Boxicity and cubicity of product graphs

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Abstract. The *boxicity* (*cubicity*) of a graph G is the minimum natural number k such that G can be represented as an intersection graph of axis-parallel rectangular boxes (axis-parallel unit cubes) in $\mathbb{R}^k$. In this article, we give estimates on the boxicity and the cubicity of *Cartesian*, *strong* and *direct products* of graphs in terms of invariants of the component graphs. In particular, we study the growth, as a function of d, of the boxicity and the cubicity of the d-th power of a graph with respect to the three products. Among others, we show a surprising result that the boxicity and the cubicity of the d-th Cartesian power of any given finite graph is in $O(\log d / \log\log d)$ and $\Theta(d/\log d)$, respectively. On the other hand, we show that there cannot exist any sublinear bound on the growth of the boxicity of powers of a general graph with respect to strong and direct products.

1 Introduction

Throughout this discussion, a *k-box* is the Cartesian product of k closed intervals on the real line $\mathbb{R}$, and a *k-cube* is the Cartesian product of k closed unit length intervals on $\mathbb{R}$. Hence both are subsets of $\mathbb{R}^k$ with edges parallel to one of the coordinate axes. All the graphs considered here are finite, undirected and simple.

Definition 1.1 (Boxicity, Cubicity). A *k-box representation* (*k-cube representation*) of a graph G is a function f that maps each vertex of G to a k-box (k-cube) such that for any two distinct vertices u and v of G, the pair uv is an edge in G if and only if the boxes $f(u)$ and $f(v)$ have a non-empty intersection. The *boxicity* (*cubicity*) of a graph G, denoted by boxicity(G) (cubicity(G)), is the smallest natural number k such that G has a k-box (k-cube) representation.

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It follows from the above definition that complete graphs have boxicity and cubicity 0 and interval graphs (unit interval graphs) are precisely the graphs with boxicity (cubicity) at most 1. The concepts of boxicity and cubicity were introduced by F.S. Roberts in 1969 [9]. He showed that every graph on n vertices has an $\lfloor n/2 \rfloor$-box and a $\lfloor 2n/3 \rfloor$-cube representation.

Given two graphs G_1 and G_2 with respective box representations f_1 and f_2, let G denote the graph on the vertex set $V(G_1) \times V(G_2)$ whose box representation is a function f defined by $f((v_1, v_2)) = f_1(v_1) \times f_2(v_2)$. It is not difficult to see that G is the usual strong product of G_1 and G_2 (cf. Definition 1.2). Hence it follows that the boxicity (cubicity) of G is at most the sum of the boxicities (cubicities) of G_1 and G_2. The interesting question here is: *can it be smaller?* We show that *it can be smaller* in general. But in the case when G_1 and G_2 have at least one universal vertex each, we show that that the boxicity (cubicity) of G is equal to the sum of the boxicities (cubicities) of G_1 and G_2 (Theorem 2.1).

Definition 1.2 (Graph products). The *strong product*, the *Cartesian product* and the *direct product* of two graphs G_1 and G_2, denoted respectively by $G_1 \boxtimes G_2$, $G_1 \square G_2$ and $G_1 \times G_2$, are graphs on the vertex set $V(G_1) \times V(G_2)$ with the following edge sets:

$$\begin{aligned} E(G_1 \boxtimes G_2) &= \{(u_1, u_2)(v_1, v_2) : (u_1 = v_1 \text{ or } u_1v_1 \in E(G_1)) \text{ and} \\ &\qquad (u_2 = v_2 \text{ or } u_2v_2 \in E(G_2))\}, \\ E(G_1 \square G_2) &= \{(u_1, u_2)(v_1, v_2) : (u_1 = v_1, u_2v_2 \in E(G_2)) \text{ or} \\ &\qquad (u_1v_1 \in E(G_1), u_2 = v_2)\}, \\ E(G_1 \times G_2) &= \{(u_1, u_2)(v_1, v_2) : u_1v_1 \in E(G_1) \text{ and } u_2v_2 \in E(G_2)\}. \end{aligned}$$

The *d-th strong power*, *Cartesian power* and *direct power* of a graph G with respect to each of these products, that is, the respective product of d copies of G, are denoted by $G^{\boxtimes d}$, $G^{\square d}$ and $G^{\times d}$, respectively. Please refer to [7] to know more about graph products.

Unlike the case in strong product, the boxicity (cubicity) of the Cartesian and direct products can have a boxicity (cubicity) larger than the sum of the individual boxicities (cubicities). For example, while the complete graph on n vertices K_n has boxicity 0, we show that the Cartesian product of two copies of K_n has boxicity at least $\log n$ and the direct product of two copies of K_n has boxicity at least $n-2$. In this note, we give estimates on boxicity and cubicity of Cartesian and direct products in terms of the boxicities (cubicities) and chromatic number of the component graphs. This answers a question raised by Douglas B. West in 2009 [10].

