Locally-maximal embeddings of graphs in orientable surfaces

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1 Introduction

The set of all orientable cellular embeddings of a graph has the intrinsic structure of adjacency between embeddings based on elementary operations on rotation schemes. Several types of elementary operations were considered in the past, usually in proofs of interpolation theorems: moving a single arc within its local rotation, moving both ends of an edge in the respective local rotations, and interchanging two arcs in the local rotation at a given vertex, see [2,6,7,10]. We call these operations *rotation moves*. Each type of a rotation move gives rise to the structure of a *stratified graph* on the set of all embeddings of a given graph. Stratified graphs were studied by Gross, Rieper, and Tucker [5,6,8], although they were implicit already in the works of Duke [2] and Stahl [10]. Very little is known about stratified graphs in general, although their structure is crucial for understanding the entire system of all embeddings of a given graph. In the present paper we focus on embeddings that correspond to local maxima in stratified graphs. We call a cellular embedding of a graph into an orientable surface *locally maximal* if its genus cannot be raised by moving a single arc within its local rotation. Somewhat surprisingly, the concept of a locally-maximal embedding does not depend on which type of a move is taken as a basis for the stratified graph, indicating its important position in the hierarchy of graph embeddings between the minimum genus and the maximum genus.

The main results of this paper are (1) a characterisation of locally-maximal embeddings, (2) analysis of their relationship to the minimum and the maximum genus of a graph, and (3) a simple greedy 2-approxi-

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mation algorithm for maximum genus based on the notion of a locally-maximal embedding. Full proofs and additional results can be found in the forthcoming papers.

2 Fundamentals

The fundamental property of rotation moves is that any rotation move changes the number of faces of an embedding by $-2, 0$, or $+2$, changing the genus by $-1, 0$, or 1. This property is well known and was used in several proofs of the interpolation theorem for the genus range of a graph.

The following result asserts that whenever a vertex v is incident with at least three faces, there is a rotation move at v that merges three faces into one, thus raising the genus. In this regard, all three types of rotation moves display a similar behaviour.

Theorem 2.1. *If a vertex v is incident with at least three faces of an embedding Π, then there exists a move of an arc at v that merges three faces at v into one while leaving all other faces of Π intact. There also exists an interchange of two arcs at v that merges three faces at v into one and leaves all other faces intact.*

A particularly useful corollary of our proof of Theorem 2.1 is that moving any arc lying on the boundary of two distinct faces into a corner belonging to a third face merges these three faces into one.

The next result is the cornerstone of our theory of locally-maximal embeddings as it shows that the concept of a locally-maximal embedding is independent on which type of a move is chosen as its basis. At the same time, it provides a characterisation of locally-maximal embeddings in terms of the multiplicity of vertex-face incidences.

Theorem 2.2. *For any orientable embedding Π of a connected graph G the following statements are equivalent.*

(i) *The embedding Π is locally maximal.*
(ii) *The genus of Π cannot be raised by interchanging two arcs in a local rotation.*
(iii) *The genus of Π cannot be raised by moving any edge in the rotation of Π.*
(iv) *Every vertex of G is incident with at most two faces of Π.*

A natural question about locally-maximal embeddings concerns their distribution within the embedding range. We therefore define the *locally-maximal genus* of a graph G, $\gamma_L(G)$, as the minimum among the genera of all locally-maximal embeddings of G. For a graph G we further denote by $\mu(G)$ the largest number of pairwise disjoint circuits contained in G; this number is sometimes known as the *cycle packing number* of G. We also define the *reduced Betti number* $\beta'(G)$ by setting $\beta'(G) = \beta(G) - \mu(G)$; note that the reduced Betti number is nonnegative since $\mu(G) \leq \beta(G)$. The importance of these two invariants is explained by the following theorem.

Theorem 2.3. *The maximum number of faces in a locally-maximal embedding of a graph G does not exceed* $\mu(G) + 1$ *or* $\mu(G)$*, depending on whether* $\beta'(G)$ *is even or odd, respectively.*

The previous result enables us to prove that the relationship between the Betti number, the reduced Betti number, and the genus parameters of a graph is governed by the following inequalities.

Theorem 2.4. *The following inequalities hold for every connected graph G:*

(i) $\gamma(G) \leq \beta'(G)/2 \leq \gamma_L(G) \leq \gamma_M(G) \leq \beta(G)/2$

(ii) $\beta'(G)/2 \leq \gamma_L(G) \leq \gamma_M(G) \leq \beta'(G)$

(iii) $\gamma_M(G)/2 \leq \gamma_L(G) \leq \gamma_M(G)$

Graphs G for which $\gamma_M(G) = \lfloor \beta(G)/2 \rfloor$ are known as *upper-embeddable* graphs. With this analogy in mind we define a graph G to be *lower-embeddable* if $\gamma_L(G) = \lceil \beta'(G)/2 \rceil$. As with upper-embeddable graphs, many important classes of graphs are lower-embeddable.

