Proof of the 1-factorization and Hamilton decomposition conjectures

Béla Csaba[1], Daniela Kühn[2], Allan Lo[2], Deryk Osthus[2] and Andrew Treglown[3]

Abstract. We prove the following results (via a unified approach) for all sufficiently large n:

(i) [*1-factorization conjecture*] Suppose that n is even and $D \geq 2\lceil n/4 \rceil - 1$. Then every D-regular graph G on n vertices has a decomposition into perfect matchings. Equivalently, $\chi'(G) = D$.

(ii) [*Hamilton decomposition conjecture*] Suppose that $D \geq \lfloor n/2 \rfloor$. Then every D-regular graph G on n vertices has a decomposition into Hamilton cycles and at most one perfect matching.

(iii) [*Optimal packings of Hamilton cycles*] Suppose that G is a graph on n vertices with minimum degree $\delta \geq n/2$. Then G contains at least $(n-2)/8$ edge-disjoint Hamilton cycles.

According to Dirac, (i) was first raised in the 1950's. (ii) and (iii) answer questions of Nash-Williams from 1970. All of the above bounds are best possible.

1 Introduction

In a sequence of four papers [5,6,11,12], we provide a unified approach towards proving three long-standing conjectures for all sufficiently large graphs. Firstly, the 1-factorization conjecture, which can be formulated as an edge-colouring problem; secondly, the Hamilton decomposition conjecture, which provides a far-reaching generalization of Walecki's result [15] that every complete graph of odd order has a Hamilton decomposition and thirdly, a best possible result on packing edge-disjoint Hamilton cycles in Dirac graphs. The latter two were raised by Nash-Williams [17–19] in 1970. A key tool is the recent result of Kühn and Osthus [13] that every dense even-regular robustly expanding graph has a Hamilton decomposition.

1.1 The 1-factorization conjecture

Vizing's theorem states that for any graph G of maximum degree Δ, its edge-chromatic number $\chi'(G)$ is either Δ or $\Delta + 1$. In general, it is a very difficult problem to determine which graphs G attain the (trivial) lower bound Δ – much of the recent book [22] is devoted to the subject. For regular graphs G, $\chi'(G) = \Delta(G)$ is equivalent to the existence of a 1-factorization: a 1-*factorization* of a graph G consists of a set of edge-

[1] Bolyai Institute, University of Szeged, H-6720 Szeged, Hungary.
Email: bcsaba@math.u-szeged.hu
[2] School of Mathematics, University of Birmingham, Birmingham B15 2TT, United Kingdom.
Email: d.kuhn@bham.ac.uk, s.a.lo@bham.ac.uk, d.osthus@bham.ac.uk
[3] School of Mathematical Sciences, Queen Mary, University of London, London E1 4NS, United Kingdom. Email: a.treglown@qmul.ac.uk

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disjoint perfect matchings covering all edges of G. The long-standing 1-factorization conjecture states that every regular graph of sufficiently high degree has a 1-factorization. It was first stated explicitly by Chetwynd and Hilton [1,3] (who also proved partial results). However, they state that according to Dirac, it was already discussed in the 1950's.

Theorem 1.1. *There exists an $n_0 \in \mathbb{N}$ such that the following holds. Let $n, D \in \mathbb{N}$ be such that $n \geq n_0$ is even and $D \geq 2\lceil n/4 \rceil - 1$. Then every D-regular graph G on n vertices has a 1-factorization. Equivalently, $\chi'(G) = D$.*

The bound on the degree in Theorem 1.1 is best possible. To see this, suppose first that $n = 2 \pmod 4$. Consider the graph which is the disjoint union of two cliques of order $n/2$ (which is odd). If $n = 0 \pmod 4$, consider the graph obtained from the disjoint union of cliques of orders $n/2-1$ and $n/2+1$ (both odd) by deleting a Hamilton cycle in the larger clique.