We also study the growth, as a function of d, of the boxicity and the cubicity of the d-th power of a graph with respect to these three products. Among others, we show a surprising result that the boxicity and the cubicity of the d-th Cartesian power of any given finite graph is in $O(\log d/\log\log d)$ and $\Theta(d/\log d)$, respectively (Corollary 2.7). To get this result, we had to obtain non-trivial estimates on boxicity and cubicity of hypercubes and Hamming graphs and a bound on boxicity and cubicity of the Cartesian product which does not involve the sum of the boxicities or cubicities of the component graphs.

The results are summarised in the next section after a brief note on notations. The proofs and figures are included in the full version of the paper [3].

1.1 Notational note

The vertex set and edge set of a graph G are denoted, respectively, by $V(G)$ and $E(G)$. A pair of distinct vertices u and v is denoted at times by uv instead of $\{u, v\}$ in order to avoid clutter. A vertex in a graph is *universal* if it is adjacent to every other vertex in the graph. If S is a subset of vertices of a graph G, the subgraph of G induced on the vertex set S is denoted by $G[S]$. If A and B are sets, then $A \triangle B$ denotes their symmetric difference and $A \times B$ denotes their Cartesian product. The set $\{1, \ldots, n\}$ is denoted by $[n]$. All logarithms mentioned are to the base 2.

2 Our Results

2.1 Strong products

Theorem 2.1. *Let G_i, $i \in [d]$, be graphs with* boxicity$(G_i) = b_i$ *and* cubicity$(G_i) = c_i$. *Then*

$$\max_{i=1}^{d} b_i \le \text{boxicity}(\boxtimes_{i=1}^{d} G_i) \le \sum_{i=1}^{d} b_i, \text{ and}$$
$$\max_{i=1}^{d} c_i \le \text{cubicity}(\boxtimes_{i=1}^{d} G_i) \le \sum_{i=1}^{d} c_i.$$

Furthermore, if each G_i, $i \in [d]$ has a universal vertex, then the second inequality in both the above chains is tight.

If we consider the strong product of a 4-cycle C_4 with a path on 3 vertices P_3, we get an example where the upper bound in Theorem 2.1 is not tight.

Corollary 2.2. *For any given graph G,* boxicity$(G^{\boxtimes d})$ *and* cubicity $(G^{\boxtimes d})$ *are in $O(d)$ and there exist graphs for which they are in $\Omega(d)$.*

2.2 Cartesian products

We show two different upper bounds on the boxicity and cubicity of Cartesian products. The first and the easier result bounds from above the boxicity (cubicity) of a Cartesian product in terms of the boxicity (cubicity) of the corresponding strong product and the boxicity (cubicity) of a Hamming graph whose size is determined by the chromatic number of the component graphs. The second bound is in terms of the maximum cubicity among the component graphs and the boxicity (cubicity) of a Hamming graph whose size is determined by the sizes of the component graphs. The second bound is much more useful to study the growth of boxicity and cubicity of higher Cartesian powers since the first term remains a constant.

Theorem 2.3. *For graphs $G_1, \ldots, G_d$,*

$$\text{boxicity}(\Box_{i=1}^d G_i) \le \text{boxicity}(\boxtimes_{i=1}^d G_i) + \text{boxicity}(\Box_{i=1}^d K_{\chi_i}) \text{ and}$$
$$\text{cubicity}(\Box_{i=1}^d G_i) \le \text{cubicity}(\boxtimes_{i=1}^d G_i) + \text{cubicity}(\Box_{i=1}^d K_{\chi_i})$$

where χ_i denotes the chromatic number of G_i, $i \in [d]$.

When $G_i = K_q$ for every $i \in [d]$, $G = \boxtimes_{i=1}^d G_i$ is a complete graph on q^d vertices and hence has boxicity and cubicity 0. In this case it is easy to see that both the bounds in Theorem 2.3 are tight.

Theorem 2.4. *For graphs $G_1, \ldots, G_d$, with $|V(G_i)| = q_i$ and* cubicity$(G_i) = c_i$, *for each $i \in [d]$,*

$$\text{boxicity}(\Box_{i=1}^d G_i) \le \max_{i \in [d]} c_i + \text{boxicity}(\Box_{i=1}^d K_{q_i}), \text{ and}$$
$$\text{cubicity}(\Box_{i=1}^d G_i) \le \max_{i \in [d]} c_i + \text{cubicity}(\Box_{i=1}^d K_{q_i}).$$

In wake of the two results above, it becomes important to have a good upper bound on the boxicity and the cubicity of Hamming graphs. The *Hamming graph K_q^d* is the Cartesian product of d copies of a complete graph on q vertices. We call the K_2^d the *d-dimensional hypercube*.

