The Carathéodory number of the P_3 convexity of chordal graphs

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Abstract. If S is a set of vertices of a graph G, then the convex hull of S in the P_3-convexity of G is the smallest set $H_G(S)$ of vertices of G that contains S such that no vertex in $V(G) \setminus H_G(S)$ has at least two neighbors in S. The Carathéodory number of the P_3 convexity of G is the smallest integer c such that for every set S of vertices of G and every vertex u in $H_G(S)$, there is a set $F \subseteq S$ with $|F| \leq c$ and $u \in H_G(F)$. We describe a polynomial time algorithm to determine the Carathéodory number of the P_3 convexity of a chordal graph.

1 Introduction

Graph convexities are a well studied topic. For a finite, simple, and undirected graph G with vertex set $V(G)$, a *graph convexity* on $V(G)$ is a collection $\mathcal{C}$ of subsets of $V(G)$ such that

- $\emptyset$, $V(G) \in \mathcal{C}$ and
- $\mathcal{C}$ is closed under intersections.

The sets in $\mathcal{C}$ are called *convex sets* and the *convex hull* in $\mathcal{C}$ of a set S of vertices of G is the smallest set $H_{\mathcal{C}}(S)$ in $\mathcal{C}$ containing S.

Several well known graph convexities $\mathcal{C}$ are defined using some set $\mathcal{P}$ of paths of the underlying graph G. In this case, a subset S of vertices of G is convex, that is, belongs to $\mathcal{C}$, if for every path P in $\mathcal{P}$ whose endvertices belong to S also every vertex of P belongs to S. When $\mathcal{P}$ is the set of all shortest paths in G, this leads to the *geodetic convexity*

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[2, 8, 12, 14]. The *monophonic convexity* is defined by considering as $\mathcal{P}$ the set of all induced paths of G [9, 10, 13]. The set of all paths of G leads to the *all path convexity* [7]. Similarly, if $\mathcal{P}$ is the set of all triangle paths in G, then $\mathcal{C}$ is the *triangle path convexity* [6]. Here we consider the P_3 *convexity* of G, which is defined when $\mathcal{P}$ is the set of all paths of length two. The P_3 convexity was first considered for directed graphs [11,15–17]. For undirected graphs, the P_3 convexity was studied in [1,4,5].

A famous result about convex sets in $\mathbb{R}^d$ is *Carathéodory's theorem* [3]. It states that every point u in the convex hull of a set $S \subseteq \mathbb{R}^d$ lies in the convex hull of a subset F of S of order at most $d+1$. Let G be a graph and let $\mathcal{C}$ be a graph convexity on $V(G)$. The *Carathéodory number* of $\mathcal{C}$ is the smallest integer c such that for every set S of vertices of G and every vertex u in $H_{\mathcal{C}}(S)$, there is a set $F \subseteq S$ with $|F| \leq c$ and $u \in H_{\mathcal{C}}(F)$. A set S of vertices of G is a *Carathéodory set* of $\mathcal{C}$ if the set $\partial H_{\mathcal{C}}(S)$ defined as $H_{\mathcal{C}}(S) \setminus \bigcup_{u \in S} H_{\mathcal{C}}(S \setminus \{u\})$ is not empty. This notion allows an alternative definition of the *Carathéodory number* of $\mathcal{C}$ as the largest cardinality of a Carathéodory set of $\mathcal{C}$.

The Carathéodory number was determined for several graph convexities. The Carathéodory number of the monophonic convexity of a graph G is 1 if G is complete and 2 otherwise [10]. The Carathéodory number of the triangle path convexity of G is 2 whenever G has at least one edge [6]. It is known that the maximum Carathéodory number of the P_3 convexity of a multipartite tournament is 3 [16]. Some general results concerning the Carathéodory number of the P_3 convexity are shown in [1]. On the one hand, [1] contains efficient algorithms to determine the Carathéodory number of the P_3 convexity of trees and, more generally, block graphs. On the other hand, it is NP-hard to determine the Carathéodory number of the P_3 convexity of bipartite graphs [1].

In the present extended abstract we exclusively study the Carathéodory number of P_3 convexities of graphs. Since a graph G uniquely determines its P_3 convexity $\mathcal{C}$, we speak of a Carathéodory set of G and the Carathéodory number $c(G)$ of G. Furthermore, we write $H_G(S)$ and $\partial H_G(S)$ instead of $H_{\mathcal{C}}(S)$ and $\partial H_{\mathcal{C}}(S)$, respectively.

Our result is a polynomial time algorithm for the computation of the Carathéodory number of a chordal graph, which extends results from [1].

2 Results

The following result from [5] plays a central role in our approach.

Theorem 2.1 (Centeno *et al.* [5]). *If u and v are two vertices at distance at most 2 in a 2-connected chordal graph G, then $H_G(\{u, v\}) = V(G)$.*

Let G be a connected chordal graph. Let r be a vertex of G. Let the graph G' arise from G by adding edges between all pairs of vertices of G that lie in the same block of G. Let T be the breadth first search tree of G' rooted in r.

