Coloring intersection graphs of arcwise connected sets in the plane

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Abstract. A family of sets in the plane is simple if the intersection of its any subfamily is arcwise connected. We prove that the intersection graphs of simple families of compact arcwise connected sets in the plane pierced by a common line have chromatic number bounded by a function of their clique number.

1 Introduction

A *proper coloring* of a graph is an assignment of colors to the vertices of the graph such that no two adjacent ones are assigned the same color. The minimum number of colors sufficient to color a graph G properly is called the *chromatic number* of G and denoted by $\chi(G)$. The maximum size of a clique (a set of pairwise adjacent vertices) in a graph G is called the *clique number* of G and denoted by $\omega(G)$. It is clear that $\chi(G) \geqslant \omega(G)$. A class of graphs is χ*-bounded* if there is a function $f : \mathbb{N} \to \mathbb{N}$ such that $\chi(G) \leqslant f(\omega(G))$ holds for any graph G in the class.

In this paper, we focus our attention on the relation between the chromatic number and the clique number for classes of graphs arising from geometry. The *intersection graph* of a family of sets $\mathcal{F}$ is the graph with vertex set $\mathcal{F}$ and edge set consisting of pairs of intersecting elements of $\mathcal{F}$. We consider families $\mathcal{F}$ of arcwise connected compact sets in the plane. For simplicity, we identify the family $\mathcal{F}$ with its intersection graph.

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In the one-dimensional case of subsets of $\mathbb{R}$, the only arcwise connected compact sets are closed intervals. They define the class of *interval graphs*, for which $\chi(G) = \omega(G)$. The study of the chromatic number of intersection graphs of geometric objects in higher dimensions was initiated in the seminal paper of Asplund and Grünbaum [1], where they proved that the families of axis-aligned rectangles in $\mathbb{R}^2$ are χ-bounded. On the other hand, Burling [2] showed that triangle-free intersection graphs of axis-aligned boxes in $\mathbb{R}^3$ can have arbitrarily large chromatic number.

Gyárfás [3] proved χ-boundedness of the class of graphs defined by intersections of chords of a circle. This was generalized by Kostochka and Kratochvíl [4], who showed that the families of convex polygons inscribed in a circle are χ-bounded. McGuinness [5] proved that the families of L-shapes (shapes consisting of a horizontal and a vertical segments of arbitrary lengths, forming the letter 'L') all of which intersect a fixed vertical line are χ-bounded. Later, McGuinness [6] showed that the simple families $\mathcal{F}$ of compact arcwise connected sets in the plane pierced by a common line with $\omega(\mathcal{F}) \leqslant 2$ have bounded chromatic number. A family is *simple* if the intersection of its any subfamily is arcwise connected, and is *pierced* by a line ℓ if the intersection of its any member with ℓ is a nonempty segment. Suk [8] proved χ-boundedness of the simple families of x-monotone curves intersecting a fixed vertical line.

We generalize the results of McGuinness, allowing any bound on the clique number, and of Suk, getting rid of the x-monotonicity assumption.

Theorem 1.1. *The class of simple families of compact arcwise connected sets in the plane pierced by a common line is χ-bounded.*

This contrasts with a recent result due to Pawlik et al. [7] that there are triangle-free intersection graphs of straight-line segments with arbitrarily large chromatic number. This explains why the assumption of Theorem 1.1 that the sets are pierced by a common line is necessary.

The ultimate goal of this quest is to understand the border line between the classes of graphs (and classes of geometric objects) that are χ-bounded and those that are not. The authors would like to share two open problems in this context.

Problem 1.2. Are the families (not necessarily simple) of x-monotone curves in the plane pierced by a common vertical line χ-bounded?

Problem 1.3. Are the families of curves in the plane pierced by a common line χ-bounded?

