The Erdős-Pósa property for long circuits

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Abstract. For an integer ℓ at least 3, we prove that if G is a graph containing no two vertex-disjoint circuits of length at least ℓ, then there is a set X of at most $\frac{5}{3}\ell + \frac{29}{2}$ vertices that intersects all circuits of length at least ℓ. Our result improves the bound $2\ell + 3$ due to Birmelé, Bondy, and Reed (The Erdős-Pósa property for long circuits, Combinatorica 27 (2007), 135-145) who conjecture that ℓ vertices always suffice.

1 Introduction

A family $\mathcal{F}$ of graphs is said to have the *Erdős-Pósa property* if there is a function $f_{\mathcal{F}} : \mathbb{N} \to \mathbb{N}$ such that for every graph G and every $k \in \mathbb{N}$, either G contains k vertex-disjoint subgraphs that belong to $\mathcal{F}$ or there is a set X of at most $f_{\mathcal{F}}(k)$ vertices of G such that $G - X$ has no subgraph that belongs to $\mathcal{F}$. The origin of this notion is [3] where Erdős and Pósa prove that the family of all circuits has this property.

Let ℓ be an integer at least 3. Let $\mathcal{F}_\ell$ denote the family of circuits of length at least ℓ. In [2] Birmelé, Bondy, and Reed show that $\mathcal{F}_\ell$ has the Erdős-Pósa property with

$$f_{\mathcal{F}_\ell}(k) \leq 13\ell(k-1)(k-2) + (2\ell+3)(k-1), \tag{1.1}$$

which improves an earlier doubly exponential bound on $f_{\mathcal{F}_\ell}(k)$ obtained by Thomassen [5]. The main contribution of Birmelé, Bondy, and Reed [2] is to prove (1.1) for $k = 2$, that is, to show

$$f_{\mathcal{F}_\ell}(2) \leq 2\ell + 3. \tag{1.2}$$

For $k \geq 3$, an inductive argument allows to deduce (1.1) from (1.2).

Birmelé, Bondy, and Reed [2] conjecture that

$$f_{\mathcal{F}_\ell}(2) \leq \ell, \tag{1.3}$$

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that is, for every graph G containing no two vertex-disjoint circuits of length at least ℓ, there is a set X of at most ℓ vertices such that $G - X$ has no circuit of length at least ℓ. In view of the complete graph of order $2\ell - 1$, (1.3) would be best possible. For $\ell = 3$, (1.3) was shown by Lovász [4] and for $\ell \in \{4, 5\}$, (1.3) was shown by Birmelé [1].

Our contribution in the present paper is the following result.

Theorem 1.1. *Let ℓ be an integer at least 3. Let G be a graph containing no two vertex-disjoint circuits of length at least ℓ.*

There is a set X of at most $\frac{5}{3}\ell + \frac{29}{2}$ vertices that intersects all circuits of length at least ℓ, that is, $f_{\mathcal{F}_\ell}(2) \leq \frac{5}{3}\ell + \frac{29}{2}$.

While Theorem 1.1 is a nice improvement of (1.2), for $k \geq 3$, the above-mentioned inductive argument still leads to an estimate of the form $f_{\mathcal{F}_\ell}(k) = O(\ell k^2)$.

The rest of this paper is devoted to the proof of Theorem 1.1.

2 Proof of Theorem 1.1

With respect to notation and terminology we follow [2] and recall some specific notions. All graphs are finite, simple, and undirected. We abbreviate *vertex-disjoint* as *disjoint*. If A and B are sets of vertices of a graph G, then an (A, B)*-path* is a path P in G between a vertex in A and a vertex in B such that no internal vertex of P belongs to $A \cup B$. If P is a path and x and y are vertices of P, then $P[x, y]$ denotes the subpath of P between x and y. Similarly, if C is a circuit endowed with an orientation and x and y are vertices of C, then $C[x, y]$ denotes the segment of C from x to y following the orientation of C. In all figures of circuits the orientations will be counterclockwise.

We fix an integer ℓ at least 3 and call a circuit of length at least ℓ *long*.

