Topological methods in algebraic geometry

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Prologue

Let me begin by citing Hermann Weyl ([93, p. 500]):

'In these days the angel of topology and the devil of abstract algebra fight for the soul of each individual mathematical domain'.

My motivation for this citation is first of all a practical reflexion on the primary role played by the field of Topology in the mathematics of the 20-th century, and the danger that among algebraic geometers this great heritage and its still vivid current interest may be not sufficiently considered.

Second, the word 'soul' used by Weyl reminds us directly of the fact that mathematics is one of the pillars of scientific culture, and that some philosophical discussion about its role in society is deeply needed. [1]

Also, dozens of years dominated by neo liberism, and all the rest, have brought many of us to accept the slogan that mathematics is a key-technology. So, the question which is too often asked is: 'for which immediate purposes is this good for?'[2] Instead of asking: 'how beautiful, important or enriching is this theory?', or 'how do all these theories contribute to deep knowledge and wisdom, and to broad scientific progress?'

While it is of course true that mathematics is extremely useful for the advancement of society and the practical well being of men, yet I would wish that culture and mathematics should be highly respected and supported, without the need of investing incredible amounts of energy devoted to make it survive. Our energy should better be reserved to the major task of making mathematical culture more unified, rather than a Babel tower where adepts of different disciplines can hardly talk to each other.

Thus, in a way, one should conclude trying to underline the fruitful interactions among several fields of mathematics, and thus paraphrase the motto by Weyl by asking: 'How can the angel of topology live happily with the devil of abstract algebra?'.

Now, the interaction of algebraic geometry and topology has been such, in the last three centuries, that it is often difficult to say when does a result belong to one discipline or to the other, the archetypical example being the Bézout theorem, first conceived through geometrical ideas, and later clarified through topology and through algebra.

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[1] Mathematics privileges problem solving and critical thinking versus passive acceptance of dogmatic 'truths'. And the peaceful survival of our current world requires men to lose their primitive nature and mentality and to become culturally more highly developed.

[2] And too often this is only measured by monetary or immediate financial success.

Thus, the ties are so many that I will have to soon converge towards my personal interests. I shall mostly consider moduli theory as the fine part of classification theory of complex varieties: and I shall try to show how in some lucky cases topology helps also for the fine classification, allowing the study of the structure of moduli spaces: as we have done quite concretely in several papers ([10–13,15,16]). Finally I shall present how the theory of moduli, guided by topological considerations, gives in return important information on the Galois group of the field $\bar{\mathbb{Q}}$.

For a broader treatment, I refer the reader to the article [38], to which this note is an invitation.

1. Applications of algebraic topology: non existence and existence of continuous maps

Algebraic topology flourished from some of its applications, inferring the non existence of certain continuous maps from the observation that their existence would imply the existence of homomorphisms satisfying algebraic properties which are manifestly impossible to be verified. The most famous such examples are Brouwer's fixed point theorem, and the theorem of Borsuk-Ulam.

Theorem 1.1 (Brouwer's fixed point theorem). *Every continuous self map $f : D^n \to D^n$, where $D^n = \{x \in \mathbb{R}^n | |x| \leq 1\}$ is the unit disk, has a fixed point,* i.e.*, there is a $x \in D$ such that $f(x) = x$.*

The proof is by contradiction:

1. Assuming that $f(x) \neq x \; \forall x$, let $\phi(x)$ be the intersection of the boundary S^{n-1} of D^n with the half line stemming from $f(x)$ in the direction of x; ϕ would be a continuous map
$$\phi : D^n \to S^{n-1}, \; s.t. \; \phi|_{S^{n-1}} = \mathrm{Id}_{S^{n-1}} .$$
2. *i.e.*, we would have a sequence of two continuous maps (ι is the inclusion) whose composition $\phi \circ \iota$ is the identity
$$\iota : S^{n-1} \to D^n, \; \phi : D^n \to S^{n-1}.$$
3. One uses then the **covariant functoriality** of reduced homology groups $H_i(X, \mathbb{Z})$: to each continuous map $f : X \to Y$ of topological spaces is associated a homomorphism of abelian groups $H_i(f, \mathbb{Z}) : H_i(X, \mathbb{Z}) \to H_i(Y, \mathbb{Z})$, and in such a way that to a composition $f \circ g$ is associated the composition of the corresponding homomorphisms. That is,
$$H_i(f \circ g, \mathbb{Z}) = H_i(f, \mathbb{Z}) \circ H_i(g, \mathbb{Z}).$$
Moreover, to the identity is associated the identity.

4. The key point is (one observes that the disc is contractible) to show that the reduced homology groups

$$H_{n-1}(S^{n-1}, \mathbb{Z}) \cong \mathbb{Z}, \quad H_{n-1}(D^n, \mathbb{Z}) = 0.$$

The functoriality of the homology groups, since $\phi \circ \iota = \mathrm{Id}_{S^{n-1}}$, would imply $0 = H_{n-1}(\phi, \mathbb{Z}) \circ H_{n-1}(\iota, \mathbb{Z}) = H_{n-1}(\mathrm{Id}_{S^{n-1}}, \mathbb{Z}) = \mathrm{Id}_{\mathbb{Z}}$, the desired contradiction.

The cohomology algebra is used instead for the Borsuk-Ulam theorem.

Theorem 1.2 (Borsuk-Ulam theorem). *There exists no odd continuous function $F : S^n \to S^m$ for $n > m$ (F is odd means that $F(-x) = -F(x), \forall x$).*

Here there are two ingredients, the main one being the cohomology algebra, and its contravariant functoriality: to any continuous map $f : X \to Y$ there corresponds an algebra homomorphism

$$f^* : H^*(Y, R) = \oplus_{i=0}^{\dim(Y)} H^i(Y, R) \to H^*(X, R),$$

for any ring R of coefficients.

In our case one takes as $X := \mathbb{P}^n_{\mathbb{R}} = S^n/\{\pm 1\}$, similarly $Y := \mathbb{P}^m_{\mathbb{R}} = S^m/\{\pm 1\}$ and lets f be the continuous map induced by F.

One needs to show that, choosing $R = \mathbb{Z}/2\mathbb{Z}$, then the cohomology algebra of real projective space is a truncated polynomial algebra, namely:

$$H^*(\mathbb{P}^n_{\mathbb{R}}, \mathbb{Z}/2\mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})[\xi_n]/(\xi_n^{n+1}).$$

The other ingredient consists in showing that

$$f^*([\xi_m]) = [\xi_n],$$

$[\xi_m]$ denoting the residue class in the quotient algebra.

One gets then the desired contradiction since, if $n > m$,

$$0 = f^*(0) = f^*([\xi_m]^{m+1}) = f^*([\xi_m])^{m+1} = [\xi_n]^{m+1} \neq 0.$$

Notice that up to now we have mainly used that f is a continuous map $f := \mathbb{P}^n_{\mathbb{R}} \to \mathbb{P}^m_{\mathbb{R}}$, while precisely in order to obtain that $f^*([\xi_m]) = [\xi_n]$ we must make use of the hypothesis that f is induced by an odd function F.

This property can be interpreted as the property that F yields a commutative diagram

$$\begin{array}{ccc} S^n & \to & S^m \\ \downarrow & & \downarrow \\ \mathbb{P}^n_{\mathbb{R}} & \to & \mathbb{P}^m_{\mathbb{R}} \end{array}$$

which exhibits the two sheeted covering of $\mathbb{P}^n_{\mathbb{R}}$ by S^n as the pull-back of the analogous two sheeted cover for $\mathbb{P}^m_{\mathbb{R}}$. Now, as we shall digress soon, any such two sheeted covering is given by a homomorphism of $H_1(X, \mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}$, *i.e.*, by an element in $H^1(X, \mathbb{Z}/2\mathbb{Z})$, and this element is trivial if and only if the covering is trivial (that is, homeomorphic to $X \times (\mathbb{Z}/2\mathbb{Z})$, in other words a disconnected cover).

This shows that the pull back of the cover, which is nontrivial, corresponds to $f^*([\xi_m])$ and is nontrivial, hence $f^*([\xi_m]) = [\xi_n]$.

In this way the proof is accomplished.

Algebraic topology attaches to a good topological space homology groups $H_i(X, R)$, which are covariantly functorial, a cohomology algebra $H^*(X, R)$ which is contravariantly functorial, and these groups can be calculated, by virtue of the Mayer-Vietoris exact sequence and of excision (see any textbook), by chopping the space in smaller pieces. In particular, these groups vanish when $i > \dim(X)$.

To X are also attached the homotopy groups $\pi_i(X)$.

Definition 1.3.

(1) Let $f, g : X \to Y$ be continuous maps. Then f and g are said to be homotopic (one writes $f \sim g$) if there is continuous map $F : X \times [0, 1] \to Y$ such that $f(x) = F(x, 0)$ and $g(x) = F(x, 1)$. Similar definition for maps of pairs $f, g : (X, X') \to (Y, Y')$, which means that $X' \subset X$ is mapped to $f(X') \subset Y' \subset Y$.

(2) $[X, Y]$ is the set of homotopy classes of continuous maps $f : X \to Y$.

(3) $\pi_i(X, x_0) := [(S^i, e_1), (X, x_0)]$ is a group for $i \geq 1$, abelian for $i \geq 2$, and independent of the point $x_0 \in X$ if X is path-connected.

(4) X is said to be homotopy equivalent to Y ($X \sim Y$) if and only if there are continuous maps $f : X \to Y$, $g : Y \to X$ such that $f \circ g$ and $g \circ f$ are both homotopic to the identity (of Y, resp. of X).

The common feature is that homotopic maps induce the same homomorphisms on homology, cohomology, and homotopy.

We are, for our purposes, more interested in the more mysterious homotopy groups, which, while not necessarily vanishing for $i > \dim(X)$, enjoy however a fundamental property.

Recall the definition due to Whitney and Steenrod ([88]) of a fibre bundle. In the words of Steenrod, the notion of a fibre bundle is a weakening of the notion of a product, since a product $X \times Y$ has two continuous projections $p_X : X \times Y \to X$, and $p_Y : X \times Y \to Y$, while a fibre bundle E over B with fibre F has only one projection, $p = p_B : E \to B$ and its similarity to a product lies in the fact that for each point $x \in B$ there is an open set U containing x, and a homeomorphism of $p_B^{-1}(U) \cong U \times F$ compatible with both projections onto U.

The fundamental property of fibre bundles is that there is a long exact sequence of homotopy groups

$$\dots \to \pi_i(F) \to \pi_i(E) \to \pi_i(B) \to \pi_{i-1}(F) \to \pi_{i-1}(E) \to \pi_{i-1}(B) \to \dots$$

where one should observe that $\pi_i(X)$ is a group for $i \geq 1$, an abelian group for $i \geq 2$, and for $i = 0$ is just the set of arc-connected components of X (we assume the spaces to be good, that is, locally arcwise connected, semilocally simply connected, see [55], and, most of the times, connected).

