On a conjecture of Graham and Häggkvist for random trees

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Abstract. A conjecture of Graham and Häggkvist says that every tree with m edges decomposes the complete bipartite graph $K_{m,m}$. By establishing some properties of random trees with the use of singularity analysis of generating functions, we prove that asymptotically almost surely a tree with m edges decomposes the complete bipartite graph $K_{2m,2m}$.

1 Introduction

Given two graphs H and G we say that H decomposes G if G is the edge–disjoint union of isomorphic copies of H. The following is a well–known conjecture of Ringel.

Conjecture 1.1 (Ringel [12]). Every tree with m edges decomposes the complete graph K_{2m+1}.

The conjecture has been verified by a number of particular classes of trees, see the dynamic survey of Gallian [5]. By using the polynomial method, the conjecture was verified by Kézdy [7] for the more general class of so–called *stunted* trees. As mentioned by the author, this class is still small among the set of all trees.

The following bipartite version of the conjecture was formulated by Graham and Häggkvist.

Conjecture 1.2 (Graham and Häggkvist [6]). Every tree with m edges decomposes the complete bipartite graph $K_{m,m}$.

Again the conjecture has been verified by a number of cases; see *e.g.* [9]. Approximate versions of the two conjectures have been also proved [6, 8–10]. However, to our knowledge, there are no results stating that

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every tree decomposes K_{cm+1} or $K_{cm,cm}$ for some absolute constant c of reasonable size. The purpose of this paper is to show such a result for almost all trees.

Let $\mathcal{T}$ denote the class of (unlabelled) trees and let $\mathcal{T}_m$ be the class of trees with m edges. By a random tree with m edges we mean a tree chosen from $\mathcal{T}_m$ with the uniform distribution. We say that a random tree satisfies a property $\mathbb{P}$ asymptotically almost surely (a.a.s) if the probability that a random tree with m edges satisfies $\mathbb{P}$ tends to one with $m \to \infty$.

Theorem 1.3. *Asymptotically almost surely a tree with m edges decomposes* $K_{2m,2m}$.

The proof of Theorem 1.3 combines a structural analysis of random trees with combinatorial techniques for graph decompositions. In the following two sections we discuss the results and tools used in this proof.

2 Stable sets of random trees

The first property we use in the proof of Theorem 1.3 concerns the number of leaves in a random tree. Robinson and Schwenk [13] proved that the average number of leaves in an (unlabelled) random tree with m edges is asymptotically cm with $c \approx 0.438$. Drmota and Gittenberger [2] showed that the distribution of the number of leaves in a random tree with m edges is asymptotically normal with variance $c_2 m$ for some positive constant c_2. Thus, asymptotically almost surely a random tree with m edges has more than $2m/5$ leaves.

The second property we use deals with the size of a stable set in the base tree of T (the tree obtained from T by deleting its leaves.) Unfortunately this is not a parameter whose analysis can be explicitly found in the literature. We prove the following result.

Theorem 2.1. *The stable sets* A, B *of the base tree of a random tree with* m *edges satisfy a.a.s.*

$$||A| - |B|| \le \epsilon m,$$

for every fixed $\epsilon > 0$.

The proof of Theorem 2.1 is based on the use of generating functions. We first consider the case of rooted trees. Let

$$t(x, w_0, w_1) = \sum_{m,k_0,k_1} t_{m,k_0,k_1} x^{m+1} w_0^{k_0} w_1^{k_1}, \tag{2.1}$$

where t_{m,k_0,k_1} denotes the number of rooted trees with m edges and k_0 inner vertices (including the root if the tree has at least one edge even if the root has degree one) with even distance to the root and k_1 inner vertices with odd distance to the root. Then, by using the recursive description of a tree as a collection of trees hanging from a root (iterated twice to get the proper alignment of stable sets), one can obtain an explicit expression for (2.1).

Recall that we are interested in the difference $|A| - |B|$ which we can do by setting $w_0 = w$ and $w_1 = w^{-1}$. Hence, if $T(x, w) = \sum_{m,\ell} T_{m,\ell} x^{m+1} w^{\ell}$ denotes the generating function, where $T_{m,\ell}$ denotes the number of rooted trees with m edges and $|A| - |B| = \ell$ (where ℓ is some – possibly negative – integer and the root is contained in A even if the root has degree one) then an epxlicit expression for $T(x, w) = t(x, w, w^{-1})$ is also obtained.

