Polynomial gap extensions of the Erdős–Pósa theorem

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Abstract. Given a graph H, we denote by $\mathcal{M}(H)$ all graphs that can be contracted to H. The following extension of the Erdős–Pósa Theorem holds: for every h-vertex planar graph H, there exists a function $f_H : \mathbb{N} \to \mathbb{N}$ such that every graph G, either contains k disjoint copies of graphs in $\mathcal{M}(H)$, or contains a set of $f_H(k)$ vertices meeting every subgraph of G that belongs in $\mathcal{M}(H)$. In this paper we prove that f_H can be polynomially (upper) bounded for every graph H of pathwidth at most 2 and, in particular, that $f_H(k) = 2^{O(h^2)} \cdot k^2 \cdot \log k$. As a main ingredient of the proof of our result, we show that for every graph H on h vertices and pathwidth at most 2, either G contains k disjoint copies of H as a minor or the treewidth of G is upper-bounded by $2^{O(h^2)} \cdot k^2 \cdot \log k$. We finally prove that the exponential dependence on h in these bounds can be avoided if $H = K_{2,r}$. In particular, we show that $f_{K_{2,r}} = O(r^2 \cdot k^2)$.

1 Introduction

In 1965, Paul Erdős and Lajos Pósa proved that every graph that does not contain k disjoint cycles, contains a set of $O(k \log k)$ vertices meeting all its cycles [6]. Moreover, they gave a construction asserting that this bound is tight. This classic result can be seen as a "loose" min-max relation between covering and packing of combinatorial objects. Various extensions of this result, referring to different notions of packing and covering, attracted the attention of many researchers in modern Graph Theory (see, *e.g.* [1,11]).

Given a graph H, we denote by $\mathcal{M}(H)$ the set of all graphs that can be contracted to H (*i.e.* if $H' \in \mathcal{M}(H)$, then H can be obtained from H' after contracting edges). We call the members of $\mathcal{M}(H)$ *models* of

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H. Then the notions of covering and packing can be extended as follows: we denote by $\mathbf{cover}_H(G)$ the minimum number of vertices that meet every model of H in G and by $\mathbf{pack}_H(G)$ the maximum number of mutually disjoint models of H in G. We say that a graph H *has the Erdős–Pósa Property* if there exists a function $f_H : \mathbb{N} \to \mathbb{N}$ such that for every graph G,

$$\text{if } k = \mathbf{pack}_H(G), \text{ then } k \leqslant \mathbf{cover}_H(G) \leqslant f_H(k) \tag{1.1}$$

We will refer to f_H as the *gap* of the Erdős–Pósa Property. Clearly, if $H = K_3$, then (1.1) holds for $f_{K_3} = O(k \log k)$ and the general question is to find, for each instantiation of H, the best possible estimation of the gap f_H, if it exists.

It turns out that H has the Erdős–Pósa Property if and only if H is a planar graph. This beautiful result appeared as a byproduct of the Graph Minors series of Robertson and Seymour. In particular, it is a consequence of the grid-exclusion theorem, proved in [14] (see also [3]).

Proposition 1.1. *There is a function $g \colon \mathbb{N} \to \mathbb{N}$ such that if a graph excludes an r-vertex planar graph R as a minor, then its treewidth is bounded by $g(r)$.*

In [14] Robertson, Seymour, and Thomas conjectured that g is a low degree polynomial function. Currently, the best known bound for g is $g(k) = 2^{O(k \log k)}$ and follows from [4] and [13] (see also [12, 14] for previous proofs and improvements). As the function g is strongly used in the construction of the function f_H in (1.1), the best, so far, estimation for f_H is far from being exponential in general. This initiated a quest for detecting instantiations of H where a polynomial gap f_H can be proved.

