On the order of cages with a given girth pair

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1 Cages with a given girth pair

The cage problem asks for the construction of regular simple graphs with specified degree k, girth g and minimum order $n(k; g)$, (see [5] for a complete survey). This problem was first considered by Tutte [10]. In 1963, Erdös and Sachs [4] proved that $(k; g)$-cages exist for any given values of k and g.

Counting the numbers of vertices in the distance partition with respect to a vertex when g is odd, and with respect to an edge when g is even, yields the lower bound $n_0(k; g)$ on the order of a $(k; g)$-cage. For $k \geq 3$ and $g \geq 5$ the order $n(k; g)$ of a cage is bounded by

$$n(k; g) \geq n_0(k; g) = \begin{cases} 1 + k \sum_{i=0}^{(g-3)/2} (k-1)^i & g \text{ odd} \\ 2 \sum_{i=0}^{(g-2)/2} (k-1)^i & g \text{ even} \end{cases} \tag{1.1}$$

This bound is called the Moore bound, it is known that the order of a $(k; g)$-cage $n(k; g) = n_0(k; g)$ only for $g = 6, 8, 12$ and $k = q + 1$ with q a prime power; and for $g = 5$ and $k = 3, 7, 57$ (*cf.* [1,3]). Therefore

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Biggs and Ito [2] defined the excess of a cage to be the number $n(k;g) - n_0(k;g)$ and proved the following theorem.

Theorem 1.1 ([2]). *[2] Let G be a a $(k;g)$-cage of girth $g = 2m \leq 6$ and excess e. If $e \leq k-2$ then G is bipartite and its diameter is $m+1$.*

By allowing a *girth pair* $g < h$ (one even and the other odd), in [6], Harary and Kovács introduced the concept of a $(k;g,h)$-*cage* as the smallest k-regular graph with girth pair g, h. They obtained the bound $n(k;g,h) \leq 2n(k;h)$ that relates the order of a cage with girth pair with the order of a cage, and showed that $n(k;h-1,h) \leq n(k;h)$, i.e. in general the bound $n(k;g,h) \leq 2n(k;h)$ is not the best and stated the following conjecture.

Conjecture 1.2 ([6]). $n(k;g,h) \leq n(k;h)$, for all $k, g \geq 3$.

Xu, Wang, Wang [11] proved the strict inequality, $n(k;h-1,h) < n(k;h)$. Kovács proved that the the Möbius ladder of order $2(h-1)$ is the unique minimal $(3;4,h)$-graph [7]. Campbell [8] studied the size of smallest cubic graphs with girth pair $(6,b)$ and constructed the cages for the exact values $(3;6,7)$, $(3;6,9)$ and $(3;6,11)$.

We obtain that the conjecture $n(k;g,h) < n(k;h)$ holds for all $(k;g,h)$-cages when g is odd. For g even, we settle the conjecture for cages of small excess, i.e. such that $n(k;g) - n_0(k;g) \leq k-1$, also we prove that $n(k;g,h) < n(k;h)$ for h sufficiently large, in both cases, for g even and g odd, under the assumption that $(k;g)$-cages are bipartite for g even. Notice that the cages of even girth and small excess are known to be bipartite [2], furthermore it is conjectured that all cages with even girth are bipartite [9,12].

2 Notation

For any graph of girth $g \geq 4$ even, $uv \in E(G)$ and $0 \leq l \leq \frac{g}{2} - 1$, let us denote the sets

$$B^l_{uv} = \{x \in V(G) : d(x,u) = l \text{ and } d(x,v) = l+1\} \text{ and } \overline{B}^l_{uv} = \bigcup_{i=0}^{l} B^i_{uv}.$$

Observe that $B^0_{uv} = \{u\} = \overline{B}^0_{uv}$ and $B^1_{uv} = N(u) - v$ while $\overline{B}^1_{uv} = (N(u) - v) \cup \{u\}$. Moreover, note that $B^l_{uv} \neq B^l_{vu}$ and $\overline{B}^l_{uv} \neq \overline{B}^l_{vu}$. Denote $T^l_{uv} = G[\overline{B}^l_{uv} \cup \overline{B}^l_{vu}]$ and observe that if $l \leq \frac{g}{2} - 2$, where g is the girth of G, then T^l_{uv} is the tree rooted in the edge uv of depth l. When $l = \frac{g}{2} - 1$ the subgraph T^l_{uv} may not be a tree, it can contain edges between vertices in B^l_{uv} and vertices in B^l_{vu}.

We will denote the set of cycles in G by $\mathcal{C}(G) = \{\alpha : \alpha \text{ is a cycle in } G\}$.

Let G be a $(k; g)$-cage of even girth $g = 2m$. The *excess* e of G with respect to an edge $uv \in E(G)$ is the cardinality of the set $X = V(G) \setminus T_{uv}^{m-1}$. Note that the order of T_{uv}^{m-1} is the same for every edge $uv \in E(G)$. Thus, $e = |X| = n(k; g) - n_0(k; g)$.

3 Constructions for g odd

Deleting from G the sets $\overline{B}_{uv}^{(\frac{h-g-1}{2}-1)} \cup \overline{B}_{vu}^{(\frac{h-g+1}{2}-1)}$ and completing the degrees of the remaining vertices we obtain Theorem 3.2. In order to keep cycles of length h after the deletion we proved the following lemma.

