Classification of k-nets embedded in a plane

Gábor Korchmáros[1]

Abstract. We present some recent, partly unpublished, results on k-nets embedded in a projective plane $PG(2, \mathbb{K})$ defined over a field $\mathbb{K}$ of any characteristic $p \geq 0$, obtained in collaboration with G.P. Nagy and N. Pace.

1 Introduction

A general problem in finite geometry is to determine geometric structures, such as graphs, designs and incidence geometries, which can be embedded in a projective plane. Here we deal with k-nets embedded in a projective plane $PG(2, \mathbb{K})$ defined over a field $\mathbb{K}$ of any characteristic $p \geq 0$. They are line configurations in $PG(2, \mathbb{K})$ consisting of k pairwise disjoint line-sets, called components, such that any two lines from distinct families are concurrent with exactly one line from each component. The size of each component of a k-net is the same, the order of the k-net. The concept of a k-net arose in classical Differential geometry, and there is a long history about finite k-nets in Combinatorics, especially for $k = 3$, related to affine planes, latin squares, loops and strictly transitive permutation sets. In recent years a strong motivation for investigation of k-nets embedded in $PG(2, \mathbb{K})$ came from Algebraic geometry and Resonance theory see [2,8,9,13,14].

2 k-nets embedded in $PG(2, \mathbb{K})$

The Stipins-Yuzvinsky theorem states that no embedded k-net for $k \geq 5$ exists when $p = 0$; see [11,14]. Our present investigation of k-nets embedded in $PG(2, \mathbb{K})$ includes groundfields $\mathbb{K}$ of positive characteristic

[1] Dipartimento di Matematica e Informatica, Università degli Studi della Basilicata, Italia.
Email: gabor.korchmaros@unibas.it

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p, and as a matter of fact, many examples. This phenomena is not unexpected since $PG(2, \mathbb{K})$ with $\mathbb{K}$ of characteristic $p > 0$ contains an affine subplane $AG(2, \mathbb{F}_p)$ of order p from which k-nets for $3 \leq k \leq p+1$ arise taking k parallel line classes as components. Similarly, if $PG(2, \mathbb{K})$ also contains an affine subplane $AG(2, \mathbb{F}_{p^h})$, in particular if $\mathbb{K} = \mathbb{F}_q$ with $q = p^r$ and $h|r$, then k-nets of order p^h for $3 \leq k \leq p^h + 1$ exist in $PG(2, \mathbb{K})$. Actually, more families of k-nets embedded in $PG(2, \mathbb{F}_q)$ when $q = p^r$ with $r \geq 3$ arise from Lunardon's work; see [5,6]. On the other hand, no 5-net of order n with $p > n$ is known to exist. This suggests that for sufficiently large p compared with n, the Stipins-Yuzvinsky theorem remains valid in $PG(2, \mathbb{K})$. Our main result [4] in this direction proves it:

Theorem 2.1. *If $p > 3^{\varphi(n^2-n)}$ where φ is the classical Euler function then no k-net with $k \geq 5$ is embedded in $PG(2, \mathbb{K})$.*

Our approach also works in zero characteristic and provides a new proof for the Stipins-Yuzvinksy theorem. A key idea in our proof is to consider the cross-ratio of four concurrent lines from different components of a 4-net. We prove that the cross-ratio remains constant when the four lines vary without changing component. In other words, every 4-net in $PG(2, \mathbb{K})$ has constant cross-ratio. In zero characteristic, and in characteristic p with $p > 3^{\varphi(n^2-n)}$, the constant cross-ratio is restricted to two values only, namely to the roots of the polynomial $X^2 - X + 1$. From this, the non-existence of k-nets for $k \geq 5$ easily follows both in zero characteristic and in characteristic p with $p > 3^{\varphi(n^2-n)}$. It should be noted that without a suitable hypothesis on n with respect to p, the constant cross-ratio of a 4-net may assume many different values, even for finite fields.