The special case where the fibre F has the discrete topology is the case of a **covering space**, which is called the universal covering if moreover $\pi_1(E)$ is trivial.

Special mention deserves the following more special case.

Definition 1.4. Assume that E is arcwise connected, contractible (hence all homotopy groups $\pi_i(E)$ are trivial), and that the fibre F is discrete, so that all the higher homotopy groups $\pi_i(B) = 0$ for $i \geq 2$, while $\pi_1(B) \cong \pi_0(F) = F$.

Then one says that B is a classifying space $K(\pi, 1)$ for the group $\pi = \pi_1(B)$.

In general, given a group π, a CW complex B is said to be a $K(\pi, 1)$ if $\pi_i(B) = 0$ for $i \geq 2$, while $\pi_1(B) \cong \pi$.

Example 1.5. The easiest examples are the following ones:

1. the real torus $T^n := \mathbb{R}^n/\mathbb{Z}^n$ is a classifying space $K(\mathbb{Z}^n, 1)$ for the group $\pi = \mathbb{Z}^n$;
2. a classifying space $K(\mathbb{Z}/2\mathbb{Z}, 1)$ is given by the inductive limit $\mathbb{P}^\infty_\mathbb{R} := \lim_{n\to\infty} \mathbb{P}^n_\mathbb{R}$.

These classifying spaces, although not unique, are unique up to **homotopy-equivalence** (we use the notation $X \sim_{h.e.} Y$ to denote homotopy equivalence, defined above and meaning that there exist continuous maps $f : X \to Y,\ g : Y \to X$ such that both compositions $f \circ g$ and $g \circ f$ are homotopic to the identity).

Therefore, given two classifying spaces for the same group, they not only do have the same homotopy groups, but also the same homology and cohomology groups. Thus the following definition is well posed.

Definition 1.6. Let Γ be a finitely presented group, and let $B\Gamma$ be a classifying space for Γ: then the homology and cohomology groups and algebra of Γ are defined as

$$H_i(\Gamma, \mathbb{Z}) := H_i(B\Gamma, \mathbb{Z}),$$
$$H^i(\Gamma, \mathbb{Z}) := H^i(B\Gamma, \mathbb{Z}),$$
$$H^*(\Gamma, \mathbb{Z}) := H^*(B\Gamma, \mathbb{Z}),$$

and similarly for other rings of coefficients instead of $\mathbb{Z}$.

We now come to the other side: algebraic topology is not only useful to detect the non existence of certain continuous maps, it is also used to assert the existence of certain continuous maps.

Indeed classifying spaces, even if often quite difficult to construct explicitly, are very important because they guarantee the existence of continuous maps! We have more precisely the following (*cf.* [87, Theorem 9, page 427, and Theorem 11, page 428]).

Theorem 1.7. *Let Y be a 'nice' topological space,* i.e., *Y is homotopy-equivalent to a CW-complex, and let X be a nice space which is a $K(\pi, 1)$ space: then, choosing base points $y_0 \in Y, x_0 \in X$, one has a bijective correspondence*

$$[(Y, y_0), (X, x_0)] \cong \mathrm{Hom}(\pi_1(Y, y_0), \pi_1(X, x_0)), [f] \mapsto \pi_1(f),$$

where $[(Y, y_0), (X, x_0)]$ denotes the set of homotopy classes $[f]$ of continuous maps $f : Y \to X$ such that $f(y_0) = x_0$ (and where the homotopies $F(y, t)$ are also required to satisfy $F(y_0, t) = x_0, \ \forall t \in [0, 1]$).

In particular, the free homotopy classes $[Y, X]$ of continuous maps are in bijective correspondence with the conjugacy classes of homomorphisms $\mathrm{Hom}(\pi_1(Y, y_0), \pi)$ *(conjugation is here inner conjugation by* $\mathrm{Inn}(\pi)$ *on the target).*

While topology deals with continuous maps, when dealing with manifolds more regularity is wished for. For instance, when we choose for Y a differentiable manifold M, and the group π is abelian and torsion free, say $\pi = \mathbb{Z}^r$, then a more precise incarnation of the above theorem is given by the De Rham theory.

We have indeed the following proposition.

Proposition 1.8. *Let Y be a differentiable manifold, and let X be a differentiable manifold that is a $K(\pi, 1)$ space: then, choosing base points $y_0 \in Y, x_0 \in X$, one has a bijective correspondence*

$$[(Y, y_0), (X, x_0)]^{\mathrm{diff}} \cong \mathrm{Hom}(\pi_1(Y), \pi), [f] \mapsto \pi_1(f),$$

where $[(Y, y_0), (X, x_0)]^{\mathrm{diff}}$ denotes the set of differential homotopy classes $[f]$ of differentiable maps $f : Y \to X$ such that $f(y_0) = x_0$.

Remark 1.9. In the case where X is a torus $T^r = \mathbb{R}^r/\mathbb{Z}^r$, then f is obtained as the projection onto T^r of

$$\tilde{\phi}(y) := \int_{y_0}^{y} (\eta_1, \dots, \eta_r), \eta_j \in H^1(Y, \mathbb{Z}) \subset H^1_{DR}(Y, \mathbb{R}).$$

Here η_j is indeed a closed 1-form, representing a certain De Rham cohomology class with integral periods (*i.e.*, $\int_\gamma \eta_j = \varphi(\gamma) \in \mathbb{Z}, \forall \gamma \in \pi_1(Y)$). Therefore f is defined by $\int_{y_0}^{y} (\eta_1, \dots, \eta_r) \mod (\mathbb{Z}^r)$. Moreover, changing η_j with another form $\eta_j + dF_j$ in the same cohomology class, one finds a homotopic map, since $\int_{y_0}^{y} (\eta_j + t dF_j) = \int_{y_0}^{y} (\eta_j) + t(F_j(y) - F_j(y_0))$.

In algebraic geometry, the De Rham theory leads to the theory of Albanese varieties, which can be understood as dealing with the case where G is free abelian and the classifying maps are holomorphic.

Before we mention other results concerning higher regularity of the classifying maps, we shall now give the basic examples of projective varieties that are classifying spaces.

2. Projective varieties which are $K(\pi, 1)$

The following are the easiest examples of projective varieties which are $K(\pi, 1)$'s.

(1) Projective curves C of genus $g \geq 2$.
By the **Uniformization theorem**, these have the Poincaré upper half plane $\mathcal{H} := \{z \in \mathbb{C} | Im(z) > 0\}$ as universal covering, hence they are compact quotients $C = \mathcal{H}/\Gamma$, where $\Gamma \subset \mathbb{P}SL(2, \mathbb{R})$ is a discrete subgroup isomorphic to the fundamental group of C, $\pi_1(C) \cong \pi_g$. Here

$$\pi_g := \langle \alpha_1, \beta_1, \dots \alpha_g, \beta_g | \Pi_1^g [\alpha_i, \beta_i] = 1 \rangle$$

contains no elements of finite order.
Moreover, the complex orientation of C induces a standard generator $[C]$ of $H_2(C, \mathbb{Z}) \cong \mathbb{Z}$, the so-called fundamental class.

(2) AV : = Abelian varieties.
More generally, a complex torus $X = \mathbb{C}^g/\Lambda$, where Λ is a discrete subgroup of maximal rank (isomorphic then to $\mathbb{Z}^{2g}$), is a Kähler classifying space $K(\mathbb{Z}^{2g}, 1)$, the Kähler metric being induced by the translation invariant Euclidean metric $\frac{i}{2} \sum_1^g dz_j \otimes d\overline{z_j}$.
For $g = 1$ one gets in this way all projective curves of genus $g = 1$; but, for $g > 1$, X is in general not projective: it is projective, and

called then an Abelian variety, if it satisfies the Riemann bilinear relations. These amount to the existence of a positive definite Hermitian form H on $\mathbb{C}^g$ whose imaginary part A (*i.e.*, $H = S + iA$), takes integer values on $\Lambda \times \Lambda$. In modern terms, there exists a positive line bundle L on X, with Chern class $A \in H^2(X, \mathbb{Z}) = H^2(\Lambda, \mathbb{Z}) = \wedge^2(\mathrm{Hom}(\Lambda, \mathbb{Z}))$, whose curvature form, equal to H, is positive (the existence of a positive line bundle on a compact complex manifold X implies that X is projective algebraic, by Kodaira's theorem, [67]).

(3) LSM : = Locally symmetric manifolds.
These are the quotients of a **bounded symmetric domain** $\mathcal{D}$ by a cocompact discrete subgroup $\Gamma \subset \mathrm{Aut}(\mathcal{D})$ acting freely. Recall that a bounded symmetric domain $\mathcal{D}$ is a bounded domain $\mathcal{D} \subset\subset \mathbb{C}^n$ such that its group $\mathrm{Aut}(\mathcal{D})$ of biholomorphisms contains, for each point $p \in \mathcal{D}$, a holomorphic automorphism σ_p such that $\sigma_p(p) = p$, and such that the derivative of σ_p at p is equal to $-Id$. This property implies that σ is an involution (*i.e.*, it has order 2), and that $\mathrm{Aut}(\mathcal{D})^0$ (the connected component of the identity) is transitive on $\mathcal{D}$, and one can write $\mathcal{D} = G/K$, where G is a connected Lie group, and K is a maximal compact subgroup.
The two important properties are:
(3.1) $\mathcal{D}$ splits uniquely as the product of irreducible bounded symmetric domains.
(3.2) each such $\mathcal{D}$ is contractible.
Bounded symmetric domains were classified by Elie Cartan in [24], and there is only a finite number of them (up to isomorphism) for each dimension n.
Recall the notation for the simplest irreducible domains:

(i) $I_{n,p}$ is the domain $\mathcal{D} = \{Z \in \mathrm{Mat}(n, p, \mathbb{C}) : \mathrm{I}_p - {}^t Z \cdot \overline{Z} > 0\}$.
(ii) II_n is the intersection of the domain $I_{n,n}$ with the subspace of skew symmetric matrices.
(iii) III_n is instead the intersection of the domain $I_{n,n}$ with the subspace of symmetric matrices.

We refer the reader to [60], Theorem 7.1, page 383 and exercise D, pages 526-527, for a list of these irreducible bounded symmetric domains.
In the case of type III domains, the domain is biholomorphic to the Siegel's upper half space:

$$\mathcal{H}_g := \{\tau \in \mathrm{Mat}(g, g, \mathbb{C}) | \tau = {}^t\tau, \mathrm{Im}(\tau) > 0\},$$

a generalisation of the upper half-plane of Poincaré.

