A taste of nonstandard methods in combinatorics of numbers

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Abstract. *By presenting the proofs of a few sample results, we introduce the reader to the use of nonstandard analysis in aspects of combinatorics of numbers.*

Introduction

In the last years, several combinatorial results about sets of integers that depend on their asymptotic densities have been proved by using the techniques of nonstandard analysis, starting from the pioneering work by R. Jin (see *e.g*. [6, 8, 9, 12–14, 16, 17]). Very recently, the hyper-integers of nonstandard analysis have also been used in Ramsey theory to investigate the partition regularity of possibly non-linear diophantine equations (see [6, 19]).

The goal of this paper is to give a soft introduction to the use of nonstandard methods in certain areas of density problems and Ramsey theory. To this end, we will focus on a few sample results, aiming to give the flavor of how and why nonstandard techniques could be successfully used in this area.

Grounding on nonstandard definitions of the involved notions, the presented proofs consist of arguments that can be easily followed by the intuition and that can be taken at first as heuristic reasonings. Subsequently, in the last foundational section, we will outline an algebraic construction of the hyper-integers, and give hints to show how those nonstandard arguments are in fact rigorous ones when formulated in the appropriate language. We will also prove that all the nonstandard definitions presented in this paper are actually equivalent to the usual "standard" ones.

Two disclaimers are in order. Firstly, this paper is not to be taken as a comprehensive presentation of nonstandard methods in combinatorics, but only as a taste of that area of research. Secondly, the presented re-

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sults are only examples of "first-level" applications of the nonstandard machinery; for more advanced results one needs higher-level nonstandard tools, such as saturation and Loeb measure, combined with other non-elementary mathematical arguments.

1 The hyper-numbers of nonstandard analysis

This introductory section contains an informal description of the basics of nonstandard analysis, starting with the hyper-natural numbers. Let us stress that what follows are not rigorous definitions and results, but only informal discussions aimed to help the intuition and provide the essential tools to understand the rest of the paper.[1]

One possible way to describe the hyper-natural numbers ${}^*\mathbb{N}$ is the following:

- The *hyper-natural numbers* ${}^*\mathbb{N}$ are the natural numbers when seen with a "telescope" which allows to also see infinite numbers beyond the usual finite ones. The structure of ${}^*\mathbb{N}$ is essentially the same as $\mathbb{N}$, in the sense that ${}^*\mathbb{N}$ and $\mathbb{N}$ cannot be distinguished by any "elementary property".

Here by *elementary property* we mean a property that talks about elements but *not* about subsets[2], and where no use of the notion of infinite or finite number is made.

In consequence of the above, the order structure of ${}^*\mathbb{N}$ is clear. After the usual finite numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$, one finds the infinite numbers $\xi > n$ for all $n \in \mathbb{N}$. Every $\xi \in {}^*\mathbb{N}$ has a successor $\xi + 1$, and every non-zero $\xi \in {}^*\mathbb{N}$ has a predecessor $\xi - 1$.

$$ {}^*\mathbb{N} = \Big\{ \underbrace{1, 2, 3, \ldots, n, \ldots}_{\text{finite numbers}} \quad \underbrace{\ldots, N-2, N-1, N, N+1, N+2, \ldots}_{\text{infinite numbers}} \Big\} $$

Thus the set of finite numbers $\mathbb{N}$ has not a greatest element and the set of infinite numbers $\mathbb{N}_\infty = {}^*\mathbb{N} \setminus \mathbb{N}$ has not a least element, and hence ${}^*\mathbb{N}$ is *not* well-ordered. Remark that being a well-ordered set is not an "elementary property" because it is about subsets, not elements.[3]

[1] A model for the introduced notions will be constructed in the last section.

[2] In logic, this kind of properties are called *first-order* properties.

[3] In logic, this kind of properties are called *second-order* properties.

- The *hyper-integers* ${}^*\mathbb{Z}$ are the discretely ordered ring whose positive part is the semiring ${}^*\mathbb{N}$.
- The *hyper-rationals* ${}^*\mathbb{Q}$ are the ordered field of fractions of ${}^*\mathbb{Z}$.

Thus ${}^*\mathbb{Z} = -{}^*\mathbb{N} \cup \{0\} \cup {}^*\mathbb{N}$, where $-{}^*\mathbb{N} = \{-\xi \mid \xi \in {}^*\mathbb{N}\}$ are the negative hyper-integers. The hyper-rational numbers $\zeta \in {}^*\mathbb{Q}$ can be represented as ratios $\zeta = \frac{\xi}{\nu}$ where $\xi \in {}^*\mathbb{Z}$ and $\nu \in {}^*\mathbb{N}$.

As the next step, one considers the hyper-real numbers, which are instrumental in nonstandard calculus.

- The *hyper-reals* ${}^*\mathbb{R}$ are an ordered field that properly extends both ${}^*\mathbb{Q}$ and $\mathbb{R}$. The structures $\mathbb{R}$ and ${}^*\mathbb{R}$ satisfy the same "elementary properties".

As a proper extension of $\mathbb{R}$, the field ${}^*\mathbb{R}$ is *not* Archimedean, *i.e.* it contains non-zero *infinitesimal* and *infinite* numbers. (Recall that a number ε is infinitesimal if $-1/n < \varepsilon < 1/n$ for all $n \in \mathbb{N}$; and a number Ω is infinite if $|\Omega| > n$ for all n.) In consequence, the field ${}^*\mathbb{R}$ is *not* complete: *e.g.*, the bounded set of infinitesimals has not a least upper bound.[4]

Each set $A \subseteq \mathbb{R}$ has its *hyper-extension* ${}^*A \subseteq {}^*\mathbb{R}$, where $A \subseteq {}^*A$. *E.g.*, one has the set of hyper-even numbers, the set of hyper-prime numbers, the set of hyper-irrational numbers, and so forth. Similarly, any function $f : A \to B$ has its *hyper-extension* ${}^*f : {}^*A \to {}^*B$, where ${}^*f(a) = f(a)$ for all $a \in A$. More generally, in nonstandard analysis one considers hyper-extensions of arbitrary sets and functions.

The general principle that hyper-extensions are indistinguishable from the starting objects as far as their "elementary properties" are concerned, is called *transfer principle*.

- *Transfer principle*: An "elementary property" P holds for the sets $A_1, \dots, A_k$ and the functions $f_1, \dots, f_h$ if and only if P holds for the corresponding hyper-extensions:

$$P(A_1, \dots, A_k, f_1, \dots, f_h) \iff P({}^*A_1, \dots, {}^*A_k, {}^*f_1, \dots, {}^*f_h)$$

Remark that all basic set properties are elementary, and so $A \subseteq B \Leftrightarrow {}^*A \subseteq {}^*B$, $A \cup B = C \Leftrightarrow {}^*A \cup {}^*B = {}^*C$, $A \setminus B = C \Leftrightarrow {}^*A \setminus {}^*B = {}^*C$, and so forth.

4 Remark that the property of completeness is *not* elementary, because it talks about subsets and not about elements of the given field. Also the Archimedean property is *not* elementary, because it requires the notion of *finite* hyper-natural number to be formulated.

