On the Bruhat-Chevalley order on fixed-point-free involutions

Mahir Bilen Can[1], Yonah Cherniavsky[2] and Tim Twelbeck[3]

1 Introduction

The purpose of this paper is twofold. First is to prove that the Bruhat-Chevalley ordering restricted to fixed-point-free involutions is a lexicographically shellable poset. Second is to prove that the Deodhar-Srinivasan poset is a graded subposet of the Bruhat-Chevalley poset structure on fixed-point-free involutions.

In this work we are concerned with the interaction between two well known subgroups of the special linear group SL_{2n}, namely a Borel subgroup and a symplectic subgroup. Without loss of generality, we choose the Borel subgroup B to be the group of invertible upper triangular matrices, and define the *symplectic group,* Sp_{2n} as the subgroup of fixed elements of the involutory automorphism $\theta : SL_{2n} \to SL_{2n}$, $\theta(g) = J(g^{-1})^\top J^{-1}$, where J denotes the skew form $J = \begin{pmatrix} 0 & \omega_0 \\ -\omega_0 & 0 \end{pmatrix}$, and ω_0 is the $n \times n$, 0/1 matrix with 1's on its main anti-diaonal.

It is clear that B acts by left-multiplication on SL_{2n}/Sp_{2n}. We investigate the covering relations of the poset F_{2n} of inclusion relations among the B-orbit closures. To further motivate our discussion and help the reader to place our work appropriately we look at a related situation. It is well known that the symmetric group of permutation matrices, S_m parameterizes the orbits of the Borel group of upper triangular matrices $B \subset SL_m$ in the flag variety SL_m/B. For $u \in S_m$, let $\dot{u}$ denote the right coset in SL_m/B represented by u. The classical *Bruhat-Chevalley ordering* is defined by $u \leq_{S_m} v \iff B \cdot \dot{u} \subseteq \overline{B \cdot \dot{v}}$ for $u, v \in S_m$.

[1] Yale University, New Haven, USA. Email: mahir.can@yale.edu

[2] Ariel University, Israel. Email: yonahch@ariel.ac.il

[3] Tulane University, New Orleans, USA. Email: ttwelbec@tulane.edu

The full-text paper can be found in [6].

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A permutation $u \in S_m$ is said to be an *involution*, if $u^2 = id$, or equivalently, its permutation matrix is a symmetric matrix. We denote by I_m the set of all involutions in S_m, and consider it as a subposet of the Bruhat-Chevalley poset $(S_m, \leq_{S_m})$. Let m be an even number, $m = 2n$. An element $x \in I_{2n}$ is called *fixed-point-free*, if the matrix of x has no non-zero diagonal entries. In [15], Example 10.4, Richardson and Springer show that there exists a poset isomorphism between F_{2n} and a subposet of fixed-point-free involutions in I_{2n}. Unfortunately, F_{2n} does not form an interval in I_{2n}, hence it does not immediately inherit nice properties therein. Let $\leq$ denote the restriction of the Bruhat-Chevalley ordering on F_{2n}. Our first main result is that $(F_{2n}, \leq)$ is "EL-shellable," which is a property that well known to be true for many other related posets. See [3–5, 8, 11, 12]. See [10], also. Notice that there are several versions of lexicographic shellability and EL-shellability is the strongest one. In [13] it is shown that F_{2n} is CL-shellable. So, our result is a strengthening of this result of [13] in the special case of the poset F_{2n}. Recall that a finite graded poset P with a maximum and a minimum element is called *EL-shellable*, if there exists a map $f = f_\Gamma : C(P) \to \Gamma$ between the set of covering relations $C(P)$ of P into a totally ordered set Γ satisfying: 1) In every interval $[x, y] \subseteq P$ of length $k > 0$ there exists a unique saturated chain $\mathfrak{c} : \ x_0 = x < x_1 < \cdots < x_{k-1} < x_k = y$ such that the entries of the sequence $f(\mathfrak{c}) = (f(x_0, x_1), f(x_1, x_2), \ldots, f(x_{k-1}, x_k))$ is weakly increasing. 2) The sequence $f(\mathfrak{c})$ of the unique chain $\mathfrak{c}$ from (1) is the smallest among all sequences of the form $(f(x_0, x_1'), f(x_1', x_2'), \ldots, f(x_{k-1}', x_k))$, where $x_0 \leq x_1' \leq \cdots \leq x_{k-1}' \leq x_k$. Recall that the *order complex* of a poset P is the abstract simplicial complex $\Delta(P)$ whose simplexes are the chains in P. For a lexicographically shellable poset the order complex is shellable, in particular it implies that $\Delta(P)$ is Cohen-Macaulay [2]. These, of course, are among the most desirable properties of a topological space. One of the reasons the EL-shellability of F_{2n} is not considered before is that there is a closely related EL-shellable partial order studied by Deodhar and Srinivasan in [7] which we denote here as $\leq_{DS}$. By some authors the Deodhar-Srinivasan's ordering is thought to be the same as Bruhat-Chevalley ordering on F_{2n}. A careful inspection of the Hasse diagrams of $(F_{2n}, \leq)$ and $(\tilde{F}_{2n}, \leq_{DS})$ reveals that these two posets are "almost" the same but different. Our second main result is that the rank functions of these posets are the same, and furthermore, the latter is a graded subposet of the former.