(4) A particular, but very explicit case of locally symmetric manifolds is given by the VIP : = Varieties isogenous to a product.
These were studied in [29], and they are defined as quotients

$$X = (C_1 \times C_2 \times \cdots \times C_n)/G$$

of the product of projective curves C_j of respective genera $g_j \geq 2$ by the action of a finite group G acting freely on the product.
In this case the fundamental group of X is not so mysterious and fits into an exact sequence

$$1 \to \pi_1(C_1 \times C_2 \times \cdots \times C_n) \cong \pi_{g_1} \times \cdots \times \pi_{g_n} \to \pi_1(X) \to G \to 1.$$

Such varieties are said to be of the **unmixed type** if the group G does not permute the factors, *i.e.*, there are actions of G on each curve such that

$$\gamma(x_1, \ldots, x_n) = (\gamma x_1, \ldots, \gamma x_n), \forall \gamma \in G.$$

Equivalently, each individual subgroup π_{g_j} is normal in $\pi_1(X)$.
(5) Hyperelliptic surfaces: these are the quotients of a complex torus of dimension 2 by a finite group G acting freely, and in such away that the quotient is not again a complex torus.
These surfaces were classified by Bagnera and de Franchis ([4], see also [51] and [5]) and they are obtained as quotients $(E_1 \times E_2)/G$ where E_1, E_2 are two elliptic curves, and G is an abelian group acting on E_1 by translations, and on E_2 effectively and in such a way that $E_2/G \cong \mathbb{P}^1$.
(6) In higher dimension we define the Generalized Hyperelliptic Varieties (GHV) as quotients A/G of an Abelian Variety A by a finite group G acting freely, and with the property that G is not a subgroup of the group of translations. Without loss of generality one can then assume that G contains no translations, since the subgroup G_T of translations in G would be a normal subgroup, and if we denote $G' = G/G_T$, then $A/G = A'/G'$, where A' is the Abelian variety $A' := A/G_T$.
We proposed instead the name **Bagnera-de Franchis (BdF) Varieties** for those quotients $X = A/G$ were G contains no translations, and G is a cyclic group of order m, with generator g (observe that, when A has dimension $n = 2$, the two notions coincide, thanks to the classification result of Bagnera-De Franchis in [4]).
A concrete description of such Bagnera-De Franchis varieties is given in [38].

2.1. Rational $K(\pi, 1)$'s: basic examples

An important role is also played by complex **Rational** $K(\pi, 1)$**'s**, *i.e.*, quasi projective varieties (or complex spaces) Z such that

$$Z = \mathcal{D}/\pi,$$

where $\mathcal{D}$ is a contractible manifold (or complex space) and the action of π on $\mathcal{D}$ is properly discontinuous but not necessarily free.

While for a $K(\pi, 1)$ we have $H^*(\pi, \mathbb{Z}) \cong H^*(Z, \mathbb{Z})$, $H_*(\pi, \mathbb{Z}) \cong H_*(Z, \mathbb{Z})$, for a rational $K(\pi, 1)$ we have $H^*(\pi, \mathbb{Q}) \cong H^*(Z, \mathbb{Q})$ and therefore also $H_*(\pi, \mathbb{Q}) \cong H_*(Z, \mathbb{Q})$.

Typical examples of such rational $K(\pi, 1)$'s are:

(1) quotients of a bounded symmetric domain $\mathcal{D}$ by a subgroup $\Gamma \subset \mathrm{Aut}(\mathcal{D})$ which is acting properly discontinuously (equivalently, Γ is discrete); especially noteworthy are the case where Γ is **cocompact**, meaning that $X = \mathcal{D}/\Gamma$ is compact, and the **finite volume** case where the volume of X via the invariant volume form for $\mathcal{D}$ is finite.

(2) the moduli space of principally polarized Abelian Varieties, where $\mathcal{D}$ is Siegel's upper half space

$$\mathcal{H}_g := \{\tau \in \mathrm{Mat}(g, g, \mathbb{C}) | \tau = {}^t\tau, \mathrm{Im}(\tau) > 0\},$$

and the group Γ is

$$\mathrm{Sp}(2g, \mathbb{Z}) := \{M \in \mathrm{Mat}(2g, \mathbb{Z}) | {}^tMIM = I\}.$$

(3) The moduli space of curves of genus $g \geq 2$, a quotient

$$(**)\ \mathfrak{M}_g = \mathcal{T}_g/\mathcal{M}ap_g$$

of a connected complex manifold $\mathcal{T}_g$ of dimension $3g - 3$, called **Teichmüller space**, by the properly discontinuous action of the **Mapping class group** $\mathcal{M}ap_g$. A key result (see [62,66,90]) is that Teichmüller space $\mathcal{T}_g$ is diffeomorphic to a ball, and the action of $\mathcal{M}ap_g$ is properly discontinuous.

Denoting as usual by π_g the fundamental group of a compact complex curve C of genus g, we have in fact a more concrete description of the mapping class group:

$$(M)\ \mathcal{M}ap_g \cong \mathrm{Out}^+(\pi_g).$$

The above superscript $^+$ refers to the orientation preserving property.

The above isomorphism (M) is of course related to the fact that C is a $K(\pi_g, 1)$, as soon as $g \geq 1$.

As we already discussed, there is a bijection between homotopy classes of self maps of C and endomorphisms of π_g, taken up to inner conjugation. Clearly a homeomorphism $\varphi : C \to C$ yields then an associated element $\pi_1(\varphi) \in \mathrm{Out}(\pi_g)$.

Teichmüller theory can be further applied in order to analyse the fixed loci of finite subgroups G of the mapping class group (see [29,66,90]).

Theorem 2.1 (Refined Nielsen realization). *Let $G \subset \mathcal{M}ap_g$ be a finite subgroup. Then $Fix(G) \subset \mathcal{T}_g$ is a non empty complex manifold, diffeomorphic to a ball. It describes the curves which admit a group of automorphisms isomorphic to G and with a given topological action.*

3. Regularity of classifying maps and fundamental groups of projective varieties

3.1. Harmonic maps

Given a continuous map $f : M \to N$ of differentiable manifolds, we can approximate it by a differentiable one, homotopic to the previous one. Indeed, we may assume that $N \subset \mathbb{R}^n$, $M \subset \mathbb{R}^m$ and, by a partition of unity argument, that M is an open set in $\mathbb{R}^h$. Convolution approximates then f by a differentiable function F_1 with values in a tubular neighbourhood $T(N)$ of N, and then the implicit function theorem applied to the normal bundle provides a differentiable retraction $r : T(N) \to N$. Then $F := r \circ F_1$ is the required approximation, and the same retraction provides a homotopy between f and F (the homotopy between f and F_1 being obvious).

If however M, N are algebraic varieties, and algebraic topology tells us about the existence of a continuous map f as above, we would wish for more regularity, possibly holomorphicity of the homotopic map F.

Now, Wirtinger's theorem characterises complex submanifolds as area minimizing ones, so the first idea is to try to deform a differentiable mapping f until it minimizes some functional.

We may take the Riemannian structure inherited form the chosen embedding, and assume that (M, g_M), (N, g_N) are Riemannian manifolds.

If we assume that M is compact, then one defines the **Energy** $\mathcal{E}(f)$ of the map as the integral:

$$\mathcal{E}(f) := 1/2 \int_M |Df|^2 d\mu_M,$$

where Df is the derivative of the differentiable map f, $d\mu_M$ is the volume element on M, and $|Df|$ is just its norm as a differentiable section of a bundle endowed with a metric:

$$Df \in H^0(M, \mathcal{C}^\infty(TM^\vee \otimes f^*(TN))).$$

These notions were introduced by Eells and Sampson in the seminal paper [50], which used the **heat flow**

$$\frac{\partial f_t}{\partial t} = \Delta(f)$$

in order to find extremals for the energy functional. These curves in the space of maps are (as explained in [50]) the analogue of gradient lines in Morse theory, and the energy functional decreases on these lines.

The obvious advantage of the flow method with respect to discrete convergence procedures ('direct methods of the calculus of variations') is that here it is clear that all the maps are homotopic to each other! [3]

The next theorem is one of the most important results, first obtained in [50]

Theorem 3.1 (Eells-Sampson). *Let M,N be compact Riemannian manifolds, and assume that the sectional curvature $\mathcal{K}_N$ of N is semi-negative ($\mathcal{K}_N \leq 0$): then every continuous map $f_0 : M \to N$ is homotopic to a harmonic map $f : M \to N$. Moreover the equation $\Delta(f) = 0$ implies, in case where M, N are real analytic manifolds, the real analyticity of f.*

Not only the condition about the curvature is necessary for the existence of a harmonic representative in each homotopy class, but moreover it constitutes the main source of connections with the concept of classifying spaces, in view of the classical (see [71], [23]) theorem of Cartan-Hadamard establishing a deep link between curvature and topology.

Theorem 3.2 (Cartan-Hadamard). *Suppose that N is a complete Riemannian manifold, with semi-negative ($\mathcal{K}_N \leq 0$) sectional curvature: then the universal covering $\tilde{N}$ is diffeomorphic to an Euclidean space, more precisely given any two points there is a unique geodesic joining them.*

In complex dimension 1 one cannot hope for a stronger result, to have a holomorphic map rather than just a harmonic one. The surprise comes from the fact that, with suitable assumptions, the hope can be realized

[3] The flow method made then its way further through the work of Hamilton, Perelman and others, leading to the solution of the three dimensional Poincaré conjecture (see for example [75] for an exposition).

in higher dimensions, with a small proviso: given a complex manifold X, one can define the conjugate manifold $\bar{X}$ as the same differentiable manifold, but where in the decomposition $TX \otimes_{\mathbb{R}} \mathbb{C} = T^{(1,0)} \oplus T^{(0,1)}$ the roles of $T^{(1,0)}$ and $T^{(0,1)}$ are interchanged (this amounts, in case where X is an algebraic variety, to replacing the defining polynomial equations by polynomials obtained from the previous ones by applying complex conjugation to the coefficients, *i.e.*, replacing each $P(x_0, \dots, x_N)$ by $\overline{P(\overline{x_0}, \dots, \overline{x_N})}$).

In this case the identity map, viewed as a map $\iota : X \to \bar{X}$ is no longer holomorphic, but antiholomorphic. Assume now that we have a harmonic map $f : Y \to X$: then also $\iota \circ f$ shall be harmonic, but a theorem implying that f must be holomorphic then necessarily implies that there is a complex isomorphism between X and $\bar{X}$. Unfortunately, this is not the case, as one sees, already in the case of elliptic curves; but then one may restrict the hope to proving that f is either holomorphic or antiholomorphic.

A breakthrough in this direction was obtained by Siu ([84]) who proved several results, that we shall discuss in the next sections.