As direct applications of *transfer* one obtains the following facts: The hyper-rationals ${}^*\mathbb{Q}$ are *dense* in the hyper-reals ${}^*\mathbb{R}$; every hyper-real number $\xi \in {}^*\mathbb{R}$ has an an *integer part*, *i.e.* there exists a unique hyper-integer $\mu \in {}^*\mathbb{Z}$ such that $\mu \leq \xi < \mu + 1$; and so forth.

As our first example of nonstandard reasoning, let us see a proof of König's Lemma, one of the oldest results in infinite combinatorics.

Theorem 1.1 (König's Lemma – 1927). *If a finite branching tree has infinitely many nodes, then it has an infinite branch.*

Nonstandard proof. Given a finite branching tree T, consider the sequence of its finite levels $\langle T_n \mid n \in \mathbb{N}\rangle$, and let $\langle T_\nu \mid \nu \in {}^*\mathbb{N}\rangle$ be its hyper-extension. By the hypotheses, it follows that all finite levels $T_n \neq \emptyset$ are nonempty. Then, by *transfer*, also all "hyper-levels" T_ν are nonempty. Pick a node $\tau \in T_\nu$ for some infinite ν. Then $\{t \in T \mid t \leq \tau\}$ is an infinite branch of T. □

2 Piecewise syndetic sets

A notion of largeness used in combinatorics of numbers is the following.

- A set of integers A is *thick* if it includes arbitrarily long intervals:

$$\forall n \in \mathbb{N}\, \exists x \in \mathbb{Z}\ \ [x, x+n) \subseteq A.$$

In the language of nonstandard analysis:

Definition 2.1 (Nonstandard). A is *thick* if $I \subseteq {}^*A$ for some infinite interval I.

By *infinite interval* we mean an interval $[\nu, \mu] = \{\xi \in {}^*\mathbb{Z} \mid \nu \leq \xi \leq \mu\}$ with infinitely many elements or, equivalently, an interval whose length $\mu - \nu + 1$ is an infinite number.

Another important notion is that of syndeticity. It stemmed from dynamics, corresponding to finite return-time in a discrete setting.

- A set of integers A is *syndetic* if it has bounded gaps:

$$\exists k \in \mathbb{N}\, \forall x \in \mathbb{Z}\ \ [x, x+k) \cap A \neq \emptyset.$$

So, a set is syndetic means that its complement is not thick. In the language of nonstandard analysis:

Definition 2.2 (Nonstandard). A is *syndetic* if ${}^*A \cap I \neq \emptyset$ for every infinite interval I.

The fundamental structural property considered in Ramsey theory is that of partition regularity.

- A family $\mathcal{F}$ of sets is *partition regular* if whenever an element $A \in \mathcal{F}$ is finitely partitioned $A = A_1 \cup \ldots \cup A_n$, then at least one piece $A_i \in \mathcal{F}$.

Remark that the family of syndetic sets fails to be partition regular.[5] However, a suitable weaking of syndeticity satisfies the property.

- A set of integers A is *piecewise syndetic* if $A = T \cap S$ where T is thick and S is syndetic; *i.e.*, A has bounded gaps on arbitrarily large intervals:

$$\exists k \in \mathbb{N}\ \forall n \in \mathbb{N}\ \exists y \in \mathbb{Z}\ \forall x \in \mathbb{Z}\ \ [x, x+k) \subseteq [y, y+n) \Rightarrow$$
$$\Rightarrow [x, x+k) \cap A \neq \emptyset.$$

In the language of nonstandard analysis:

Definition 2.3 (Nonstandard). A is *piecewise syndetic* (PS for short) if there exists an infinite interval I such that ${}^*A \cap I$ has only finite gaps, *i.e.* ${}^*A \cap J \neq \emptyset$ for every infinite subinterval $J \subseteq I$.

Several results suggest the notion of piecewise syndeticity as a relevant one in combinatorics of numbers. *E.g.*, the sumset of two sets of natural numbers having positive density is piecewise syndetic[6]; every piecewise syndetic set contains arbitrarily long arithmetic progressions; a set is piecewise syndetic if and only if it belongs to a minimal idempotent ultrafilter[7].

Theorem 2.4. *The family of PS sets is partition regular.*

[5] *E.g.*, consider the partition of the integers determined by

$$A = \bigcup_{n\in\mathbb{N}} [-2^{2n}, -2^{2n-1}) \cup \bigcup_{n\in\mathbb{N}} [2^{2n-1}, 2^{2n})$$

and its complement $\mathbb{Z} \setminus A$, neither of which is syndetic.

[6] This is *Jin's theorem*, proved in 2000 by using nonstandard analysis (see [13]).

[7] See [11, Section 4.4].

Nonstandard proof. By induction, it is enough to check the property for 2-partitions. So, let us assume that $A = \text{BLUE} \cup \text{RED}$ is a PS set; we have to show that RED or BLUE is PS. We proceed as follows:

- Take the hyper-extensions ${}^*A = {}^*\text{BLUE} \cup {}^*\text{RED}$.
- By the hypothesis, we can pick an infinite interval I where ${}^*A$ has only finite gaps.
- If the *blue elements of ${}^*A$ have only finite gaps in I, then BLUE is piecewise syndetic.
- Otherwise, there exists an infinite interval $J \subseteq I$ that only contains *red elements of ${}^*A$. But then *RED has only finite gaps in J, and hence RED is piecewise syndetic. □

3 Banach and Shnirelmann densities

An important area of research in number theory focuses on combinatorial properties of sets which depend on their density. Recall the following notions:

- The *upper asymptotic density* $\overline{d}(A)$ of a set $A \subseteq \mathbb{N}$ is defined by putting:
$$\overline{d}(A) \;=\; \limsup_{n\to\infty} \frac{|A \cap [1,n]|}{n}\,.$$
- The *upper Banach density* $\text{BD}(A)$ of a set of integers $A \subseteq \mathbb{Z}$ generalizes the upper density by considering arbitrary intervals in place of just initial intervals:
$$\begin{aligned}\text{BD}(A) \;&=\; \lim_{n\to\infty}\left(\max_{x\in\mathbb{Z}} \frac{|A\cap[x+1,x+n]|}{n}\right)\\ &=\; \inf_{n\in\mathbb{N}}\left\{\max_{x\in\mathbb{Z}} \frac{|A\cap[x+1,x+n]|}{n}\right\}.\end{aligned}$$

In order to translate the above definitions in the language of nonstandard analysis, we need to introduce new notions.

In addition to hyper-extensions, a larger class of well-behaved subsets of ${}^*\mathbb{Z}$ that is considered in nonstandard analysis is the class of *internal* sets. All sets that can be "described" without using the notions of finite or infinite number are internal. Typical examples are the intervals

$$[\xi,\zeta] = \{x \in {}^*\mathbb{Z} \mid \xi \le x \le \zeta\}\,;\quad [\xi,+\infty) = \{x \in {}^*\mathbb{Z} \mid \xi \ge x\}\,;\quad \textit{etc.}$$

Also finite subsets $\{\xi_1, \dots, \xi_n\} \subset {}^*\mathbb{Z}$ are internal, as they can be described by simply giving the (finite) list of their elements. Internal subsets of ${}^*\mathbb{Z}$ share the same "elementary properties" of the subsets of $\mathbb{Z}$. *E.g.*, every nonempty internal subset of ${}^*\mathbb{Z}$ that is bounded below has a least element; in consequence, the set $\mathbb{N}_\infty$ of infinite hyper-natural numbers is *not* internal. Internal sets are closed under unions, intersections, and relative complements. So, also the set of finite numbers $\mathbb{N}$ is *not* internal, as otherwise $\mathbb{N}_\infty = {}^*\mathbb{N} \setminus \mathbb{N}$ would be internal.

