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# The Seventh European Conference on Combinatorics, Graph Theory and Applications

EuroComb 2013

edited by  
Jaroslav Nešetřil and Marco Pellegrini



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# Foreword

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We are presenting in this volume the collection of extended abstracts that were selected for presentation at the European Conference on Combinatorics, Graph Theory and Applications - Eurocomb 2013, held in Pisa on September 9-13, 2013. Eurocomb conferences are organized biannually, the series was started in Barcelona 2001, and continued with Prague 2003, Berlin 2005, Seville 2007, Bordeaux 2009 and Budapest 2011. Since Prague Eurocomb 2003 the European Prize in Combinatorics is awarded at the meeting. The prize is established to recognize excellent contributions in Combinatorics, Discrete Mathematics and their Applications by young European researchers (eligibility of EU) not older than 35. It is supported by DIMATIA and by private sources.

The role of combinatorics is growing in the contemporary research in mathematics, computer science, engineering, also in physics, life and social sciences. This is reflected in the growing interest in Eurocomb conferences, which became an established forum of the frontier research in these fields. This year a total of 181 contributions were submitted and the Program Committee selected 95 extended abstracts and 9 posters to be presented at the conference. These figures, similar to those of the previous Eurocomb conference, indicate that Eurocomb meeting and its format got firmly established in the spectrum of high level European conferences. We thank all members of the Program Committee for their work. We also thank to all members of the Organizing Committee of Eurocomb 2013 for all their work in preparing the meeting in lovely Pisa.

The conference is organized by the Institute for Informatics and Telematics of the National Research Council of Italy (CNR), with the generous support of the Scuola Normale Superiore - Centro di Ricerca Matematica Ennio De Giorgi, the Dipartimento di Informatica - Università di Pisa, and the Center for Discrete Mathematics, Theoretical Computer Science and Applications (DIMATIA) of Charles University.

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## Erdős problems



# A problem of Erdős and Sós on 3-graphs

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**Abstract.** We show that for every  $\varepsilon > 0$  there exist  $\delta > 0$  and  $n_0 \in \mathbb{N}$  such that every 3-uniform hypergraph on  $n \geq n_0$  vertices with the property that every  $k$ -vertex subset, where  $k \geq \delta n$ , induces at least  $(\frac{1}{4} + \varepsilon) \binom{k}{3}$  edges, contains  $K_4^-$  as a subgraph, where  $K_4^-$  is the 3-uniform hypergraph on 4 vertices with 3 edges. This question was originally raised by Erdős and Sós. The constant  $1/4$  is the best possible.

## 1 Introduction

One of the most influential results in the extremal graph theory is the celebrated theorem of Turán [14] that determines the largest possible number of edges in an  $n$ -vertex graph without a complete subgraph of a given size. Erdős, Simonovits, and Stone [5, 7] generalized Turán's theorem by showing that the *extremal number* of an arbitrary graph  $F$ , defined as

$\text{ex}(n, F) := \max\{|E(G)| : G \text{ is a graph on } n \text{ vertices with no copy of } F\}$ ,

is asymptotically determined by its chromatic number. Specifically, for every graph  $F$  with at least one edge,

$$\text{ex}(n, F) = \left( \frac{\chi(F) - 2}{\chi(F) - 1} + o(1) \right) \binom{n}{2}.$$

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Note that this problem is dual to determining the minimum number of edges  $m(F)$  that guarantees an  $n$ -vertex graph  $G$  with at least  $m(n, F)$  edges to contain a copy of  $F$ , since  $m(n, F) = ex(n, F) + 1$ .

In Ramsey-Turán type problems, which were introduced by Sós in [13], we are again interested in the largest possible number of edges of an  $n$ -vertex graph without a copy of  $F$ , but under the additional restriction that the graph has to be somewhat “far” from the Turán graph. Typically, such a restriction is expressed by requiring that the graph has only sublinear independence number. For more details on Ramsey-Turán theory, see the survey by Simonovits and Sós [12].

In our problem, we are dealing with an even stronger restriction; we require every linear-size subset of vertices not only to induce at least one edge, but actually to induce a positive proportion of all possible edges. We define the  $\delta$ -linear density of a graph  $G$  to be the smallest density induced on an  $\delta$ -fraction of vertices, *i.e.*,

$$d(G, \delta) := \min \left\{ \frac{|E(G[A])|}{\binom{|A|}{2}} : A \subseteq V(G), |A| \geq \delta|V(G)| \right\}.$$

Note that for any graph  $G$  the function  $d(G, \delta)$  is a non-decreasing function of  $\delta$  taking values in  $[0, 1]$ .

However, requiring a positive  $\delta$ -linear density immediately forces large graphs to behave similarly to random graphs in the sense that they contain every given graph as a subgraph, which follows, *e.g.*, from [11, Theorem 1].

**Observation 1.1.** For every  $\varepsilon > 0$  and a fixed graph  $F$ , there exist  $\delta > 0$  and  $n_0 \in \mathbb{N}$  such that every graph  $G$  on at least  $n_0$  vertices with  $d(G, \delta) \geq \varepsilon$  contains  $F$  as a subgraph.