3.2. Kähler manifolds and some archetypal theorem

The assumption that a complex manifold X is a Kähler manifold is that there exists a Hermitian metric on the tangent bundle $T^{(1,0)}$ whose associated $(1,1)$ form ξ is closed. In local coordinates the metric is given by

$$h = \Sigma_{i,j} g_{i,j} dz_i d\bar{z_j}, \text{ with } d\xi = 0, \xi := (\Sigma_{i,j} g_{i,j} dz_i \wedge d\bar{z_j}).$$

Hodge theory shows that the cohomology of a compact Kähler manifold X has a Hodge-Kähler decomposition, where $H^{p,q}$ is the space of harmonic forms of type (p,q), which are in particular d-closed (and d^*-closed):

$$H^m(X, \mathbb{C}) = \oplus_{p,q \geq 0, p+q=m} H^{p,q}, \;\; H^{q,p} = \overline{H^{p,q}}, H^{p,q} \cong H^q(X, \Omega^p_X).$$

We give just an elementary application of the above theorem, a characterization of complex tori (see [28,32,35] for other characterizations)

Theorem 3.3. *Let X be a cKM,* i.e., *a compact Kähler manifold X, of dimension n. Then X is a complex torus if and only if it has the same integral cohomology algebra of a complex torus,* i.e. $H^*(X, \mathbb{Z}) \cong \wedge^* H^1(X, \mathbb{Z})$. *Equivalently, if and only if $H^*(X, \mathbb{C}) \cong \wedge^* H^1(X, \mathbb{C})$ and $H^{2n}(X, \mathbb{Z}) \cong \wedge^{2n} H^1(X, \mathbb{Z})$*

Proof. Since $H^{2n}(X,\mathbb{Z}) \cong \mathbb{Z}$, it follows that $H^1(X,\mathbb{Z})$ is free of rank equal to $2n$, therefore $dim_{\mathbb{C}}(H^{1,0}) = n$. We consider then, chosen a base point $x_0 \in X$, the Albanese map

$$a_X : X \to \mathrm{Alb}(X) := H^0(\Omega^1_X)^\vee / \mathrm{Hom}(H^1(X,\mathbb{Z}),\mathbb{Z}),\ x \mapsto \int_{x_0}^{x}.$$

Therefore we have a map between X and the complex torus $T := \mathrm{Alb}(X)$, which induces an isomorphism of first cohomology groups, and has degree 1, in view of the isomorphism

$$H^{2n}(X,\mathbb{Z}) \cong \Lambda^{2n}(H^1(X,\mathbb{Z})) \cong H^{2n}(T,\mathbb{Z}).$$

In view of the normality of X, it suffices to show that a_X is finite. Let Y be a subvariety of X of dimension $m > 0$ mapping to a point: then the cohomology (or homology class, in view of Poincaré duality) class of Y is trivial, since the cohomology algebra of X and T are isomorphic. But since X is Kähler, if ξ is the Kähler form, $\int_Y \xi^m > 0$, a contradiction, since this integral depends only (by the closedness of ξ) on the homology class of Y. □

One can conjecture that a stronger theorem holds, namely

Conjecture 3.4. Let X be a cKM, *i.e.*, a compact Kähler manifold X, of dimension n. Then X is a complex torus if and only if it has the same rational cohomology algebra of a complex torus, *i.e.* $H^*(X,\mathbb{Q}) \cong \wedge^* H^1(X,\mathbb{Q})$. Equivalently, if and only if $H^*(X,\mathbb{C}) \cong \wedge^* H^1(X,\mathbb{C})$.

3.3. Siu's results on harmonic maps

The result by Siu that is the simplest to state is the following

Theorem 3.5.

(I) *Assume that $f : M \to N$ is a harmonic map between two compact Kähler manifolds and that the curvature tensor of N is strongly negative. Assume further that the real rank of the derivative Df is at least 4 in some point of M. Then f is either holomorphic or antiholomorphic.*

(II) *In particular, if* $\dim_{\mathbb{C}}(N) \geq 2$ *and M is homotopy equivalent to N, then M is either biholomorphic or antibiholomorphic to N.*

Let us try however to describe precisely the main hypothesis of strong negativity of the curvature, which is a stronger condition than the strict negativity of the sectional curvature.

As we already mentioned, the assumption that N is a Kähler manifold is that there exists a Hermitian metric on the tangent bundle $T^{(1,0)}$ whose associated $(1, 1)$ form is closed. In local coordinates the metric is given by

$$\Sigma_{i,j} g_{i,j} dz_i d\bar{z_j}, \text{ with } d(\Sigma_{i,j} g_{i,j} dz_i \wedge d\bar{z_j}) = 0.$$

The curvature tensor is a $(1, 1)$ form with values in $(T^{(1,0)})^\vee \otimes T^{(1,0)}$, and using the Hermitian metric to identify $(T^{(1,0)})^\vee \cong \overline{T^{(1,0)}} = T^{(0,1)}$, and their conjugates $((T^{(0,1)})^\vee = \overline{(T^{(0,1)})} \cong T^{(1,0)}$) we write as usual the curvature tensor as a section R of

$$(T^{(1,0)})^\vee \otimes (T^{(0,1)})^\vee \otimes (T^{(1,0)})^\vee \otimes (T^{(0,1)})^\vee.$$

Then seminegativity of the sectional curvature is equivalent to

$$-R(\xi \wedge \bar{\eta} - \eta \wedge \bar{\xi}, \overline{\xi \wedge \bar{\eta} - \eta \wedge \bar{\xi}}) \leq 0,$$

for all pairs of complex tangent vectors ξ, η (here one uses the isomorphism $T^{(1,0)} \cong TN$, and one sees that the expression depends only on the real span of the two vectors ξ, η).

Strong negativity means instead that

$$-R(\xi \wedge \bar{\eta} - \zeta \wedge \bar{\theta}, \overline{\xi \wedge \bar{\eta} - \zeta \wedge \bar{\theta}}) < 0,$$

for all 4-tuples of complex tangent vectors ξ, η, ζ, θ.

The geometrical meaning is the following (see [1, page 71]): the sectional curvature is a quadratic form on $\wedge^2(TN)$, and as such it extends to the complexified bundle $\wedge^2(TN) \otimes \mathbb{C}$ as a Hermitian form. Then strong negativity in the sense of Siu is also called negativity of the **Hermitian sectional curvature** $R(v, w, \bar{v}, \bar{w})$ for all vectors $v, w \in (TN) \otimes \mathbb{C}$.

Then a reformulation of the result of Siu ([84]) and Sampson ([81]) is the following:

Theorem 3.6. *Let M be a compact Kähler manifold, and N a Riemannian manifold with semi-negative Hermitian sectional curvature. Then every harmonic map $f : M \to N$ is pluri-harmonic.*

Now, examples of varieties N with a strongly negative curvature are the balls in $\mathbb{C}^n$, *i.e.*, the BSD of type $I_{n,1}$; Siu finds out that ([84]) for the irreducible bounded symmetric domains of type

$$I_{p,q}, \text{ for } pq \geq 2, \; II_n, \forall n \geq 3, III_n, \forall n \geq 2, IV_n, \forall n \geq 3,$$

the metric is not strongly negative, but just very strongly seminegative, where very strong negativity simply means negativity of the curvature as a Hermitian form on $T^{1,0} \otimes T^{0,1} = T^{1,0} \otimes \overline{T^{0,1}}$. This gives rise to several technical difficulties, where the bulk of the calculations is to see that there is an upper bound for the nullity of the Hermitian sectional curvature, *i.e.* for the rank of the real subbundles of TM where the Hermitian sectional curvature restricts identically to zero.

Siu derives then several results, and we refer the reader to the book [1] for a nice exposition of these results of Siu.

3.4. Hodge theory and existence of maps to curves

Siu also used harmonic theory in order to construct holomorphic maps from Kähler manifolds to projective curves. This the theorem of [86]

Theorem 3.7 (Siu). *Assume that a compact Kähler manifold X is such that there is a surjection $\phi : \pi_1(X) \to \pi_g$, where $g \geq 2$ and, as usual, π_g is the fundamental groups of a projective curve of genus g. Then there is a projective curve C of genus $g' \geq g$ and a fibration $f : X \to C$ (*i.e.*, the fibres of f are connected) such that ϕ factors through $\pi_1(f)$.*

In this case the homomorphism leads to a harmonic map to a curve, and one has to show that the Stein factorization yields a map to some Riemann surface which is holomorphic for some complex structure on the target.

In this case it can be seen more directly how the Kähler assumption, which boils down to Kähler identities, is used.

Recall that Hodge theory shows that the cohomology of a compact Kähler manifold X has a Hodge-Kähler decomposition, where $H^{p,q}$ is the space of harmonic forms of type (p, q):

$$H^m(X, \mathbb{C}) = \oplus_{p,q \geq 0, p+q=m} H^{p,q}, \;\; H^{q,p} = \overline{H^{p,q}}, H^{p,q} \cong H^q(X, \Omega^p_X).$$

The Hodge-Kähler decomposition theorem has a long story, and was proven by Picard in special cases. It entails the following consequence:

Holomorphic forms are closed, *i.e.*, $\eta \in H^0(X, \Omega^p_X) \Rightarrow d\eta = 0$.

At the turn of last century this fact was then used by Castelnuovo and de Franchis ([25,46]):

Theorem 3.8 (Castelnuovo-de Franchis). *Assume that X is a compact Kähler manifold, $\eta_1, \eta_2 \in H^0(X, \Omega^1_X)$ are $\mathbb{C}$-linearly independent, and the wedge product $\eta_1 \wedge \eta_2$ is d-exact. Then $\eta_1 \wedge \eta_2 \equiv 0$ and there exists a fibration $f : X \to C$ such that $\eta_1, \eta_2 \in f^* H^0(C, \Omega^1_C)$. In particular, C has genus $g \geq 2$.*

From it one gets the following simple theorem ([27]):

Theorem 3.9 (Isotropic subspace theorem). *On a compact Kähler manifold X there is a bijection between isomorphism classes of fibrations* $f : X \to C$ *to a projective curve of genus* $g \geq 2$*, and real subspaces* $V \subset H^1(X, \mathbb{C})$ *('real' means that V is self conjugate,* $\overline{V} = V$*) which have dimension* $2g$ *and are of the form* $V = U \oplus \bar{U}$*, where U is a maximal isotropic subspace for the wedge product*

$$H^1(X, \mathbb{C}) \times H^1(X, \mathbb{C}) \to H^2(X, \mathbb{C}).$$

Another result in this vein is (*cf.* [34]).