The first result in the direction of proving polynomial gaps for the Erdős–Pósa Property appeared in [9] where H is the graph θ_c consisting of two vertices connected by c multiple edges (also called *c-pumpkin graph*). In particular, in [9] it was proved that $f_{\theta_c}(k) = O(c^2 k^2)$. More recently Fiorini, Joret, and Sau optimally improved this bound by proving that $f_{\theta_c}(k) \leqslant c_t \cdot k \cdot \log k$ for some computable constant c_t depending on c [8]. In [15] Fiorini, Joret, and Wood proved that if T is a tree, then $f_T(k) \leqslant c_T \cdot k$ where c_T is some computable constant depending on T. Finally, very recently, Fiorini [7] proved that $f_{K_4} = O(k \log k)$.

Our main result is a polynomial bound on f_H for a broad family of planar graphs, namely those of pathwidth at most 2. We prove the following:

Theorem 1.2. *If H is an h-vertex graph of pathwidth at most* 2 *and $h > 5$, then* (1.1) *holds for $f_H(k) = 2^{O(h^2)} \cdot k^2 \cdot \log k$.*

Note that the contribution of h in f_H is exponential. However, such a dependence can be waived when we restrict H to be $K_{2,r}$. Our second result is the following:

Theorem 1.3. *If* $H = K_{2,r}$*, then* (1.1) *holds for* $f_H(k) = O(r^2 \cdot k^2)$*.*

Both results above are based on a proof of Proposition 1.1, with polynomial g, for the cases where R consists of k disjoint copies of H and H is either a graph of pathwidth at most 2 or $H = K_{2,3}$ (Theorems 2.1 and 2.2 respectively). For this, we follow an approach that makes strong use of the k-mesh structure introduced by Diestel *et al*. [4] in their proof of Proposition 1.1. Our proof indicates that, when excluding copies of some graph of pathwidth at most 2, the entangled machinery of [4] can be partially modified so that polynomial bounds on treewidth are possible. Finally, these bounds are then "translated" to polynomial bounds for the Erdős–Pósa gap using a technique developed in [10] (see also [9]).

Definitions and preliminaries. All graphs in this paper are simple, finite and undirected and logarithms are binary. We use standard notation in Graph Theory. We define $k \cdot H$ as the graph obtained if we take k disjoint copies of H.

Treewidth. A *tree decomposition* of a graph G is a tree T whose vertices are some subsets of $V(G)$ such that: (i) $\bigcup_{X \in V(T)} X = V(G)$, (ii) for every edge e of G there is a vertex of T containing both end of e, and (iii) for all $v \in V(G)$, the subgraph of T induced by $\{X \in V(T),\ v \in X\}$ is connected. The *width* of a tree decomposition T is defined as equal to $\max_{X \in V(T)} |X| - 1$. The *treewidth* of G, written $\mathbf{tw}(G)$, is the minimum width of any of its tree decompositions. The *pathwidth* of G, written $\mathbf{pw}(G)$, is defined as the treewidth if we consider paths instead of trees.

2 Excluding packings of planar graphs

Theorems 1.2 and 1.3 follow combining the two following results with the machinery introduced in [10] (see also [9]). They have independent interest as they detect cases of Theorem 1.1 where g depends polynomially on k. In this extended abstract we only sketch the proof of Theorem 2.1.

Theorem 2.1. *Let* H *be a graph of pathwidth at most 2 on* $r > 5$ *vertices. If* G *does not contain* k *disjoint copies of* H *as minor then* $\mathbf{tw}(G) \leqslant 2^{O(r^2)} \cdot k^2 \cdot \log 2k$.

Theorem 2.2. *For every positive integer r, if G does not contain k disjoint copies of $K_{2,r}$ as minor then* $\mathbf{tw}(G) = O(r^2k^2)$.

