Tight minimum degree conditions forcing perfect matchings in uniform hypergraphs

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Abstract. Given positive integers k and ℓ where $k/2 \leq \ell \leq k-1$, we give a minimum ℓ-degree condition that ensures a perfect matching in a k-uniform hypergraph. This condition is best possible and improves on work of Pikhurko [12] who gave an asymptotically exact result, and extends work of Rödl, Ruciński and Szemerédi [15] who determined the threshold for $\ell = k-1$. Our approach makes use of the absorbing method.

1 Introduction

A central question in graph theory is to establish conditions that ensure a (hyper)graph H contains some spanning (hyper)graph F. Of course, it is desirable to fully characterize those (hyper)graphs H that contain a spanning copy of a given (hyper)graph F. Tutte's theorem [18] characterizes those graphs with a perfect matching. (A *perfect matching* in a (hyper)graph H is a collection of vertex-disjoint edges of H which cover the vertex set $V(H)$ of H.) However, for some (hyper)graphs F it is unlikely that such a characterization exists. Indeed, for many (hyper)graphs F the decision problem of whether a (hyper)graph H contains F is NP-complete. For example, in contrast to the graph case, the decision problem whether a k-uniform hypergraph contains a perfect matching is NP-complete for $k \geq 3$ (see [4, 6]). Thus, it is desirable to find sufficient conditions that ensure a perfect matching in a k-uniform hypergraph.

Given a k-uniform hypergraph H with an ℓ-element vertex set S (where $0 \leq \ell \leq k-1$) we define $d_H(S)$ to be the number of edges containing S. The *minimum ℓ-degree $\delta_\ell(H)$* of H is the minimum of $d_H(S)$ over all

ℓ-element sets S of vertices in H. Clearly $\delta_0(H)$ is the number of edges in H. We also refer to $\delta_1(H)$ as the *minimum vertex degree* of H and $\delta_{k-1}(H)$ the *minimum codegree* of H.

Over the last few years there has been a strong focus in establishing minimum ℓ-degree thresholds that force a perfect matching in a k-uniform hypergraph. See [13] for a survey on matchings (and Hamilton cycles) in hypergraphs. In particular, Rödl, Ruciński and Szemerédi [15] determined the minimum codegree threshold that ensures a perfect matching in a k-uniform hypergraph on n vertices for all $k \geq 3$. The threshold is $n/2 - k + C$, where $C \in \{3/2, 2, 5/2, 3\}$ depends on the values of n and k. This improved bounds given in [9,14].

Less is known about minimum vertex degree thresholds that force a perfect matching. One of the earliest results on perfect matchings was given by Daykin and Häggkvist [3], who showed that a k-uniform hypergraph H on n vertices contains a perfect matching provided that $\delta_1(H) \geq (1 - 1/k)\binom{n-1}{k-1}$. Hàn, Person and Schacht [5] determined, asymptotically, the minimum vertex degree that forces a perfect matching in a 3-uniform hypergraph. Kühn, Osthus and Treglown [10] and independently Khan [7] made this result exact. Khan [8] has also determined the exact minimum vertex degree threshold for 4-uniform hypergraphs. For $k \geq 5$, the precise minimum vertex degree threshold that ensures a perfect matching in a k-uniform hypergraph is not known.

The situation for ℓ-degrees where $1 < \ell < k - 1$ is also still open. Hàn, Person and Schacht [5] provided conditions on $\delta_\ell(H)$ that ensure a perfect matching in the case when $1 \leq \ell < k/2$. These bounds were subsequently lowered by Markström and Ruciński [11]. Alon et al. [1] gave a connection between the minimum ℓ-degree that forces a perfect matching in a k-uniform hypergraph and the minimum ℓ-degree that forces a *perfect fractional matching*. As a consequence of this result they determined, asymptotically, the minimum ℓ-degree which forces a perfect matching in a k-uniform hypergraph for the following values of (k, ℓ): $(4, 1), (5, 1), (5, 2), (6, 2)$, and $(7, 3)$.

Pikhurko [12] showed that if $\ell \geq k/2$ and H is a k-uniform hypergraph whose order n is divisible by k then H has a perfect matching provided that $\delta_\ell(H) \geq (1/2 + o(1))\binom{n}{k-\ell}$. This result is best possible up to the $o(1)$-term (see the constructions in $\mathcal{H}_{\mathrm{ext}}(n, k)$ below).