Theorem 3.10. *Let X be a compact Kähler manifold, and let* $f : X \to C$ *be a fibration onto a projective curve C, of genus g, and assume that there are exactly r fibres which are multiple with multiplicities* $m_1, \dots m_r \geq 2$*. Then f induces an orbifold fundamental group exact sequence*

$$\pi_1(F) \to \pi_1(X) \to \pi_1(g; m_1, \dots m_r) \to 0,$$

where F is a smooth fibre of f, and

$$\pi_1(g; m_1, \dots m_r) = \langle \alpha_1, \beta_1, \dots, \alpha_g, \beta_g, \gamma_1, \dots \gamma_r | \Pi_1^g [\alpha_j, \beta_j] \Pi_1^r \gamma_i \\ = \gamma_1^{m_1} = \dots = \gamma_r^{m_r} = 1 \rangle.$$

Conversely, let X be a compact Kähler manifold and let $(g, m_1, \dots m_r)$ *be a hyperbolic type,* i.e., *assume that* $2g - 2 + \Sigma_i (1 - \frac{1}{m_i}) > 0$.

Then each epimorphism $\phi : \pi_1(X) \to \pi_1(g; m_1, \dots m_r)$ *with finitely generated kernel is obtained from a fibration* $f : X \to C$ *of type* $(g; m_1, \dots m_r)$.

The following (see [29] and [30]) is the main result concerning surfaces isogenous to a product.

Theorem 3.11.

a) *A projective smooth surface S is isogenous to a product of two curves of respective genera* $g_1, g_2 \geq 2$ *, if and only if the following two conditions are satisfied:*

   1) *there is an exact sequence*

   $$1 \to \pi_{g_1} \times \pi_{g_2} \to \pi = \pi_1(S) \to G \to 1,$$

   where G is a finite group and where π_{g_i} *denotes the fundamental group of a projective curve of genus* $g_i \geq 2$*;*

   2) $e(S) (= c_2(S)) = \frac{4}{|G|}(g_1 - 1)(g_2 - 1)$.

b) *Write* $S = (C_1 \times C_2)/G$*. Any surface* X *with the same topological Euler number and the same fundamental group as* S *is diffeomorphic to* S *and is also isogenous to a product. There is a smooth proper family with connected smooth base manifold* T, $p : \mathcal{X} \to T$ *having two fibres respectively isomorphic to* X, *and* Y, *where* Y *is one of the 4 surfaces* $S = (C_1 \times C_2)/G$, $S_{+-} := (\overline{C_1} \times C_2)/G$, $\bar{S} = (\overline{C_1} \times \overline{C_2})/G$, $S_{-+} := (C_1 \times \overline{C_2})/G = \overline{S_{+-}}$.

c) *The corresponding subset of the moduli space of surfaces of general type* $\mathfrak{M}_S^{\mathrm{top}} = \mathfrak{M}_S^{\mathrm{diff}}$, *corresponding to surfaces orientedly homeomorphic, resp. orientedly diffeomorphic to* S, *is either irreducible and connected or it contains two connected components which are exchanged by complex conjugation.*
In particular, if S' *is orientedly diffeomorphic to* S, *then* S' *is deformation equivalent to* S *or to* $\bar{S}$.

4. Inoue type varieties

While a couple of hundreds examples are known today of families of minimal surfaces of general type with geometric genus $p_g(S) := \dim H^0(\mathcal{O}_S(K_S)) = 0$ (observe that for these surfaces $1 \leq K_S^2 \leq 9$), for the value $K_S^2 = 7$ there are only two examples known (*cf.* [63] and [44]), and for a long time only one family of such surfaces was known, the one constructed by Masahisa Inoue (*cf.* [63]).

The attempt to prove that Inoue surfaces form a connected component of the moduli space of surfaces of general type proved to be successful ([15]), and was based on a weak rigidity result: the topological type of an Inoue surface determines an irreducible connected component of the moduli space (a phenomenon similar to the one which was observed in several papers, as [10,11,16,43]).

The starting point was the calculation of the fundamental group of an Inoue surface with $p_g = 0$ and $K_S^2 = 7$: it sits in an extension (π_g being as usual the fundamental group of a projective curve of genus g):

$$1 \to \pi_5 \times \mathbb{Z}^4 \to \pi_1(S) \to (\mathbb{Z}/2\mathbb{Z})^5 \to 1.$$

This extension is given geometrically, *i.e.*, stems from the observation ([15]) that an Inoue surface S admits an unramified $(\mathbb{Z}/2\mathbb{Z})^5$ - Galois covering $\hat{S}$ which is an ample divisor in $E_1 \times E_2 \times D$, where E_1, E_2 are elliptic curves and D is a projective curve of genus 5; while Inoue described $\hat{S}$ as a complete intersection of two non ample divisors in the product $E_1 \times E_2 \times E_3 \times E_4$ of four elliptic curves.

It turned out that the ideas needed to treat this special family of Inoue surfaces could be put in a rather general framework, valid in all dimensions, setting then the stage for the investigation and search for a new class of varieties, which we proposed to call Inoue-type varieties.

Definition 4.1 ([15]). Define a complex projective manifold X to be an **Inoue-type manifold** if

(1) $\dim(X) \geq 2$;
(2) there is a finite group G and an unramified G-covering $\hat{X} \to X$, (hence $X = \hat{X}/G$) such that
(3) $\hat{X}$ is an ample divisor inside a $K(\Gamma, 1)$-projective manifold Z, (hence by the theorems of Lefschetz, $\pi_1(\hat{X}) \cong \pi_1(Z) \cong \Gamma$) and moreover
(4) the action of G on $\hat{X}$ yields a faithful action on $\pi_1(\hat{X}) \cong \Gamma$: in other words the exact sequence

$$1 \to \Gamma \cong \pi_1(\hat{X}) \to \pi_1(X) \to G \to 1$$

gives an injection $G \to \mathrm{Out}(\Gamma)$, induced by conjugation by lifts of elements of G.
(5) the action of G on $\hat{X}$ is induced by an action of G on Z.

Similarly one defines the notion of an **Inoue-type variety**, by requiring the same properties for a variety X with canonical singularities.

The above definition of Inoue type manifold, although imposing a strong restriction on X, is too general, and in order to get weak rigidity type results it is convenient to impose restrictions on the fundamental group Γ of Z, for instance the most interesting case is the one where Z is a product of Abelian varieties, curves, and other locally symmetric varieties with ample canonical bundle.

Definition 4.2. We shall say that an Inoue-type manifold X is

(1) a **special Inoue type manifold** if moreover

$$Z = (A_1 \times \cdots \times A_r) \times (C_1 \times \cdots \times C_h) \times (M_1 \times \cdots \times M_s)$$

where each A_i is an Abelian variety, each C_j is a curve of genus $g_j \geq 2$, and M_i is a compact quotient of an irreducible bounded symmetric domain of dimension at least 2 by a torsion free subgroup;
(2) a **classical Inoue type manifold** if moreover $Z = (A_1 \times \cdots \times A_r) \times (C_1 \times \cdots \times C_h)$ where each A_i is an Abelian variety, each C_j is a curve of genus $g_j \geq 2$;

(3) a special Inoue type manifold is said to be **diagonal** if moreover:

(I) the action of G on $\hat{X}$ is induced by a diagonal action on Z, *i.e.*,

$$G \subset \prod_{i=1}^{r} \mathrm{Aut}(A_i) \times \prod_{j=1}^{h} \mathrm{Aut}(C_j) \times \prod_{l=1}^{s} \mathrm{Aut}(M_l) \qquad (4.1)$$

and furthermore:

(II) the faithful action on $\pi_1(\hat{X}) \cong \Gamma$, induced by conjugation by lifts of elements of G in the exact sequence

$$\begin{aligned} 1 \to \Gamma = \Pi_{i=1}^{r}(\Lambda_i) \times \Pi_{j=1}^{h}(\pi_{g_j}) \times \Pi_{l=1}^{s}(\pi_1(M_l)) \\ \to \pi_1(X) \to G \to 1 \end{aligned} \qquad (4.2)$$

(observe that each factor Λ_i, resp. π_{g_j}, $\pi_1(M_l)$ is a normal subgroup), satisfies the **Schur property**

$$(SP) \quad \mathrm{Hom}(V_i, V_j)^G = 0, \forall i \neq j.$$

Here $V_j := \Lambda_j \otimes \mathbb{Q}$ and, in order that the Schur property holds, it suffices for instance to verify that for each Λ_i there is a subgroup H_i of G for which $\mathrm{Hom}(V_i, V_j)^{H_i} = 0, \forall j \neq i$.

The Schur property (SP) plays an important role in order to show that an Abelian variety with such a G-action on its fundamental group must split as a product.

Before stating the main general result of [15] we need the following definition, which was already used in 3.3 for the characterization of complex tori among Kähler manifolds.

Definition 4.3. Let Y, Y' be two projective manifolds with isomorphic fundamental groups. We identify the respective fundamental groups $\pi_1(Y) = \pi_1(Y') = \Gamma$. Then we say that the condition **(SAME HOMOLOGY)** is satisfied for Y and Y' if there is an isomorphism $\Psi : H_*(Y', \mathbb{Z}) \cong H_*(Y, \mathbb{Z})$ of homology groups which is compatible with the homomorphisms

$$u\colon H_*(Y, \mathbb{Z}) \to H_*(\Gamma, \mathbb{Z}), u'\colon H_*(Y', \mathbb{Z}) \to H_*(\Gamma, \mathbb{Z}),$$

i.e., Ψ satisfies $u \circ \Psi = u'$.

We can now state the following

Theorem 4.4. *Let X be a diagonal special Inoue type manifold, and let X' be a projective manifold with $K_{X'}$ nef and with the same fundamental group as X, which moreover either*

(A) *is homotopically equivalent to* X*;*
or satisfies the following weaker property:
(B) *let* $\hat{X}'$ *be the corresponding unramified covering of* X'*. Then* $\hat{X}$ *and* $\hat{X}'$ *satisfy the condition* **(SAME HOMOLOGY)**.
Setting $W := \hat{X}'$*, we have that*

(a) $X' = W/G$ *where* W *admits a generically finite morphism* $f : W \to Z'$*, and where* Z' *is also a* $K(\Gamma, 1)$ *projective manifold, of the form* $Z' = (A'_1 \times \cdots \times A'_r) \times (C'_1 \times \cdots \times C'_h) \times (M'_1 \times \cdots \times M'_s)$*. Moreover here* M'_i *is either* M_i *or its complex conjugate, and the product decomposition corresponds to the product decomposition* (4.2) *of the fundamental group of* Z*.*
The image cohomology class $f_*([W])$ *corresponds, up to sign, to the cohomology class of* $\hat{X}$*.*
(b) *The morphism* f *is finite if* $n = \dim X$ *is odd, and it is generically injective if*
(**) *the cohomology class of* $\hat{X}$ *(in* $H^*(Z, \mathbb{Z})$*) is indivisible, or if every strictly submultiple cohomology class cannot be represented by an effective* G*-invariant divisor on any pair* (Z', G) *homotopically equivalent to* (Z, G)*.*
(c) f *is an embedding if moreover* $K_{X'}$ *is ample,*
(*) *every such divisor* W *of* Z' *is ample, and*
(***) $K^n_{X'} = K^n_X$.[4]

In particular, if $K_{X'}$ *is ample and* (*), (**) *and* (***) *hold, also* X' *is a diagonal SIT (special Inoue type) manifold.*
A similar conclusion holds under the alternative assumption that the homotopy equivalence sends the canonical class of W *to that of* $\hat{X}$*: then* X' *is a minimal resolution of a diagonal SIT (special Inoue type) variety.*

For the proof of Theorem 4.4 the first step consists in showing that $W := \hat{X}'$ admits a holomorphic mapping to a manifold Z' of the above type $Z' = (A'_1 \times \cdots \times A'_r) \times (C'_1 \times \cdots \times C'_h) \times (M'_1 \times \cdots \times M'_s)$, where M'_i is either M_i or its complex conjugate.

