Fractional and integer matchings in uniform hypergraphs

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Abstract. A conjecture of Erdős from 1965 suggests the minimum number of edges in a k-uniform hypergraph on n vertices which forces a matching of size t, where $t \leq n/k$. Our main result verifies this conjecture asymptotically, for all $t < 0.48n/k$. This gives an approximate answer to a question of Huang, Loh and Sudakov, who proved the conjecture for $t \leq n/3k^2$. As a consequence of our result, we extend bounds of Bollobás, Daykin and Erdős by asymptotically determining the minimum vertex degree which forces a matching of size $t < 0.48n/(k-1)$ in a k-uniform hypergraph on n vertices. We also obtain further results on d-degrees which force large matchings. In addition we improve bounds of Markström and Ruciński on the minimum d-degree which forces a perfect matching in a k-uniform hypergraph on n vertices. Our approach is to inductively prove fractional versions of the above results and then translate these into integer versions.

Large matchings in hypergraphs with many edges

A *k-uniform hypergraph* is a pair $G = (V, E)$ where V is a finite set of vertices and the edge set E consists of unordered k-tuples of elements of V. A *matching* (or *integer matching*) M in G is a set of disjoint edges of G. The *size* of M is the number of edges in M. M is *perfect* if it has size $|V|/k$.

A classical theorem of Erdős and Gallai [6] determines the number of edges in a graph which forces a matching of a given size. In 1965, Erdős [5] made a conjecture which would generalize this to k-uniform hypergraphs.

Conjecture 1. Let $n, k \geq 2$ and $1 \leq s \leq n/k$ be integers. The minimum number of edges in a k-uniform hypergraph on n vertices which forces a matching of size s is

$$\max\left\{\binom{ks-1}{k}, \binom{n}{k} - \binom{n-s+1}{k}\right\} + 1.$$

It is easy to see that the conjecture would be best possible: the first expression in the lower bound is obtained by considering the k-uniform

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clique $K^{(k)}_{ks-1}$ (complemented by $n - ks + 1$ isolated vertices); the second expression in the lower bound is obtained as follows. Let $H(s)$ be a k-uniform hypergraph on n vertices with edge set consisting of all k-element subsets of $V(H(s))$ intersecting a given subset of $V(H(s))$ of size $s - 1$, that is $H(s) = K^{(k)}_n - K^{(k)}_{n-s+1}$.

The case $s = 2$ of Conjecture 1 corresponds to the Erdős-Ko-Rado Theorem on intersecting families [7]. The conjecture also has applications to the Manickam-Mikós-Singhi conjecture in number theory (for details see *e.g.* [2]). Despite its seeming simplicity Conjecture 1 is still wide open in general. For the cases $k \leq 4$, it was verified asymptotically by Alon, Frankl, Huang, Rödl, Ruciński and Sudakov [1]. For $k = 3$, it was recently proved by Frankl [8]. Bollobás, Daykin and Erdős [3] proved Conjecture 1 for general k whenever $s < n/(2k^3)$, which extended earlier results of Erdős [5]. Huang, Loh and Sudakov [10] proved it for $s < n/(3k^2)$. The main result in this paper verifies Conjecture 1 asymptotically for matchings of any size up to almost half the size of a perfect matching. This gives an asymptotic answer to a question in [10].

Theorem 2. [15] *Let n, $k \geq 2$ and $0 \leq a < 0.48/k$ be such that n, k, $an \in \mathbb{N}$. The minimum number of edges in a k-uniform hypergraph on n vertices which forces a matching of size an is*

$$\left(1 - (1-a)^k + o(1)\right)\binom{n}{k}.$$

Large matchings in hypergraphs with large degrees

It is also natural to consider degree conditions that force matchings in uniform hypergraphs. Given a k-uniform hypergraph $G = (V, E)$ and $S \in \binom{V}{d}$, where $0 \leq d \leq k-1$, let $deg_G(S) = |\{e \in E : S \subseteq e\}|$ be the *degree* of S in G. Let $\delta_d(G) = \min_{S \in \binom{V}{d}}\{deg_G(S)\}$ be the *minimum d-degree* of G. When $d = 1$, we refer to $\delta_1(G)$ as the *minimum vertex degree* of G. Note that $\delta_0(G) = |E|$.