As usual we denote by $a_n = [x^n]\, a(x)$ the n-th coefficient of a power series $a(x) = \sum_{n \geq 0} a_n x^n$. With the help of this notation it follows that

$$\mathbb{E}\, w^{|A|-|B|} = \frac{[x^{m+1}]\, T(x, w)}{[x^{m+1}]\, T(x, 1)}.$$

This magnitude can be determined asymptotically if w is close to 1 with the help of standard singularity analysis tools.

From a version of Hwang's Quasi-Power-Theorem (see [1]), one can deduce that the variable $Z = |A| - |B|$ follows a normal distribution which in our case has zero mean and variance linear in m from which

$$\Pr\left(||A| - |B|| \geq \varepsilon m\right) \leq C e^{-c\varepsilon^2 m}$$

for some positive constants c and C and for sufficiently small $\varepsilon > 0$. Of course this is precisely the statement that we want to prove for unlabelled trees. We translate the above analysis to unrooted unlabeled trees via Otter's bijection (see [11] or [1]). Unfortunately we cannot prove something like a central limit theorem for $|A| - |B|$ in this case, but it is still possible to keep track of the second moment which, by using Chebyshev inequality, provides a proof of Theorem 2.1.

3 The embedding

The general approach to show that a tree T decomposes a complete graph or a complete bipartite graph consists in showing that T cyclically decomposes the corresponding graphs. We next recall the basic principle behind this approach in slightly different terminology.

A rainbow embedding of a graph H into an oriented arc–colored graph X is an injective homomorphism f of some orientation $\vec{H}$ of H in X such that no two arcs of $f(\vec{H})$ have the same color.

Let $X = Cay(G, S)$ be a Cayley digraph of an abelian group G with respect to an antisymmetric subset $S \subset G$ (that is, $S \cap -S = \emptyset$). We consider X as an arc–colored oriented graph, by giving to each arc $(x, x+s)$, $x \in G, s \in S$, the color s. Suppose that H admits a rainbow embedding f in X. For each $a \in G$ the translation $x \to x + a$, $x \in G$, is an automorphism of X which preserves the colors and has no fixed points. Therefore, each translation sends $f(\vec{H})$ to an isomorphic copy which is edge disjoint from it. Thus the sets of translations for all $a \in G$ give rise to $n := |G|$ edge–disjoint copies of $\vec{H}$ in X. By ignoring orientations and colors, we thus have n edge disjoint copies of H in the underlying graph of X.

We will use the above approach with the Cayley graph $X = Cay(\mathbb{Z}_m \times \mathbb{Z}_4, \mathbb{Z}_m \times \{1\})$. We note that the underlying graph of X is isomorphic to $K_{2m,2m}$. The strategy of the proof is to show first that the base tree T_0 of a random tree with m edges admits a rainbow embedding f into X in such a way that $f(T_0) \subset \mathbb{Z}_m \times \{1, 2\}$. This can actually be achieved greedily as shown in the proof of next Lemma.

Lemma 3.1. *Let m be a positive integer. Let T be a tree with $n < 3m/5$ edges and stable sets A, B. If $||A| - |B|| \leq m/10$ then there is a rainbow embedding f of T into $X = Cay(\mathbb{Z}_m \times \mathbb{Z}_4, \mathbb{Z}_m \times \{1\})$ such that $f(V(T)) \subset (\mathbb{Z}_m \times \{1\}) \cup (\mathbb{Z}_m \times \{2\})$.*

The second step involves a proper embedding of the leaves of T. For this we use Häggkvist [6, Corolary 2.8] to get:

Lemma 3.2. *Let T be a tree with m edges. If the base tree T_0 of T admits a rainbow embedding f in $X = Cay(\mathbb{Z}_m \times \mathbb{Z}_4, \mathbb{Z}_m \times \{1\})$ such that $f(V(T_0)) \subset (\mathbb{Z}_m \times \{1\}) \cup (\mathbb{Z}_m \times \{2\})$ then T decomposes $K_{2m,2m}$.*

The proof of Theorem 1.3 follows now directly from Lemma 3.1 and Lemma 3.2 and the results on random trees from Section 2.

Proof. As it has been mentioned in Section 2, a random tree T with m edges has a.a.s. more than $2m/5$ leaves. Furthermore, by Theorem 2.1, the cardinalities of the stable sets of the base tree of T differ less than $m/10$ in absolute value a.a.s. By Lemma 3.1, the base tree of T admits a.a.s. a rainbow embedding in $Cay(\mathbb{Z}_m \times \mathbb{Z}_4, \mathbb{Z}_m \times \{1\})$ in such a way that the image of the embedding sits in $\mathbb{Z}_m \times \{1, 2\}$. In that case, Lemma 3.2 ensures that the tree T decomposes $K_{2m,2m}$. □

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