First of all, by the results of Siu and others ([31, 84, 85], [34, Theorem 5.14]) cited in Section 3, W admits a holomorphic map to a product manifold of the desired type

$$Z'_2 \times Z'_3 = (C'_1 \times \cdots \times C'_h) \times (M'_1 \times \cdots \times M'_s).$$

[4] This last property for algebraic surfaces follows automatically from homotopy invariance.

Then one looks at the Albanese variety $\mathrm{Alb}(W)$ of the Kähler manifold W, whose fundamental group is the quotient of the Abelianization of $\Gamma = \pi_1(Z)$ by its torsion subgroup.

Then the cohomological assumptions and adjunction theory are used to complete the result.

The study of moduli spaces of Inoue type varieties, and their connected and irreducible components, relies very much on the study of moduli spaces of varieties X endowed with the action of a finite group G: and it is for us a strong motivation to pursue this line of research.

This topic will occupy a central role in the following sections, first in general, and then in the special case of algebraic curves.

5. Moduli spaces of symmetry marked varieties

5.1. Moduli marked varieties

We give now the definition of a symmetry marked variety for projective varieties, but one can similarly give the same definition for complex or Kähler manifolds; to understand the concept of a marking, it suffices to consider a cyclic group acting on a variety X. A marking consists in this case of the choice of a generator for the group acting on X. The marking is very important when we have several actions of a group G on some projective varieties, and we want to consider the diagonal action of G on their product.

Definition 5.1.

(1) A *G-marked (projective) variety* is a triple (X, G, η) where X is a projective variety, G is a group and $\eta\colon G \to \mathrm{Aut}(X)$ is an injective homomorphism

(2) equivalently, a marked variety is a triple (X, G, α) where $\alpha\colon X \times G \to X$ is a faithful action of the group G on X.

(3) Two marked varieties (X, G, α), (X', G, α') are said to be *isomorphic* if there is an isomorphism $f\colon X \to X'$ transporting the action $\alpha\colon X \times G \to X$ into the action $\alpha'\colon X' \times G \to X'$, *i.e.*, such that

$$f \circ \alpha = \alpha' \circ (f \times \mathrm{id}) \Leftrightarrow \eta' = Ad(f) \circ \eta, \quad Ad(f)(\phi) := f\phi f^{-1}.$$

(4) If G is a subset of $\mathrm{Aut}(X)$, then the natural marked variety is the triple (X, G, i), where $i\colon G \to \mathrm{Aut}(X)$ is the inclusion map, and it shall sometimes be denoted simply by the pair (X, G).

(5) A marked curve (D, G, η) consisting of a smooth projective curve of genus g and a faithful action of the group G on D is said to be a *marked triangle curve of genus g* if $D/G \cong \mathbb{P}^1$ and the quotient morphism $p\colon D \to D/G \cong \mathbb{P}^1$ is branched in three points.

Remark 5.2. Observe that:

1) we have a natural action of Aut(G) on G-marked varieties, namely, if $\psi \in \mathrm{Aut}(G)$,
$$\psi(X, G, \eta) := (X, G, \eta \circ \psi^{-1}).$$
The corresponding equivalence class of a G-marked variety is defined to be a *G-(unmarked) variety.*
2) the action of the group Inn(G) of inner automorphisms does not change the isomorphism class of (X, G, η) since, for $\gamma \in G$, we may set $f := \eta(\gamma)$, $\psi := Ad(\gamma)$, and then $\eta \circ \psi = Ad(f) \circ \eta$, since $\eta(\psi(g)) = \eta(\gamma g \gamma^{-1}) = \eta(\gamma)\eta(g)(\eta(\gamma)^{-1}) = Ad(f)(\eta(g))$.
3) In the case where $G = \mathrm{Aut}(X)$, we see that Out(G) acts simply transitively on the isomorphism classes of the Aut(G)-orbit of (X, G, η).

Let us see now how the picture works in the case of curves: this case is already very enlightening and intriguing.

5.2. Moduli of curves with automorphisms

There are several 'moduli spaces' of curves with automorphisms. First of all, given a finite group G, we define a subset $\mathfrak{M}_{g,G}$ of the moduli space $\mathfrak{M}_g$ of smooth curves of genus $g > 1$: $\mathfrak{M}_{g,G}$ is the locus of the curves that admit an effective action by the group G. It turns out that $\mathfrak{M}_{g,G}$ is a Zariski closed algebraic subset. The description of these Zariski closed subsets is related to the description of the singular locus of the moduli space $\mathfrak{M}_g$ (for instance of its irreducible components, see [45]), and of its compactification $\overline{\mathfrak{M}_g}$, (see [36]).

In order to understand the irreducible components of $\mathfrak{M}_{g,G}$ we have seen that Teichmüller theory plays an important role: it shows the connectedness, given an injective homomorphism $\rho\colon G \to \mathrm{Map}_g$, of the locus
$$\mathcal{T}_{g,\rho} := Fix(\rho(G)).$$
Its image $\mathfrak{M}_{g,\rho}$ in $\mathfrak{M}_{g,G}$ is a Zariski closed irreducible subset (as observed in [39]). Recall that to a curve C of genus g with an action by G we can associate several discrete invariants that are constant under deformation.

The first is the above *topological type* of the G-action: it is a homomorphism $\rho\colon G \to \mathrm{Map}_g$, which is well-defined up to inner conjugation (induced by different choices of an isomorphism $\mathrm{Map}(C) \cong \mathrm{Map}_g$).

We immediately see that the locus $\mathfrak{M}_{g,\rho}$ is first of all determined by the subgroup $\rho(G)$ and not by the marking. Moreover, this locus remains the same not only if we change ρ modulo the action by Aut(G), but also if we change ρ by the adjoint action by Map_g.

Definition 5.3.

1) The moduli space of G-marked curves of a certain topological type ρ is the quotient of the Teichmüller submanifold $\mathcal{T}_{g,\rho}$ by the centralizer subgroup $\mathcal{C}_{\rho(G)}$ of the subgroup $\rho(G)$ of the mapping class group. We get a normal complex space which we shall denote $\mathfrak{M}_g[\rho]$. $\mathfrak{M}_g[\rho] = \mathcal{T}_{g,\rho}/\mathcal{C}_{\rho(G)}$ is a finite covering of a Zariski closed subset of the usual moduli space (its image $\mathfrak{M}_{g,\rho}$), therefore it is quasi-projective, by the theorem of Grauert and Remmert.
2) Defining $\mathfrak{M}_g(\rho)$ as the quotient of $\mathcal{T}_{g,\rho}$ by the normalizer $\mathcal{N}_{\rho(G)}$ of $\rho(G)$, we call it the moduli space of curves with a G-action of a given topological type. It is again a normal quasi-projective variety.

Remark 5.4.

1) If we consider $G' := \rho(G)$ as a subgroup $G' \subset \mathrm{Map}_g$, then we get a natural G'-marking for any $C \in Fix(G') = \mathcal{T}_{g,\rho}$.
2) As we said, $Fix(G') = \mathcal{T}_{g,\rho}$ is independent of the chosen marking, moreover the projection $Fix(G') = \mathcal{T}_{g,\rho} \to \mathfrak{M}_{g,\rho}$ factors through a finite map $\mathfrak{M}_g(\rho) \to \mathfrak{M}_{g,\rho}$.

The next question is whether $\mathfrak{M}_g(\rho)$ maps 1-1 into the moduli space of curves. This is not the case, as we shall easily see. Hence one gives the following definition.

Definition 5.5. Let $G \subset \mathrm{Map}_g$ be a finite group, and let C represent a point in $Fix(G)$. Then we have a natural inclusion $G \to A_C := \mathrm{Aut}(C)$, and C is a fixed point for the subgroup $A_C \subset \mathrm{Map}_g$: A_C is indeed the stabilizer of the point C in Map_g, so that locally (at the point of $\mathfrak{M}_g$ corresponding to C) we get a complex analytic isomorphism $\mathfrak{M}_g = \mathcal{T}_g/A_C$.

We define $H_G := \cap_{C \in Fix(G)} A_C$ and we shall say that G is a **full subgroup** if $G = H_G$. Equivalently, H_G is the largest subgroup H such that $Fix(H) = Fix(G)$.

This implies that H_G is a full subgroup.

Then we have:

Proposition 5.6. *If H is a full subgroup $H \subset \mathrm{Map}_g$, and $\rho : H \subset \mathrm{Map}_g$ is the inclusion homomorphism, then $\mathfrak{M}_g(\rho)$ is the normalization of $\mathfrak{M}_{g,\rho}$.*

5.3. Numerical and homological invariants of group actions on curves

As already mentioned, given an effective action of a finite group G on C, we set $C' := C/G$, $g' := g(C')$, and we have the quotient morphism $p\colon C \to C/G =: C'$, a G-cover.

The geometry of p encodes several numerical invariants that are constant on $M_{g,\rho}(G)$: first of all the genus g' of C', then the number d of branch points $y_1, \dots, y_d \in C'$.

We call the set $B = \{y_1, \dots, y_d\}$ the branch locus, and for each y_i we denote by m_i the multiplicity of y_i (the greatest number dividing the divisor $p^*(y_i)$). We choose an ordering of B such that $m_1 \leq \cdots \leq m_d$.

These numerical invariants $g', d, m_1 \leq \cdots \leq m_d$ form the so-called **primary numerical type**.

$p\colon C \to C'$ is determined (Riemann's existence theorem) by the monodromy, a surjective homomorphism:

$$\mu\colon \pi_1(C' \setminus B) \to G\,.$$

We have:

$$\pi_1(C' \setminus B) \cong \Pi_{g',d}$$
$$:= \langle \gamma_1, \dots, \gamma_d, \alpha_1, \beta_1, \dots, \alpha_{g'}, \beta_{g'} | \prod_{i=1}^{d} \gamma_i \prod_{j=1}^{g'} [\alpha_j, \beta_j] = 1\rangle.$$

We set then $c_i := \mu(\gamma_i)$, $a_j := \mu(\alpha_j)$, $b_j := \mu(\beta_j)$, thus obtaining a Hurwitz generating vector, *i.e.* a vector

$$v := (c_1, \dots, c_d, a_1, b_1, \dots, a_{g'}, b_{g'}) \in G^{d+2g'}$$

s.t.