2 Preliminaries

We denote the set $\{1,\dots,m\}$ by $[m]$. In this work, all posets are assumed to be finite and assumed to have a minimal and a maximal element, denoted by $\hat{0}$ and $\hat{1}$, respectively. Recall that in a poset P, an element y is said to *cover* another element x, if $x < y$ and if $x \leq z \leq y$ for some $z \in P$, then either $z = x$ or $z = y$. In this case, we write $y \rightarrow x$. Given P, we denote by $C(P)$ the set of all covering relations of P. An (increasing) *chain* in P is a sequence of distinct elements such that $x = x_1 < x_2 < \cdots < x_{n-1} < x_n = y$. A chain in a poset P is called *saturated* (or, *maximal*), if it is of the form $x = x_1 \leftarrow x_2 \leftarrow \cdots \leftarrow x_{n-1} \leftarrow x_n = y$. Recall also that a poset is called *graded* if all maximal chains between any two comparable elements $x \leq y$ have the same length. This amounts to the existence of an integer valued function $\ell_P : P \rightarrow \mathbb{N}$ satisfying 1) $\ell_P(\hat{0}) = 0$, 2) $\ell_P(y) = \ell_P(x) + 1$ whenever y covers x in P. ℓ_P is called the *length function* of P.

Let Sym_n denote the affine space of symmetric matrices and let Sym_n^0 denote its closed subset consisting of symmetric matrices with determinant 1. Similarly, let $Skew_{2n}$ denote the affine space of skew-symmetric matrices, and let $Skew_{2n}^0$ denote its closed subset consisting of elements with determinant 1. Let X denote any of the spaces Sym_n, Sym_n^0, $Skew_{2n}$, or $Skew_{2n}^0$. Then the special linear group of appropriate rank acts on X via $g \cdot A = (g^{-1})^\top A g^{-1}$. Define $SO_n := \{g \in SL_n : gg^\top = id_n\}$. The symmetric spaces $\mathrm{SL}_{2n}/\mathrm{Sp}_{2n}$ and $\mathrm{SL}_n/\mathrm{SO}_n$ can be canonically identified with the spaces $Skew_{2n}^0$ and Sym_n^0, respectively (for details see [9]). Recall that an $n \times n$ *partial permutation matrix* (or, a *rook matrix*) is a 0/1 matrix with at most one 1 in each row and each column. The set of all $n \times n$ rook matrices is denoted by R_n. In [14], Renner shows that R_n parameterizes the $B \times B$-orbits on the monoid of $n \times n$ matrices. It is known that the partial ordering on R_n induced from the containment relations among the $B \times B$-orbit closures is a lexicographically shellable poset (see [4]). On the other hand, for the purposes of this paper, it is more natural for us to look at the inclusion poset of $B^\top \times B$-orbit closures in R_n, which we denote by $(R_n, \leq_{Rook})$. A symmetric rook matrix is called a *partial involution*. The set of all partial involutions in R_n is denoted by PI_n. It is known that each Borel orbit in Sym_n contains a unique element of PI_n. A rook matrix is called a *partial fixed-point-free involution*, if it is symmetric and does not have any non-zero entry on its main diagonal. We denote by PF_{2n} the set of all partial fixed-point-free involutions. It is known that PF_{2n} parameterizes the Borel orbits in $Skew_{2n}$.

Containment relations among the closures of Borel orbits in $Skew_{2n}$ define a partial ordering on PF_{2n}. We denote its opposite by $\leq_{Skew}$. Similarly, on PI_n we have the opposite of the partial ordering induced from the containment relations among the Borel orbit closures in Sym_n. We denote this opposite partial ordering by $\leq_{Sym}$. In [11], Incitti, studying the restriction of the partial order $\leq_{Sym}$ on I_n, finds an EL-labeling for I_n. Let us mention that in a recent preprint, Can and Twelbeck, using an extension of Incitti's edge-labeling show that PI_n is EL-shellable. See [5].

