Snarks with large oddness and small number of vertices

Robert Lukot'ka[1], Edita Máčajová[2], Ján Mazák[1]
and Martin Škoviera[2]

Abstract. We estimate the minimum number of vertices of a cubic graph with given oddness and cyclic connectivity. We show that a 2-connected cubic graph G with oddness $\omega(G)$ different from the Petersen graph has order at least $5.41\,\omega(G)$, and for any integer k with $2 \leq k \leq 6$ we construct an infinite family of cubic graphs with cyclic connectivity k and small oddness ratio $|V(G)|/\omega(G)$. For cyclic connectivity 2, 4, 5, and 6 we improve the upper bounds on the oddness ratio of snarks to 7.5, 13, 25, and 99 from the known values 9, 15, 76, and 118, respectively. We also construct a cyclically 4-connected snark of girth 5 with oddness 4 and order 44, improving the best previous value of 46.

1 Introduction

Snarks are connected bridgeless cubic graphs with chromatic index 4, sometimes required to satisfy additional conditions, such as cyclic 4-edge-connectivity and girth at least five, to avoid triviality. There are several important conjectures in graph theory where snarks are the principal obstacle: if true for snarks, they would hold for all graphs. Some of the conjectures have been verified for snarks that are close to 3-edge-colourable graphs. For example, the 5-flow conjecture is known to hold for snarks with oddness at most 2, and the cycle double cover conjecture has been verified for snarks with oddness at most 4 (see [4,5]). However, snarks with large oddness remain potential counterexamples to these and other conjectures, and therefore merit further investigation.

[1] Trnavská univerzita. Priemyselná 4, 918 43 Trnava.
Email: robert.lukotka@truni.sk, jan.mazak@truni.sk

[2] Univerzita Komenského, Mlynská dolina, 842 48 Bratislava.
Email: macajova@dcs.fmph.uniba.sk, skoviera@dcs.fmph.uniba.sk

The authors acknowledge partial support from the APVV grant ESF-EC-0009-10 within the EUROCORES Programme EUROGIGA (project GReGAS) of the European Science Foundation, from APVV-0223-10, and from VEGA 1/1005/12.

J. Nešetřil et al. (eds.), *The Seventh European Conference on Combinatorics, Graph Theory and Applications*

The *oddness* of a cubic graph G, denoted by $\omega(G)$, is the minimum number of odd circuits in a 2-factor of G. A cubic graph is 3-edge-colourable if and only if its oddness is 0, so oddness provides a natural measure of uncolourability of a cubic graph. Another common measure of uncolourability is the *resistance* of G, $\rho(G)$, the minimum number of edges whose removal from G yields a 3-edge-colourable graph. Since $\rho(G) \leq \omega(G)$, resistance provides a practical lower bound for oddness [8].

Our aim to provide bounds on the the ratio $|V(G)|/\omega(G)$ for a snark G within the class of cyclically k-connected snarks. So far, only trivial lower bounds for oddness ratio have been known; as regards the upper bounds, there are various constructions, probably not optimal. Since the oddness ratio of the Petersen graph is 5, it is meaningless to attempt improving this absolute lower bound. In Section 22 we therefore take an asymptotic approach similar to that found in [3] and [8]. We summarise the known results and our improvements in Table 11; we only consider cyclic connectivity $k \leq 6$ since no cyclically 7-connected snarks are known. In fact, they are believed not to exist.

connectivity k	LB	current UB	previous UB	
2	5.41	7.5	9	(Steffen [8])
3	5.52	9	9	(Steffen [8])
4	5.52	13	15	(Hägglund [3])
5	5.83	25	76	(Steffen [8])
6	7	99	118	(Kochol [6])

Table 1. Upper (UB) and lower (LB) bounds on oddnes ratio $|V|/\omega$.

Besides general bounds, we are also interested in identifying the smallest snarks with oddness 4, addressing a long-standing open problem restated as Problem 4 in [1]. Our best results in this direction are shown in Figure 4.1 and described in Section 44. For more details and full proofs see [7].

2 Oddness and resistance ratios

The *oddness ratio* of a snark G is the quantity $|V(G)|/\omega(G)$, and its *resistance ratio* is $|V(G)|/\rho(G)$. We also examine asymptotic quantities

$$A_\omega = \liminf_{|V(G)|\to\infty} \frac{|V(G)|}{\omega(G)} \qquad \text{and} \qquad A_\rho = \liminf_{|V(G)|\to\infty} \frac{|V(G)|}{\rho(G)}.$$

Since the oddness ratio of a graph is at least as large as its resistance ratio, we have $A_\omega \leq A_\rho$. The oddness and resistance ratios heavily depend on the cyclic connectivity of a graph in question. This suggests to study analogous values A^k_ω and A^k_ρ obtained under the assumption that the class of snarks is restricted to those with cyclic connectivity at least k. Note that $A^2_\omega = A_\omega$, $A^2_\rho = A_\rho$, and $A^k_\omega \le A^k_\rho$ for every $k \ge 2$. Similar ideas were pursued by Steffen [8] who proved that $8 \le A_\rho \le 9$ and therefore $A_\omega \le A_\rho \le 9$. Since snarks constructed in [8] are cyclically 3-connected, we also have $A_\omega = A^2_\omega \le A^3_\omega \le A^3_\rho \le 9$.

