Extended abstract for structure results for multiple tilings in 3D

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Abstract. We study multiple tilings of 3-dimensional Euclidean space by a convex body. In a multiple tiling, a convex body P is translated with a discrete multiset Λ in such a way that each point of $\mathbb{R}^d$ gets covered exactly k times, except perhaps the translated copies of the boundary of P. It is known that all possible multiple tilers in $\mathbb{R}^3$ are zonotopes. In $\mathbb{R}^2$ it was known by the work of M. Kolountzakis [9] that, unless P is a parallelogram, the multiset of translation vectors Λ must be a finite union of translated lattices (also known as quasi periodic sets). In that work [9] the author asked whether the same quasi-periodic structure on the translation vectors would be true in $\mathbb{R}^3$. Here we prove that this conclusion is indeed true for $\mathbb{R}^3$.

Namely, we show that if P is a convex multiple tiler in $\mathbb{R}^3$, with a discrete multiset Λ of translation vectors, then Λ has to be a finite union of translated lattices, unless P belongs to a special class of zonotopes. This exceptional class consists of two-flat zonotopes P, defined by the Minkowski sum of two 2-dimensional symmetric polygons in $\mathbb{R}^3$, one of which may degenerate into a single line segment. It turns out that rational two-flat zonotopes admit a multiple tiling with an aperiodic (non-quasi-periodic) set of translation vectors Λ. We note that it may be quite difficult to offer a visualization of these 3-dimensional non-quasi-periodic tilings, and that we discovered them by using Fourier methods.

The study of multiple tilings of Euclidean space began in 1936, when the famous Minkowski facet-to-facet conjecture [15] for classical tilings was extended to the setting of k-tilings with the unit cube, by Furtwängler [3]. Minkowski's facet-to-facet conjecture states that for any *lattice* tiling of $\mathbb{R}^d$ by translations of the unit cube, there exist at least two translated cubes that share a facet (face of co-dimension 1). The conjecture was strengthened by Furtwängler [3] who conjectured the same conclusion for any *multiple* lattice tiling.

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To define a multiple tiling, suppose we translate a convex body P with a discrete multiset Λ, in such a way that each point of $\mathbb{R}^d$ gets covered exactly k times, except perhaps the translated copies of the boundary of P. We then call such a body a k-tiler, and such an action has been given the following names in the literature: a **k-tiling**, a **tiling at level** k, a **tiling with multiplicity** k, and sometimes simply a **multiple tiling**. We may use any of these synonyms here, and we immediately point out, for polytopes P, a trivial but useful algebraic equivalence for a tiling at level k:

$$\sum_{\lambda \in \Lambda} \mathbf{1}_{P+\lambda}(x) = k, \tag{1}$$

for almost all $x \in \mathbb{R}^d$, where $\mathbf{1}_P$ is the indicator function of the polytope P.

Furtwängler's conjecture was disproved by Hajós [7] for dimension larger than 3 and for $k \geq 9$ while Furtwängler himself [3] proved it for dimension at most 3. Hajós [8] also proved Minkowski's conjecture in all dimensions. The ideas of Furtwängler were subsequently extended (but still restricted to cubes) by the important work of Perron [16], Robinson [17], Szabó [21], Gordon [4] and Lagarias and Shor [11]. These authors showed that for some levels k and dimensions d and under the lattice assumption as well as not, a facet-to-facet conclusion for k-tilings is true in $\mathbb{R}^d$, while for most values of k and d it is false.

It was known to Bolle [2] that in $\mathbb{R}^2$, every k-tiling convex polytope has to be a centrally symmetric polygon, and using combinatorial methods Bolle [2] gave a characterization for all polygons in $\mathbb{R}^2$ that admit a k-tiling with a *lattice* Λ of translation vectors. Kolountzakis [10] proved that if a convex polygon P tiles $\mathbb{R}^2$ multiply with *any* discrete multiset Λ, then Λ must be a *finite union of two-dimensional lattices*. The ingredients of Kolountzakis' proof include the idempotent theorem for the Fourier transform of a measure. Roughly speaking, the idempotent theorem of Meyer [14] tells us that if the square of the Fourier transform of a measure is itself, then the support of the measure is contained in a finite union of lattices.

A multiple tiling is called **quasi-periodic** if its multiset of discrete translation vectors Λ is a finite union of translated lattices, not necessarily all of the same dimension.

Theorem (Kolountzakis, 2002 [9]). *Suppose that K is a symmetric convex polygon which is not a parallelogram. Then K admits only quasi-periodic multiple tilings if any.*

In this work, we extend this result to $\mathbb{R}^3$, and we also find a fascinating class of polytopes analogous to the parallelogram of the theorem above. To describe this class, we first recall the definition of a **zonotope**, which is the Minkowski sum of a finite number of line segments. In other words, a zonotope equals a translate of $[-\mathbf{v}_1, \mathbf{v}_1] + \cdots + [-\mathbf{v}_N, \mathbf{v}_N]$, for some positive integer N and vectors $\mathbf{v}_1, \ldots, \mathbf{v}_N \in \mathbb{R}^d$. A zonotope may equivalently be defined as the projection of some l-dimensional cube. A third equivalent condition is that for a d-dimensional zonotope, all of its k-dimensional faces are centrally symmetric, for $1 \leq k \leq d$. For example, the zonotopes in $\mathbb{R}^2$ are the centrally symmetric polygons.

We shall say that a polytope $P \subseteq \mathbb{R}^3$ is a **two-flat zonotope** if P is the Minkowski sum of $n + m$ line segments which lie in the union of two different two-dimensional subspaces H_1 and H_2. In other words, H_1 contains n of the segments and H_2 contains m of the segments (if one of the segments belongs to both H_1 and H_2 we list it twice, once for each plane). Equivalently, P may be thought of as the Minkowski sum of two 2-dimensional symmetric polygons one of which may degenerate into a single line segment.