If C is a circuit and P and P' are disjoint $(V(C), V(C))$-paths such that P is between u and v and P' is between u' and v', then

- P and P' are called *parallel (with respect to C)* if u, u', v', v appear in the given cyclic order on C and
- P and P' are called *crossing (with respect to C)* if u, u', v, v' appear in the given cyclic order on C.

See Figure 2.1.

In the proof of Theorem 1.1 below we consider three cases according to the length L of a shortest long circuit. If L is less than $3\ell/2$, the result is trivial. For L between $3\ell/2$ and 2ℓ the following lemma implies the desired bound. Finally, for L larger than 2ℓ, Lemma 2.2 implies the desired bound.

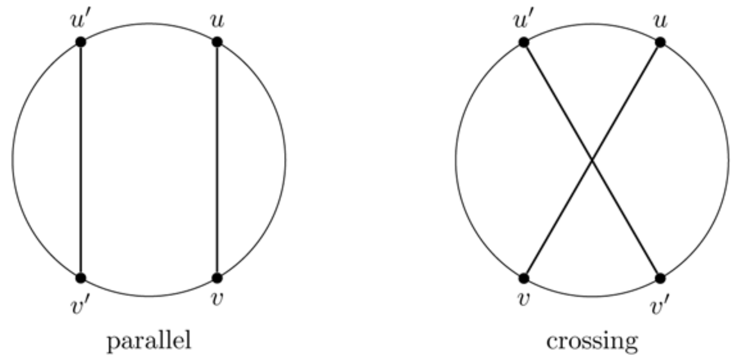

Figure 2.1. Parallel and crossing pairs of paths.

Lemma 2.1. *Let G be a graph containing no two disjoint long circuits.*

If the shortest long circuit of G has length L with $L \geq 3\left(\left\lceil \frac{1}{2}\ell \right\rceil - 2\right)$, then there is a set X of at most $\frac{1}{3}L + \ell + \frac{14}{3}$ vertices that intersects all long circuits.

Proof. Let C be a shortest long circuit of G. We endow C with an orientation. We decompose C into 6 cyclically consecutive and internally disjoint segments $C_1, \dots, C_6$ such that C_1, C_3, and C_5 have length $\left\lceil \frac{1}{2}\ell \right\rceil - 2$ and C_2, C_4, and C_6 have lengths between $\left\lfloor \frac{1}{3}L - \left(\left\lceil \frac{1}{2}\ell \right\rceil - 2\right) \right\rfloor$ and $\left\lceil \frac{1}{3}L - \left(\left\lceil \frac{1}{2}\ell \right\rceil - 2\right) \right\rceil$, that is, the six segments cover all of C and C_i and C_{i+1} overlap in exactly one vertex for every $i \in [6]$ where we identify indices modulo 6.

Let $X_1 = V(C_1) \cup V(C_3) \cup V(C_5)$. See the left part of Figure 2.2.

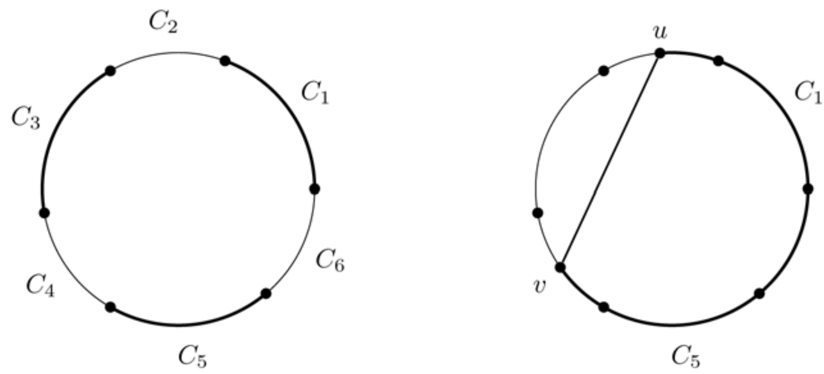

Figure 2.2. On the left the six segments of C and the set X in bold. On the right a long circuit formed by a $(V(C_2), V(C_4))$-path between u and v in $G - (X_1 \cup V(C_6))$.

