Degenerated induced subgraphs of planar graphs

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Abstract. A graph G is *k-degenerated* if it can be deleted by subsequent removals of vertices of degree k or less. We survey known results on the size of maximal k-degenerated induced subgraph in a planar graph. In addition, we sketch the proof that every planar graph of order n has a 4-degenerated induced subgraph of order at least $8/9 \cdot n$. We also show that in every planar graph with at least 7 vertices, deleting a suitable vertex allows us to subsequently remove at least 6 more vertices of degree four or less.

1 Introduction

A graph G is *k-degenerated* if every subgraph of G has a vertex of degree k or less. Equivalently, a graph is k-degenerated if we can delete the whole graph by subsequently removing vertices of degree at most k. The reverse of this sequence of removed vertices can be used to colour (or even list-colour) G with $k + 1$ colours in a greedy fashion. Graph degeneracy is therefore a natural bound on both chromatic number and list chromatic number. In fact, for some problems graph degeneracy provides the best known bounds on the choice number [3].

Every planar graph has a vertex of degree at most 5. Since a subgraph of a planar graph is planar, it also has a vertex of degree at most 5. Therefore, every planar graph is 5-degenerated. If $k < 5$, we can still choose at least some of the vertices of G, and if these vertices induce a

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k-degenerated graph, then they can be greedily coloured by $k+1$ colours. Thus an interesting question is how large k-degenerated subgraph can be guaranteed in a planar graph G.

Without the restriction to planar graphs Alon, Kahn, and Seymour [2] determined exactly how large k-degenerated induced subgraph one can guarantee depending only on the degree sequence of G. For planar graphs the only settled case is $k = 0$, due to four colour theorem. Except for $k = 4$ all other values of k were examined. Most attention is devoted to the case $k = 1$. The Albertson-Berman conjecture [1] asserts that every planar graph has an induced forest with at least half of the vertices. The best known bound, guaranteeing a forest of size at least $2/5 \cdot |V(G)|$, is implied by the fact that planar graphs are acyclic 5-colourable [4].

We do not know of any results on maximum 4-degenerated induced subgraphs of planar graphs. A likely reason is that such a bound is not interesting for list-colouring applications: Thomassen [5] proved that every planar graph is 5-choosable. The rest of this paper focuses on degeneracy 4.

We define two operations for vertex removal: deletion and collection. To *delete* a vertex v, we remove v and its incident edges from the graph. To *collect* a vertex v is the same as to delete v, but to be able to collect v we require v to be of degree at most 4. The collected vertices induce a 4-degenerated subgraph. Vertices that are deleted or collected are collectively called *removed*.

Our main results are the following two theorems.

Theorem 1.1. *In every planar graph G we can delete at most* $1/9$ *of its vertices in such a way that we can collect all the remaining ones.*

Theorem 1.2. *In every planar graph with at least* 7 *vertices we can delete a vertex in such a way that we can subsequently collect at least* 6 *vertices.*

These results are probably not the best possible. The worst example known to us is the icosahedron from which we need to delete one vertex out of twelve to be able to collect the remaining eleven. We believe that this is the worst case possible.

Conjecture 1.3. In each planar graph G we can delete at most $1/12$ of its vertices in such a way that we can collect all the remaining ones.

Conjecture 1.4. In each planar graph with at least 12 vertices we can delete a vertex in such a way that we can subsequently collect at least 11 vertices.

2 Sketch of the proof

To prove Theorem 1.1 we incorporate the degrees of the vertices of G to the statement; let

$$\Phi(G) = \sum_{v \in V(G)} (\deg(v) - 5). \tag{2.1}$$

We prove the following theorem by induction on the number of vertices of G. The function $\mathrm{tc}(G)$ stands for the number of tree components of G.

Theorem 2.1. *If G is a planar graph, then there is a set $S \subset V(G)$ with at most $\Gamma(G) = |V(G)|/12 + 1/36 \cdot \Phi(G) + 1/18 \cdot \mathrm{tc}(G)$ vertices such that if we delete S we can subsequently collect all the vertices of G.*

Since $\Phi(G) + 2\,\mathrm{tc}(G) \leq |V(G)|$, Theorem 2.1 implies Theorem 1.1. Theorem 1.2 can be proved alongside Theorem 2.1.

We prove Theorem 2.1 by a discharging procedure. Let G be a minimal counterexample to Theorem 2.1 (with respect to the number of vertices); it can be easily shown that G is connected with minimal degree 5. We embed G into the plane (for now, we let the embedding be arbitrary).

Type (t)	Degree (deg)	Min. number of non-tr. faces ($n_{\sqcup}$)	Max. number of V_5 neigh. (n_5)	Maximal charge (mc)
10a	10+	0	3	1
10b	10+	0	∞	1/2
9a	9	1	3	1
9b	9	0	2	1
9c	9	0	3 if consecutive	9/10, 1, 9/10
9d	9	0	9	1/2
8a	8	0	1	1
8b	8	1	2	1
8c	8	2	3 if consecutive	9/10, 1, 9/10
8d	8	0	2	9/10
8e	8	0	8	1/2
7a	7	0	1	4/5
7b	7	1	2	13/20
7c	7	0	2	2/5
7d	7	0	7	1/3
6a	6	1	1	2/5
6b	6	0	6	0

Table 1. Maximal charges that can be send to a vertex.

Each vertex of degree at least 6 is assigned a certain *type* according to Table 1. If w is of degree d, is contained on at least $n_{\sqcup}$ non-triangular faces, and has at most n_5 neighbours from V_5, then w can be of type t. If w can have more than one type, then the type of w is the type that occurs first in the table. Let vw be an edge such that v is of degree 5 and w is of degree at least 6. For every such edge we define the maximal charge $\mathrm{mc}(v, w)$ that v can send to w. This maximal charge is given in the last column of Table 1.

