Regular hypergraphs: asymptotic counting and loose Hamilton cycles

Andrzej Dudek[1], Alan Frieze[2], Andrzej Ruciński[3] and Matas Šileikis[4]

Abstract. We present results from two papers by the authors on analysis of d-regular k-uniform hypergraphs, when k is fixed and the number n of vertices tends to infinity. The first result is approximate enumeration of such hypergraphs, provided $d = d(n) = o(n^{\kappa})$, where $\kappa = \kappa(k) = 1$ for all $k \geq 4$, while $\kappa(3) = 1/2$. The second result is that a random d-regular hypergraph contains as a dense subgraph the uniform random hypergraph (a generalization of the Erdős-Rényi uniform graph), and, in view of known results, contains a loose Hamilton cycle with probability tending to one.

1. Regular k-graphs and k-multigraphs. We consider *k-uniform hypergraphs* (or *k-graphs*, for short) on the vertex set $V = [n] := \{1, \ldots, n\}$, that is, families of k-element subsets of V. A k-graph H is *d-regular*, if the degree of every vertex $v \in V$, $\deg_H(v) := \deg(v) := |\{e \in H : v \in e\}|$ equals d.

Let $\mathcal{H}^{(k)}(n, d)$ be the class of all d-regular k-graphs on $[n]$. Note that each $H \in \mathcal{H}^{(k)}(n, d)$ has $M := nd/k$ edges (throughout, we implicitly assume that k divides nd). Let $\mathbb{H}^{(k)}(n, d)$ be a k-graph chosen from $\mathcal{H}^{(k)}(n, d)$ uniformly at random. We treat d as a function of n (possibly constant) and study $\mathcal{H}^{(k)}(n, d)$ as well as $\mathbb{H}^{(k)}(n, d)$ as n tends to infinity.

By a *k-multigraph* on the vertex set $[n]$ we mean a multiset of k-element multisubsets of $[n]$. An edge is called a *loop* if it contains more than one copy of some vertex and otherwise it is called *a proper edge*. A k-multigraph is *simple*, if it is a k-graph.

A standard tool to study regular (hyper)graphs is the so called *configuration model* of a random k-multigraph (see [10] for $k = 2$; its generalization to every k is straightforward). We use a slightly different model yielding the same distribution of k-multigraphs. Let $\mathcal{S} \subset [n]^{nd}$ be the

[1] Department of Mathematics, Western Michigan University, Kalamazoo, MI.
Email: andrzej.dudek@wmich.edu

[2] Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, PA.
Email: alan@random.math.cmu.edu

[3] Department of Discrete Mathematics, Adam Mickiewicz University, Poznań, Poland.
Email: andrzej@mathcs.emory.edu

[4] Department of Mathematics, Uppsala University, Sweden. Email: matas.sileikis@gmail.com

J. Nešetřil et al. (eds.), *The Seventh European Conference on Combinatorics, Graph Theory and Applications*

family of all sequences in which every value $i \in [n]$ occurs precisely d times. Let $\mathbf{Y} = (Y_1, \dots, Y_{nd})$ be a sequence chosen from $\mathcal{S}$ uniformly at random, and define $\mathbb{H}_*^{(k)}(n, d)$ as a k-multigraph with the edge set

$$\{Y_{ki+1} \dots Y_{ki+k} : i = 0, \dots, nd - 1\}.$$

2. The switching. The essential tool in both papers presented here is the so called *switching* technique, introduced by McKay [8] for asymptotic enumeration of regular graphs. McKay and Wormald [9] improved McKay's result by applying a more advanced version of switching, which we extend to hypergraphs as follows.

Let us view sequences $\mathcal{S}$ as ordered k-multigraphs (that is, k-multigraphs with an ordering of edges). Let $\mathcal{E}_l \subset \mathcal{S}$ be the family of sequences with no multiple edges and exactly l loops, but no loops with less than $k - 1$ distinct vertices. Thus, every loop in such a k-multigraph has only one multiple vertex, which has multiplicity two.