The cubicity of hypercubes is known to be in $\Theta\left(\frac{d}{\log d}\right)$. The lower bound is due to Chandran, Mannino and Oriolo [4] and the upper bound is due to Chandran and Sivadasan [6]. But we do not have such tight estimates on the boxicity of hypercubes. The only explicitly known upper bound is one of $O(d/\log d)$ which follows from the bound on cubicity since boxicity is bounded above by cubicity for all graphs. The only non-trivial lower bound is one of $\frac{1}{2}(\lceil \log \log d \rceil + 1)$ due to Chandran, Mathew and Sivadasan [5].

We make use of a non-trivial upper bound shown by Kostochka on the dimension of the partially ordered set (poset) formed by two neighbouring levels of a Boolean lattice [8] and a connection between boxicity and poset dimension established by Adiga, Bhowmick and Chandran in [1] to obtain the following result.

Theorem 2.5. *Let b_d be the largest dimension possible of a poset formed by two adjacent levels of a Boolean lattice over a universe of d elements. Then*

$$\tfrac{1}{2}b_d \leq \text{boxicity}(K_2^d) \leq 3b_d.$$

Furthermore, $\text{boxicity}(K_2^d) \leq 12\log d/\log\log d$.

Below, we extend the results on hypercubes to Hamming graphs.

Theorem 2.6. *Let K_q^d be the d-dimensional Hamming graph on the alphabet $[q]$ and let K_2^d be the d-dimensional hypercube. Then for $d \geq 2$,*

$$\begin{aligned}\log q &\leq \text{boxicity}(K_q^d) \leq \lceil 10\log q\rceil\, \text{boxicity}(K_2^d), \text{ and}\\ \log q &\leq \text{cubicity}(K_q^d) \leq \lceil 10\log q\rceil\, \text{cubicity}(K_2^d).\end{aligned}$$

Theorem 2.4, along with the bounds on boxicity and cubicity of Hamming graphs, gives the following corollary which is the main result in this article. The lower bound on the order of growth is due to the presence of K_2^d as an induced subgraph in the d-the Cartesian power of any non-trivial graph.

Corollary 2.7. *For any given graph G with at least one edge,*

$$\begin{aligned}&\text{boxicity}(G^{\square d}) \in O(\log d/\log\log d) \cap \Omega(\log\log d), \text{ and}\\ &\text{cubicity}(G^{\square d}) \in \Theta(d/\log d).\end{aligned}$$

2.3 Direct products

Theorem 2.8. *For graphs $G_1, \ldots, G_d$,*

$$\begin{aligned}\text{boxicity}(\times_{i=1}^d G_i) &\leq \text{boxicity}(\boxtimes_{i=1}^d G_i) + \text{boxicity}(\times_{i=1}^d K_{\chi_i}) \text{ and}\\ \text{cubicity}(\times_{i=1}^d G_i) &\leq \text{cubicity}(\boxtimes_{i=1}^d G_i) + \text{cubicity}(\times_{i=1}^d K_{\chi_i})\end{aligned}$$

where χ_i denotes the chromatic number of G_i, $i \in [d]$.

In the wake of Theorem 2.8, it is useful to estimate the boxicity and the cubicity of the direct product of complete graphs. Before stating our result on the same, we would like to discuss a few special cases. If $G = \times_{i=1}^d K_2$ then G is a perfect matching on 2^d vertices and hence has boxicity and cubicity equal to 1. If $G = K_q \times K_2$, then it is isomorphic to a graph obtained by removing a perfect matching from the complete bipartite graph with q vertices on each part. This is known as the *crown graph* and its boxicity is known to be $\lceil q/2\rceil$ [2].

Theorem 2.9. *Let $q_i \geq 2$ for each $i \in [d]$. Then,*

$$\frac{1}{2}\sum_{i=1}^{d}(q_i - 2) \leq \text{boxicity}\left(\times_{i=1}^{d} K_{q_i}\right) \leq \sum_{i=1}^{d} q_i, \text{ and}$$
$$\frac{1}{2}\sum_{i=1}^{d}(q_i - 2) \leq \text{cubicity}\left(\times_{i=1}^{d} K_{q_i}\right) \leq \sum_{i=1}^{d} q_i \log(n/q_i),$$

where $n = \Pi_{i=1}^{d} q_i$ is the number of vertices in $\times_{i=1}^{d} K_{q_i}$.

Corollary 2.10. *For graphs $G_1, \ldots, G_d$,*

$$\text{boxicity}(\times_{i=1}^{d} G_i) \leq \sum_{i=1}^{d}(\text{boxicity}(G_i) + \chi(G_i)).$$

Corollary 2.11. *For any given graph G, $boxicity(G^{\times d})$ is in $O(d)$ and there exist graphs for which it is in $\Omega(d)$.*

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