We start by assigning initial charges to the vertices and faces of G. Each vertex v of degree d receives charge $6-d$ and each face of length ℓ receives charge $2(3-\ell)$. According to Euler's theorem, the total initial charge is equal to 12. In the discharging procedure, we redistribute the charges between vertices and faces in a certain way such that no charge is created or lost. The discharging procedure consists of the following three steps.

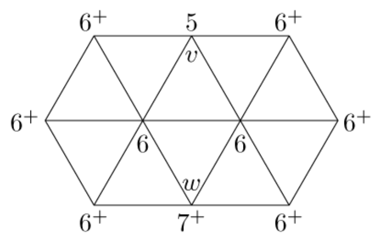

Figure 2.1. Distance discharging (d^+ denotes a vertex of degree at least d).

Step 1: Discharging to faces. For each vertex v and for each non-triangular face that contains v, send $1/2$ from v to f, except for two cases: if v is of degree 6, then send $2/5$; and if v is not of degree 6 but both its neighbours on f are of degree 6, then send $3/5$.

Step 2: Distance discharging. In every subgraph of G isomorphic to the configuration in Figure 2.1 send $1/5$ from vertex v to vertex w (vertices are denoted as in Figure 2.1; the depicted vertices are pairwise distinct; the numbers indicate degrees).

Step 3: Final discharging of the vertices of degree five. For each vertex v of degree 5 carry out the following procedure. Order the neighbours w of v which have degree at least 6 according to the value of $\mathrm{mc}(v, w)$ starting with the largest value; let $w_1, w_2, \ldots$ be the resulting ordering. If the value of $\mathrm{mc}(v, w)$ is the same for two neighbours of v, then we order them arbitrarily. For $i = 1, 2, \ldots$, send $\max\{\mathrm{mc}(v, w_i), \mathrm{ch}_a(v)\}$ from v to w_i, where $\mathrm{ch}_a(v)$ denotes the current charge of v.

Since no face can have positive final charge, there exists a vertex with positive charge. A very technical examination allows us to show that if we start with a certain embedding of G, then there is a vertex with positive final charge not contained in any C_3-cut or C_4-cut of G.

We examine v together with its neighbourhood; this gives rise to a number of cases, each of them leading to a contradiction. We demonstrate this analysis on Configuration 7-3 shown in Figure 2.2 (the numbers in the right part of the figure indicate the degrees of the vertices shown in the left part). Since G contains no C_3-cut, the triangles depicted in Figure 2.2 are faces and contain no vertices inside. The vertex v is of type $7d$.

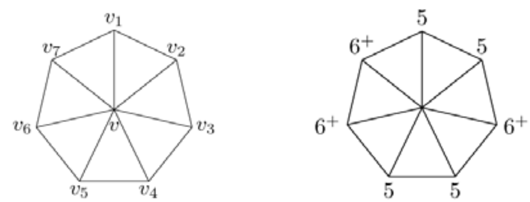

Figure 2.2. Configuration 7-3.

Configuration 7-3: Suppose first that the vertex v_7 has degree at most 7. Delete v_6. Then we can collect v_5, v_4, v, v_1, v_2, and v_7. We get a new graph G' smaller than G, so there is a set S' in G' with at most $\Gamma(G')$ vertices such that if we delete S', we can collect the rest of G'. If $\Gamma(G) - \Gamma(G') \geq 1$, we can extend the set S' by v_6 and obtain a contradiction with the fact that G is a smallest counterexample to Theorem 2.1.

We want show that $\Gamma(G) - \Gamma(G') \geq 1$. The hardest part is typically to compute $\Phi(G) - \Phi(G')$. Among the vertices removed from G, four vertices have degree at least 5, two vertices have degree at least 6, and one vertex has degree at least 7. After removing these vertices from the sum (2.1) the value of Φ decreases by at least 4. Moreover, all the neighbours of the removed vertices have smaller degree in G' than in G. From the fact that v is in no C_3-cuts we know that there is no extra edge between the neighbours of v; all such edges are shown in Figure 2.2. This decreases Φ further by at least 17. Therefore $\Phi(G) - \Phi(G') \geq 21$. It can be shown that no new tree components are created, so together $\Gamma(G) - \Gamma(G') \geq 42/36 \geq 1$.

We are left with the case where v_7 has degree at least 8. If v_7 has another neighbour w of degree 5 besides v_1, then we can delete v_7 and collect v_1, v_2, v, v_4, v_5, and w (again we need to check that Γ decreases by at least 1). Otherwise, v_7 has only one neighbour of degree 5, the vertex v_1. According to Table 1, $\mathrm{mc}(v_1, v_7) = 1$, so v_1 discharges nothing into v in Step 3 and the final charge of v is at most 0. This contradicts the fact that v has positive charge after the discharging.

References

[1] M. O. Albertson and D. M. Berman, *A conjecture on planar graphs*, In: “Graph Theory and Related Topics” J. A. Bondy and U. S. R. Murty, (eds.), Academic Press, 1979, 357.

[2] N. Alon, J. Kahn and P. D. Seymour, *Large induced degenerate subgraphs*, Graphs and Combinatorics **3** (1987), 203–211.

[3] J. Barát, G. Joret and D. R. Wood, *Disproof of the list Hadwiger Conjecture*, Electron. J. Combin. **18** (2011), #P232.

[4] O. V. Borodin, *On acyclic colorings of planar graphs*, Discrete Math. **25** (1979), 211–236.

[5] C. Thomassen, *Every planar graph is* 5*-choosable*, Journal of Comb. Theory Ser B **62** (1994), 180–181.