3 Lower bounds on oddness ratio

A snark with girth at least 5 has oddness ratio at least 5. This bound is best possible because of the Petersen graph, but it can be improved for any other graph. Our approach is based on the following key observation.

Proposition 3.1. *Let $\mathcal{C}$ be a set of* 5*-circuits of a bridgeless cubic graph G. Then G has a* 2*-factor that contains at most* $1/6$ *of* 5*-circuits from* $\mathcal{C}$.

Proof. Let $\mathcal{P}$ be the perfect matching polytope of G. For a vector $\mathbf{x} \in \mathbb{R}^{|E(G)|}$ let $\mathbf{x}(e)$ denote the entry corresponding to an edge $e \in E(G)$, and let $\delta(U)$ be the set of all edges with precisely one end in the subgraph U of G. Since G is cubic and bridgeless, we have $\mathcal{P} \neq \emptyset$. Note that the vector $\mathbf{t} = (1/3, 1/3, \ldots, 1/3)$ always belongs to $\mathcal{P}$. Consider the function

$$f(\mathbf{x}) = \sum_{C \in \mathcal{C}} \sum_{e \in \delta(C)} \mathbf{x}(e)$$

defined for each $\mathbf{x} \in \mathcal{P}$. Since f is linear, there is a vertex of $\mathcal{P}$ where f reaches its minimum. Since $\mathbf{t} \in \mathcal{P}$, we have $f(\mathbf{x}_0) \le 5/3 \cdot |\mathcal{C}|$.

Let M be the perfect matching corresponding to $\mathbf{x}_0$ and let F be the 2-factor complementary to M. Assume that F contains k circuits from $\mathcal{C}$. If a 5-circuit $C \in \mathcal{C}$ belongs to F, it adds 5 to the sum in $f(\mathbf{x}_0)$. If C does not belong to F, it adds at least 1. Altogether $f(\mathbf{x}_0) \ge 5k + (|\mathcal{C}| - k) = |\mathcal{C}| + 4k$. Since $f(\mathbf{x}_0) \le 5/3 \cdot |\mathcal{C}|$, we obtain $k \le |\mathcal{C}|/6$. □

Proposition 3.1 has an interesting corollary: If G is a snark different from the Petersen graph, then for every vertex v of G there exists a 2-factor F of G such that every 5-circuit of F misses v. This gives a much shorter alternative way of proving the result of DeVos [2] that the Petersen graph is the only cubic graph having each 2-factor composed only of 5-cycles.

The following lemma provides the main tool for bounding oddness from above.

Lemma 3.2. *Let G be a snark of order n with girth at least* 4*. If G has q circuits of length* 5*, then* $\omega(G) \le 3n + q/21$.

4 Constructions

Cyclic connectivity 2. Figure 4.1 (top) displays the smallest snarks of oddness 4. Their order is 28 and cyclic connectivity is 2 and 3, respectively. The proof of minimality is computer assisted. We also create an infinite family of snarks with oddness ratio approaching 7.5 from below and conjecture that every snark with oddness ω has at least $7.5\,\omega - 5$ vertices.

Cyclic connectivity 4. Let P_4^v and P_4^e be the Petersen graph with either two adjacent vertices removed or two non-adjacent edges disconnected, respectively, and the dangling edges retained. There are two pairs of dangling edges in both. By the Parity Lemma, in P_4^v the dangling edges of each pair must have the same colour for every 3-edge-colouring, while in P_4^e they must have different colours. If we join a pair of dangling edges from P_4^v to a pair of dangling edges from P_4^e, we get an uncolourable 4-pole N_1 with 18 vertices. The 4-pole N_2 with 26 vertices arises from P_4^e and two distinct copies of P_4^v by joining each pair of dangling edges of P_4^e to a pair of edges edges in a different copy of P_4^v. The 4-pole N_2 is uncolourable even after the removal of a vertex w. This is clearly true if w belongs to a copy of P_4^v. If it was false for some w from a copy of P_4^e, then P_4^e would have a 3-edge-colouring where the edges in both pairs of dangling edges have the same colour. This would yield a 3-edge-colouring of the Petersen graph minus a vertex, but no such colouring exists.