Internal sets are either *hyper-infinite* or *hyper-finite*; for instance, all intervals $[\xi, +\infty)$ are hyper-infinite, and all intervals $[\xi, \zeta]$ are hyper-finite. Every nonempty hyper-finite set $A \subset {}^*\mathbb{Z}$ has its *internal cardinality* $\|A\| \in {}^*\mathbb{N}$; for instance $\|[\xi, \zeta]\| = \zeta - \xi + 1$. Internal cardinality and the usual cardinality agree on finite sets.

If $\xi, \zeta \in {}^*\mathbb{R}$ are hyperreal numbers, we write $\xi \sim \zeta$ when ξ and ζ are *infinitely close*, *i.e.* when their distance $|\xi - \zeta|$ is infinitesimal. Remark that if $\xi \in {}^*\mathbb{R}$ is finite (*i.e.*, not infinite), then there exists a unique real number $r \sim \xi$, namely $r = \inf\{x \in \mathbb{R} \mid x > \xi\}$.[8]

We are finally ready to formulate the definitions of density in nonstandard terms.

Definition 3.1 (Nonstandard). For $A \subseteq \mathbb{N}$, its *upper asymptotic density* $\overline{d}(A) = \beta$ is the greatest real number β such that there exists an infinite $\nu \in {}^*\mathbb{N}$ with

$$\|{}^*A \cap [1, \nu]\|/\nu \ \sim\ \beta\,.$$

Definition 3.2 (Nonstandard). For $A \subseteq \mathbb{Z}$, its *upper Banach density* $\mathrm{BD}(A) = \beta$ is the greatest real number β such that there exists an infinite interval I with

$$\|{}^*A \cap I\|/\|I\| \ \sim\ \beta\,.$$

Another notion of density that is widely used in number theory is the following.

- The *Schnirelmann density* $\sigma(A)$ of a set $A \subseteq \mathbb{N}$ is defined by

$$\sigma(A) \ =\ \inf_{n \in \mathbb{N}} \frac{|A \cap [1, n]|}{n}.$$

8 Such a real number r is usually called the *standard part* of ξ.

Clearly $\mathrm{BD}(A) \geq \overline{d}(A) \geq \sigma(A)$, and it is easy to find examples where inequalities are strict. Remark that $\sigma(A) = 1 \Leftrightarrow A = \mathbb{N}$, and that $\mathrm{BD}(A) = 1 \Leftrightarrow A$ is thick. Moreover, if A is piecewise syndetic then $\mathrm{BD}(A) > 0$, but not conversely.

Let us now recall a natural notion of embeddability for the combinatorial structure of sets:[9]

- We say that X is *finitely embeddable* in Y, and write $X \leq_{\mathrm{fe}} Y$, if every finite $F \subseteq X$ has a shifted copy $t + F \subseteq Y$.

It is readily seen that transitivity holds: $X \leq_{\mathrm{fe}} Y$ and $Y \leq_{\mathrm{fe}} Z$ imply $X \leq_{\mathrm{fe}} Z$. Notice that a set is $\leq_{\mathrm{fe}}$-maximal if and only if it is thick. Finite embeddability preserves fundamental combinatorial notions:

- If $X \leq_{\mathrm{fe}} Y$ and X is PS, then also Y is PS.
- If $X \leq_{\mathrm{fe}} Y$ and X contains an arithmetic progression of length k, then also Y contains an arithmetic progression of length k.
- If $X \leq_{\mathrm{fe}} Y$ then $\mathrm{BD}(X) \leq \mathrm{BD}(Y)$.

Remark that while piecewise syndeticity is preserved under $\leq_{\mathrm{fe}}$, the property of being syndetic is *not*. Similarly, the upper Banach density is preserved or increased under $\leq_{\mathrm{fe}}$, but upper asymptotic density is *not*.

Other properties that suggest finite embeddability as a useful notion are the following:

- If $X \leq_{\mathrm{fe}} Y$ then $X - X \subseteq Y - Y$;
- If $X \leq_{\mathrm{fe}} Y$ and $X' \leq_{\mathrm{fe}} Y'$ then $X - X' \leq_{\mathrm{fe}} Y - Y'$.

In the nonstandard setting, $X \leq_{\mathrm{fe}} Y$ means that a shifted copy of the whole X is found in the hyper-extension ${}^*Y$.

Definition 3.3 (Nonstandard). $X \leq_{\mathrm{fe}} Y$ if $\nu + X \subseteq {}^*Y$ for a suitable $\nu \in {}^*\mathbb{N}$.

Remark that the key point here is that the shift ν could be an infinite number.

The sample result that we present below, due to R. Jin [12], allows to extend results that hold for sets with positive Schnirelmann density to sets with positive upper Banach density.

[9] This notion is implicit in I.Z. Ruzsa's paper [20], and has been explicitly considered in [6, Section 4]. As natural as it is, it is well possible that finite embeddability has been also considered by other authors, but I am not aware of it.

Theorem 3.4. *Let $BD(A) = \beta > 0$. Then there exists a set $E \subseteq \mathbb{N}$ with $\sigma(E) \geq \beta$ and such that $E \leq_{fe} A$.*

Nonstandard proof. By the nonstandard definition of Banach density, there exists an infinite interval I such that the relative density $\|{}^*A \cap I\|/\|I\| \sim \beta$. By translating if necessary, we can assume without loss of generality that $I = [1, M]$ where $M \in \mathbb{N}_\infty$. By a straight counting argument, we will prove the following:

- **Claim.** *For every $k \in \mathbb{N}$ there exists $\xi \in [1, M]$ such that for all $i = 1, \ldots, k$, the relative density $\|{}^*A \cap [\xi, \xi + i)\|/i \geq \beta - 1/k$.*

We then use an important principle of nonstandard analysis, namely:

- *Overflow*: If $A \subseteq {}^*\mathbb{N}$ is internal and contains all natural numbers, then it also contains all hyper-natural numbers up to an infinite ν:

$$A \text{ internal } \& \ \mathbb{N} \subset A \implies \exists \nu \in \mathbb{N}_\infty \ [1, \nu] \subseteq A.$$

By the Claim, the internal set below includes $\mathbb{N}$:

$$A = \{\nu \in {}^*\mathbb{N} \mid \exists \xi \in [1, M] \ \forall i \leq \nu \ \|{}^*A \cap [\xi, \xi + i)\|/i \geq \beta - 1/\nu\}.$$

Then, by *overflow*, there exists an infinite $\nu \in {}^*\mathbb{N}$ and $\xi \in [1, M]$ such that $\|{}^*A \cap [\xi, \xi + i)\|/i \geq \beta - 1/\nu$ for all $i = 1, \ldots, \nu$. In particular, for all finite $n \in \mathbb{N}$, the real number $\|{}^*A \cap [\xi, \xi + n)\|/n \geq \beta$ because it is not smaller than $\beta - 1/\nu$, which is infinitely close to β. If we denote by $E = \{n \in \mathbb{N} \mid \xi + n \in {}^*A\}$, this means that $\sigma(E) \geq \beta$. The thesis is reached because $\xi + E \subseteq {}^*A$, and hence $E \leq_{\text{fe}} A$, as desired.