In [16,17] we make Pikhurko's result exact. In order to state this result, we need some more definitions. Fix a set V of n vertices. Given a partition of V into non-empty sets A, B, let $E_{\mathrm{odd}}^k(A, B)$ ($E_{\mathrm{even}}^k(A, B)$) denote the family of all k-element subsets of V that intersect A in an odd (even) number of vertices. (Notice that the ordering of the vertex classes A, B is important.) Define $\mathcal{B}_{n,k}(A, B)$ to be the k-uniform hypergraph with ver-

tex set $V = A \cup B$ and edge set $E^k_{\text{odd}}(A, B)$. Note that the complement $\overline{\mathcal{B}}_{n,k}(A, B)$ of $\mathcal{B}_{n,k}(A, B)$ has edge set $E^k_{\text{even}}(A, B)$.

Suppose $n, k \in \mathbb{N}$ such that k divides n. Define $\mathcal{H}_{\text{ext}}(n, k)$ to be the collection of the following hypergraphs: $\mathcal{H}_{\text{ext}}(n, k)$ contains all hypergraphs $\overline{\mathcal{B}}_{n,k}(A, B)$ where $|A|$ is odd. Further, if n/k is odd then $\mathcal{H}_{\text{ext}}(n, k)$ also contains all hypergraphs $\mathcal{B}_{n,k}(A, B)$ where $|A|$ is even; if n/k is even then $\mathcal{H}_{\text{ext}}(n, k)$ also contains all hypergraphs $\mathcal{B}_{n,k}(A, B)$ where $|A|$ is odd.

It is easy to see that no hypergraph in $\mathcal{H}_{\text{ext}}(n, k)$ contains a perfect matching. Indeed, first assume that $|A|$ is even and n/k is odd. Since every edge of $\mathcal{B}_{n,k}(A, B)$ intersects A in an odd number of vertices, one cannot cover A with an odd number of disjoint odd sets. Similarly $\mathcal{B}_{n,k}(A, B)$ does not contain a perfect matching if $|A|$ is odd and n/k is even. Finally, if $|A|$ is odd then since every edge of $\overline{\mathcal{B}}_{n,k}(A, B)$ intersects A in an even number of vertices, $\overline{\mathcal{B}}_{n,k}(A, B)$ does not contain a perfect matching.

Given $\ell \in \mathbb{N}$ such that $k/2 \le \ell \le k-1$ define $\delta(n, k, \ell)$ to be the maximum of the minimum ℓ-degrees among all the hypergraphs in $\mathcal{H}_{\text{ext}}(n, k)$. For example, it is not hard to see that

$$\delta(n,k,k-1) = \begin{cases} n/2 - k + 2 & \text{if } k/2 \text{ is even and } n/k \text{ is odd} \\ n/2 - k + 3/2 & \text{if } k \text{ is odd and } (n-1)/2 \text{ is odd} \\ n/2 - k + 1/2 & \text{if } k \text{ is odd and } (n-1)/2 \text{ is even} \\ n/2 - k + 1 & \text{otherwise.} \end{cases} \quad (1.1)$$

In [16, 17] we prove the following exact version of Pikhurko's result.

Theorem 1.1. *Let $k, \ell \in \mathbb{N}$ such that $k \ge 3$ and $k/2 \le \ell \le k - 1$. Then there exists an $n_0 \in \mathbb{N}$ such that the following holds. Suppose H is a k-uniform hypergraph on $n \ge n_0$ vertices where k divides n. If*

$$\delta_\ell(H) > \delta(n, k, \ell)$$

then H contains a perfect matching.

In [16] we prove Theorem 1.1 for k divisible by 4 and then in [17] we extend this result to all values of k. Independent to our work, Czygrinow and Kamat [2] have proven Theorem 1.1 in the case when $k = 4$ and $\ell = 2$.

As explained before, the minimum ℓ-degree condition in Theorem 1.1 is best possible. Theorem 1.1 and (1.1) together give the aforementioned result of Rödl, Ruciński and Szemerédi [15].

In general, the precise value of $\delta(n, k, \ell)$ is unknown because it is not known what value of $|A|$ maximizes the minimum ℓ-degree of $\mathcal{B}_{n,k}(A, B)$ (or $\overline{\mathcal{B}}_{n,k}(A, B)$). (See [16] for a discussion on this.) However, in [16] we gave a tight upper bound on $\delta(n, 4, 2)$.

2 Overview of the proof of Theorem 1.1

The proof of Theorem 1.1 follows the so-called *stability approach*. We first prove that

  (i) H has a perfect matching or;
 (ii) H is 'close' to one of the hypergraphs $\mathcal{B}_{n,k}(A, B)$ or $\overline{\mathcal{B}}_{n,k}(A, B)$ in $\mathcal{H}_{\text{ext}}(n, k)$.