2 Preliminaries

First we simplify the setting of Theorem 1.1. Let $\mathcal{F}$ be a simple family of compact arcwise connected sets in the plane pierced by a common line with $\omega(\mathcal{F}) \leqslant k$. We can assume without loss of generality that this piercing line is the horizontal axis $\mathbb{R} \times \{0\}$. Call it the *baseline*. The *base* of a set X, denoted by base(X), is the intersection of X with the baseline. The intersection graph of the bases of the members of $\mathcal{F}$ is an interval graph and thus can be properly colored with k colors. To find a proper coloring of $\mathcal{F}$, we can restrict our attention to one color class in the coloring of this interval graph. Thus we assume that no two members of $\mathcal{F}$ intersect on the baseline and show that $\mathcal{F}$ can be colored properly with a bounded number of colors.

Let $\mathcal{F}^+ = \{X \cap (\mathbb{R} \times [0, +\infty)) : X \in \mathcal{F}\}$ and $\mathcal{F}^- = \{X \cap (\mathbb{R} \times (-\infty, 0]) : X \in \mathcal{F}\}$. Clearly, $\mathcal{F}^+$ and $\mathcal{F}^-$ are simple families of arcwise connected sets. If we find proper colorings ϕ^+ and ϕ^- of $\mathcal{F}^+$ and $\mathcal{F}^-$, respectively, with bounded numbers of colors, then the coloring of $\mathcal{F}$ by pairs of colors (ϕ^+, ϕ^-) is proper on $\mathcal{F}$. Thus we assume that $\mathcal{F} = \mathcal{F}^+$ (the other case $\mathcal{F} = \mathcal{F}^-$ is symmetric). All geometric objects that we consider from now on are contained in $\mathbb{R} \times [0, +\infty)$. Thus we consider families of compact arcwise connected subsets of $\mathbb{R} \times [0, +\infty)$ all of which are pierced by the baseline. We call such families *attached*.

Theorem 2.1. *For $k \geqslant 1$, there is ξ_k such that $\chi(\mathcal{F}) \leqslant 2^{\xi_k}$ holds for any attached family $\mathcal{F}$ with $\omega(\mathcal{F}) \leqslant k$.*

The case $k = 1$ is trivial, and the case $k = 2$ is the result of McGuinness [6]. Our proof of Theorem 2.1 depends heavily on the techniques developed by McGuinness [6] and Suk [8].

Let $X \prec Y$ denote that base(X) is entirely to the left of base(Y). The relation $\prec$ is a total order on any attached family $\mathcal{F}$. For attached sets X_1 and X_2 such that $X_1 \prec X_2$, define $\mathcal{F}(X_1, X_2) = \{Y \in \mathcal{F} : X_1 \prec Y \prec X_2\}$. For an attached family $\mathcal{X}$, we define ext$(\mathcal{X})$ to be the only unbounded arcwise connected component of $(\mathbb{R} \times [0, +\infty)) \setminus \bigcup \mathcal{X}$.

Lemma 2.2 ([5]). *Let $\mathcal{F}$ be an attached family, let $a, b \geqslant 0$, and suppose $\chi(\mathcal{F}) > 2^{a+b+1}$. Then there exists a subfamily $\mathcal{H}$ of $\mathcal{F}$ such that $\chi(\mathcal{H}) > 2^a$ and for any intersecting $H_1, H_2 \in \mathcal{H}$ we have $\chi(\mathcal{F}(H_1, H_2)) \geqslant 2^b$.*

A subfamily $\mathcal{G}$ of an attached family $\mathcal{F}$ is *externally supported* in $\mathcal{F}$ if for any $X \in \mathcal{G}$ there exists $Y \in \mathcal{F}$ such that $Y \cap X \neq \emptyset$ and $Y \cap \text{ext}(\mathcal{G}) \neq \emptyset$.