Let $i \in [6]$ be even. Let $\mathcal{P}_i$ denote the set of $(V(C_i), V(C_{i+2}))$-paths in $G - (X_1 \cup V(C_{i+4}))$. The choice of C implies that every path P in $\mathcal{P}_i$

has length at least $\frac{1}{2}\ell$; otherwise P together with a segment of C avoiding $V(C_{i+1})$ forms a long circuit that is shorter than C. See the right part of Figure 2.2. This implies that for every path P in $\mathcal{P}_i$, P together with a segment of C containing $V(C_{i+1})$ forms a long circuit.

Let $\mathcal{P} = \mathcal{P}_2 \cup \mathcal{P}_4 \cup \mathcal{P}_6$. Since G has no two disjoint long circuits, it follows that $\mathcal{P}$ contains no two disjoint parallel paths and no four disjoint crossing paths. See Figure 2.3.

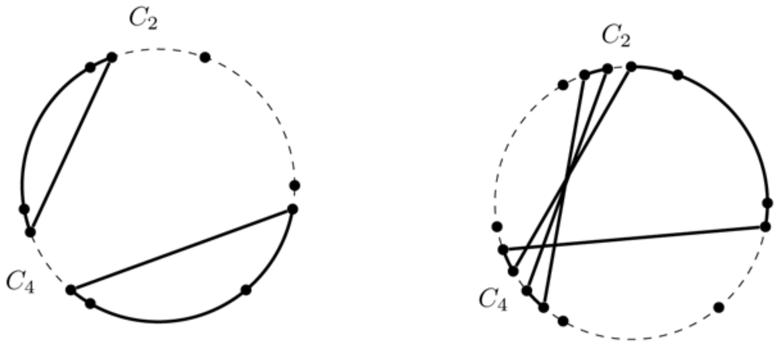

Figure 2.3. Two disjoint long circuits formed by two disjoint parallel paths in $\mathcal{P}$ or by four disjoint crossing paths in $\mathcal{P}$.

Let X_2 be a smallest set of vertices separating $V(C_2)$ and $V(C_4) \cup V(C_6)$ in $G - X_1$. Let X_3 be a smallest set of vertices separating $V(C_4)$ and $V(C_6)$ in $G - (X_1 \cup X_2)$. By the above observations and Menger's theorem, $|X_2| \leq 3$ and $|X_3| \leq 3$.

There is some even $j \in [6]$ such that in $G - (X_1 \cup X_2 \cup X_3)$, all long circuits intersect C only in $V(C_j)$; otherwise there is a $(V(C_i), V(C_{i+2}))$-path in $G - (X_1 \cup X_2 \cup X_3)$ for some even $i \in [6]$. This implies that $X_1 \cup X_2 \cup X_3 \cup V(C_j)$ intersects all long circuits of G. Since

$$\begin{aligned} |X_1 \cup X_2 \cup X_3 \cup V(C_j)| &\leq 3\left(\left\lceil \frac{1}{2}\ell \right\rceil - 2\right) + 3 + 3 \\ &\quad + \left\lceil \frac{1}{3}L - \left(\left\lceil \frac{1}{2}\ell \right\rceil - 2\right) \right\rceil + 1 \\ &\leq \frac{1}{3}L + \ell + \frac{14}{3} \end{aligned}$$

we obtain the desired result. □

Lemma 2.2. *Let G be a graph containing no two disjoint long circuits.*

If the shortest long circuit of G has length at least $2\ell - 3$, then there is a set X of at most $\frac{3}{2}\ell + \frac{29}{2}$ vertices that intersects all long circuits.

Sketch of the proof. We only sketch the proof. Specifically, we do not give the proofs of the claims. Let C be shortest long circuit of G. Let L denote the length of C. We endow C with an orientation.

As in [2], a path between two vertices x and y of C that is internally disjoint from C is called *long*, if the segments $C[x, y]$ and $C[y, x]$ both have length at least $\frac{1}{2}\ell$.

Claim A. *Every long path has length at least* $\ell - 1$.

Choose a long circuit D of G distinct from C and a segment $C[x, y]$ of C such that $C[x, y]$ contains $V(C) \cap V(D)$ and has minimum possible length. Note that $x, y \in V(C) \cap V(D)$.