The *switching* is an operation which maps a sequence $\mathbf{x} \in \mathcal{E}_l$ to $\mathbf{y} \in \mathcal{E}_{l-1}$ as follows. Choose a loop f and two proper edges e_1, e_2. Select a vertex $v \in e_1 \setminus e_2$ as well as a vertex $w \in e_2 \setminus e_1$. Suppose that u is the multiple vertex of f. Swap v with one copy of u and w with the other. The effect of this is that edges f, e_1, e_2 in $\mathbf{x}$ are replaced with the following three edges in $\mathbf{y}$

$$f \setminus \{u, u\} \cup \{v, w\}, \quad e_1 \setminus \{v\} \cup \{u\}, \quad e_2 \setminus \{w\} \cup \{u\}.$$

3. Counting Regular Hypergraphs. In [5] we approximately count d-regular k-graphs. Since $|\mathcal{S}| = (nd)!/(d!)^n$ and every every simple k-graph is given by exactly $M!(k!)^M$ sequences in $\mathcal{S}$, the number of d-regular k-graphs is precisely

$$|\mathcal{H}^{(k)}(n, d)| = \frac{(nd)!}{M!(k!)^M (d!)^n} \mathbb{P}\left(\mathbb{H}_*^{(k)}(n, d) \text{ is simple}\right).$$

Therefore the problem of asymptotic enumeration reduces to a the analysis of the probability. For graphs, that is, $k = 2$, this has been well studied (see [10]). For general k but fixed d, Cooper, Frieze, Molloy and Reed [1] showed that the probability converges to $\exp\{-(k-1)(d-1)/2\}$. In [5] we extend this to the following formula. Let $\kappa(k) = 1$, if $k \geq 4$ and $\kappa(3) = 1/2$.

Theorem 1. *For $k \geq 3$ and $1 \leq d = o(n^{\kappa(k)})$ we have*

$$\mathbb{P}\left(\mathbb{H}_*^{(k)}(n, d) \text{ is simple}\right) = \exp\left\{-\frac{(k-1)(d-1)}{2} + O\left(\frac{d^2}{n} + \sqrt{\frac{d}{n}}\right)\right\}$$

The proof of Theorem 1 is simplified by the fact (shown in [5]) that a randomly chosen sequence $\mathbf{Y} \in \mathcal{S}$ with probability tending to one belongs to $\mathcal{E}_l$ with a reasonably small l. This allows us to reduce the analysis of the probability to estimating the ratios $|\mathcal{E}_l|/|\mathcal{E}_{l-1}|$. This is done by bounding the number of ways one can apply switching to a sequence. The proof works because, as it turns out, the number of possible switchings depends essentially on the number of loops, but not on the structure of the sequence.

4. Hamilton Cycles in Regular Hypergraphs. Let us recall that for integer $m \in [0, \binom{n}{k}]$, $\mathbb{H}^{(k)}(n, m)$ is the random graph chosen uniformly at random among k-graph on $[n]$ with precisely m edges.

Our main result in [6] is that we can couple $\mathbb{H}^{(k)}(n, d)$ and $\mathbb{H}^{(k)}(n, m)$ so that the latter is a subgraph of the former with probability tending to one.

Theorem 2. *For every $k \geq 3$, there are positive constants c and C such that if $d \geq C \log n$, $d = o(n^{1/2})$ and $m = \lfloor cM \rfloor = \lfloor cnd/k \rfloor$, then one can define a joint distribution of random graphs $\mathbb{H}^{(k)}(n, d)$ and $\mathbb{H}^{(k)}(n, m)$ in such a way that*

$$\mathbb{P}\left(\mathbb{H}^{(k)}(n, m) \subset \mathbb{H}^{(k)}(n, d)\right) \to 1, \qquad n \to \infty.$$