Lemma 2.3. *Let $\mathcal{F}$ be an attached family, let $a \geqslant 0$, and suppose $\chi(\mathcal{F}) > 2^{a+1}$. Then there exists a subfamily $\mathcal{G}$ of $\mathcal{F}$ that is externally supported in $\mathcal{F}$ and satisfies $\chi(\mathcal{G}) > 2^a$.*

Let $\mathcal{F}$ be an attached family. A *k-clique* in $\mathcal{F}$ is a family of k pairwise intersecting members of $\mathcal{F}$. For a k-clique $\mathcal{K}$, define $\operatorname{int}(\mathcal{K})$ to be the only arcwise connected component of $(\mathbb{R}\times[0,+\infty))\setminus\bigcup\mathcal{K}$ containing the part of the baseline between the two least members of $\mathcal{K}$. A *k-bracket* in $\mathcal{F}$ is a family $\mathcal{B} \subseteq \mathcal{F}$ consisting of a k-clique $\mathcal{K}$, a set $P \subseteq \operatorname{int}(\mathcal{K})$ called *hook*, and a set S called *support* such that $S \prec \mathcal{K}$ or $\mathcal{K} \prec S$ and $S \cap P \neq \emptyset$. For such a k-bracket $\mathcal{B}$, define $\operatorname{int}(\mathcal{B})$ to be the only arcwise connected component of $(\mathbb{R} \times [0,+\infty)) \setminus \bigcup\mathcal{B}$ containing the part of the baseline between S and $\mathcal{K}$. The following lemma exhibits a crucial property of these two constructs.

Lemma 2.4. *If $\mathcal{S}$ is an attached clique or bracket, then any closed curve c such that $\mathcal{S}\cup\{c\}$ is simple, $\operatorname{int}(\mathcal{S})\cap c \neq \emptyset$, and $\operatorname{ext}(\mathcal{S})\cap c \neq \emptyset$ intersects all members of $\mathcal{S}$.*

3 Proof sketch of Theorem 2.1

The proof of Theorem 2.1 proceeds by induction on k. The case $k = 1$ is trivial. Thus assume for the remainder of this section that $k \geqslant 2$ and the statement of the theorem holds for $k - 1$. A typical application of the induction hypothesis looks as follows: if $\mathcal{F}$ is an attached family with $\omega(\mathcal{F}) \leqslant k$, $\mathcal{G} \subseteq \mathcal{F}$, and there is $X \in \mathcal{F} \setminus \mathcal{G}$ intersecting all members of $\mathcal{G}$, then $\omega(\mathcal{G}) \leqslant k-1$ and thus $\chi(\mathcal{G}) \leqslant 2^{\xi_{k-1}}$.

Define $\beta_k = 5\xi_{k-1} + \xi_2 + k + 7$, $\gamma_k = 2\xi_{k-1} + k + 5$, $\delta_{k,1} = \xi_{k-1} + \beta_k + \gamma_k + 2$, $\delta_{k,i} = \delta_{k,i-1} + \beta_k + \gamma_k + 2$ for $i \geqslant 2$, and finally $\xi_k = \delta_{k,k+1}$.

For a k-clique or k-bracket $\mathcal{S}$, define $\mathcal{F}(\mathcal{S}) = \{X \in \mathcal{F} : \operatorname{base}(X) \subseteq \operatorname{int}(\mathcal{S})\}$. The following technical fact (considered in the induction context) is a generalization of an analogous statement in [8], with a similar proof.

Lemma 3.1. *Let $\mathcal{F}$ be an attached family with $\omega(\mathcal{F}) \leqslant k$ and $\mathcal{S}$ be a k-clique or k-bracket in $\mathcal{F}$. Let $\mathcal{R} = \{R \in \mathcal{F}(\mathcal{S}) : R \cap \operatorname{ext}(\mathcal{S}) \neq \emptyset\}$ and $\mathcal{D} = \{D \in \mathcal{F}(\mathcal{S}) : D \cap \bigcup(\mathcal{R} \cup \mathcal{S}) \neq \emptyset\}$. Then $\chi(\mathcal{D}) \leqslant 2^{\beta_k}$.*

A *fancy k-clique* in an attached family $\mathcal{F}$ consists of a k-clique $\mathcal{K}$, a set $P \subseteq \operatorname{int}(\mathcal{K})$ called *hook*, two intersecting sets $X_1, X_2 \in \mathcal{F}(-\infty, K_1)$ called *left guards*, and two intersecting sets $Y_1, Y_2 \in \mathcal{F}(K_k, +\infty)$ called *right guards*, where K_1 and K_k are the least and the greatest elements of $\mathcal{K}$, respectively.