Note that Theorem 1.1 implies that for every regular graph G on an even number of vertices, either G or its complement has a 1-factorization. Also, Theorem 1.1 has an interpretation in terms of scheduling round-robin tournaments (where n players play all of each other in $n-1$ rounds): one can schedule the first half of the rounds arbitrarily before one needs to plan the remainder of the tournament.

The best previous result towards Theorem 1.1 is due to Perkovic and Reed [20], who proved an approximate version, *i.e.* they assumed that $D \geq n/2 + \varepsilon n$. This was generalized by Vaughan [23] to multigraphs of bounded multiplicity. Indeed, he proved an approximate version of the following multigraph version of the 1-factorization conjecture which was raised by Plantholt and Tipnis [21]: Let G be a regular multigraph of even order n with multiplicity at most r. If the degree of G is at least $rn/2$ then G is 1-factorizable.

In 1986, Chetwynd and Hilton [2] made the following 'overfull subgraph' conjecture, which also generalizes the 1-factorization conjecture. Roughly speaking, this says that a dense graph satisfies $\chi'(G) = \Delta(G)$ unless there is a trivial obstruction in the form of a dense subgraph H on an odd number of vertices. Formally, we say that a subgraph H of G is *overfull* if $e(H) > \Delta(G)\lfloor |H|/2 \rfloor$ (note this requires $|H|$ to be odd).

Conjecture 1.2. A graph G on n vertices with $\Delta(G) \geq n/3$ satisfies $\chi'(G) = \Delta(G)$ if and only if G contains no overfull subgraph.

This conjecture is still wide open – partial results are discussed in [22], which also discusses further results and questions related to the 1-factorization conjecture.

1.2 The Hamilton decomposition conjecture

Rather than asking for a 1-factorization, Nash-Williams [17,19] raised the more difficult problem of finding a Hamilton decomposition in an even-regular graph. Here, a *Hamilton decomposition* of a graph G consists of a set of edge-disjoint Hamilton cycles covering all edges of G. A natural extension of this to regular graphs G of odd degree is to ask for a decomposition into Hamilton cycles and one perfect matching (*i.e.* one perfect matching M in G together with a Hamilton decomposition of $G - M$). The following result solves the problem of Nash-Williams for all large graphs.

Theorem 1.3. *There exists an $n_0 \in \mathbb{N}$ such that the following holds. Let $n, D \in \mathbb{N}$ be such that $n \geq n_0$ and $D \geq \lfloor n/2 \rfloor$. Then every D-regular graph G on n vertices has a decomposition into Hamilton cycles and at most one perfect matching.*

Again, the bound on the degree in Theorem 1.3 is best possible. Previous results include the following: Nash-Williams [16] showed that the degree bound in Theorem 1.3 ensures a single Hamilton cycle. Jackson [8] showed that one can ensure close to $D/2 - n/6$ edge-disjoint Hamilton cycles. Christofides, Kühn and Osthus [4] obtained an approximate decomposition under the assumption that $D \geq n/2 + \varepsilon n$. Under the same assumption, Kühn and Osthus [14] obtained an exact decomposition (as a consequence of their main result in [13] on Hamilton decompositions of robustly expanding graphs).

Note that Theorem 1.3 does not quite imply Theorem 1.1, as the degree threshold in the former result is slightly higher.

A natural question is whether one can extend Theorem 1.3 to sparser (quasi)-random graphs. Indeed, for random regular graphs of bounded degree this was proved by Kim and Wormald [9] and for (quasi-)random regular graphs of linear degree this was proved in [14] as a consequence of the main result in [13]. However, the intermediate range remains open.

1.3 Packing Hamilton cycles in graphs of large minimum degree

Although Dirac's theorem is best possible in the sense that the minimum degree condition $\delta \geq n/2$ is best possible, the conclusion can be strengthened considerably: a remarkable result of Nash-Williams [18] states that every graph G on n vertices with minimum degree $\delta(G) \geq n/2$ contains $\lfloor 5n/224 \rfloor$ edge-disjoint Hamilton cycles. He raised the question of finding the best possible bound, which we answer below for all large graphs.