Proof of theorem 2.1. (sketch) We prove the contrapositive. Let k be a integer, H a graph on $r > 5$ vertices and of pathwidth at most 2 and G a graph. It can easily be proved that $H \leqslant_{\mathrm{m}} \Xi_r$ where Ξ_r is the graph obtained by a $(r \times 2)$ if we subdivide once the "horizontal" edges. If we show that G contains k disjoint copies of Ξ_r as minors then we are done. Let $g\colon \mathbb{N} \to \mathbb{N}$ such that $g(k,r) = k^2 \log 2k \left(180 \cdot 2^{r(r-2)} - 24 \cdot 2^{\frac{1}{2}r(r-2)}\right) + 6 \cdot 2^{\frac{1}{2}r(r-2)} - 1$. We prove that for all graph G, $\mathbf{tw}(G) \geqslant g(k,r)$ implies that $G \geqslant_{\mathrm{m}} k \cdot \Xi_r$. Let k and $r > 5$ be two positive integers and assume that $\mathbf{tw}(G) \geqslant g(k,r)$. We examine below the case where $\delta_c(G) < c \cdot 3rk\sqrt{\log 3rk}$ (the case $\delta_c(G) \geqslant c \cdot 3rk\sqrt{\log 3rk}$ is ommited). Observe that $c \cdot 3rk\sqrt{\log 3rk} < c \cdot 3r\sqrt{\log 6r} \cdot k\sqrt{\log 2k}$. Let $k_0 = k\sqrt{\log 2k}$ and $r_0 = 3 \cdot 2^{\frac{r(r-2)}{2}}$, and remark that $k_0 \geqslant k$ and, $r_0 \geqslant c \cdot 3r\sqrt{\log 6r}$ (remember that $c \leqslant 648$ and $r > 5$). With these notations, we have $\delta_c(G) < 2k_0r_0$. We will show that $G \geqslant_{\mathrm{m}} k_0 \cdot \Xi_r$ from which yields that $G \geqslant_{\mathrm{m}} k \cdot \Xi_r$. By assumption, $\mathbf{tw}(G) \geqslant g(k,r)$. Using the results of [4], we can prove that G contains $2k_0$ subsets $X_1, \ldots, X_{2k_0}$ of $V(G)$ and a set $\mathcal{P}$ of $k_0r_0 = 3k_0 \cdot 2^{\frac{r(r-2)}{2}}$ disjoint paths of length at least 2 in G such that (i) $\forall i \in [\![1, 2k_0]\!]$, X_i is of size $r_0 = 3 \cdot 2^{\frac{r(r-2)}{2}}$ and is connected in G by a tree T_i using the elements of some set $A \subseteq V(G)$, (ii) any path in $\mathcal{P}$ has one of its ends in some X_i with $i \in [\![1, k_0]\!]$, the other end in X_{2i} and its internal vertices are in none of the X_l, for all $l \in [\![1, 2k_0]\!]$, nor in A, and (iii) $\forall i, j \in [\![1, 2k_0]\!]$, $i \neq j \Rightarrow T_i \cap T_j = \emptyset$.

We assume that for all $i \in [\![1, 2k_0]\!]$, $X_i = \{v \in V(T_i),\ \deg_T(v) \leqslant 2\}$. It is easy to come down to this case by considering the minor of G obtained after deleting in T_i the leaves that are not in X_i and contracting one edge meeting a vertex of degree 2 which is not in X while such a vertex exists. As T_i is a ternary tree, one can easily prove that for all $i \in [\![1, 2k_0]\!]$, T_i contains a path containing $2\log\frac{2}{3}|X_i| = (r-1)^2 + 1$ vertices of X_i. Let us call P_i such a path whose two ends are in X_i. Let us consider now the paths $\{P_i\}_{i \in [\![1, 2k_0]\!]}$ and the paths that link the elements of different P_i's. For each path $i \in [\![1, 2k_0]\!]$, we choose in P_i one end vertex (remember that both are in X_i) that we name $p_{i,0}$. We follow P_i from this vertex and we denote the other vertices of $P_i \cap X_i$ by $p_{i,1}, p_{i,1}, \ldots, p_{i,(r-1)^2}$ in this order. The *corresponding vertex* of some vertex $p_{i,j}$ of $P_i \cap X_i$ (for $i \in [\![1, k_0]\!]$) is defined as the vertex of $P_{2i} \cap X_{2i}$ to which $p_{i,j}$ is linked to by a path of $\mathcal{P}$. As said before, the sets $\{P_i \cap X_i\}_{i \in [\![1, 2k_0]\!]}$ are of size $(r-1)^2 + 1$. According to [5], one can find for all $i \in [\![1, k_0]\!]$ a subsequence of length r in $p_{i,0}, p_{i,1}, \ldots, p_{i,(r-1)^2}$, such that the corresponding vertices in X_{2i} of this sequence are either in the same order (with respect to the subscripts of the names of the vertices), or in reverse order. For all $i \in [\![1, k_0]\!]$, this subsequence, its corresponding vertices and the vertices of the paths that link them together forms a Ξ_r model. We have thus k_0 models of Ξ_r in G, that gives us k disjoint models of Ξ_r in G (since $k \leqslant k_0$). We showed that for all k and $r > 5$ positive integers, if a graph G has $\mathbf{tw}(G) \geqslant g(k,r)$, then $G \geqslant_{\mathrm{m}} k \cdot \Xi_r$. Consequently, if G has treewidth at least $g(k,r)$, then G contains k disjoint copies of H and we are done. □