Claim 3.2. Any attached family $\mathcal{F}$ with $\omega(\mathcal{F}) \leqslant k$ and $\chi(\mathcal{F}) > 2^{\gamma_k}$ contains a fancy k-clique.

Proof. Find in $\mathcal{F}$ sets $X_1 \prec X_2 \prec Y_1 \prec Y_2$ so that $X_1 \cap X_2 \neq \emptyset$, $Y_1 \cap Y_2 \neq \emptyset$, and $\chi(\mathcal{F}(X_2, Y_1)) \geqslant \chi(\mathcal{F}) - 4 > 2^{2\xi_{k-1}+k+1}$. Apply Lemma 2.2 to find $\mathcal{H} \subseteq \mathcal{F}(X_2, Y_1)$ such that $\chi(\mathcal{H}) > 2^{\xi_{k-1}}$ and for any intersecting $H_1, H_2 \in \mathcal{H}$ we have $\chi(\mathcal{F}(H_1, H_2)) \geqslant 2^{\xi_{k-1}+k}$. It follows that $\mathcal{H}$ contains a k-clique $\mathcal{K}$ such that $\chi(\mathcal{F}(\mathcal{K})) \geqslant 2^{\xi_{k-1}+k} > 2^{\xi_{k-1}}k$. Since $|\mathcal{K}| = k$, the members of $\mathcal{F}(\mathcal{K})$ that intersect $\bigcup\mathcal{K}$ can be properly colored with $2^{\xi_{k-1}}k$ colors. Hence there exists $P \in \mathcal{F}(\mathcal{K})$ disjoint from $\bigcup\mathcal{K}$, so that $P \subseteq \operatorname{int}(\mathcal{K})$. The clique $\mathcal{K}$ with hook P, left guards X_1, X_2, and right guards Y_1, Y_2 forms a fancy k-clique in $\mathcal{F}$. □

A (k, i)-*bracket system* in an attached family $\mathcal{F}$ consists of k-brackets $\mathcal{B}_1, \ldots, \mathcal{B}_i$ with pairwise intersecting supports, and two intersecting sets $X_1, X_2 \in \mathcal{F}(\mathcal{B}_1) \cap \ldots \cap \mathcal{F}(\mathcal{B}_i)$ called *guards*.

Claim 3.3. Any attached family $\mathcal{F}$ with $\omega(\mathcal{F}) \leqslant k$ and $\chi(\mathcal{F}) > 2^{\delta_{k,i}}$ contains a (k, i)-bracket system.