We are left to prove the Claim. Given k, assume by contradiction that for every $\xi \in [1, M]$ there exists $i \leq k$ such that $\|{}^*A \cap [\xi, \xi + i)\| < i \cdot (\beta - 1/k)$. By "hyper-induction" on ${}^*\mathbb{N}$, define $\xi_1 = 1$, and $\xi_{s+1} = \xi_s + n_s$ where $n_s \leq k$ is the least natural number such that $\|{}^*A \cap [\xi_s, \xi_s + n_s)\| < n_s \cdot (\beta - 1/k)$; and stop at step N when $M - k \leq \xi_N < M$. Since k is finite, we have $k/M \sim 0$ and $\xi_N/M \sim 1$. Then:

$$\begin{aligned}
\beta &\sim \frac{1}{M} \cdot \left\|{}^*A \cap [1, M]\right\| \sim \frac{1}{M} \cdot \left\|{}^*A \cap [\xi_1, \xi_N)\right\| \\
&= \frac{1}{M} \cdot \sum_{s=1}^{N-1} \left\|{}^*A \cap [\xi_s, \xi_{s+1})\right\| \\
&< \frac{1}{M} \cdot \left(\sum_{s=1}^{N-1} n_s \cdot \left(\beta - \frac{1}{k}\right)\right) = \frac{\xi_N - 1}{M} \cdot \left(\beta - \frac{1}{k}\right) \sim \beta - \frac{1}{k},
\end{aligned}$$

a contradiction. □

The previous theorem can be strengthened in several directions. For instance, one can find E to be "densely" finitely embedded in A, in the sense that for every finite $F \subseteq X$ one has "densely-many" shifted copies included in Y, *i.e.* $\mathrm{BD}(\{t \in \mathbb{Z} \mid t + F \subseteq Y\}) > 0$.[10]

4 Partition regularity problems

In this section we focus on the use of hyper-natural numbers in partition regularity problems.

The notion of partition regularity for families of sets given in Section 2, is sometimes weakened as follows:

- A family $\mathcal{F}$ of sets is *weakly partition regular* on X if for every finite partition $X = C_1 \cup \ldots \cup C_n$ there exists $F \in \mathcal{F}$ which is contained in one piece $F \subseteq C_i$.

Differently from the usual approach to nonstandard analysis, here it turns out useful to work in a framework where hyper-extensions can be iterated, so that one can consider, *e.g.*:

- The hyper-hyper-natural numbers ${}^{**}\mathbb{N}$;
- The hyper-extension ${}^*\xi \in {}^{**}\mathbb{N}$ of an hyper-natural number $\xi \in {}^*\mathbb{N}$;

and so forth. We remark that working with iterated hyper-extensions requires caution, because of the existence of different levels of extensions.[11] Here, it will be enough to notice that, by *transfer*, one has that ${}^*\mathbb{N} \subsetneq {}^{**}\mathbb{N}$, and if $\xi \in {}^*\mathbb{N} \setminus \mathbb{N}$ then ${}^*\xi \in {}^{**}\mathbb{N} \setminus {}^*\mathbb{N}$; and similarly for n-th iterated hyper-extensions.[12]

Let us start with a nonstandard proof of the classic Ramsey theorem for pairs.

Theorem 4.1 (Ramsey – 1928). *Given a finite colouring* $[\mathbb{N}]^2 = C_1 \cup \ldots \cup C_r$ *of the pairs of natural numbers, there exists an infinite set* H *whose pairs are monochromatic:* $[H]^2 \subseteq C_i$.[13]

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[10] See [6,9] for more on this topic.

[11] See [7] for a discussion of the foundations of iterated hyper-extensions.

[12] Notice also that ${}^*\mathbb{N}$ is an initial segment of ${}^{**}\mathbb{N}$, *i.e.* $\xi < \nu$ for every $\xi \in {}^*\mathbb{N}$ and for every $\nu \in {}^{**}\mathbb{N} \setminus {}^*\mathbb{N}$ (such a property is not used in this paper).

[13] In other words, the family $\mathcal{F} = \{[H]^2 \mid H \text{ infinite}\}$ is weakly partition regular on $[\mathbb{N}]^2$.

Nonstandard proof. Take hyper-hyper-extensions and get the finite coloring

$$[{}^{**}\mathbb{N}]^2 \;=\; {}^{**}([\mathbb{N}]^2) \;=\; {}^{**}C_1 \cup \ldots \cup {}^{**}C_r.$$

Pick an infinite $\xi \in {}^*\mathbb{N}$, let i be such that $\{\xi, {}^*\xi\} \in {}^{**}C_i$, and consider the set $A = \{x \in \mathbb{N} \mid \{x, \xi\} \in {}^*C_i\}$. Then $\xi \in \{x \in {}^*\mathbb{N} \mid \{x, {}^*\xi\} \in {}^{**}C_i\} = {}^*A$. Now inductively define the sequence $\{a_1 < a_2 < \ldots < a_n < \ldots\}$ as follows:

- Pick any $a_1 \in A$, and let $B_1 = \{x \in \mathbb{N} \mid \{a_1, x\} \in C_i\}$. Then $\{a_1, \xi\} \in {}^*C_i$ and $\xi \in {}^*B_1$.
- $\xi \in {}^*A \cap {}^*B_1 \Rightarrow A \cap B_1$ is infinite.[14] Then pick $a_2 \in A \cap B_1$ with $a_2 > a_1$.
- $a_2 \in B_1 \Rightarrow \{a_1, a_2\} \in C_i$.
- $a_2 \in A \Rightarrow \{a_2, \xi\} \in {}^*C_i \Rightarrow \xi \in {}^*\{x \in \mathbb{N} \mid \{a_2, x\} \in {}^*C_1\} = {}^*B_2$.
- $\xi \in {}^*A \cap {}^*B_1 \cap {}^*B_2 \Rightarrow$ we can pick $a_3 \in A \cap B_1 \cap B_2$ with $a_3 > a_2$.
- $a_3 \in B_1 \cap B_2 \Rightarrow \{a_1, a_3\}, \{a_2, a_3\} \in C_i$, and so forth.

Then the infinite set $H = \{a_n \mid n \in \mathbb{N}\}$ is such that $[H]^2 \subseteq C_i$. □

We now give some hints on how iterated hyper-extensions can be used in partition regularity of equations. Recall that:

- An equation $E(X_1, \ldots, X_n) = 0$ is [injectively] *partition regular* over $\mathbb{N}$ if the set of [distinct] solutions is weakly partition regular on $\mathbb{N}$, *i.e.*, for every finite coloring $\mathbb{N} = C_1 \cup \ldots \cup C_r$ one finds [distinct] monochromatic $a_1, \ldots, a_n \in C_i$ such that $E(a_1, \ldots, a_n) = 0$.