For integers n, k, d, s satisfying $0 \leq d \leq k-1$ and $0 \leq s \leq n/k$, we let $m^s_d(k, n)$ denote the minimum integer m such that every k-uniform hypergraph G on n vertices with $\delta_d(G) \geq m$ has a matching of size s. So the results discussed in the previous section correspond to the case $d = 0$. The following degree condition for forcing perfect matchings has been conjectured in [9,14] and also received much attention recently.

Conjecture 3. Let n and $1 \leq d \leq k-1$ be such that $n, d, k, n/k \in \mathbb{N}$. Then

$$m^{n/k}_d(k, n) = \left(\max\left\{\frac{1}{2}, 1 - \left(\frac{k-1}{k}\right)^{k-d}\right\} + o(1)\right)\binom{n-d}{k-d}.$$

The lower bound here is given by the hypergraph $H(n/k)$ defined after Conjecture 1 and the following parity-based construction from [13]. For any integers n, k, let H' be a k-uniform hypergraph on n vertices with vertex partition $A \cup B = V(H')$, such that $||A| - |B|| \leq 2$ and $|A|$ and n/k have different parity. Let H' have edge set consisting of all k-element subsets of $V(H')$ that intersect A in an odd number of vertices. Observe that H' has no perfect matching, and that for every $1 \leq d \leq k-1$ we have that $\delta_d(H') = (1/2 + o(1))\binom{n-d}{k-d}$.

For $d = k-1$, $m_{k-1}^{n/k}(k, n)$ was determined exactly by Rödl, Ruciński and Szemerédi [19]. This was generalized by Treglown and Zhao [20], who determined the extremal families for all $d \geq k/2$. The extremal constructions are similar to the parity based one of H' above. For $d < k/2$ less is known. In [1] Conjecture 3 was proved for $k-4 \leq d \leq k-1$, by reducing it to a probabilistic conjecture of Samuels. In particular, this implies Conjecture 3 for $k \leq 5$. Khan [11], and independently Kühn, Osthus and Treglown [16], determined $m_1^{n/k}(k, n)$ exactly for $k = 3$. Khan [12] also determined $m_1^{n/k}(k, n)$ exactly for $k = 4$. As a consequence of these results, $m_1^s(k, n)$ is determined exactly whenever $s \leq n/k$ and $k \leq 4$ (for details see the concluding remarks in [16]). More generally, we propose the following version of Conjecture 3 for non-perfect matchings.

Conjecture 4. For all $\varepsilon > 0$ and all integers n, d, k, s with $1 \leq d \leq k-1$ and $0 \leq s \leq (1-\varepsilon)n/k$ we have

$$m_d^s(k, n) = \left(1 - \left(1 - \frac{s}{n}\right)^{k-d} + o(1)\right)\binom{n-d}{k-d}.$$

In fact it may be that the bound holds for all $s \leq n - C$, for some C depending only on d and k. The lower bound here is given by $H(s)$. The case $d = k-1$ of Conjecture 4 follows easily from the determination of $m_{k-1}^s(k, n)$ for s close to n/k in [19]. Bollobás, Daykin and Erdős [3] determined $m_1^s(k, n)$ for small s, *i.e.* whenever $s < n/2k^3$. As a consequence of our main result, for $1 \leq d \leq k-2$ we are able to determine $m_d^s(k, n)$ asymptotically for non-perfect matchings of any size less than $0.48n/(k-d)$. Note that this proves Conjecture 4 in the case $0.53k \leq d \leq k-2$, say.

Theorem 5. [15] *Let* $\varepsilon > 0$ *and let* n, k, d *be integers with* $1 \leq d \leq k-2$, *and let* $0 \leq a < \min\{0.48/(k-d), (1-\varepsilon)/k\}$ *be such that* $an \in \mathbb{N}$. *Then*

$$m_d^{an}(k, n) = \left(1 - (1-a)^{k-d} + o(1)\right)\binom{n-d}{k-d}.$$

We now focus again on the case $s = n/k$, *i.e.* perfect matchings. It was shown by Hàn, Person and Schacht [9] that for $k \geq 3$, $1 \leq d < k/2$ we have $m_d^{n/k}(k,n) \leq ((k-d)/k + o(1))\binom{n-d}{k-d}$. (The case $d = 1$ of this is already due to Daykin and Häggkvist [4].) These bounds were slightly improved by Markström and Ruciński [17], using similar techniques, to

$$m_d^{n/k}(k,n) \leq \left(\frac{k-d}{k} - \frac{1}{k^{k-d}} + o(1)\right)\binom{n-d}{k-d}.$$

Using similar methods to those developed to prove Theorem 5, we are also able to slightly improve on this bound.