Lemma 3.1. *Let G be a $(k; h)$-cage with $k \geq 3$ and even girth $h \geq 6$. Then G contains a cycle β of length h such that $V(\beta) \cap B_{uv}^{\frac{h}{4}-1} = \emptyset$ or $V(\beta) \cap B_{vu}^{\frac{h}{4}-1} = \emptyset$.*

Theorem 3.2. *Let $h \geq 6$ even and $k \geq 3$. Suppose that there is a bipartite $(k; h)$-cage. If $g \geq 5$ is an odd number such that $\frac{h}{2} + 1 \leq g < h$, then*

$$n(k; g, h) \leq n(k; h) - 2 \sum_{i=0}^{\frac{h-g-3}{2}} (k-1)^i - (k-1)^{\frac{h-g-1}{2}}.$$

Theorem 3.3. *Let $h \geq 6$ even and $k \geq 3$. Suppose that there is a bipartite $(k; h)$-cage. If g is an odd number such that $g < h$, then $n(k; g, h) < n(k; h)$.*

4 Constructions for g even and h odd

First of all we introduce a construction that we will use later for breaking short odd cycles while preserving the regularity and the even girth.

Definition 4.1. Let G, H be two vertex-disjoint graphs, $uv \in E(G)$ and $st \in E(H)$. We will define a new graph $G^{uv}\Gamma_{st}H$, that we will call the *insertion* of (G, uv) into (H, st) by letting:

- $V(G^{uv}\Gamma_{st}H) = V(G) \cup V(H)$
- $E(G^{uv}\Gamma_{st}H) = (E(G) \setminus \{uv\}) \cup (E(H) \setminus \{st\}) \cup \{us, vt\}$.

See Figure 4.1, for an example illustrating this definition.

The first basic result with respect to g even and h odd is the following theorem, the bound is proved inserting a graph (G, uv) into a copy $(G', u'v')$, and performing some edge operations in order to obtain cycles of length $2g - 1$.

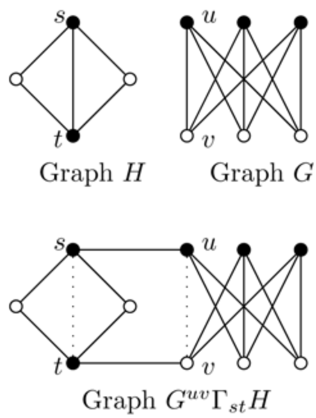

Figure 4.1. The insertion $G^{uv}\Gamma_{st}H$.

Theorem 4.2. *Let* $k \geq 3$ *and* $g \geq 6$ *even. Then* $n(k; g, 2g-1) \leq 2n(k; g)$ *provided that there is a bipartite* $(k; g)$*-cage.*

To prove the corresponding result for $n(k; g, g+r)$ we will use the following remark.

Remark 4.3. Let G, H be graphs with girths g, h, respectively, such that $g \leq h$, and let $G^{uv}\Gamma_{st}H$ be the insertion of (G, uv) into (H, st). Then the set of cycles in $G^{uv}\Gamma_{st}H$ is:

$$\begin{aligned}\mathcal{C}(G^{uv}\Gamma_{st}H) = {} & (\mathcal{C}(G) \setminus \{\alpha \in \mathcal{C}(G) : uv \in E(\alpha)\}) \\ & \cup (\mathcal{C}(H) \setminus \{\beta \in \mathcal{C}(H) : st \in E(\beta)\}) \\ & \cup \{\gamma = P_1 vt P_2 su : P_1 \text{ is a } uv\text{-path in } G - uv \\ & \qquad \text{and } P_2 \text{ is a } ts\text{-path in } H - st\}.\end{aligned}$$

This means that if there were cycles of lengths c_1 and c_2 in graphs G and H that used the edges uv and st, respectively, they are removed in the new graph $G^{uv}\Gamma_{st}H$ and new cycles of length $c_1 + c_2$ are created.

Theorem 4.4. *Let* $k \geq 3$*,* g *even such that* $6 \leq g$ *and* r *an odd number such that* $1 \leq r \leq g-3$*. Then* $n(k; g, g+r) \leq 4n(k; g)$*, provided that there is a bipartite* $(k; g)$*-cage.*

By applying the insertion to the graph obtained in Theorem 4.4 on specific edges, we obtain the following lemma.

Lemma 4.5. *Let* $k \geq 3$*,* $g \geq 6$ *even and suppose that there is a bipartite* $(k; g)$*-cage. Then* $n(k; g, mg+r) \leq 4n(k; g) + k(m-1)n(k; g)$*, for* $m \geq 1$ *and* r *any odd number such that* $1 \leq r \leq g-1$*. In particular when* $r = g-1$*, from Theorem 4.2, we have* $n(k; g, (m+1)g-1) \leq 2mn(k; g)$*.*

Theorem 4.6. *Suppose there is a bipartite $(k; g)$-cage with degree $k \geq 3$ and even girth $g \geq 6$. Then $n(k; g, h) < n(k, h)$, for h sufficiently large.*

So, we have proved Conjecture 1.2 for even girth g in general but asymptotically. For specific values of k and g, the Conjecture 1.2 can be completely settled, as we will show next.

Corollary 4.7. *For every $(k; g)$-cage of even girth g, degree $k \geq 3$ and excess $e \leq k - 2$, we have $n(k + 1; g, h) < n(k + 1; h)$.*

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