A useful nonstandard notion in this context is the following:

Definition 4.2. We say that two hyper-natural numbers $\xi, \zeta \in {}^*\mathbb{N}$ are *indiscernible*, and write $\xi \simeq \zeta$, if they cannot be distinguished by any hyper-extension, *i.e.* if for every $A \subseteq \mathbb{N}$ one has either $\xi, \zeta \in {}^*A$ or $\xi, \zeta \notin {}^*A$.[15]

[14] Here we use the fact that the hyper-extension ${}^*X$ of a set $X \subseteq \mathbb{N}$ contains infinite numbers if and only if X is infinite.

[15] The name "indiscernible" is borrowed from mathematical logic. Recall that in model theory two elements are named *indiscernible* if they cannot be distinguished by any first-order formula.

Notice that indiscernibility coincides with equality on finite numbers, because if $k \in \mathbb{N}$ is finite and $\xi \neq k$, then trivially $k \in \{k\} = {}^*\{k\}$ and $\xi \notin {}^*\{k\}$. Notice also that if $k > 1$ is any natural number, then $k\xi \not\simeq \xi$. Indeed, if A is the set of those natural numbers n with the property that the largest exponent a such that k^a divides n is even, then $\xi \in {}^*A \Leftrightarrow k\xi \notin {}^*A$. A useful property that one can easily prove is the following: "If $\xi \simeq \zeta$, then for every $f : \mathbb{N} \to \mathbb{N}$ one has ${}^*f(\xi) \simeq {}^*f(\zeta)$."

By using the notion of indiscernibility, one can reformulate in nonstandard terms:

Definition 4.3 (Nonstandard). An equation $E(X_1, \dots, X_n) = 0$ is [injectively] *partition regular* on $\mathbb{N}$ if there exist [distinct] hyper-natural numbers $\xi_1 \simeq \dots \simeq \xi_n$ such that $E(\xi_1, \dots, \xi_n) = 0$.

The following result recently appeared in [5].

Theorem 4.4. *The equation $X + Y = Z^2$ is* not *partition regular on $\mathbb{N}$, except for the trivial solution $X = Y = Z = 2$.*

Nonstandard proof. Assume by contradiction that there exist $\alpha \simeq \beta \simeq \gamma$ in ${}^*\mathbb{N}$ such that $\alpha + \beta = \gamma^2$. Notice that α, β, γ are infinite, as otherwise $\alpha = \beta = \gamma = 2$ would be the trivial solution. By the hypothesis of indiscernibility, α, β, γ belong to the same congruence class modulo 5, say $\alpha \equiv \beta \equiv \gamma \equiv i \mod 5$ with $0 \leq i \leq 4$. Now write the numbers in the forms:

$$\alpha = 5^a \cdot \alpha_1 + i\,; \quad \beta = 5^b \cdot \beta_1 + i\,; \quad \gamma = 5^c \cdot \gamma_1 + i$$

where $a, b, c > 0$ and $\alpha_1, \beta_1, \gamma_1$ are not divisible by 5. Pick a function $f : \mathbb{N} \to \mathbb{N}$ such that, for $n \geq 5$, the value $f(n)$ is the unique $k \not\equiv 0 \mod 5$ such that $n = 5^h k + i$ for suitable $h > 0$ and $0 \leq i \leq 4$. Observe that $\alpha_1, \beta_1, \gamma_1$ are the images under ${}^*f$ of α, β, γ respectively; so, $\alpha \simeq \beta \simeq \gamma$ implies that $\alpha_1 \simeq \beta_1 \simeq \gamma_1$, and therefore $\alpha_1 \equiv \beta_1 \equiv \gamma_1 \equiv j \not\equiv 0 \mod 5$.

The equality $\alpha + \beta = \gamma^2$ implies that either $i = 0$ or $i = 2$. Assume first that $i = 0$. In this case $\gamma^2 = 5^{2c}\gamma_1^2$ where $\gamma_1^2 \equiv j^2 \not\equiv 0 \mod 5$. If $a < b$ then $\alpha + \beta = 5^a(\alpha_1 + 5^{b-a}\beta_1)$ where $\alpha_1 + 5^{b-a}\beta_1 \equiv j \not\equiv 0 \mod 5$. It follows that $2c = a \simeq c$, a contradiction. If $a > b$ the proof is similar. If $a = b$ then $\alpha + \beta = 5^a(\alpha_1 + \beta_1)$ where $\alpha_1 + \beta_1 \equiv 2j \not\equiv 0 \mod 5$, and also in this case we would have $2c = a \simeq c$, a contradiction. If $i = 2$ then $\gamma^2 - 4 = 5^c(5^c\gamma_1^2 + 4\gamma_1)$ where $5^c\gamma_1^2 + 4\gamma_1 \equiv 4j \not\equiv 0 \mod 5$. Now, in case $a < b$, one has that $\alpha + \beta - 4 = 5^a(\alpha_1 + 5^{b-a}\beta_1)$ where $\alpha_1 + 5^{b-a}\beta_1 \equiv j \not\equiv 0 \mod 5$, and so it would follow that $5^c\gamma_1^2 + 4\gamma_1 =$

$\alpha_1 + 5^{b-a}\beta_1$. But then we would have $4j \equiv j$, which is not possible because $j \not\equiv 0$. The case $a > b$ is similar. Finally, if $a = b$ then $\alpha + \beta - 4 = 5^a(\alpha_1 + \beta_1)$ where $\alpha_1 + \beta_1 \equiv 2j \not\equiv 0 \mod 5$, and it would follow that $4j \equiv 2j$, again reaching the contradiction $j \equiv 0$. □

The notion of indiscernibility naturally extends to the iterated hyper-extensions of the natural numbers. *E.g.*, if $\Omega, \Xi \in {}^{**}\mathbb{N}$ then $\Omega \simeq \Xi$ means that for every $A \subseteq \mathbb{N}$ one has either $\Omega, \Xi \in {}^{**}A$ or $\Omega, \Xi \notin {}^{**}A$. Notice that $\alpha \simeq {}^*\alpha$ for every $\alpha \in {}^*\mathbb{N}$.

In the sequel, a fundamental role will be played by the following special numbers.

Definition 4.5. A hyper-natural number $\xi \in {}^*\mathbb{N}$ is *idempotent* if $\xi \simeq \xi + {}^*\xi$.[16]

Recall van der Waerden Theorem: *"Arbitrarily large monochromatic arithmetic progressions are found in every finite coloring of* $\mathbb{N}$*"*. Here we prove a weakened version about 3-term arithmetic progressions, by showing the partition regularity of a suitable equation.

