Discontinuous bootstrap percolation in power-law random graphs

Hamed Amini[1], Nikolaos Fountoulakis[2]
and Konstantinos Panagiotou[3]

1 Introduction

Bootstrap percolation was introduced by Chalupa, Leath and Reich [6] during the 1970's in the context of magnetic disordered systems and has been re-discovered since then by several authors mainly due to its connections with various physical models. A *bootstrap percolation process* with *activation threshold* an integer $r \geq 2$ on a graph $G = G(V, E)$ is a deterministic process which evolves in rounds. Every vertex has two states: it is either *infected* or *uninfected*. Initially, there is a subset $\mathcal{A}_0 \subseteq V$ which consists of infected vertices, whereas every other vertex is uninfected. Subsequently, in each round, if an uninfected vertex has at least r of its neighbours infected, then it also becomes infected and remains so forever. This is repeated until no more vertices become infected. We denote the final infected set by $\mathcal{A}_f$. Our general assumption will be that the initial set of infected vertices $\mathcal{A}_0$ is chosen randomly among all subsets of vertices of a certain size.

These processes have been studied on a variety of graphs, such as trees, grids, hypercubes, as well as on several distributions of random graphs. A short survey regarding applications of bootstrap percolation processes can be found in [1].

During the last decade, there has been significant experimental evidence on the structural characteristics of networks that arise in applications such as the Internet, the World Wide Web as well as social networks or even biological networks. One of the fundamental features is their

[1] Swiss Finance Institute, EPFL, Lausanne, Switzerland. Email: hamed.amini@epfl.ch

[2] School of Mathematics, University of Birmingham, United Kingdom.
Email: n.fountoulakis@bham.ac.uk. This research has been supported by the EPSRC Grant EP/K019740/1.

[3] Mathematishes Institut, LMU, Munich, Germany. Email: kpanagio@math.lmu.de. The author was partially supported by DFG grant PA 2080/1.

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degree distribution, which is most cases appears to follow a power law with exponent between 2 and 3 (see for example the article of Albert and Barabási [2]). The theme of this contribution is the study of the evolution of bootstrap percolation processes on random graphs which exhibit this characteristic. We show that this boosts the evolution of the process, resulting in large infected sets starting from a small set of infected vertices.

2 Models and results

The random graph model that we consider is asymptotically equivalent to a model considered by Chung and Lu [7], and is a special case of the so-called *inhomogeneous random graph*, which was introduced by Söderberg [9] and was studied in detail by Bollobás, Janson and Riordan in [5].

2.1 Inhomogeneous random graphs – The Chung-Lu model

In order to define the model we consider for any $n \in \mathbb{N}$ the vertex set $[n] := \{1, \ldots, n\}$. Each vertex i is assigned a positive weight $w_i(n)$, and we will write $\mathbf{w} = \mathbf{w}(n) = (w_1(n), \ldots, w_n(n))$. We assume in the remainder that the weights are deterministic, and we will suppress the dependence on n, whenever this is obvious from the context. However, note that the weights could be random variables; we will not consider this case here, although it is very likely that under suitable technical assumptions our results generalize to this case as well. For any $S \subseteq [n]$, set

$$W_S(\mathbf{w}) := \sum_{i \in S} w_i.$$

In our random graph model, the event of including the edge $\{i, j\}$ in the resulting graph is independent of the events of including all other edges, and its probability equals

$$p_{ij}(\mathbf{w}) = \min \left\{ \frac{w_i w_j}{W_{[n]}(\mathbf{w})}, 1 \right\}. \tag{2.1}$$

We will refer to this model as the *Chung-Lu* model, and we shall write $CL(\mathbf{w})$ for a random graph in which each possible edge $\{i, j\}$ is included independently with probability as in (2.1).

2.2 Power-law degree distributions

Following van der Hofstad [10], we write for any $n \in \mathbb{N}$ and any sequence of weights $\mathbf{w} = (w_1(n), \ldots, w_n(n))$

$$F_n(x) = n^{-1} \sum_{i=1}^{n} \mathbf{1}[w_i(n) < x], \quad \forall x \in [0, \infty)$$

for the empirical distribution function of the weight of a vertex chosen uniformly at random. We will assume that F_n satisfies the following two conditions.

Definition 2.1. We say that $(F_n)_{n\geq 1}$ is *regular*, if it has the following two properties.

- **[Weak convergence of weight]** There is a distribution function $F : [0, \infty) \to [0, 1]$ such that for all x at which F is continuous $\lim_{n\to\infty} F_n(x) = F(x)$;
- **[Convergence of average weight]** Let W_n be a random variable with distribution function F_n, and let W_F be a random variable with distribution function F. Then we have $\lim_{n\to\infty} \mathbb{E}[W_n] = \mathbb{E}[W_F]$.

The regularity of $(F_n)_{n\geq 1}$ guarantees two important properties. Firstly, the weight of a random vertex is approximately distributed as a random variable that follows a certain distribution. Secondly, this variable has finite mean and therefore the resulting graph has bounded average degree. Apart from regularity, our focus will be on weight sequences that give rise to power-law degree distributions.