Postscript. Very recently, the general open problem of estimating $f_H(k)$ when H is a general planar graph has been tackled in [2]. Moreover, very recently, using the results of [13] we were able to improve both Theorems 2.1 and 1.3 by proving polynomial (on both k and $|V(H)|$) bounds for more general instantiations of H.

References

[1] E. Birmelé, J. A. Bondy and B. A. Reed, *The Erdős–Pósa property for long circuits*, Combinatorica **27** (2007), 135–145.

[2] C. Chekuri and J. Chuzhoy, *Large-treewidth graph decompositions and applications*, In: “45st Annual ACM Symposium on Theory of Computing”, (STOC 2013), 2013.

[3] R. Diestel, “Graph Theory”, volume 173 of *Graduate Texts in Mathematics*, Springer-Verlag, Heidelberg, fourth edition, 2010.

[4] R. Diestel, T. R. Jensen, K. Yu. Gorbunov and C. Thomassen, *Highly connected sets and the excluded grid theorem*, J. Combin. Theory Ser. B **75** (1) (1999), 61–73.

[5] P. Erdős and G. Szekeres, *A combinatorial problem in geometry*, In: “Classic Papers in Combinatorics”, Ira Gessel and Gian-Carlo Rota (eds.), Modern Birkhäuser Classics, Birkhäuser Boston, 1987, 49–56.

[6] P. Erdős and L. Pósa, *On independent circuits contained in a graph*, Canad. J. Math. **17** (1965), 347–352.

[7] S. Fiorini, T. Huyhn and G. Joret, personal communication, 2013.

[8] S. Fiorini, G. Joret and I. Sau, *Optimal Erdős–Pósa property for pumpkins*, Manuscript, 2013.

[9] F. V. Fomin, D. Lokshtanov, N. Misra, G. Philip and S. Saurabh, *Quadratic upper bounds on the Erdős–Pósa property for a generalization of packing and covering cycles*, Journal of Graph Theory, to appear in 2013.

[10] F. V. Fomin, S. Saurabh and D. M. Thilikos, *Strengthening Erdős–Pósa property for minor-closed graph classes*, Journal of Graph Theory **66** (3) (2011), 235–240.
[11] J. Geelen and K. Kabell, *The Erdős–Pósa property for matroid circuits*, J. Comb. Theory Ser. B **99** (2) (2009), 407–419.
[12] Ken-ichi Kawarabayashi and Y. Kobayashi, *Linear min-max relation between the treewidth of H-minor-free graphs and its largest grid*, In: "29th Int. Symposium on Theoretical Aspects of Computer Science (STACS 2012)", Vol. 14 of *LIPIcs*, Dagstuhl, Germany, 2012, 278–289.
[13] A. Leaf and P. Seymour, *Treewidth and planar minors*, Manuscript, 2012.
[14] N. Robertson and P. D. Seymour, *Graph minors. V. Excluding a planar graph*, J. Combin. Theory Series B **41** (2) (1986), 92–114.
[15] D. R. Wood Samuel Fiorini and G. Joret, *Excluded forest minors and the Erdős-Pósa property*, Technical report, Cornell University, 2012.