For a vertex u of G, let $V(u)$ denote the set of vertices of G that are either u or a descendant of u in T. By the definition of T, we have that $V(r) = V(G)$ and if u is a vertex of G that is distinct from r, then $V(u)$ is the union of $\{u\}$ and all vertex sets of components of G that do not contain r. Furthermore, the set of children of u in T is the set of all vertices of G that belong to $V(u)$ and lie in a common block of G with u. Note that a vertex of G that is distinct from r is a leaf of T if and only if it is no cut vertex of G.

In order to determine the Carathéodory number of G, we consider the following three values for every vertex u of G:

- $c_{(G,r)}(u)$ is the maximum cardinality of a set S with
  - $S \subseteq V(u)$ and
  - $u \in \partial H_{G[V(u)]}(S)$.
- $c'_{(G,r)}(u)$ is the maximum cardinality of a set S with
  - $S \subseteq V(u)$,
  - $u \notin H_{G[V(u)]}(S)$,
  - $|H_{G[V(u)]}(S) \cap N_G(u)| = 1$, and
  - $H_{G[V(u)]}(S) \cap N_G(u) \subseteq \partial H_{G[V(u)]}(S)$.
- $c''_{(G,r)}(u)$ is the maximum cardinality of a set S with
  - $S \subseteq V(u)$,
  - $u \in H_{G[V(u)]}(S)$, and
  - $H_{G[V(u)]}(S) \cap N_G[u] \subseteq \partial H_{G[V(u)]}(S)$.

Let $S_{(G,r)}(u)$, $S'_{(G,r)}(u)$, and $S''_{(G,r)}(u)$ denote sets of maximum cardinality satisfying the conditions in the above definitions of $c_{(G,r)}(u)$, $c'_{(G,r)}(u)$, and $c''_{(G,r)}(u)$, respectively, that is, for instance, $c_{(G,r)}(u) = |S_{(G,r)}(u)|$. Note that if u is a leaf of T, then no set as in the definition of $c'_{(G,r)}(u)$ exists. In this case, let $c'_{(G,r)}(u) = -\infty$ and let $S'_{(G,r)}(u)$ be undefined.

The following lemma describes recursions for $c_{(G,r)}(u)$, $c'_{(G,r)}(u)$, and $c''_{(G,r)}(u)$, which allow their efficient recursive computation.

Lemma 2.2. *Let G, r, and T be as above. Let u be a vertex of G.*

(a) $c_{(G,r)}(u)$ *is the maximum of the following values:*

(i) 1.

(ii) $c_{(G,r)}(v) + c_{(G,r)}(w)$, *where v and w are children of u in T and* $\mathrm{dist}_G(v, w) = 2$.

(iii) $c_{(G,r)}(v) + c'_{(G,r)}(w)$, *where v and w are children of u in T and* $\mathrm{dist}_G(v, w) = 1$.

(iv) $c''_{(G,r)}(v) + c''_{(G,r)}(w)$, *where v and w are children of u in T and* $\mathrm{dist}_G(v, w) = 1$.

(b) $c'_{(G,r)}(u)$ *is the maximum of the following values:*

(i) $-\infty$.

(ii) $c_{(G,r)}(u')$, *where u' is a child of u in T that is a neighbor of u in G.*

(c) $c''_{(G,r)}(u)$ *is the maximum of the following values:*

(i) 1.

(ii) $c_{(G,r)}(v) + c_{(G,r)}(w)$, *where v and w are children of u in T such that v and w are no neighbors of u in G and* $\mathrm{dist}_G(v, w) = 2$.

(iii) $c_{(G,r)}(v) + c'_{(G,r)}(w)$, *where v and w are children of u in T such that v is no neighbor of u in G and* $\mathrm{dist}_G(v, w) = 1$.

(iv) $c''_{(G,r)}(v) + c''_{(G,r)}(w)$, *where v and w are children of u in T such that v and w are no neighbors of u in G and* $\mathrm{dist}_G(v, w) = 1$.

Theorem 2.3. *The Carathéodory number of a chordal graph can be determined in polynomial time.*

Proof. Let G be a chordal graph. Since the Carathéodory number of G is the maximum of the Carathéodory numbers of the components of G, we may assume that G is connected. Since $c(G) = \max\{c_{(G,r)}(r) : r \in V(G)\}$, it suffices to argue that, for every vertex r of G, the value $c_{(G,r)}(r)$ can be determined in polynomial time. In fact, using the recursions given in Lemma 2.2, it is possible to determine $c_{(G,r)}(r)$ in polynomial time calculating the values $c_{(G,r)}(u)$, $c'_{(G,r)}(u)$, and $c''_{(G,r)}(u)$ for all vertices u of G in a bottom up fashion along the corresponding breadth first search tree T. This completes the proof. □

While the running time of the algorithm described in the proof of Theorem 2.3 is obviously polynomial, it is possible to reduce it below some immediate estimates, because many values are essentially calculated several times, that is, there are many triples (r, s, u) of vertices of G with $c_{(G,r)}(u) = c_{(G,s)}(u)$, $c'_{(G,r)}(u) = c'_{(G,s)}(u)$, and $c''_{(G,r)}(u) = c''_{(G,s)}(u)$.

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