- G is generated by the entries $c_1, \dots, c_d, a_1, b_1, \dots, a_{g'}, b_{g'}$,
- $c_i \neq 1_G, \forall i$, and
- $\prod_{i=1}^{d} c_i \prod_{j=1}^{g'} [a_j, b_j] = 1$.

We see that the monodromy μ is completely equivalent, once an isomorphism $\pi_1(C' \setminus B) \cong \Pi_{g',d}$ is chosen, to the datum of a Hurwitz generating vector (we also call the sequence $c_1, \dots, c_d, a_1, b_1, \dots, a_{g'}, b_{g'}$ of the vector's coordinates a *Hurwitz generating system*).

A second numerical invariant of these components of $\mathfrak{M}_g(G)$ is obtained from the monodromy $\mu\colon \pi_1(C' \setminus \{y_1, \dots, y_d\}) \to G$ of the restriction of p to $p^{-1}(C' \setminus \{y_1, \dots, y_d\})$, and is called the *$\nu$-type* or Nielsen function of the covering.

The Nielsen function ν is a function defined on the set of conjugacy classes in G which, for each conjugacy class $\mathcal{C}$ in G, counts the number $\nu(\mathcal{C})$ of local monodromies $c_1, \dots, c_d$ which belong to $\mathcal{C}$ (observe that the numbers $m_1 \leq \cdots \leq m_d$ are just the orders of the local monodromies).

Observe in fact that the generators γ_j are well defined only up to conjugation in the group $\pi_1(C' \setminus \{y_1, \dots, y_d\})$, hence the local monodromies are well defined only up to conjugation in the group G.

We have already observed that the irreducible closed algebraic sets $M_{g,\rho}(G)$ depend only upon what we call the 'unmarked topological type', which is defined as the conjugacy class of the subgroup $\rho(G)$ inside Map_g. This concept remains however still mysterious, due to the complicated nature of the group Map_g. Therefore one tries to use more geometry to get a grasp on the topological type.

The following is immediate by Riemann's existence theorem and the irreducibility of the moduli space $\mathfrak{M}_{g',d}$ of d-pointed curves of genus g'. Given g' and d, the unmarked topological types whose primary numerical type is of the form $g', d, m_1, \dots, m_d$ are in bijection with the quotient of the set of the corresponding monodromies μ modulo the actions by $\mathrm{Aut}(G)$ and by $\mathrm{Map}(g', d)$.
Here $\mathrm{Map}(g', d)$ is the full mapping class group of genus g' and d unordered points.

Thus Riemann's existence theorem shows that the components of the moduli space

$$\mathfrak{M}(G) := \cup_g \mathfrak{M}_g(G)$$

with numerical invariants g', d correspond to the following quotient set.

Definition 5.7.

$$\mathcal{A}(g', d, G) := \mathrm{Epi}(\Pi_{g',d}, G)/\mathrm{Map}_{g',d} \times \mathrm{Aut}(G)\,.$$

Thus a first step toward the general problem consists in finding a fine invariant that distinguishes these orbits.

In the paper [39] we introduced a new homological invariant $\hat{\epsilon}$ for G-actions on smooth curves and showed that, in the case where G is the dihedral group D_n of order $2n$, $\hat{\epsilon}$ is a fine invariant since it distinguishes the different unmarked topological types.

This invariant generalizes the classical homological invariant in the unramified case.

Definition 5.8. Let $p\colon C \to C/G =: C'$ be unramified, so that $d = 0$ and we have a monodromy $\mu\colon \pi_1(C') \to G$.

Since C' is a classifying space for the group $\pi_{g'}$, we obtain a continuous map

$$m : C' \to BG, \ \pi_1(m) = \mu.$$

Moreover, $H_2(C', \mathbb{Z})$ has a natural generator $[C']$, the fundamental class of C' determined by the orientation induced by the complex structure of C'.

The homological invariant of the G-marked action is then defined as:

$$\epsilon := H_2(m)([C']) \in H_2(BG, \mathbb{Z}) = H_2(G, \mathbb{Z}).$$

If we forget the marking we have to take ϵ as an element in $H_2(G, \mathbb{Z})/\mathrm{Aut}(G)$.

In the ramified case, one needs also the following definition.

Definition 5.9. An element

$$\nu = (n_{\mathcal{C}})_{\mathcal{C}} \in \bigoplus_{\mathcal{C} \neq \{1\}} \mathbb{N}\langle \mathcal{C} \rangle$$

is **admissible** if the following equality holds in the $\mathbb{Z}$-module G^{ab}:

$$\sum_{\mathcal{C}} n_{\mathcal{C}} \cdot [\mathcal{C}] = 0$$

(here $[\mathcal{C}]$ denotes the image element of $\mathcal{C}$ in the abelianization G^{ab}).

The main result of [40] is the following 'genus stabilization' theorem.

Theorem 5.10. *There is an integer h such that for $g' > h$*

$$\hat{\epsilon} \colon \mathcal{A}(g', d, G) \to (K^{\cup})/_{\mathrm{Aut}(G)}$$

induces a bijection onto the set of admissible classes of refined homology invariants.

In particular, if $g' > h$, and we have two Hurwitz generating systems v_1, v_2 having the same Nielsen function, they are equivalent if and only if the 'difference' $\hat{\epsilon}(v_1)\hat{\epsilon}(v_2)^{-1} \in H_{2,\Gamma}(G)$ is trivial.

The above result extends a nice theorem of Livingston, Dunfield and Thurston ([47, 69]) in the unramifed case, where also the statement is simpler.

Theorem 5.11. *For $g' >> 0$*

$$\hat{\epsilon} \colon \mathcal{A}(g', 0, G) \to H_2(G, \mathbb{Z})/_{\mathrm{Aut}(G)}$$

is a bijection.

Remark 5.12. Unfortunately the integer h in Theorem 5.10, which depends on the group G, is not explicit.

A key concept used in the proof is the concept of genus stabilization of a covering, which we now briefly explain.

Definition 5.13. Consider a group action of G on a projective curve C, and let $C \to C' = C/G$ the quotient morphism, with monodromy

$$\mu\colon \pi_1(C' \setminus B) \to G$$

(here B is as usual the branch locus). Then the first genus stabilization of the differentiable covering is defined geometrically by simply adding a handle to the curve C', on which the covering is trivial.

Algebraically, given the monodromy homomorphism

$$\mu : \pi_1(C' \setminus B) \cong \Pi_{g',d}$$
$$:= \left\langle \gamma_1, \dots, \gamma_d, \alpha_1, \beta_1, \dots, \alpha_{g'}, \beta_{g'} | \prod_{i=1}^{d} \gamma_i \prod_{j=1}^{g'} [\alpha_j, \beta_j] = 1 \right\rangle \to G,$$

we simply extend μ to $\mu^1\colon \Pi_{g'+1,d} \to G$ setting

$$\mu^1(\alpha_{g'+1}) = \mu^1(\beta_{g'+1}) = 1_G.$$

In terms of Hurwitz vectors and Hurwitz generating systems, we replace the vector

$$v := (c_1, \dots, c_d, a_1, b_1, \dots, a_{g'}, b_{g'}) \in G^{d+2g'}$$

by

$$v^1 := (c_1, \dots, c_d, a_1, b_1, \dots, a_{g'}, b_{g'}, 1, 1) \in G^{d+2g'+2}.$$

The operation of first genus stabilization generates then an equivalence relation among monodromies (equivalently, Hurwitz generating systems), called **stable equivalence**.

The most important step in the proof, the geometric understanding of the invariant $\epsilon \in H_2(G, \mathbb{Z})$ was obtained by Livingston [69].

Theorem 5.14. *Two monodromies μ_1, μ_2 are stably equivalent if and only if they have the same invariant $\epsilon \in H_2(G, \mathbb{Z})$.*

A purely algebraic proof of Livingston's theorem was given by Zimmermann in [96], while a nice sketch of proof was given by Dunfield and Thurston in [47].

5.4. Classification results for certain concrete groups

The first result in this direction was obtained by Nielsen ([78]) who proved that ν determines ρ if G is cyclic (in fact in this case $H_2(G, \mathbb{Z}) = 0$!).

In the cyclic case the Nielsen function for $G = \mathbb{Z}/n$ is simply a function $\nu : (\mathbb{Z}/n) \setminus \{0\} \to \mathbb{N}$, and admissibility here simply means that

$$\sum_i i \cdot \nu(i) \equiv 0 \pmod{n}.$$

The class of ν is just the equivalence class for the equivalence relation $\nu(i) \sim \nu_r(i), \forall r \in (\mathbb{Z}/n)^*$, where $\nu_r(i) := \nu(ri), \forall i \in (\mathbb{Z}/n)$.

From the refined Nielsen realization theorem of [29] (2.1) it follows that the components of $\mathfrak{M}_g(\mathbb{Z}/n)$ are in bijection with the classes of Nielsen functions (see also [36] for an elementary proof).

The genus g' and the Nielsen class (which refine the primary numerical type), and the homological invariant $h \in H_2(G/H, \mathbb{Z})$ (here H is again the subgroup generated by the local monodromies) determine the connected components of $\mathfrak{M}_g(G)$ under some restrictions: for instance when G is abelian or when G acts freely and is the semi-direct product of two finite cyclic groups (as it follows by combining results from [29,36,48] and [49]).

Theorem 5.15 (Edmonds). *ν and $h \in H_2(G/H, \mathbb{Z})$ determine ρ for G abelian. Moreover, if G is split-metacyclic and the action is free, then h determines ρ.*

However, in general, these invariants are not enough to distinguish unmarked topological types, as one can see already for non-free D_n-actions (see [39]). Already for dihedral groups, one needs the refined homological invariant $\hat{\epsilon}$.

Theorem 5.16 ([39]). *For the dihedral group $G = D_n$ the connected components of the moduli space $\mathfrak{M}_g(D_n)$ are in bijection, via the map $\hat{\epsilon}$, with the admissible classes of refined homology invariants.*

The above result completes the classification of the unmarked topological types for $G = D_n$; moreover this result entails the classification of the irreducible components of the loci $\mathfrak{M}_{g,D_n}$ (see the appendix to [39]).

It is an interesting question: for which groups G does the refined homology invariant $\hat{\epsilon}$ determine the connected components of $\mathfrak{M}_g(G)$?

In view of Edmonds' result in the unramified case, it is reasonable to expect a positive answer for split metacyclic groups.

