trigonometric Fourier series which *connect* the various sectorial normalisations of f with their immediate neighbours. Although these invariant *connectors* are totally independent and mutually unrelated, they all derive from a more basic object, the **collector** $\mathfrak{p}(z)$, which is unique and "of one piece", but unfortunately not invariant. The *collector*, with its natural expansions into series of multitangents *or* monotangents, is a natural intermediary between f and the invariant-carrying *connectors*.

10.2 Affiliates, generators, mediators

The *analytic* invariants $A_\omega(f)$ are also *holomorphic* in f as long as f ranges through a fixed formal conjugacy class $\mathbb{G}^{(p,\rho)}$ of $\mathbb{G}$, where $p \in \mathbb{N}^*$ is the *tangency order* and $\rho \in \mathbb{C}$ the *iteration residue*. Thus, for elements of the prototypal class $\mathbb{G}^{(1,0)}$, which may be written as $f = l \circ g$ with $l(z) = z + 1$ and $g(z) = z + \mathcal{O}(z^{-2})$, the invariants $A_\omega(f)$ as well as the connector $\pi(z)$ and collector $\mathfrak{p}(z)$ that carry them, must be *entire functions* of g, hence of each of g's coefficients g_n.

Now, given any analytic function $\gamma(t) := \sum_{0 \leq r} \gamma_r t^r$, we can associate with $f, g, \pi, \mathfrak{p}$ the so-called *affiliates* $f_\diamondsuit, g_\diamondsuit, \pi_\diamondsuit, \mathfrak{p}_\diamondsuit$ defined via the corresponding substitution operators $F, G, \Pi, \mathfrak{P}$.[99]

Three types of affiliates are of special relevance:

(i) the *infinitesimal generators* $f_*, g_*, \pi_*, \mathfrak{p}_*$, with $\gamma(t) = \log(1+t)$.

(ii) the *first* or *main mediators* $f_\sharp, g_\sharp, \pi_\sharp, \mathfrak{p}_\sharp$, with $\gamma(t) = \frac{t}{1+\frac{1}{2}t}$.

(iii) the *second mediators* $f_{\sharp\sharp}, g_{\sharp\sharp}, \pi_{\sharp\sharp}, \mathfrak{p}_{\sharp\sharp}$, with $\gamma(t) = \frac{(1+t)^2-1}{(1+t)^2+1}$.

Each of the three series $f_*, f_\sharp, f_{\sharp\sharp}$ is resurgent and verifies resurgence equations ruled by (and yielding) the invariants $A_\omega(f)$. Here, f_* is by far the best choice.

The three series $g_*, g_\sharp, g_{\sharp\sharp}$ are resurgent, too, but with resurgence coefficients $A_\omega(g)$ totally unrelated to the $A_\omega(f)$. The usefulness of $g_*, g_\sharp, g_{\sharp\sharp}$, however, lies elsewhere – namely in their providing a bridge, first to the collectors $\mathfrak{p}_*, \mathfrak{p}_\sharp, \mathfrak{p}_{\sharp\sharp}$ and then to the connectors $\pi_*, \pi_\sharp, \pi_{\sharp\sharp}$. Here, the best choice is not g_*, but $g_\sharp$, with $g_{\sharp\sharp}$ the second best choice.

As for the three connectors $\pi_*, \pi_\sharp, \pi_{\sharp\sharp}$, each is as good as the other, since their Fourier coefficients stand in bi-polynomial correspondence with one another.

10.3 Main alien operators

To each type of *affiliate* $f_\diamondsuit$ there naturally corresponds a specific system of *alien operators* $\{\Delta_\omega^\diamondsuit,\ \omega \in \mathbb{C}_\bullet\}$.

The alien counterpart of the infinitesimal *generators* f_* is the system $\{\Delta_\omega,\ \omega \in \mathbb{C}_\bullet\}$ of (standard) *alien derivations* .

The alien counterparts of the *mediators* $f_\sharp$ and $f_{\sharp\sharp}$ are the systems of so-called *medial alien operators*[100] $\{\Delta_\omega^\sharp,\ \omega \in \mathbb{C}_\bullet\}$ and $\{\Delta_\omega^{\sharp\sharp},\ \omega \in \mathbb{C}_\bullet\}$. Although these medial operators are not exact derivations (they possess

[99] Thus $f_\diamondsuit(z) := F_\diamondsuit.z$ with $F_\diamondsuit := \gamma(F - 1)$.