To construct a cyclically 4-connected snark with arbitrarily large oddness we take a number of copies of N_1 and a number of N_2, arrange them into a circuit, and join one pair of dangling edges from each copy to a pair of dangling edges of its predecessor and another pair of dangling edges to a pair of dangling edges of its successor. The way in which copies of N_1 and N_2 are arranged is not unique, therefore we may get several non-isomorphic graphs even if we only use copies of one of N_1 and N_2.

In this construction, each copy of N_1 adds 1 and each copy of N_2 adds 2 to the resistance of the resulting graph. Thus if we take r copies of N_2, we get a cyclically 4-connected snark of order $26r$ with resistance $2r$ and oddness at least $2r$. If we take r copies of N_2 and one copy of N_1 we get a cyclically 4-connected snark of order $26r + 18$ with resistance $2r + 1$ and oddness at least $2r + 2$. This shows, in particular, that $A_\omega^4 \leq A_\rho^4 \leq 13$.

For $r = 1$ we obtain a cyclically 4-connected snark of order 44 with resistance 3 and oddness 4 shown in Figure 4.1, currently the smallest known non-trivial snark of oddness 4, improving the previous value of 46 [3].

Cyclic connectivity 6. Let P_3 be the Petersen graph with one vertex removed and the dangling edges retained. Take r copies $Q_1, Q_2, \ldots, Q_r$ of P_3. Arrange them into a circuit and, for each $i \in \{1, 2, \ldots, r\}$, join one dangling edge of Q_i to a dangling edge of Q_{i-1} and do the same for Q_{i+1} (indices reduced modulo r). There are $r/2$ pairs of oppositely positioned copies of P_3; join the remaining dangling edges for each such pair. The result of is a cubic graph L_r of order $9r$ and resistance r.

To obtain a snark with cyclic connectivity 6 we apply superposition to L_r (see [6] for details). Nontrivial supervertices will be copies of the 7-pole X having a vertex incident with three edges belonging to different connectors, plus two more edges joining the first two connectors of size 3. Nontrivial superedges will be copies of a 6-pole Y with 18 vertices created from the flower snark J_5 by removing two nonadjacent vertices, one from the single 5-circuit of J_5. We choose a circuit C in L_r which passes through five vertices of each copy of P_3 and finish the superposition by replacing each vertex on C with a copy of X and each edge on C with a copy of Y; we use trivial supervertices and superedges everywhere outside C. The resulting graph has order $99r$. Its resistance is at least r due to the following proposition.

Proposition 4.1. *Let $\tilde{G}$ be a snark resulting from a proper superposition of a snark G. Then $\rho(\tilde{G}) \geq \rho(G)$.*

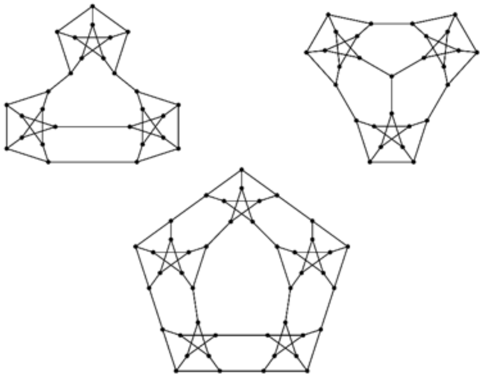

Figure 4.1. Smallest snarks of oddness 4 with cyclic connectivity 2, 3, 4.

References

[1] G. BRINKMANN, J. GOEDGEBEUR, J. HÄGGLUND and K. MARKSTRÖM, *Generation and properties of snarks*, accessed 7th June 2012, http://abel.math.umu.se/ klasm/Uppsatser/snarks genprop.pdf,

[2] M. DeVos, *Petersen graph conjecture*, http://garden.irmacs.sfu.ca/?q=op/petersen_graph_conjecture.
[3] J. Hägglund, *On snarks that are far from being 3-edge-colorable*, http://arxiv.org/pdf/1203.2015.pdf.
[4] R. Häggkvist and S. McGuinness, *Double covers of cubic graphs of oddness 4*, J. Combin. Theory Ser. B **93** (2005), 251–277.
[5] A. Huck and M. Kochol, *Five cycle double covers of some cubic graphs*, J. Combin. Theory Ser. B **64** (1995), 119–125.
[6] M. Kochol, *Superposition and constructions of graphs without nowhere-zero k-flows*, European J. Combin. **23** (2002), 281–306.
[7] R. Lukot'ka, E. Máčajová, J. Mazák and M. Škoviera, *Small snarks with large oddness*, http://arxiv.org/pdf/1212.3641v1.pdf.
[8] E. Steffen, *Measurements of edge-uncolorability*, Discrete Math. **280** (2004), 191–214.