In the complex plane, there is known only one 4-net up to projectivity; see [11–14]. This 4-net, called the classical 4-net, has order 3 and it exists since $PG(2, \mathbb{C})$ contains an affine subplane $AG(2, \mathbb{F}_3)$ of order 3, unique up to projectivity, and the four parallel line classes of $AG(2, \mathbb{F}_3)$ are the components of a 4-net in $PG(2, \mathbb{K})$. It has been conjectured that the classical 4-net is the only 4-net embedded in $PG(2, \mathbb{C})$.

3 3-nets embedded in $PG(2, \mathbb{K})$

There are known plenty of 3-nets embedded in $PG(2, \mathbb{K})$. One infinite family arises from plane cubic curves. More precisely, let $\mathcal{C}$ be a plane (possible reducible) cubic curve equipped with its abelian group $(G, +)$ defined on the set of the nonsingular points of $\mathcal{C}$, take three distinct cosets $H + a,\ H + b,\ H + c$ of a subgroup H of G of order n, such that $a+b+c=0$. Then, in the dual plane, the lines corresponding to the points in these three cosets are the component of a 3-net. Such embedded 3-nets in $PG(2,\mathbb{K})$ are called *algebraic*. The isotopy class of quasigroups coordinatizing an algebraic 3-net contains a group isomorphic to H. Another infinite family arise from tetrahedrons of $PG(3,\mathbb{K})$ by projection, and called of *tetrahedron type*. In the dual plane, the components $\Lambda_1, \Lambda_2, \Lambda_3$ of a tetrahedron type embedded 3-net lie on the six sides (diagonals) of a non-degenerate quadrangle such a way that $\Lambda_i = \Delta_i \cup \Gamma_i$ with Δ_i and Γ_i lying on opposite sides, for $i = 1, 2, 3$. More precisely, $(\Lambda_1, \Lambda_2, \Lambda_3)$ can be lifted to the fundamental tetrahedron of $PG(3,\mathbb{K})$ so that the projection π from the point $P_0 = (1,1,1,1)$ on the plane $X_4 = 0$ returns $(\Lambda_1, \Lambda_2, \Lambda_3)$. For this purpose, it is enough to define the sets lying on the edges of the fundamental tetrahedron:

$$\begin{array}{ll} \Gamma'_1 = \{(\xi,0,1,0) | \xi \in L_1\}, & \Gamma'_2 = \{(0,\eta,1,0) | \eta \in L_2\}, \\ \Gamma'_3 = \{(1,-\zeta,0,0) | \zeta \in L_3\}, & \Delta'_1 = \{(0,\alpha-1,0,-1) | \alpha \in M_1\}, \\ \Delta'_2 = \{(\beta-1,0,0,-1) | \beta \in M_2\}, & \Delta'_3 = \{(0,0,\gamma-1,-1) | \gamma \in M_3\}, \end{array}$$

and observe that $\pi(\Gamma'_i) = \Gamma_i$ and $\pi(\Delta'_i) = \Delta_i$ for $i = 1, 2, 3$. Moreover, a triple (P_1, P_2, P_3) of points with $P_i \in \Gamma_i \cup \Delta_i$ consists of collinear points if and only if if their projection does. Hence, $(\Gamma'_1 \cup \Gamma'_2, \Gamma'_3 \cup \Delta'_1, \Delta'_2 \cup \Delta'_3)$ can be viewed as a "spatial" dual 3-net realizing the same group H. Clearly, $(\Gamma'_1 \cup \Gamma'_2, \Gamma'_3 \cup \Delta'_1, \Delta'_2 \cup \Delta'_3)$ is contained in the sides of the fundamental tetrahedron. We claim that these sides minus the vertices form an infinite spatial dual 3-net realizing the dihedral group $2.\mathbb{K}^*$. To prove this, parametrize the points as follows.