Theorem 1.4. *There exists an $n_0 \in \mathbb{N}$ such that the following holds. Suppose that G is a graph on $n \geq n_0$ vertices with minimum degree $\delta \geq n/2$. Then G contains at least $(n-2)/8$ edge-disjoint Hamilton cycles.*

The following construction (which is based on a construction of Babai, see [17]) shows that the bound is best possible for $n = 8k + 2$, where $k \in \mathbb{N}$. Consider the graph G consisting of one empty vertex class A of size $4k$, one vertex class B of size $4k + 2$ containing a perfect matching and no other edges, and all possible edges between A and B. Thus G has order $n = 8k + 2$ and minimum degree $4k + 1 = n/2$. Any Hamilton cycle in G must contain at least two edges of the perfect matching in B, so G contains at most $\lfloor |B|/4 \rfloor = k = (n-2)/8$ edge-disjoint Hamilton cycles.

A more general question is to ask for the number of edge-disjoint Hamilton cycles one can guarantee in a graph G of minimum degree δ. This number has been determined exactly by Kühn, Lapinskas and Osthus [10] unless G is close to one of the extremal graphs for Dirac's theorem (*i.e.* unless G is close to the complete balanced bipartite graph or close to the union of two disjoint copies of a clique). In particular, the number of edge-disjoint Hamilton cycles one can guarantee is known exactly whenever $\delta \geq n/2 + \varepsilon n$. This improves earlier results of Christofides, Kühn and Osthus [4] as well as Hartke and Seacrest [7]. Actually, our proof of Theorem 1.4 also settles the cases when G is close to the extremal graphs for Dirac's theorem. So altogether this solves the problem for all values of δ.

2 Overview of the proofs of Theorems 1.1 and 1.3

The proofs develop methods established by Kühn and Osthus [13], who proved a generalization of Kelly's conjecture that every regular tournament has a Hamilton decomposition (for large tournaments). For all three of our main results, we split the argument according to the structure of the graph G under consideration:

(i) G is close to the complete balanced bipartite graph $K_{n/2,n/2}$;
(ii) G is close to the union of two disjoint copies of a clique $K_{n/2}$;
(iii) G is a 'robust expander'.

Informally, a graph G is a robust expander if for every set $S \subseteq V(G)$ which is not too large or too small, its neighbourhood is substantially larger than $|S|$, even if we delete a small proportion of the edges of G. In other words, G is an expander graph which is 'locally resilient'. The main result of [13] states that every dense regular robust expander has a Hamilton decomposition. This immediately implies Theorems 1.1 and 1.3 in Case (iii).

Suppose we are going to prove Theorem 1.3 in the case when D is even. So our aim is to decompose G into $D/2$ edge-disjoint Hamilton cycles. As mentioned above, we may assume that G is in either Case (i) or Case (ii). In [6], we find an approximate Hamilton decomposition of G in both cases, *i.e.* a set of edge-disjoint Hamilton cycles covering almost all edges of G. However, one does not have any control over the 'leftover' graph H, which makes a complete decomposition seem infeasible. This problem was overcome in [13] by introducing the concept of a 'robustly decomposable graph' $G^{\rm rob}$. Roughly speaking, this is a sparse regular graph with the following property: given *any* very sparse regular graph H with $V(H) = V(G^{\rm rob})$ which is edge-disjoint from $G^{\rm rob}$, one can guarantee that $G^{\rm rob} \cup H$ has a Hamilton decomposition. This leads to a natural (and very general) strategy to obtain a decomposition of G:

(1) find a (sparse) robustly decomposable graph $G^{\rm rob}$ in G and let G' denote the leftover;
(2) find an approximate Hamilton decomposition of G' and let H denote the (very sparse) leftover;
(3) find a Hamilton decomposition of $G^{\rm rob} \cup H$.

It is of course far from clear that one can always find such a graph $G^{\rm rob}$, especially in Case (ii) where G is close to being disconnected. In [5], we find $G^{\rm rob}$ for Case (i). In [11,12], we find $G^{\rm rob}$ for Case (ii).

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