The extremal situation (ii) is then dealt with separately. For example, suppose H is 'close' to an element $\mathcal{B}_{n,k}(A, B)$ from $\mathcal{H}_{\text{ext}}(n, k)$. (So we can view A, B as a partition of $V(H)$.) The minimum ℓ-degree condition on H ensures that H contains an edge e that intersects A in an even number of vertices. Recall that no such edge exists in $\mathcal{B}_{n,k}(A, B)$; this is the 'reason' why $\mathcal{B}_{n,k}(A, B)$ does not have a perfect matching. Thus, e acts as a 'parity breaking' edge and can be used to form part of a perfect matching in H.

Almost perfect matchings

To show that (i) or (ii) holds, we apply the following result of Markström and Ruciński [11] to ensure an 'almost' perfect matching in H.

Theorem 2.1 (Lemma 2 in [11]). *For each integer $k \geq 3$, every $1 \leq \ell \leq k - 2$ and every $\gamma > 0$ there exists an $n_0 \in \mathbb{N}$ such that the following holds. Suppose that H is a k-uniform hypergraph on $n \geq n_0$ vertices such that*

$$\delta_\ell(H) \geq \left(\frac{k - \ell}{k} - \frac{1}{k^{(k-\ell)}} + \gamma \right) \binom{n - \ell}{k - \ell}.$$

Then H contains a matching covering all but at most $\sqrt{n}$ vertices.

  (In [11], Markström and Ruciński only stated Theorem 2.1 for $1 \leq \ell < k/2$. In fact, their proof works for all values of ℓ such that $1 \leq \ell \leq k-2$.) In the case when $\ell = k - 1$, we need a result of Rödl, Ruciński and Szemerédi [15, Fact 2.1]: Suppose H is a k-uniform hypergraph on n vertices. If $\delta_{k-1}(H) \geq n/k$, then H contains a matching covering all but at most k^2 vertices in H. Note that this minimum codegree condition is substantially smaller than the corresponding condition in Theorem 1.1. Further, if $k/2 \leq \ell < k - 1$ then the minimum ℓ-degree condition in Theorem 2.1 is also substantially smaller than the minimum ℓ-degree in Theorem 1.1.

Absorbing sets

Given a k-uniform hypergraph H, a set $S \subseteq V(H)$ is called an *absorbing set for* $Q \subseteq V(H)$, if both $H[S]$ and $H[S \cup Q]$ contain perfect matchings.

If the hypergraph H in Theorem 1.1 contains a 'small' set S which is an absorbing set for *any* set $Q \subseteq V(H)$ where $|Q| \leq \sqrt{n}$ is divisible by k, then it is easy to find a perfect matching in H. Indeed, in this case the minimum ℓ-degree of $H - S$ satisfies the hypothesis of Theorem 2.1 (or the hypothesis of Fact 2.1 in [15] if $\ell = k - 1$). Thus, $H - S$ contains a matching M covering all but a set Q of at most $\sqrt{n}$ vertices. Then since $H[S \cup Q]$ contains a perfect matching M', $M \cup M'$ is a perfect matching in H.

We give two conditions that ensure such an absorbing set S exists in H (and thus guarantee a perfect matching in H). Roughly speaking, the first condition asserts that $V(H)$ contains 'many' ℓ-tuples whose ℓ-degree is 'significantly' larger than $\delta(n, k, \ell)$. The second condition concerns a certain 'common neighbourhood' property. (Fixing $r := \lceil k/2 \rceil$, this condition roughly asserts that for any r-tuple $P \in \binom{V(H)}{r}$, more than half of the r-tuples P' in $\binom{V(H)}{r}$ are such that P and P' have a common neighbourhood which is not 'too small'.) We will refer to these properties as (α) and (β) respectively.

The auxiliary graph G

We then show that if H does not satisfy (α) and (β), then (ii) must be satisfied. That is, H is 'close' to one of the hypergraphs $\mathcal{B}_{n,k}(A, B)$ or $\overline{\mathcal{B}}_{n,k}(A, B)$ in $\mathcal{H}_{\text{ext}}(n, k)$. For this, we consider an auxiliary bipartite graph G defined as follows: Set $r := \lceil k/2 \rceil, r' := \lfloor k/2 \rfloor, X^r := \binom{V(H)}{r}$ and $Y^{r'} := \binom{V(H)}{r'}$. Further, let $N := \binom{n}{r}$ and $N' := \binom{n}{r'}$. G has vertex classes X^r and $Y^{r'}$. Two vertices $x_1 \ldots x_r \in X^r$ and $y_1 \ldots y_{r'} \in Y^{r'}$ are adjacent in G if and only if $x_1 \ldots x_r y_1 \ldots y_{r'} \in E(H)$.

We show that, since H fails to satisfy (α) and (β), G is 'close' to the disjoint union of two copies of $K_{N/2, N'/2}$. Once we have this information, we give direct arguments on G to show that this implies that (ii) is indeed satisfied.

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