As mentioned in [47, page 499], the group $G = \mathbb{P}SL(2, \mathbb{F}_{13})$ shows that, for $g' = 2$, in the unramified case there are different components with trivial homology invariant $\epsilon \in H_2(G, \mathbb{Z})$: these topological types of coverings are therefore stably equivalent but not equivalent.

6. Connected components of moduli spaces and the action of the absolute Galois group

Let X be a complex projective variety: let us quickly recall the notion of a conjugate variety.

Remark 6.1.

1) $\phi \in \mathrm{Aut}(\mathbb{C})$ acts on $\mathbb{C}[z_0, \dots z_n]$, by sending $P(z) = \sum_{i=0}^{n} a_i z^i \mapsto \phi(P)(z) := \sum_{i=0}^{n} \phi(a_i) z^i$.
2) Let X be as above a projective variety

$$X \subset \mathbb{P}^n_{\mathbb{C}}, X := \{z | f_i(z) = 0 \ \forall i\}.$$

The action of ϕ extends coordinatewise to $\mathbb{P}^n_{\mathbb{C}}$, and carries X to another variety, denoted X^ϕ, and called the **conjugate variety**. Since $f_i(z) = 0$ implies $\phi(f_i)(\phi(z)) = 0$, we see that

$$X^\phi = \{w | \phi(f_i)(w) = 0 \ \forall i\}.$$

If ϕ is complex conjugation, then it is clear that the variety X^ϕ that we obtain is diffeomorphic to X; but, in general, what happens when ϕ is not continuous?

Observe that, by the theorem of Steiniz, one has a surjection $\mathrm{Aut}(\mathbb{C}) \to \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, and by specialization the heart of the question concerns the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on varieties X defined over $\bar{\mathbb{Q}}$.

For curves, since in general the dimensions of spaces of differential forms of a fixed degree and without poles are the same for X^ϕ and X, we shall obtain a curve of the same genus, hence X^ϕ and X are diffeomorphic.

6.1. Galois conjugates of projective classifying spaces

General questions of which the first is answered in the positive in most concrete cases, and the second is answered in the negative in many cases, as we shall see, are the following.

Question 6.2. Assume that X is a projective $K(\pi, 1)$, and assume $\phi \in \mathrm{Aut}(\mathbb{C})$.

A) Is then the conjugate variety X^ϕ still a classifying space $K(\pi', 1)$?

B) Is then $\pi_1(X^\phi) \cong \pi \cong \pi_1(X)$?

Since ϕ is never continuous, there would be no reason to expect a positive answer to both questions A) and B), except that Grothendieck showed ([58]).

Theorem 6.3. *Conjugate varieties X, X^ϕ have isomorphic algebraic fundamental groups*

$$\pi_1(X)^{alg} \cong \pi_1(X^\phi)^{alg},$$

where $\pi_1(X)^{alg}$ is the profinite completion of the topological fundamental group $\pi_1(X)$).

We recall once more that the profinite completion of a group G is the inverse limit

$$\hat{G} = \lim_{K \trianglelefteq_f G} (G/K),$$

of the factor groups G/K, K being a normal subgroup of finite index in G; and since finite index subgroups of the fundamental group correspond to finite unramified (étale) covers, Grothendieck defined in this way the algebraic fundamental group for varieties over other fields than the complex numbers, and also for more general schemes.

The main point of the proof of the above theorem is that if we have $f : Y \to X$ which is étale, also the Galois conjugate $f^\phi : Y^\phi \to X^\phi$ is étale (f^ϕ is just defined taking the Galois conjugate of the graph of f, a subvariety of $Y \times X$).

Since Galois conjugation gives an isomorphism of natural cohomology groups, which respects the cup product, as for instance the Dolbeault cohomology groups $H^p(\Omega^q_X)$, we obtain interesting consequences in the direction of question A) above. Recall the following definition.

Definition 6.4. Two varieties X, Y are said to be isogenous if there exist a third variety Z, and étale finite morphisms $f_X : Z \to X$, $f_Y : Z \to Y$.

Remark 6.5. It is obvious that if X is isogenous to Y, then X^ϕ is isogenous to Y^ϕ.

Theorem 6.6.

i) *If X is an Abelian variety, or isogenous to an Abelian variety, the same holds for any Galois conjugate X^ϕ.*
ii) *If S is a Kodaira fibred surface, then any Galois conjugate S^ϕ is also Kodaira fibred.*
iii) *If X is isogenous to a product of curves, the same holds for any Galois conjugate X^ϕ.*

Proof.

i) X is an Abelian variety if and only it is a projective variety and there is a morphism $X \times X \to X$, $(x, y) \mapsto (x \cdot y^{-1})$, which makes X a group (see [77], it follows indeed that the group is commutative). This property holds for X if and only if it holds for X^ϕ.

ii) The hypothesis is that there is $f : S \to B$ such that all the fibres are smooth and not all isomorphic: obviously the same property holds, after Galois conjugation, for $f^\phi : S^\phi \to B^\phi$.
iii) It suffices to show that the Galois conjugate of a product of curves is a product of curves. But since $X^\phi \times Y^\phi = (X \times Y)^\phi$ and the Galois conjugate of a curve C of genus g is again a curve of the same genus g, the statement follows. □

Proceeding with other projective $K(\pi, 1)$'s, the question becomes more subtle and we have to appeal to a famous theorem by Kazhdan on arithmetic varieties (see [41,42,64,65,70,91]).

Theorem 6.7. *Assume that X is a projective manifold with K_X ample, and that the universal covering $\tilde{X}$ is a bounded symmetric domain.*
Let $\tau \in \mathrm{Aut}(\mathbb{C})$ be an automorphism of $\mathbb{C}$.
Then the conjugate variety X^τ has universal covering $\tilde{X^\tau} \cong \tilde{X}$.

Simpler proofs follow from recent results obtained together with Antonio Di Scala, and based on the Aubin-Yau theorem. These results yield a precise characterization of varieties possessing a bounded symmetric domain as universal cover, and can be rather useful in view of the fact that our knowledge and classification of these fundamental groups is not so explicit.

We just mention the simplest result (see [41]).

Theorem 6.8. *Let X be a compact complex manifold of dimension n with K_X ample.*
Then the following two conditions (1) *and* (1′)*, resp.* (2) *and* (2′) *are equivalent:*

(1) *X admits a slope zero tensor $0 \neq \psi \in H^0(S^{mn}(\Omega^1_X)(-mK_X))$, (for some positive integer m);*
(1′) *$X \cong \Omega/\Gamma$, where Ω is a bounded symmetric domain of tube type and Γ is a cocompact discrete subgroup of $\mathrm{Aut}(\Omega)$ acting freely.*
(2) *X admits a semi special tensor $0 \neq \phi \in H^0(S^n(\Omega^1_X)(-K_X) \otimes \eta)$, where η is a 2-torsion invertible sheaf, such that there is a point $p \in X$ for which the corresponding hypersurface $F_p := \{\phi_p = 0\} \subset \mathbb{P}(TX_p)$ is reduced.*
(2′) *The universal cover of X is a polydisk.*

Moreover, in case (1)*, the degrees and the multiplicities of the irreducible factors of the polynomial ψ_p determine uniquely the universal covering $\widetilde{X} = \Omega$.*

6.2. Arithmetic of moduli spaces and faithful actions of the absolute Galois group

A basic remark is that all the schemes involved in the construction of the Gieseker moduli space are defined by equations involving only $\mathbb{Z}$-coefficients.

It follows that the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}, \mathbb{Q})$ acts on the Gieseker moduli space $\mathfrak{M}_{a,b}$. In particular, it acts on the set of its irreducible components, and on the set of its connected components.

After an incomplete initial attempt in [8] in joint work with Ingrid Bauer and Fritz Grunewald, we were able in [9] to show:

Theorem 6.9. *The absolute Galois group* $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ *acts faithfully on the set of connected components of the Gieseker moduli space of surfaces of general type,*

$$\mathfrak{M} := \cup_{x,y \in \mathbb{N}, x,y \geq 1} \mathfrak{M}_{x,y}.$$

The main ingredients for the proof of theorem 6.9 are the following ones.

(1) Define, for any complex number $a \in \mathbb{C} \setminus \{-2g, 0, 1, \ldots, 2g-1\}$, C_a as the hyperelliptic curve of genus $g \geq 3$ which is the smooth complete model of the affine curve of equation

$$w^2 = (z-a)(z+2g)\Pi_{i=0}^{2g-1}(z-i).$$

Consider then two complex numbers a, b such that $a \in \mathbb{C} \setminus \mathbb{Q}$: then $C_a \cong C_b$ if and only if $a = b$.

(2) If $a \in \bar{\mathbb{Q}}$, then by Belyi's theorem ([18]) there is a morphism $f_a : C_a \to \mathbb{P}^1$ which is branched only on three points, $0, 1, \infty$.

(3) The normal closure D_a of f_a yields a triangle curve, *i.e.*, a curve D_a with the action of a finite group G_a such that $D_a/G_a \cong \mathbb{P}^1$, and $D_a \to \mathbb{P}^1$ is branched only on three points.

(4) Take surfaces isogenous to a product $S = (D_a \times D')/G_a$ where the action of G_a on D' is free. Denote by $\mathcal{N}_a$ the union of connected components of the moduli space parametrizing such surfaces.

(5) Take all the possible twists of the G_a-action on $D_a \times D'$ via an automorphism $\psi \in \mathrm{Aut}(G_a)$ (*i.e.*, given the action $(x, y) \mapsto (\gamma x, \gamma y)$, consider all the actions of the form

$$(x, y) \mapsto (\gamma x, \psi(\gamma) y).$$

One observes that, for each ψ as above, we get more connected components in $\mathcal{N}_a$.

(6) Then an explicit calculation (using (4) and (5)) shows that the subgroup of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acting trivially on the set of connected components of the moduli space would be a normal and abelian subgroup.
(7) Finally, this contradicts a known theorem (*cf.* [53]).

6.3. Change of fundamental group

Jean Pierre Serre proved in the 60's ([82]) the existence of a field automorphism $\phi \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, and a variety X defined over $\bar{\mathbb{Q}}$ such that X and the Galois conjugate variety X^{ϕ} have non isomorphic fundamental groups.

In [9] this phenomenon is vastly generalized, thus answering question B) in the negative.

Theorem 6.10. *If $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ is not in the conjugacy class of complex conjugation, then there exists a surface isogenous to a product X such that X and the Galois conjugate surface X^{σ} have non-isomorphic fundamental groups.*

Since the argument for the above theorem is not constructive, let us observe that, in work in collaboration with Ingrid Bauer and Fritz Grunewald ([7, 9]), we discovered wide classes of explicit algebraic surfaces isogenous to a product for which the same phenomenon holds.

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