Theorem 4.6. *The diophantine equation* $X_1 - 2X_2 + X_3 = 0$ *is injectively partition regular on* $\mathbb{N}$*, which means that for every finite coloring of* $\mathbb{N}$ *there exists a non-constant monochromatic 3-term arithmetic progression.*

Nonstandard proof. Pick an idempotent number $\xi \in {}^*\mathbb{N}$. The following three distinct numbers in ${}^{***}\mathbb{N}$ are a solution of the given equation:

$$\nu = 2\xi + 0 + {}^{**}\xi\ ;\ \mu = 2\xi + {}^*\xi + {}^{**}\xi\ ;\ \lambda = 2\xi + 2{}^*\xi + {}^{**}\xi.$$

That $\nu \simeq \mu \simeq \lambda$ are indiscernible is proved by a direct computation. Precisely, notice that ${}^*\xi \simeq \xi + {}^*\xi$ by the idempotency hypothesis, and so, for every $A \subseteq \mathbb{N}$ and for every $n \in \mathbb{N}$, we have that

$${}^*\xi \in {}^{**}A - n = {}^{**}(A - n) \Leftrightarrow \xi + {}^*\xi \in {}^{**}(A - n).$$

In consequence, the properties listed below are equivalent to each other:

[16] The name "idempotent" is justified by its characterization in terms of ultrafilters: "$\xi \in {}^*\mathbb{N}$ *is idempotent if and only if the corresponding ultrafilter* $\mathfrak{U}_\xi = \{A \subseteq \mathbb{N} \mid \xi \in {}^*A\}$ *is idempotent with respect to the "pseudo-sum" operation:*

$$A \in \mathcal{U} \oplus \mathcal{V} \iff \{n \mid A - n \in \mathcal{V}\} \in \mathcal{U}$$

where $A - n = \{m \mid m + n \in A\}$". The algebraic structure $(\beta\mathbb{N}, \oplus)$ on the space of ultrafilters $\beta\mathbb{N}$ and its related generalizations have been deeply investigated during the last forty years, revealing a powerful tool for applications in Ramsey theory and combinatorial number theory (see the comprehensive monography [11]). In this area of research, idempotent ultrafilters are instrumental.

- $2\xi + {}^{*}\xi + {}^{**}\xi \in {}^{***}A$
- $2\xi \in ({}^{***}A - {}^{**}\xi - {}^{*}\xi) \cap {}^{*}\mathbb{N} = {}^{*}[({}^{**}A - {}^{*}\xi - \xi) \cap \mathbb{N}]$
- $2\xi \in {}^{*}\{n \in \mathbb{N} \mid \xi + {}^{*}\xi \in {}^{**}(A - n)\}$
- $2\xi \in {}^{*}\{n \in \mathbb{N} \mid {}^{*}\xi \in {}^{**}(A - n)\}$
- $2\xi \in {}^{*}[({}^{**}A - {}^{*}\xi) \cap \mathbb{N}] = ({}^{***}A - {}^{**}\xi) \cap {}^{*}\mathbb{N}$
- $2\xi + {}^{**}\xi \in {}^{***}A$.

This shows that $\nu \simeq \mu$. The other relation $\mu \simeq \lambda$ is proved in the same fashion.[17] □

One can elaborate on the previous nonstandard proof and generalize the technique. Notice that the considered elements μ, ν, λ were linear combinations of iterated hyper-extensions of a fixed idempotent number ξ, and so they can be described by the corresponding finite strings of coefficients in the following way:

- $\nu = 2\xi + 0 + {}^{**}\xi \ \rightsquigarrow\ \langle 2, 0, 1\rangle$
- $\mu = 2\xi + {}^{*}\xi + {}^{**}\xi \ \rightsquigarrow\ \langle 2, 1, 1\rangle$
- $\lambda = 2\xi + 2{}^{*}\xi + {}^{**}\xi \ \rightsquigarrow\ \langle 2, 2, 1\rangle$

Indiscernibility of such linear combinations is characterized by means of a suitable equivalence relation $\approx$ on the finite strings, so that, *e.g.*, $\langle 2, 0, 1\rangle \approx \langle 2, 1, 1\rangle \approx \langle 2, 2, 1\rangle$.

Definition 4.7. The equivalence $\approx$ between (finite) strings of integers is the smallest equivalence relation such that:

- The empty string $\approx \langle 0\rangle$.
- $\langle a\rangle \approx \langle a, a\rangle$ for all $a \in \mathbb{Z}$.
- $\approx$ is coherent with *concatenations*, *i.e.*

$$\sigma \approx \sigma' \text{ and } \tau \approx \tau' \implies \sigma^\frown\tau \approx \sigma'^\frown\tau'.$$

So, $\approx$ is preserved by inserting or removing zeros, by repeating finitely many times a term or, conversely, by shortening a block of consecutive equal terms. The following characterization is proved in [7]:

- Let $\xi \in {}^{*}\mathbb{N}$ be idempotent. Then the following are equivalent:

  1. $a_0\xi + a_1{}^{*}\xi + \ldots + a_k \cdot {}^{k*}\xi \simeq b_0\xi + b_1{}^{*}\xi + \ldots + b_h \cdot {}^{h*}\xi$

[17] Here we actually proved the following result ([3] Th. 2.10): *"Let $\mathcal{U}$ be any idempotent ultrafilter. Then every set $A \in 2\mathcal{U} \oplus \mathcal{U}$ contains a 3-term arithmetic progression"*.

2. $\langle a_0, a_1, \ldots, a_k\rangle \approx \langle b_0, b_1, \ldots, b_h\rangle$.

Recall Rado theorem: *"The diophantine equation $c_1X_1+\ldots+c_nX_n=0$ ($c_i\neq 0$) is partition regular if and only if $\sum_{i\in F} c_i = 0$ for some nonempty $F \subseteq \{1,\ldots,n\}$"*. By using the above equivalence, one obtains a nonstandard proof of a modified version of Rado theorem, with a stronger hypothesis and a stronger thesis.

Theorem 4.8. *Let $c_1X_1+\ldots+c_nX_n = 0$ be a diophantine equation with $n\geq 3$. If $c_1+\ldots+c_n=0$ then the equation is injectively partition regular on $\mathbb{N}$.*

Nonstandard proof. Fix $\xi\in {}^*\mathbb{N}$ an idempotent element, and for simplicity denote by $\xi_i = {}^{i*}\xi$ the i-th iterated hyper-extension of ξ. For arbitrary $a_1,\ldots,a_{n-1}$, consider the following numbers in ${}^{n*}\mathbb{N}$:

$$\begin{array}{llllllll}
\mu_1 & = a_1\xi & + a_2\xi_1 & + a_3\xi_2 & + \ldots & + a_{n-2}\xi_{n-3} & + a_{n-1}\xi_{n-2} & + a_{n-1}\xi_{n-1}\\
\mu_2 & = a_1\xi & + a_2\xi_1 & + a_3\xi_2 & + \ldots & + a_{n-2}\xi_{n-3} & + 0 & + a_{n-1}\xi_{n-1}\\
\mu_3 & = a_1\xi & + a_2\xi_1 & + a_3\xi_2 & + \ldots & + 0 & + a_{n-2}\xi_{n-2} & + a_{n-1}\xi_{n-1}\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
\mu_{n-2} & = a_1\xi & + a_2\xi_1 & + 0 & + a_3\xi_3 & + \ldots & + a_{n-2}\xi_{n-2} & + a_{n-1}\xi_{n-1}\\
\mu_{n-1} & = a_1\xi & + 0 & + a_2\xi_2 & + a_3\xi_3 & + \ldots & + a_{n-2}\xi_{n-2} & + a_{n-1}\xi_{n-1}\\
\mu_n & = a_1\xi & + a_1\xi_1 & + a_2\xi_2 & + a_3\xi_3 & + \ldots & + a_{n-2}\xi_{n-2} & + a_{n-1}\xi_{n-1}
\end{array}$$