Theorem 2.5. *All graphs in the following classes are lower embeddable: complete graphs* K_n *for all* $n \geq 1$*, complete bipartite graphs* $K_{m,n}$ *for all* $m, n \geq 1$*, complete equipartite graphs* $K_{n,\ldots,n}$ *for all* $n \geq 1$*, and hypercubes* Q_n *for all* $n \geq 1$.

A connected graph is called a *cycle-tree* if any two of its cycles are vertex disjoint. It turns out that planar locally-maximal embeddings admit a simple characterisation: A connected graph G has a planar locally-maximal embedding if and only if G is a cycle-tree.

3 Constructions

The proof of the next theorem is based on the well-known edge-addition technique of raising the genus by adding a pair of adjacent edges; for technical details of the method see for example [9], [3], or [1].

We need the following definitions to state and prove our result. A cycle-tree graph is called a *k-cycle-tree* if it contains exactly k cycles. A component of a graph is called *even* if it has even number of edges.

A face of an embedding is *spanning* if it is incident with each vertex of the graph. The idea of a spanning face was used in [1] to construct embeddings with genus $\gamma_M(G) - 1$;

Theorem 3.1. *Let G be a connected graph. If G has a spanning k-cycle-tree S such that $G - E(S)$ has only even components, then G has a locally-maximal embedding with $k + 1$ faces. In particular, $\gamma_L(G) \leq (\beta(G) - k)/2$.*

Sketch of a proof. Since each component of $G - E(S)$ is even, $G - E(S)$ has a partition $\mathcal{P}$ into pairs of adjacent edges (see for example [9, Lemma 4]). Let us arrange the pairs from $\mathcal{P}$ into a linear order, and let $\{e_i, f_i\}$ be the i-th pair. Consider the graphs

$$\begin{aligned} G_0 &= S, \\ G_i &= G_{i-1} \cup \{e_i, f_i\} \quad \text{for } i \geq 1. \end{aligned}$$

Assuming that the number of pairs in $\mathcal{P}$ is n, we get $G_n = G$. The proof is finished by using the edge-addition technique and employing induction to prove that for each $i \in \{0, 1, \ldots, n\}$ the graph G_i has a locally-maximal embedding with $i + 1$ faces, at least one of them being spanning. □

Although Theorem 3.1 cannot be used to construct all locally-maximal embeddings, it is often useful in determining the locally-maximal genus provided that one can obtain good lower bounds. In particular, Theorem 3.1 can be used to prove lower-embeddability in Theorem 2.5, where lower bounds follow from Theorem 2.3. The idea is as follows. Every lower-embeddable graph G satisfies $\gamma_L(G) = \lceil \beta'(G)/2 \rceil$. Therefore, it suffices to construct a spanning cycle-tree S of G with $\mu(G)$ or $\mu(G) - 1$ cycles such that $G - E(S)$ consists of even components. In many cases, the graph $G - E(S)$ is connected. This method is in its nature similar to the one used in the proofs of upper-embeddability, which are often carried out by finding a connected cotree of the given graph.

4 Algorithms

Let us start with Greedy-Max-Genus Algorithm described in Figure 4.1. The algorithm repeatedly increases the genus by employing suitable rotation moves. By part (iv) of Theorem 2.2, the genus of an embedding can be raised by a rotation move at a vertex v if and only if v is incident with at least three faces. Note that testing whether a vertex is incident with at least three faces, as well as finding and performing a rotation move increasing the genus if such a move exists, can be easily done in polynomial time. It follows that the running time of Greedy-Max-Genus Algorithm is polynomial. Moreover, by Theorem 2.2, the embedding output by Greedy-Max-Genus Algorithm is locally maximal. Since the genus of any locally-maximal embedding is at least $\gamma_L(G)$, and $\gamma_L(G) \geq \gamma_M(G)/2$ by Theorem 2.4, we obtain the following theorem.

Greedy-Max-Genus Algorithm	
Input:	A connected graph G
Output:	An embedding of G and its genus
	randomly choose an embedding Π of G
	while there is a rotation move increasing the genus of Π
	apply one of such rotation moves to Π
	output Π and the genus of Π

Figure 4.1. 2-approximation algorithm for maximum genus.

Theorem 4.1. *The Greedy-Max-Genus Algorithm from Figure 4.1 is a polynomial-time* 2*-approximation algorithm for maximum genus. Furthermore, the embeddings output by Greedy-Max-Genus Algorithm are precisely the locally-maximal embeddings.*

Part (iii) of Theorem 2.4 enables us to efficiently approximate also the locally-maximal genus using a polynomial-time algorithm computing the maximum genus of an arbitrary graph by Glukhov [4], or Furst et al. [3].

Theorem 4.2. *There is a polynomial-time* 2*-approximation algorithm for locally-maximal genus.* □

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