We consider the two cases $x \neq y$ and $x = y$. Here we only sketch the more general case $x \neq y$. Let X_1 denote the set of $\lceil \frac{1}{2}\ell \rceil - 1$ vertices immediately preceeding x and let X_2 denote the set of $\lceil \frac{1}{2}\ell \rceil - 1$ vertices immediately following y. Let $A = V(C) \setminus (X_1 \cup X_2 \cup V(C[x, y]))$ and $B = V(C[x, y])$. In $G - (X_1 \cup X_2)$, there are no two disjoint parallel (A, B)-paths and no four disjoint crossing (A, B)-paths; otherwise there would be two disjoint long circuits. Hence, by Menger's theorem, there is a set X_3 of at most 3 vertices separating A and B in $G - (X_1 \cup X_2)$. The circuit D uniquely decomposes into a set $\mathcal{P}$ of at least two (B, B)-paths of length at least 1.

Claim B. *If $\mathcal{P}$ contains a path P between x and y, then in $G - (X_1 \cup X_2 \cup X_3)$, there are at most $\lceil \frac{1}{2}\ell \rceil + 1$ disjoint $(A, V(P))$-paths.*

Claim C. *If $\mathcal{P}$ contains two paths, say P and P', between x and y, then in $G - (X_1 \cup X_2 \cup X_3)$, there are at most $\lceil \frac{1}{2}\ell \rceil + 1$ disjoint $(A, V(D))$-paths.*

Claim D. *If P is a path in $\mathcal{P}$ that is not a path between x and y, then in $G - (X_1 \cup X_2 \cup X_3)$, there are at most 3 disjoint $(A, V(P))$-paths.*

Claim E. *If $P_1, \ldots, P_4$ are four distinct paths in $\mathcal{P}$ that are no paths between x and y, then in $G - (X_1 \cup X_2 \cup X_3)$, there are no four disjoint paths $Q_1, \ldots, Q_4$ such that Q_i is a $(A, V(P_i))$-path for $i \in [4]$.*

Let V_1 denote the set of vertices r of D such that $\mathcal{P}$ contains a path between x and y that contains r and let V_2 denote the set of vertices s of D such that $\mathcal{P}$ contains a path not between x and y that contains s. Clearly, $V_1 \cup V_2 = V(D)$. By Claims B and C and Menger's theorem, there is a set X_4 of at most $\lceil \frac{1}{2}\ell \rceil + 1$ vertices separating A and V_1 in $G - (X_1 \cup X_2 \cup X_3)$. By Claims D and E and Menger's theorem, there is a set X_5 of at most 9 vertices separating A and V_2 in $G - (X_1 \cup X_2 \cup X_3)$. Let $X = \{x, y\} \cup X_1 \cup X_2 \cup X_3 \cup X_4 \cup X_5$.

If $G - X$ contains a long circuit, say D', then D' intersects A. Since D and D' intersect, there is an $(A, V(D))$-path P in $G - X$. In view of

X_3, P cannot end in B; in view of X_4, P cannot end in V_1; and, in view of X_5, P cannot end in V_2, which is a contradiction. Hence X intersects all long circuits. Since

$$|X| \leq 2 + |X_1| + |X_2| + |X_3| + |X_4| + |X_5| \leq \frac{3}{2}\ell + \frac{29}{2},$$

this completes the proof in the case $x \neq y$. □

Proof of Theorem 1.1. Let C be shortest long circuit of G. Let L denote the length of C.

If L is at most $\frac{5}{3}\ell + \frac{29}{2}$, then let $X = V(C)$. If L is larger than $\frac{5}{3}\ell + \frac{29}{2}$ but less than $2\ell - 4$, then Lemma 2.1 implies the existence of a set X with the desired properties. If L is at least $2\ell - 3$, then Lemma 2.2 implies the existence of a set X with the desired properties. □

Our main interest was to improve the factor of ℓ in the bound in Theorem 1.1 and not the additive constant, which can easily be improved slightly.

The main open problem remains the conjectured inequality (1.3). We believe that further ideas are needed for its proof. Furthermore, it is unclear whether the quadratic dependence on k in (1.1) is best possible. For $\ell = 3$, that is, the classical case considered by Erdős and Pósa [3], it is known that $f_{\mathcal{F}_3}(k) = O(k \log k)$.

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