Notice that $\mu_1\simeq\ldots\simeq\mu_n$ because the corresponding strings of coefficients are all equivalent to $\langle a_1,\ldots,a_{n-1}\rangle$. Moreover, it can be easily checked that the μ_is are distinct. To complete the proof, we need to find suitable coefficients $a_1,\ldots,a_{n-1}$ in such a way that $c_1\mu_1+\ldots+c_n\mu_n = 0$. It is readily seen that this happens if the following conditions are fulfilled:

$$\begin{cases}
(c_1+\ldots+c_n)\cdot a_1 = 0\\
(c_1+\ldots+c_{n-2})\cdot a_2 + c_n\cdot a_1 = 0\\
(c_1+\ldots+c_{n-3})\cdot a_3 + (c_{n-1}+c_n)\cdot a_2 = 0\\
\qquad\vdots\\
c_1\cdot a_{n-1} + (c_3+\ldots+c_n)\cdot a_{n-2} = 0\\
(c_1+\ldots+c_n)\cdot a_{n-1} = 0
\end{cases}$$

Finally, observe that the first and last equations are trivially satisfied because of the hypothesis $c_1+\ldots+c_n = 0$; and the remaining $n-$

2 equations are satisfied by infinitely many choices of the coefficients $a_1, \ldots, a_{n-1}$, which can be taken in $\mathbb{N}$.[18] □

More results in this direction, including partition regularity of non-linear diophantine equations, have been recently obtained by L. Luperi Baglini (see [19]).

5 A model of the hyper-integers

In this final section we outline a construction for a model where one can give an interpretation to all nonstandard notions and principles that were considered in this paper.

The most used single construction for models of the hyper-real numbers, and hence of the hyper-natural and hyper-integer numbers, is the *ultrapower*.[19] Here we prefer to use the purely algebraic construction of [2], which is basically equivalent to an ultrapower, but where only the notion of quotient field of a ring modulo a maximal ideal is assumed.

- Consider $\text{Fun}(\mathbb{N}, \mathbb{R})$, the ring of real sequences $\varphi : \mathbb{N} \to \mathbb{R}$ where the sum and product operations are defined pointwise.
- Let $\mathfrak{I}$ be the ideal of the sequences that eventually vanish:

$$\mathfrak{I} = \{\varphi \in \text{Fun}(\mathbb{N}, \mathbb{R}) \mid \exists k \, \forall n \geq k \; \varphi(n) = 0\}.$$

- Pick a maximal ideal $\mathfrak{M}$ extending $\mathfrak{I}$, and define the hyper-real numbers as the quotient field:

$$^*\mathbb{R} = \text{Fun}(\mathbb{N}, \mathbb{R})/\mathfrak{M}.$$

- The *hyper-integers* are the subring of $^*\mathbb{R}$ determined by the sequences that take values in $\mathbb{Z}$:

$$^*\mathbb{Z} = \text{Fun}(\mathbb{N}, \mathbb{Z})/\mathfrak{M} \subset {}^*\mathbb{R}.$$

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[18] Here we actually proved the following result ([7] Th.1.2): *"Let $c_1 X_1 + \ldots + c_n X_n = 0$ be a diophantine equation with $c_1 + \ldots + c_n = 0$ and $n \geq 3$. Then there exists $a_1, \ldots, a_{n-1} \in \mathbb{N}$ such that for every idempotent ultrafilter $\mathcal{U}$ and for every $A \in a_1\mathcal{U} \oplus \ldots \oplus a_{n-1}\mathcal{U}$ there exist distinct $x_i \in A$ such that $c_1 x_1 + \ldots + c_n x_n = 0$".*

[19] For a comprehensive exposition of nonstandard analysis grounded on the ultrapower construction, see R. Goldblatt's textbook [10].

- For every subset $A \subset \mathbb{R}$, its hyper-extension is defined by:

$$ {}^*A = \mathrm{Fun}(\mathbb{N}, A)/\mathfrak{M} \subset {}^*\mathbb{R}. $$

  So, *e.g.*, the *hyper-natural numbers* ${}^*\mathbb{N}$ are the cosets $\varphi + \mathfrak{M}$ of sequences $\varphi : \mathbb{N} \to \mathbb{N}$ of natural numbers; the hyper-prime numbers are the cosets of sequences of prime numbers, and so forth.
- For every function $f : A \to B$ (where $A, B \subseteq \mathbb{R}$), its hyper-extension ${}^*f : {}^*A \to {}^*B$ is defined by putting for every $\varphi : \mathbb{N} \to A$:

$$ {}^*f(\varphi + \mathfrak{M}) = (f \circ \varphi) + \mathfrak{M}. $$

- For every sequence $\langle A_n \mid n \in \mathbb{N}\rangle$ of nonempty subsets of $\mathbb{R}$, its hyper-extension $\langle A_\nu \mid \nu \in {}^*\mathbb{N}\rangle$ is defined by putting for every $\nu = \varphi + \mathfrak{M} \in {}^*\mathbb{N}$:

$$ A_\nu = \{\psi + \mathfrak{M} \mid \psi(n) \in A_{\varphi(n)} \text{ for all } n\} \subseteq {}^*\mathbb{R}. $$

It can be directly verified that ${}^*\mathbb{R}$ is an ordered field whose positive elements are ${}^*\mathbb{R}^+ = \mathrm{Fun}(\mathbb{N}, \mathbb{R}^+)/\mathfrak{M}$. By identifying each $r \in \mathbb{R}$ with the coset $c_r + \mathfrak{M}$ of the corresponding constant sequence, one obtains that ${}^*\mathbb{R}$ is a proper superfield of $\mathbb{R}$. The subset ${}^*\mathbb{Z}$ defined as above is a discretely ordered ring having all the desired properties.

Remark that in the above model, one can interpret all notions used in this paper. We itemize below the most relevant ones.

Denote by $\alpha = \imath + \mathfrak{M} \in {}^*\mathbb{N}$ the infinite hyper-natural number corresponding to the identity sequence $\imath : \mathbb{N} \to \mathbb{N}$.