[100] These *medial* operators bear no relation to the so-called *median* convolution average.

more complex co-products), they are in a sense more basic than the alien derivations Δ_ω, and simpler too, at least in many respects, such as numerical computations. They occur naturally in several unrelated contexts and deserve to have their own niche within alien calculus.

10.4 Main moulds

To each type of *affiliate* $f_\diamondsuit$ there also correspond specific mouldian *symmetry types* which extend the familiar four-type landscape of *alternal/symmetral* and *alternel/symmetrel*. In the present instance, they also bring order and structure into the plethora of auxiliary moulds required for expanding the invariants $A_\omega(f)$. Here are the main moulds:[101]

(i) The scalar multizetas $ze^\bullet$, $za^\bullet$, $zo^\bullet$. They are the mainstay of this investigation, being the transcendental ingredient of the $A_\omega(f)$.

(ii) The multitangents $Tee^\bullet(z)$, $Taa^\bullet(z)$, $Too^\bullet(z)$. They are meromorphic, 1-periodic functions of z. It is through their Fourier coefficients that the multizetas smuggle their way into invariant analysis.

(iii) The multizetaic resurgence monomials $\widetilde{Se}^\bullet(z)$, $\widetilde{Sa}^\bullet(z)$, $\widetilde{So}^\bullet(z)$, which are related – *in several ways* – to both the scalar multizetas and the multitangents.

These very basic moulds give rise to interesting combinatorial developments, such as the conversion formulae from $Taa^\bullet$ and $Too^\bullet$ to $Tee^\bullet$. We may note that, here again, the multitangents $Too^\bullet$, *i.e.* precisely the ones associated with an 'exotic' symmetry type, turn out to be the most useful.

10.5 Main results

Half the results presented in this paper deal with somewhat tangential issues – the mould machinery, the alien operators, the attendant combinatorics, etc. Regarding the core concern of the investigation – the expansion-description of the holomorphic invariants – we may point to the following:

We derive explicit and optimal[102] expansions for the collectors and connectors of $f = l \circ g$ in their three main variants: first directly from g to π, $\mathfrak{p}$, next from g_* to π_*, $\mathfrak{p}_*$, lastly from $g_\sharp$ to $\pi_\sharp$, $\mathfrak{p}_\sharp$. We even examine the general, affiliate-based scheme, from $g_\diamondsuit$ to $\pi_\diamondsuit$, $\mathfrak{p}_\diamondsuit$, the better to bring out the 'specialness' of the three main schemes.

[101] The vowels 'e' and 'a' connote, as usual, alternelity/symmetrelity or alternality/symmetrality, whereas the vowel 'o' points to less common symmetry types, related to the mediators.

[102] *Optimal* in the sense of *incapable of further simplification*.

We also detain ourselves over the ramified case ($p > 1$) and the far-going changes it brings: the *finite* reduction of multitangents to monotangents breaks down; the procedure for recovering the multitangents from their singular parts completely changes; the Fourier coefficients of the multitangents are no longer expressible as finite sums of multizetas, not even $\mathbb{Q}$-indexed ones.

We describe the growth properties of each invariant $A_\omega(f)$ as an entire function of exponential type in the Taylor coefficients of f.

We review various natural groups of formal germs, strictly larger than the group $\mathbb{G}_0$ of analytic germs, yet close enough to $\mathbb{G}_0$ to possess non-trivial analytic classes and holomorphic invariants $A_\omega(f)$. We characterize $\mathbb{G}_{0++}$, the largest of all such groups; and $\mathbb{G}_{0+}$, the largest of all *self-replicating groups*, whose elements produce connectors which, after rescaling, still belong to the group, and in turn produce their own connectors, *ad infinitum*. These developments may be taken as an introduction to the subject of *phantom holomorphic dynamics*.

We also stress the distinction between the *arithmetical* and *dynamical* monics. They are the same objects, but viewed differently:
(i) the former as ingredients of the Stokes constants, in which capacity they are rigidly determined.
(ii) the latter as ingredients of the holomorphic invariants, the sole demand on them being that of *making the invariants invariant*.

We show how the systems of (finite or infinite) relations that constrain the monics *change* depending on which perspective we adopt. Most noticeably, the finite, algebraic constraints on the *dynamical* monics turn out to be significantly weaker than those on their *arithmetical* counterpart.

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