$$\begin{array}{l} \Sigma_1 = \{x_1 = (x,0,1,0), (\varepsilon x)_1 = (0,1,0,x) \mid x \in \mathbb{K}^*\}, \\ \Sigma_2 = \{y_2 = (1,y,0,0), (\varepsilon y)_2 = (0,0,1,y) \mid y \in \mathbb{K}^*\}, \\ \Sigma_3 = \{z_3 = (0,-z,1,0), (\varepsilon z)_3 = (1,0,0,-z) \mid z \in \mathbb{K}^*\}. \end{array} \tag{3.1}$$

Then,

$$\begin{array}{rl} x_1, y_2, z_3 \text{ are collinear} & \Leftrightarrow z = xy, \\ (\varepsilon x)_1, y_2, (\varepsilon z)_3 \text{ are collinear} & \Leftrightarrow z = xy \Leftrightarrow \varepsilon z = (\varepsilon x)y, \\ x_1, (\varepsilon y)_2, (\varepsilon z)_3 \text{ are collinear} & \Leftrightarrow z = x^{-1}y \Leftrightarrow \varepsilon z = x(\varepsilon y), \\ (\varepsilon x)_1, (\varepsilon y)_2, z_3 \text{ are collinear} & \Leftrightarrow z = x^{-1}y \Leftrightarrow z = (\varepsilon x)(\varepsilon y). \end{array}$$

Thus, $(\Gamma'_1 \cup \Gamma'_2, \Gamma'_3 \cup \Delta'_1, \Delta'_2 \cup \Delta'_3)$ is a dual 3-subnet of $(\Sigma_1, \Sigma_2, \Sigma_3)$ and H is a subgroup of the dihedral group $2.\mathbb{K}^*$. As H is not cyclic but it has a cyclic subgroup of index 2, we conclude that H is itself dihedral.

The isotopy class of quasigroups coordinatizing a tetrahedron type 3-net is a dihedral group. We also know a sporadic example; namely the

Urzua 3-net of order 8 coordinatized by the quaternion group of order 8; see [12].

In [3] we are dealt with 3-nets embedded in $PG(2, \mathbb{K})$ which are coordinatized by groups. Our main result is a complete classification for $p = 0$:

Theorem 3.1. *In the projective plane $PG(2, \mathbb{K})$ defined over an algebraically closed field $\mathbb{K}$ of characteristic $p \geq 0$, let Λ be am embedded 3-net of order $n \geq 4$ coordinatized by a group G. If either $p = 0$ or $p > n$ then one of the following holds.*

(I) *G is either cyclic or the direct product of two cyclic groups, and Λ is algebraic.*
(II) *G is dihedral and Λ is of tetrahedron type.*
(III) *G is the quaternion group of order 8, and Λ is the Urzúa 3-net of order 8..*
(IV) *G has order 12 and is isomorphic to Alt_4.*
(V) *G has order 24 and is isomorphic to Sym_4.*
(VI) *G has order 60 and is isomorphic to Alt_5.*

A computer aided exhaustive search shows that if $p = 0$ then (IV) (and hence (V), (VI)) does not occur, see [7]. Theorem 3.1 shows that every realizable finite group can act in $PG(2, \mathbb{K})$ as a projectivity group. This confirms Yuzvinsky's conjecture for $p = 0$.

A combinatorial characterization of algebraic 3-nets contained in a reducible plane cubic is given in [1]:

Theorem 3.2. *Let $p > n$ or $p = 0$. If a component of a dual 3-net is contained in a pencil, then the other two components consist of lines tangents to a unique conic, and the converse also holds.*

The proofs of Theorems 3.1 and 3.2 use several results on collineation groups of $PG(2, \mathbb{K})$ together with the classification of subgroups of $PGL(2, \mathbb{K})$ due to Dickson. Furthermore, the proof of Theorem 3.2 also uses the "Rédei polynomial approach" of Szőnyi for the study of blocking-sets in $PG(2, q)$; see [10].

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