Definition 2.2. We say that a regular sequence $(F_n)_{n\geq 1}$ is *of power law with exponent* β, if there are $0 < \gamma_1 < \gamma_2, x_0 > 0$ and $0 < \zeta \leq 1/(\beta - 1)$ such that for all $x_0 \leq x \leq n^\zeta$

$$\gamma_1 x^{-\beta+1} \leq 1 - F_n(x) \leq \gamma_2 x^{-\beta+1},$$

and $F_n(x) = 0$ for $x < x_0$, but $F_n(x) = 1$ for $x > n^\zeta$.

We consider the random graph $CL(\mathbf{w})$ where the weight sequence $\mathbf{w} = \mathbf{w}(n)$ gives rise to a regular sequence of empirical distribution functions that are of power law with exponent β. We assume that a random set of $a(n)$ vertices is initially infected. We say that an event occurs *asymptotically almost surely (a.a.s.)*, if it occurs with probability$\to$ 1 as $n \to \infty$, in the product space of the random graph and the choice of the initially infected vertices.

2.3 Results

We determine explicitly a critical function which we denote by $a_c(n)$ such that when we infect randomly $a(n)$ vertices in $[n]$, then the following threshold phenomenon occurs. If $a(n) \ll a_c(n)$, then a.a.s. the infection spreads no further than $\mathcal{A}_0$, but when $a(n) \gg a_c(n)$, then a linear number of vertices become eventually infected. We remark that $a_c(n) = o(n)$. We define the function $\psi_r(x)$ for $x \geq 0$ to be equal to the probability that a Poisson-distributed random variable with parameter x is at least r.

Also, for a random variable X with finite expected value and distribution function F, we (informally) say that X^* follows the *F-size-biased* distribution function, if the distribution of X^* is weighted by the value of X.

Theorem 2.3. *For any $\beta \in (2,3)$ and any integer $r \geq 2$, we let $a_c(n) = n^{\frac{r(1-\zeta)+\zeta(\beta-1)-1}{r}}$ for all $n \in \mathbb{N}$. Let $a : \mathbb{N} \to \mathbb{N}$ be a function such that $a(n) \to \infty$, as $n \to \infty$, but $a(n) = o(n)$. Let also $\zeta \leq \frac{1}{\beta-1}$. If we initially infect uniformly at random $a(n)$ vertices in $[n]$, then the following holds:*

- *if $a(n) \ll a_c(n)$, then a.a.s. $\mathcal{A}_f = \mathcal{A}_0$;*
- *if $a(n) \gg a_c(n)$ and also $\frac{r-1}{2r-\beta+1} < \zeta \leq \frac{1}{\beta-1}$, then*

$$\frac{|\mathcal{A}_f|}{n} \xrightarrow{p} \mathbb{E}\left[\psi_r(U\hat{y})\right], \text{ as } n \to \infty,$$

where U is a random variable with F as its distribution function and $\hat{y}$ is the smallest positive solution of

$$y = \mathbb{E}\left[\psi_r(Wy)\right],$$

with W being a random variable whose law follows the F-size-biased distribution function.

When $0 < \zeta \leq \frac{r-1}{2r-\beta+1}$ the second part of the above statement holds with $a_c^+(n) = n^{1-\zeta\ \frac{r-\beta+2}{r-1}}$ instead of $a_c(n)$.

Note that the above theorem implies that when the maximum weight of the sequence is $n^{1/(\beta-1)}$, then the threshold function becomes equal to $n^{\frac{\beta-2}{\beta-1}}$ and does not depend on r.

This result is in sharp contrast with the behaviour of the bootstrap percolation process in $G(n,p)$ random graphs, where every edge on a set of n vertices is included independently with probability p. Recently, Janson, Łuczak, Turova and Vallier [8] (see Theorem 5.2 there) showed that when $p = d/n$, with $d > 0$ fixed, if $|\mathcal{A}_0| = o(n)$, then typically no evolution occurs. In other words, the density of the initially infected vertices must be positive in order for the density of infected vertices to grow. We note that similar behavior to the case of $G(n,p)$ has been observed in the case of random regular graphs [4], and in random graphs with given vertex degrees constructed through the configuration model, studied by the first author in [3], when the sum of the square of degrees scales linearly with n, the size of the graph. The later case includes random graphs with power-law degree sequence with exponent $\beta > 3$. Our results imply that the two regimes $2 < \beta < 3$ and $\beta > 3$ have completely different behaviors.

The next theorem complements the above theorem, as it gives a law of large numbers for the size $\mathcal{A}_f$ when a positive fraction of vertices are initially infected.

Theorem 2.4. *Let $2 < \beta < 3$ and $r \geq 2$. If $a(n) = pn$, where $p \in (0, 1)$ is fixed, then*

$$\frac{|\mathcal{A}_f|}{n} \xrightarrow{p} (1-p)\mathbb{E}\left[\psi_r(U\hat{y})\right] + p, \ \text{as } n \to \infty,$$

where U is a random variable having F as its distribution function with $\hat{y}$ being the smallest positive solution of

$$y = (1-p)\mathbb{E}\left[\psi_r(Wy)\right] + p$$

and W is a random variable whose law follows the F-size-biased distribution function.

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