Theorem 6 ([15]). *Let n and $1 \leq d < k/2$ be such that $n, k, d, n/k \in \mathbb{N}$. Then*

$$m_d^{n/k}(k,n) \leq \left(\frac{k-d}{k} - \frac{k-d-1}{k^{k-d}} + o(1)\right)\binom{n-d}{k-d}.$$

Large fractional matchings

Our approach to proving these results uses the concepts of fractional matchings and fractional vertex covers. A *fractional matching* in a k-uniform hypergraph $G = (V, E)$ is a function $w : E \to [0,1]$ of weights of edges, such that for each $v \in V$ we have $\sum_{e \in E : v \in e} w(e) \leq 1$. The *size* of w is $\sum_{e \in E} w(e)$. w is *perfect* if it has size $|V|/k$. A *fractional vertex cover* in G is a function $w : V \to [0,1]$ of weights of vertices, such that for each $e \in E$ we have $\sum_{v \in e} w(v) \geq 1$. The *size* of w is $\sum_{v \in V} w(v)$.

A key idea (already used *e.g.* in [1,18]) is that we can switch between considering the largest fractional matching and the smallest fractional vertex cover of a hypergraph. The determination of these quantities are dual linear programming problems, and hence by the Duality Theorem they have the same size.

For $s \in \mathbb{R}$ we let $f_d^s(k,n)$ denote the minimum integer m such that every k-uniform hypergraph G on n vertices with $\delta_d(G) \geq m$ has a fractional matching of size s. It was shown in [18] that $f_{k-1}^{n/k}(k,n) = \lceil n/k \rceil$. Similarly to [1], we now formulate the fractional version of Conjecture 1.

Conjecture 7. For all integers n,k,s with $k \geq 2$ and $1 \leq s \leq n/k$ we have

$$f_0^s(k,n) = \max\left\{\binom{ks-1}{k}, \binom{n}{k} - \binom{n-s+1}{k}\right\} + 1.$$

As discussed in [1], this conjecture has applications to a problem on information storage and retrieval. To prove Theorems 2 and 5, we first prove Conjecture 7 asymptotically for fractional matchings of any size up to $0.48n/k$.

Theorem 8 ([15]). *Let $n, k \geq 2$ be integers and let $0 \leq a \leq 0.48/k$. Then*

$$f_0^{an}(k,n) = \left(1 - (1-a)^k + o(1)\right)\binom{n}{k}.$$

We use Theorem 8, along with methods similar to those developed in [1], to convert our edge-density conditions for the existence of fractional matchings into corresponding minimum degree conditions. For $1 \leq d \leq k-2$ the following theorem asymptotically determines $f_d^s(k,n)$ for fractional matchings of any size up to $0.48n/(k-d)$. Note that this determines $f_d^s(k,n)$ asymptotically for all $s \in (0, n/k)$ whenever $d \geq 0.52k$.

Theorem 9 ([15]). *Let $n, k \geq 3$, and $1 \leq d \leq k-2$ be integers and let $0 \leq a \leq \min\{0.48/(k-d), 1/k\}$. Then*

$$f_d^{an}(k,n) = \left(1 - (1-a)^{k-d} + o(1)\right)\binom{n-d}{k-d}.$$

We then use Theorem 8 and a variant of Theorem 9, along with the Weak Hypergraph Regularity Lemma, to prove Theorems 2 and 5 respectively, by converting our fractional matchings into integer ones. We prove Theorem 6 in a similar fashion, via the following two theorems.

Theorem 10 ([15]). *Let $n, k \geq 2$, $d \geq 1$ be integers. Then*

$$f_0^{n/(k+d)}(k,n) \leq \left(\frac{k}{k+d} - \frac{k-1}{(k+d)^k} + o(1)\right)\binom{n}{k}.$$

Theorem 11 ([15]). *Let $n, k \geq 3$, $1 \leq d \leq k-2$ be integers. Then*

$$f_d^{n/k}(k,n) \leq \left(\frac{k-d}{k} - \frac{k-d-1}{k^{k-d}} + o(1)\right)\binom{n-d}{k-d}.$$

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