There is a combinatorial method for deciding when two elements x and y from $(R_n, \leq_{Rook})$ (respectively, from $(PI_n, \leq_{Sym})$, or from $(PF_{2n}, \leq_{Skew})$) are comparable with respect to $\leq_{Rook}$ (respectively, with respect to $\leq_{Sym}$, or $\leq_{Skew}$). Denote by $Rk(x)$ the matrix whose i, j-th entry is the rank of the upper left $i \times j$ submatrix of x. We call $Rk(x)$, the *rank-control matrix* of x. Let $A = (a_{i,j})$ and $B = (b_{i,j})$ be two matrices of the same size with real number entries. We write $A \leq B$ if $a_{i,j} \leq b_{i,j}$ for all i and j. Then $x \leq_{Rook} y \iff Rk(y) \leq Rk(x)$. The same criterion holds for the posets $\leq_{Sym}$ and $\leq_{Skew}$. Now, suppose x is an $m \times m$ matrix with the rank-control matrix $Rk(x) = (r_{i,j})_{i,j=1}^{m}$. Set $r_{0,i} = 0$ for $i = 0, \ldots, m$, and define $\rho_{\leq}(x) = \#\{(i, j) : 1 \leq i \leq j \leq 2n \text{ and } r_{i,j} = r_{i-1,j-1}\}$, $\rho_{<}(x) = \#\{(i, j) : 1 \leq i < j \leq 2n \text{ and } r_{i,j} = r_{i-1,j-1}\}$. Then the length function $\ell_{PF_{2n}}$ of the poset PF_{2n} is equal to the restriction of $\rho_<$ to PF_{2n}. Furthermore, y covers x if and only if $Rk(y) \leqslant Rk(x)$ and $\ell_{PF_{2n}}(y) - \ell_{PF_{2n}}(x) = 1$. Similarly, $\ell_{PI_{2n}}$ is the restriction of $\rho_{\leq}$ to PI_{2n}, and that y covers x if and only if $Rk(y) \leqslant Rk(x)$ and $\ell_{I_{2n}}(y) - \ell_{I_{2n}}(x) = 1$. For details, see [1].

3 Results

It turns out that the intersection $PF_{2n} \cap I_{2n}$ is equal to F_{2n}, and furthermore, $(F_{2n}, \leq_{Sym})$ and $(F_{2n}, \leq_{Skew})$ are isomorphic. The relationships between the posets PI_{2n}, PF_{2n},I_{2n} and F_{2n} are as follows.

Let $w_0 \in PI_{2n}$ denote the "longest permutation," namely, the $2n \times 2n$ anti-diagonal permutation matrix, and let $j_{2n} \in F_{2n}$ denote the $2n \times 2n$ fixed-point-free involution having non-zero entries at the positions

$$(1, 2), (2, 1), (3, 4), (4, 3), \ldots, (2n - 1, 2n), (2n, 2n - 1), \text{ only} .$$

In other words, j_{2n} is the fixed-point-free involution with the only non-zero entries along its super-diagonals. Then I_{2n} is an interval in PI_{2n} with the smallest element id_{2n} and the largest element w_0. Similarly, F_{2n} is an interval in PF_{2n} with the smallest element j_{2n} and the largest element w_0.

Consider F_{2n} as a subposet of I_{2n} and let $x, y \in F_{2n}$ be two elements such that $x \leq y$. It turns out there exits a saturated chain in I_{2n} from x to y consisting of fixed-point-free involutions only. With the help of this observation, we prove

Theorem 3.1. *F_{2n} is an EL-shellable poset.*

Sketch of the proof. Recall that F_{2n} is a connected graded subposet of I_{2n}. Therefore, its covering relations are among the covering relations of I_{2n} described in [11]. Let x and y be two fixed-point-free involutions. We know the existence of a saturated chain between x and y that is entirely contained in F_{2n}. Since lexicographic ordering is a total order on maximal chains, there exists a unique largest such chain, say $\mathfrak{c}$. The idea of the proof is showing that $\mathfrak{c}$ is the unique decreasing chain. Once this is done, by switching the order of our totally ordered set $\mathbb{Z}^2$, we obtain the lexicographically smallest chain, which is the unique increasing chain.

As an important consequence of Theorem 3.1, we further show that

Theorem 3.2. *The order complex $\Delta(F_{2n})$ triangulates a ball of dimension $n(n-1)-2$.*

As it is mentioned in the introduction, the posets $(\tilde{F}_{2n}, \leq_{DS})$ and $(F_{2n}, \leq)$ are different. Indeed, for $2n = 6$ the Hasse diagrams of these two posets differ by an edge. Contrary to this observation, we have the following

Theorem 3.3. *The length functions of $(F_{2n}, \leq)$ and $(\tilde{F}_{2n}, \leq_{DS})$ are the same. Covering relations of the poset $\tilde{F}_{2n}$ are among the covering relations of F_{2n}.*

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