An analogue of the Erdős-Ko-Rado theorem for multisets

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Abstract. We verify a conjecture of Meagher and Purdy [4] by proving that if $1 \le t \le k$, $2k - t \le n$ and $\mathcal{F}$ is a family of *t-intersecting k-multisets* of $[n]$, then

$$|\mathcal{F}| \le AK(n + k - 1, k, t),$$

where $AK(n, k, t) := max_i |\mathcal{A}_{n,k,t,i}|$ with $\mathcal{A}_{n,k,t,i} := \{A : A \subseteq [n], |A| = k, |A \cap [t + 2i]| \ge t + i\}$.

1 Introduction

1.1 Definitions, notation

Let n and l be positive integers and let

$$M(n, l) = \{(i, j) : 1 \le i \le n, 1 \le j \le l\}$$

be an $n \times l$ rectangle. We call $A \subseteq M(n, l)$ a *k-multiset* if the cardinality of A is k and $(i, j) \in A$ implies $(i, j') \in A$ for all $j' \le j$. We think of multisets as sets with multiplicities, but it helps finding short and precise notation if we identify them with these special subsets of the rectangle. We denote the multiplicity of i in F by $m(i, F)$, *i.e.* $m(i, F) := \max\{s : (i, s) \in F\}$ ($m(i, F) \le l$ by definition).

Let $\mathcal{F}$ be a family of *k-multisets* of $M(n, l)$. We call $\mathcal{F}$ *t-intersecting* if $t \le |F_1 \cap F_2|$ for all $F_1, F_2 \in \mathcal{F}$. Let

$$\mathcal{M}(n, l, k, t) = \{\mathcal{F} : \mathcal{F} \text{ is t-intersecting set of k-multisets of } M(n, l)\},$$

i.e. the class of t-intersecting families of k-multisets.

Let $\mathcal{F} \in \mathcal{M}(n, l, k, t)$. We call $T \subseteq M(n, l)$ a *t-kernel* for $\mathcal{F}$ if $|F_1 \cap F_2 \cap T| \ge t$ for all $F_1, F_2 \in \mathcal{F}$.

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1.2 History

Let us briefly summarize some results using our notation.

Theorem 1.1 (Erdős, Ko, Rado, [3]). *If $n \geq 2k$ and $\mathcal{F} \in \mathcal{M}(n, 1, k, 1)$ then*

$$|\mathcal{F}| \leq \binom{n-1}{k-1}.$$

If $n > 2k$, then equality holds if and only if all members of $\mathcal{F}$ contain a fixed element of $[n]$.

They also proved that if n is large enough, every member of the largest t-intersecting family of sets contains a fixed t-element set, but did not give the optimal threshold. Frankl [5] showed for $t \geq 15$ and Wilson [6] for every t that the optimal threshold is $n = (k - t + 1)(t + 1)$. Finally, Ahlswede and Khachatrian [1] determined the maximum families for all values of n.

Theorem 1.2 (Ahlswede, Khachatrian [1]). *Let $t \leq k \leq n$ and $\mathcal{A}_{n,k,t,i} = \{A : A \subseteq [n], |A| = k, |A \cap [t + 2i]| \geq t + i\}$. If $\mathcal{F} \in \mathcal{M}(n, 1, k, t)$, then*

$$|\mathcal{F}| \leq max_i |\mathcal{A}_{n,k,t,i}| = AK(n, k, t).$$

Theorem 1.3 (Meagher, Purdy [4]). *If $n \geq k+1$ and $\mathcal{F} \in \mathcal{M}(n, k, k, 1)$, then*

$$|\mathcal{F}| \leq \binom{n+k-2}{k-1}.$$

If $n > k + 1$, then equality holds if and only if all members of $\mathcal{F}$ contain a fixed element of $M(n, k)$.

1.3 Conjectures

Brockman and Kay stated the following conjecture [2]:

Conjecture 1.4 ([2], Conjecture 5.2). There is $n_0(k, t)$ such that if $n \geq n_0(k, t)$ and $\mathcal{F} \in \mathcal{M}(n, k, k, t)$, then

$$|\mathcal{F}| \leq \binom{n+k-t-1}{k-t}.$$

Furthermore, equality is achieved if and only if each member of $\mathcal{F}$ contains a fixed t-multiset of $M(n, k)$.

Meagher and Purdy also gave a possible candidate for the threshold $n_0(k, t)$.

Conjecture 1.5 ([4], Conjecture 4.1). Let k, n and t be positive integers with $t \leq k, t(k-t)+2 \leq n$ and $\mathcal{F} \in \mathcal{M}(n,k,k,t)$, then

$$|\mathcal{F}| \leq \binom{n+k-t-1}{k-t}.$$

Moreover, if $n > t(k-t)+2$, then equality holds if and only if all members of $\mathcal{F}$ contain a fixed t-multiset of $M(n,k)$.

Note that if $n < t(k-t)+2$, then the family consisting of all multisets which contain a fixed t-multiset of $M(n,k)$ still has cardinality $\binom{n+k-t-1}{k-t}$, but cannot be the largest. Indeed, if we fix a $t+2$-multiset T and consider the family of the multisets F with$|F \cap T| \geq t+1$, we get a larger family.

1.4 Results

The main idea of our proof is the following: instead of the well-known *left-compression* operation, which is a usual method in the theory of intersecting families, we define (in two different ways) an operation on $\mathcal{M}(n,l,k,t)$ which can be called a kind of *down-compression*.

Theorem 1.6. *Let* $1 \leq t \leq k$, $2k-t \leq n$ *and* l *be arbitrary. There exists*

$$f : \mathcal{M}(n,l,k,t) \to \mathcal{M}(n,l,k,t)$$

satisfying the following properties:

(i) $|\mathcal{F}| = |f(\mathcal{F})|$ *for all* $\mathcal{F} \in \mathcal{M}(n,l,k,t)$;
(ii) $M(n,1)$ *is a* t*-kernel for* $f(\mathcal{F})$.

Using Theorem 1.6 we prove the following theorem which not only verifies Conjecture 1.5, but also gives the maximum cardinality of t-intersecting families of multisets in the case $2k-t \leq n \leq t(k-t)+2$.

Theorem 1.7. *Let* $1 \leq t \leq k$ *and* $2k-t \leq n$. *If* $\mathcal{F} \in \mathcal{M}(n,k,k,t)$ *then*

$$|\mathcal{F}| \leq AK(n+k-1,k,t).$$

2 Concluding remarks

Note that for a family $\mathcal{A}_{n+k-1,k,t,i}$ we can define a t-intersecting family of k-multisets in $M(n,k)$, hence the bound given in Theorem 1.7 is sharp. However, we do not know any nontrivial bounds in case $n < 2k-t$.

Another interesting problem is the case $l < k$. Theorem 1.6 gives us a small t-kernel, but the proof of Theorem 1.7 does not work in this case.

References

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