- The nonempty *internal sets* $B \subseteq {}^*\mathbb{R}$ are the sets of the form $B = A_\alpha$ where $\langle A_n \mid n \in \mathbb{N}\rangle$ is a sequence of nonempty sets. When all A_n are finite, $B = A_\alpha$ is called *hyper-finite*; and when all A_n are infinite, $B = A_\alpha$ is called *hyper-infinite*.[20]
- If $B = A_\alpha$ is the hyper-finite set corresponding to the sequence of nonempty finite sets $\langle A_n \mid n \in \mathbb{N}\rangle$, then its *internal cardinality* is defined by setting $\|B\| = \vartheta + \mathfrak{M} \in {}^*\mathbb{N}$ where $\vartheta(n) = |A_n| \in \mathbb{N}$ is the sequence of cardinalities.
- If $\varphi, \psi : \mathbb{N} \to \mathbb{Z}$ and the corresponding hyper-integers $\nu = \varphi + \mathfrak{M}$ and $\mu = \psi + \mathfrak{M}$ are such that $\nu < \mu$, then the (internal) interval

20 It is proved that any internal set $A \subseteq {}^*\mathbb{R}$ is either hyper-finite or hyper-infinite.

$[\nu, \mu] \subseteq {}^*\mathbb{Z}$ is defined as A_α where $\langle A_n \mid n \in \mathbb{N}\rangle$ is any sequence of sets such that $A_n = [\varphi(n), \psi(n)]$ whenever $\varphi(n) < \psi(n)$.[21]

In full generality, one can show that the *transfer* principle holds. To show this in a rigorous manner, one needs first a precise definition of "elementary property", which requires the formalism of first-order logic. Then, by using a procedure known in logic as "induction on the complexity of formulas", one proves that the equivalences $P(A_1, \ldots, A_k, f_1, \ldots, f_h) \Leftrightarrow P({}^*A_1, \ldots, {}^*A_k, {}^*f_1, \ldots, {}^*f_h)$ hold for all elementary properties P, sets A_i, and functions f_j.

Remark that all the nonstandard definitions given in this paper are actually equivalent to the usual "standard" ones. As examples, let us prove some of those equivalences in detail.

Let us start with the definition of a *thick set* $A \subseteq \mathbb{Z}$. Assume first that there exists a sequence of intervals $\langle [a_n, a_n + n] \mid n \in \mathbb{N}\rangle$ which are included in A. If $\langle [a_\nu, a_\nu + \nu] \mid \nu \in {}^*\mathbb{N}\rangle$ is its hyper-extension then, by *transfer*, every $[a_\nu, a_\nu + \nu] \subseteq {}^*A$, and hence ${}^*A$ includes infinite intervals. Conversely, assume that A is not thick and pick $k \in \mathbb{N}$ such that for every $x \in \mathbb{Z}$ the interval $[x, x + k] \nsubseteq A$. Then, by *transfer*, for every $\xi \in {}^*\mathbb{Z}$ the interval $[\xi, \xi + k] \nsubseteq {}^*A$, and hence ${}^*A$ does not contain any infinite interval.

We now focus on the nonstandard definition of *upper Banach density*. Let $\mathrm{BD}(A) \geq \beta$. Then for every $k \in \mathbb{N}$, there exists an interval $I_k \subset \mathbb{Z}$ of length $|I_k| \geq k$ and such that $|A \cap I_k|/|I_k| > \beta - 1/k$. By *overflow*, there exists an infinite $\nu \in {}^*\mathbb{N}$ and an interval $I \subset {}^*\mathbb{Z}$ of internal cardinality $\|I\| \geq \nu$ such that the ratio $\|{}^*A \cap I\|/\|I\| \geq \beta - 1/\nu \sim \beta$. Conversely, let I be an infinite interval such that $\|{}^*A \cap I\|/\|I\| \sim \beta$. Then, for every given $k \in \mathbb{N}$, the following property holds: "There exists an interval $I \subset {}^*\mathbb{Z}$ of length $\|I\| \geq k$ and such that $\|{}^*A \cap I\|/\|I\| \geq \beta - 1/k$". By *transfer*, we obtain the existence of an interval $I_k \subset \mathbb{Z}$ of length $|I_k| \geq k$ and such that $|A \cap I_k|/|I_k| \geq \beta - 1/k$. This shows that $\mathrm{BD}(A) \geq \beta$, and the proof is complete.

Let us now turn to *finite embeddability*. Assume that $X \leq_{\mathrm{fe}} Y$, and enumerate $X = \{x_n \mid n \in \mathbb{N}\}$. By the hypothesis, $\bigcap_{i=1}^n (Y - x_i) \neq \emptyset$ for every $n \in \mathbb{N}$ and so, by *overflow*, there exists an infinite $\mu \in {}^*\mathbb{N}$ such that the hyper-finite intersection $\bigcap_{i=1}^\mu ({}^*Y - x_i) \neq \emptyset$. If ν is any

[21] One can prove that this definition is well-posed. Indeed, if $\varphi + \mathfrak{M} < \psi + \mathfrak{M}$ and $\langle A_n \mid n \in \mathbb{N}\rangle$ and $\langle A'_n \mid n \in \mathbb{N}\rangle$ are two sequences of nonempty sets such that $A_n = A'_n$ whenever $\varphi(n) < \psi(n)$, then $A_\alpha = A'_\alpha$.

hyper-integer in that intersection, then $\nu + X \subseteq {}^*Y$. Conversely, let us assume that $\nu + X \subseteq {}^*Y$ for a suitable $\nu \in {}^*\mathbb{Z}$. Then for every finite $F = \{x_1, \ldots, x_k\} \subset X$ one has the elementary property: "$\exists \nu \in {}^*\mathbb{Z}\ (\nu + x_1 \in {}^*Y\ \&\ \ldots\ \&\ \nu + x_k \in {}^*Y)$". By *transfer*, it follows that "$\exists t \in \mathbb{Z}\ (t + x_1 \in Y\ \&\ \ldots\ \&\ t + x_k \in Y)$", *i.e.* $t + F \subseteq Y$.[22]

We finish this paper with a few suggestions for further readings. A rigorous formulation and a detailed proof of the *transfer principle* can be found in Chapter 4 of the textbook [10], where the *ultrapower* model is considered.[23] See also Section 4.4 of [4] for the foundations of nonstandard analysis in its full generality. A nice introduction of nonstandard methods for number theorists, including a number of examples, is given in [15] (see also [12]). Finally, a full development of nonstandard analysis can be found in several monographies of the existing literature; see *e.g.* the classical H. J. Keisler's book [18], or the comprehensive collections of surveys in [1].

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[22] For the equivalence of the nonstandard definition of partition regularity of an equation, one needs a richer model than the one presented here. Precisely, one needs the so-called $\mathfrak{c}^+$*-enlargement property*, that can be obtained in models of the form ${}^*\mathbb{R} = \mathrm{Fun}(\mathbb{R}, \mathbb{R})/\mathfrak{M}$ where $\mathfrak{M}$ is a maximal ideals of a special kind (see [2]).

[23] Remark that our algebraic model is basically equivalent to an ultrapower. Indeed, for any maximal ideal $\mathfrak{M}$ of the ring $\mathrm{Fun}(\mathbb{N}, \mathbb{R})$, the family $\mathcal{U} = \{Z(\varphi) \mid \varphi \in \mathfrak{M}\}$ where $Z(f) = \{n \in \mathbb{N} \mid \varphi(n) = 0\}$ is an ultrafilter on $\mathbb{N}$. By identifying each coset $\varphi + \mathfrak{M}$ with the corresponding $\mathcal{U}$-equivalence class $[\varphi]$, one obtains that the quotient field $\mathrm{Fun}(\mathbb{N}, \mathbb{R})/\mathfrak{M}$ and the ultrapower $\mathbb{R}^{\mathbb{N}}/\mathcal{U}$ are essentially the same object.

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