The idea of the proof is as follows. Since m is a fraction of M, we are able to couple $\mathbb{H}^{(k)}(n, m)$ with $\mathbb{H}_*^{(k)}(n, d)$ (treated as an ordered k-multigraph) in such a way that with probability tending to one $\mathbb{H}^{(k)}(n, m)$ is contained in an initial segment of $\mathbb{H}_*^{(k)}(n, d)$, which we colour *red*. Then we swap all red loops of $\mathbb{H}_*^{(k)}(n, d)$ with randomly selected non-red (*green*) proper edges. Finally, we destroy the green loops of $\mathbb{H}_*^{(k)}(n, d)$ one by one applying a randomly chosen switching which involves green edges only. This does not destroy the previously embedded copy of $\mathbb{H}^{(k)}(n, m)$. Moreover, it transforms $\mathbb{H}_*^{(k)}(n, d)$ into a k-graph $\tilde{\mathbb{H}}^{(k)}(n, d)$, which is distributed approximately as $\mathbb{H}^{(k)}(n, d)$, that is, almost uniformly. Theorem 2 then follows by a (maximal) coupling of $\tilde{\mathbb{H}}^{(k)}(n, d)$ and $\mathbb{H}^{(k)}(n, d)$.

A *loose Hamilton cycle* on a vertex set V is a set of edges $e_1, \ldots, e_s$ such that for some cyclic order of V each e_i consists of k consecutive vertices and $|e_i \cap e_{i+1}| = 1$ for $i = 1, \ldots, s$, with $e_{s+1} = e_1$. For $k = 2$ this coincides with the standard notion of a Hamilton cycle.

Asymptotic hamiltonicity for graphs has been intensely investigated since 1978 and rather recently was established in full generality for every $d \geq 3$, both fixed and growing with n (see [2] and references in it).

As for general k, this has been known for some simpler models of random k-graphs. From results of Frieze [7], Dudek and Frieze [3] as well as Dudek, Frieze, Loh and Speiss [4], it follows that $\mathbb{H}^{(k)}(n, m)$ contains a loose Hamilton cycle when the expected degree of a vertex grows faster than $\log n$. This, combined with Theorem 2, implies the following fact.

Corollary 3. *Suppose that $d = o(n^{1/2})$. If $k = 3$ and $d \geq C \log n$ for large constant C or $k \geq 4$ and $\log n = o(d)$, then*

$$\mathbb{P}\left(\mathbb{H}^{(k)}(n, d) \text{ contains a loose Hamilton cycle}\right) \to 1, \qquad n \to \infty.$$

References

[1] C. COOPER, A. FRIEZE, M. MOLLOY and B. REED, *Perfect matchings in random r-regular, s-uniform hypergraphs*, Combin. Probab. Comput. **5** (1) (1996), 1–14.

[2] C. COOPER, A. FRIEZE and B. REED, *Random regular graphs of non-constant degree: connectivity and Hamiltonicity*, Combin. Probab. Comput. **11** (3) (2002), 249–261.

[3] A. DUDEK and A. FRIEZE, *Loose Hamilton cycles in random uniform hypergraphs*, Electron. J. Combin. **18** (1) (2011), Paper 48, pp. 14.

[4] A. DUDEK, A. FRIEZE, P.-S. LOH and S. SPEISS, *Optimal divisibility conditions for loose Hamilton cycles in random hypergraphs*, Electron. J. Combin. **19** (4) (2012), Paper 44, pp. 17.

[5] A. DUDEK, A. FRIEZE, A. RUCIŃSKI and M. ŠILEIKIS, *Approximate counting of regular hypergraphs*, preprint, 2013. http://arxiv.org/abs/1303.0400.

[6] A. DUDEK, A. FRIEZE, A. RUCIŃSKI and M. ŠILEIKIS, *Loose hamilton cycles in regular hypergraphs*, preprint, 2013. http://sites.google.com/site/matassileikis/DFRS20132.pdf.

[7] A. FRIEZE, *Loose Hamilton cycles in random 3-uniform hypergraphs*, Electron. J. Combin. **17** (1) (2010), Note 28, pp. 4.

[8] B. D. MCKAY, *Asymptotics for symmetric* 0-1 *matrices with prescribed row sums*, Ars Combin. **19** (A) (1985), 15–25.

[9] B. D. MCKAY and N. C. WORMALD, *Uniform generation of random regular graphs of moderate degree*, J. Algorithms **11** (1) (1990), 52–67.

[10] N. C. WORMALD, *Models of random regular graphs*, In: "Surveys in combinatorics, 1999 (Canterbury)", volume 267 of London Math. Soc. Lecture Note Ser., Cambridge Univ. Press, Cambridge, 1999, 239–298.