Proof. The proof goes by induction on i. We start with $i = 1$. First, apply Lemma 2.3 to find $\mathcal{G} \subseteq \mathcal{F}$ that is externally supported in $\mathcal{F}$ and satisfies $\chi(\mathcal{G}) > 2^{\xi_{k-1}+\beta_k+\gamma_k+1}$. Next, apply Lemma 2.2 to find $\mathcal{H} \subseteq \mathcal{G}$ such that $\chi(\mathcal{H}) > 2^{\xi_{k-1}}$ and for any intersecting $H_1, H_2 \in \mathcal{H}$ we have $\chi(\mathcal{G}(H_1, H_2)) \geqslant 2^{\beta_k+\gamma_k}$. It follows that $\mathcal{H}$ contains a k-clique $\mathcal{K}$ such that $\chi(\mathcal{G}(\mathcal{K})) \geqslant 2^{\beta_k+\gamma_k}$. Let $\mathcal{R} = \{R \in \mathcal{F}(\mathcal{K}) : R \cap \operatorname{ext}(\mathcal{K}) \neq \emptyset\}$ and $\mathcal{D} = \{D \in \mathcal{G}(\mathcal{K}) : D \cap \bigcup(\mathcal{K} \cup \mathcal{R}) \neq \emptyset\}$. Lemma 3.1 yields $\chi(\mathcal{D}) \leqslant 2^{\beta_k}$. Let $\mathcal{G}' = \mathcal{G}(\mathcal{K}) \setminus \mathcal{D}$. It follows that $\chi(\mathcal{G}') \geqslant 2^{\beta_k+\gamma_k} - 2^{\beta_k} > 2^{\gamma_k}$. Claim 3.2 guarantees a fancy k-clique $\mathcal{K}'$ with hook P, left guards X_1, X_2, and right guards Y_1, Y_2 in $\mathcal{G}'$. Since $\mathcal{G}$ is externally supported in $\mathcal{F}$, there exists $S \in \mathcal{F}$ that intersects P and $\operatorname{ext}(\mathcal{G})$. Since $P \notin \mathcal{D}$ and $S \cap P \neq \emptyset$, we have $S \notin \mathcal{R}$. This and $S \cap \operatorname{ext}(\mathcal{K}) \supseteq S \cap \operatorname{ext}(\mathcal{G}) \neq \emptyset$ imply $S \notin \mathcal{F}(\mathcal{K})$. Therefore, $\mathcal{K}'$ with hook P, support S, and guards X_1, X_2 or Y_1, Y_2 forms a $(k, 1)$-bracket system in $\mathcal{F}$.

Now, suppose $i \geqslant 2$. As above, find $\mathcal{G} \subseteq \mathcal{F}$ externally supported in $\mathcal{F}$ and $\mathcal{H} \subseteq \mathcal{G}$ such that $\chi(\mathcal{H}) > 2^{\delta_{k,i-1}}$ and for any intersecting $H_1, H_2 \in \mathcal{H}$ we have $\chi(\mathcal{G}(H_1, H_2)) \geqslant 2^{\beta_k+\gamma_k}$. By the induction hypothesis, $\mathcal{H}$ contains a $(k, i-1)$-bracket system with brackets $\mathcal{B}_1, \ldots, \mathcal{B}_{i-1}$ and guards X_1, X_2. Thus $\chi(\mathcal{G}(X_1, X_2)) \geqslant 2^{\beta_k+\gamma_k}$. Again, $\mathcal{G}$ has a fancy k-clique $\mathcal{K}'$ with hook P, left guards X_1, X_2, and right guards Y_1, Y_2, and there exists $S \in \mathcal{F} \setminus \mathcal{F}(X_1, X_2)$ intersecting P and $\operatorname{ext}(\mathcal{G})$. By Lemma 2.4, S intersects all the supports of $\mathcal{B}_1, \ldots, \mathcal{B}_{i-1}$ (it cannot intersect the k-cliques of $\mathcal{B}_1, \ldots, \mathcal{B}_{i-1}$ as $\omega(\mathcal{F}) \leqslant k$). Thus the k-bracket $\mathcal{B}_i$ with k-clique $\mathcal{K}'$, hook P and support S together with $\mathcal{B}_1, \ldots, \mathcal{B}_{i-1}$ and guards X_1, X_2 or Y_1, Y_2 forms a (k, i)-bracket system in $\mathcal{F}$. □

To complete the proof of Theorem 2.1, observe that if $\chi(\mathcal{F}) > 2^{\xi_k} = 2^{\delta_{k,k+1}}$, then by Claim 3.3 $\mathcal{F}$ contains a $(k, k+1)$-bracket system, which contains $k+1$ pairwise intersecting supports, contradicting $\omega(\mathcal{F}) \leqslant k$.

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