It turns out that if P is a rational two-flat zonotope, then P admits a k-tiling with a non-quasi-periodic set of translation vectors Λ. For some of the classical study of 1-tilings, and their interesting connections to zonotopes, the reader may refer to the work of [12,13,19,23], and [1]. Here we find it very useful to use the intuitive language of distributions [18,20] in order to think – and indeed discover – facts about k-tilings. To that end we introduce the distribution (which is locally a measure)

$$\delta_\Lambda := \sum_{\lambda \in \Lambda} \delta_\lambda, \tag{2}$$

where δ_λ is the Dirac delta function at $\lambda \in \mathbb{R}^d$. To develop some intuition, we may check formally that

$$\delta_\Lambda * \mathbf{1}_P = \sum_{\lambda \in \Lambda} \delta_\lambda * \mathbf{1}_P = \sum_{\lambda \in \Lambda} \mathbf{1}_{P+\lambda},$$

so that from the first definition (1) of k-tiling, we see that a polytope P is a k-tiler if and only if

$$\delta_\Lambda * \mathbf{1}_P = k. \tag{3}$$

Suppose the polytope P tiles multiply with the translates $\Lambda \subseteq \mathbb{R}^d$. We will need to understand some basic facts about how the Λ points are distributed.

For any symmetric polytope P, and any face $F \subset P$, we define F^- to be the face of P symmetric to F with respect to P's center of symmetry. We call F^- the **opposite face** of F. We use the standard convention of boldfacing all vectors, to differentiate between $\mathbf{v}$ and v, for example. We furthermore use the convention that $[\mathbf{e}]$ denotes the 1-dimensional line segment from 0 to the endpoint of the vector $\mathbf{e}$. Whenever it is clear from context, we will also write $[e]$ to denote the same line segment - for example, in the case that e denotes an edge of a polytope.

Suppose $P \in \mathbb{R}^3$ is a zonotope (symmetric polytope with symmetric facets). A collection of four edges of P is called a **4-legged-frame** if whenever e is one of the edges then there exist two vectors $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ such that the four edges are

$$[e],\ [e] + \boldsymbol{\tau}_1, [e] + \boldsymbol{\tau}_2 \text{ and } [e] + \boldsymbol{\tau}_1 + \boldsymbol{\tau}_2,$$

and such that the edges $[e]$ and $[e] + \boldsymbol{\tau}_1$ belong to the same face of P and the edges $[e] + \boldsymbol{\tau}_2$ and $[e] + \boldsymbol{\tau}_1 + \boldsymbol{\tau}_2$ belong to the opposite face.

With the notion of 4-legged frames, we can introduce the following so-called **intersection property**, which plays an important role in the proof of the main theorem.

Suppose P is a k-tiler with a discrete multiset Λ, in $\mathbb{R}^3$. We say that the intersection property holds, if

$$\bigcap_{e,\tau_1.\tau_2} \left(\mathbf{e}^\perp \cup \boldsymbol{\tau}_1^\perp \cup \boldsymbol{\tau}_2^\perp\right) = \{0\}, \tag{4}$$

where the intersection above is taken over all sets of 4-legged frames of P.

Recently, a structure theorem for convex k-tilers in $\mathbb{R}^d$ was found, and is as follows.

Theorem (Gravin, Robins, Shiryaev 2012 [6]). *If a convex polytope k-tiles $\mathbb{R}^d$ by translations, then it is centrally symmetric and its facets are centrally symmetric.*

This theorem generalizes the theorem for 1-tilers by Minkowski [15]. One-tiler case was extensively studied in the past, and the complete characterization for 1-tilers was given independently by Venkov [22] and McMullen [13].

It follows immediately from the latter theorem that a k-tiler $P \subset \mathbb{R}^3$ is necessarily a zonotope. The following theorem extends the result of Kolountzakis [9] from $\mathbb{R}^2$ to $\mathbb{R}^3$, providing a structure theorem for multiple tilings by polytopes in three dimensions.

The main result here is the following theorem [5]:

Main Theorem. *Suppose a polytope P k-tiles $\mathbb{R}^3$ with a discrete multiset Λ, and suppose that P is not a two-flat zonotope. Then Λ is a finite union of translated lattices.*

Although the proof is involved, one of the main ideas is to compute the Fourier transform of any 4-legged frame of a polytope, and show that its zeros form a certain countable union of hyperplanes. Another main idea is to show that if the intersection property holds for P, then δ_Λ has discrete support.

We also show that each rational two-flat zonotope admits a very peculiar non-quasi-periodic k-tiling. We note that it may be quite difficult to offer a visualization of these 3-dimensional non-quasi-periodic tilings, and that we discovered them by using Fourier methods.

Open Questions

The proof of the main theorem does not directly generalize to dimensions higher that 3, since k-tilers in these dimensions are no longer all zonotopes. So it might require some new ideas and methods to deal with the higher dimension case, and we propose this as a primary direction for future work.

It would also be very helpful to generalize Bolle's characterization of 2-dimensional lattice k-tilers to higher dimensions.

Another important topic of future research would be to generalize the Venkov-McMullen characterization [13] from 1-tilers in $\mathbb{R}^n$ to k-tilers. It is already established that any k-tiler in $\mathbb{R}^n$ is centrally symmetric and has centrally symmetric facets, and it is reasonable to conjecture that it is enough to add one more condition on co-dimension 2 facets to get a complete characterization.

Finally, it would be interesting to prove or disprove the following conjecture: if a polytope has a quasi-periodic multiple tiling, then it also has a tiling with just one lattice.

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