The notion of the  $\delta$ -linear density has a natural generalization to a  $k$ -uniform hypergraph  $H$ , where we define

$$d(H, \delta) := \min \left\{ \frac{|E(H_n[A])|}{\binom{|A|}{k}} : A \subseteq V(H), |A| \geq \delta|V(H)| \right\}.$$

We now focus on 3-uniform hypergraphs. Let  $K_4$  be the complete hypergraph on 4 vertices, and  $K_4^-$  the hypergraph on 4 vertices with 3 edges. Erdős and Sós [6, Problem 5] asked whether, analogously to Observation 1.1 for graphs, a sufficiently large hypergraph with positive  $\delta$ -linear density contains a copy of  $K_4$ , or at least a copy of  $K_4^-$ . However, Füredi observed that the following construction of Erdős and Hajnal [4] shows that the situation in 3-uniform hypergraphs is completely different (hence giving a negative answer to the question above).

**Example 1.2.** Consider a random tournament  $T_n$  on  $n$  vertices. Let  $H_n$  be the 3-uniform hypergraph on the same vertex set consisting of exactly those triples that span an oriented cycle in  $T_n$ .

One can check that in every hypergraph obtained in this way, any four vertices span at most two edges, *i.e.*,  $H_n$  does not contain  $K_4^-$  for every  $n$ . On the other hand, for every  $\delta > 0$ , the  $\delta$ -linear density of  $H_n$  tends to  $1/4$  when  $n$  goes to infinity. Additional information about this problem and its history can be found in [12, Section 5]. It is also worth noting that in [11], Rödl presented a sequence of  $K_4$ -free (but not  $K_4^-$ -free) 3-uniform hypergraphs with  $\delta$ -linear density tending to  $1/2$ .

Our main result is the following theorem stating that the bound  $1/4$  on  $\delta$ -linear density for  $K_4^-$ -free sequences is in fact the best possible, which answers the explicit question of Erdős [3] positively.

**Theorem 1.3.** *For every  $\varepsilon > 0$  there exist  $\delta > 0$  and  $n_0 \in \mathbb{N}$  such that every 3-uniform hypergraph  $H$  on at least  $n_0$  vertices with  $d(H, \delta) \geq 1/4 + \varepsilon$  contains  $K_4^-$  as a subgraph.*

## 2 Sketch of the proof

Suppose that Theorem 1.3 were false, then the following would be true.

*There exists  $\varepsilon > 0$  and a sequence  $(H_n)_{n \in \mathbb{N}}$  of  $K_4^-$ -free hypergraphs with order tending to infinity such that  $d(H_n, 1/n) \geq \frac{1}{4} + \varepsilon$ .* (★)

We say that a sequence of 3-uniform hypergraphs  $(H_n)_{n \in \mathbb{N}}$  is *convergent* if for every fixed hypergraph  $F$  the probabilities  $p(F, H_n)$  that random  $|F|$  vertices of  $V(H_n)$  induce a copy of  $F$  converge. We refer to  $p(F, H_n)$  as to the *induced density* of  $F$  in  $H_n$ . By a standard compactness argument, every sequence of hypergraphs have a convergent subsequence. Therefore, we may assume that the sequence given in (★) is convergent.

Our aim is to show that the edge density of the sequence given in (★) must tend to zero, which would contradict having positive  $\delta$ -linear density.

The main tool used for the proof of Theorem 1.3 is the framework of flag algebras. The framework was introduced by Razborov [10], who was inspired by the theory of dense graph limits from Borgs *et al.* [1, 2] and Lovász and Szegedy [9]. However, unlike just applying the standard flag algebra approach, in order to prove Theorem 1.3 we need to find a way of expressing the  $\delta$ -linear density condition in this framework. Instead of using the assumption on  $\delta$ -linear density itself, we establish a certain type of inequalities that are valid for any convergent sequence of  $K_4^-$ -free hypergraphs  $(H_n)$  such that  $d(H_n, f(n)) \geq 1/4$ , where  $f(n) \rightarrow 0$ . Let

us be more specific about that. Fix a  $K_4^-$ -free hypergraph  $F$ , and consider an arbitrary  $K_4^-$ -free superhypergraph  $F'$  of  $F$  on  $|F| + 1$  vertices. For any such a choice of a pair  $(F, F')$ , we obtain one inequality as follows. Let  $S$  be a fixed copy of  $F$  in  $H_n$ , and let  $U(S)$  be the set of vertices  $v$  of  $V(H_n) \setminus V(S)$  such that  $S \cup \{v\}$  induces  $F'$  in  $H_n$ . Let  $T$  be now three random vertices from  $H_n$  not contained in  $S$ . The following two outcomes are possible:

- If the size of  $U(S)$  is small, *i.e.*,  $o(|V(H_n)|)$ , then both the probability that  $T$  is a subset of  $U(S)$  and the probability that  $T$  is a subset of  $U(S)$  and spans an edge in  $H_n$ , are  $o(1)$ .
- If the size of  $U(S)$  is large, *i.e.*,  $\Omega(|V(H_n)|)$ , then, by the assumption on the  $\delta$ -linear density, the probability that  $T \subseteq U(S)$  and span an edge is at least quarter of the probability that  $T \subseteq U(S)$ .

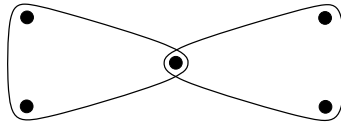
Therefore, we get that the following holds for every choice of  $S$

$$4 \cdot P[T \subseteq U(S) \wedge T \in E(H_n)] - P[T \subseteq U(S)] \geq -o(1). \quad (2.1)$$

Note that the little  $o$  goes to zero with  $n$  tending to infinity. The sought inequality corresponding to the pair  $(F, F')$  is then obtained by taking the expectation of (2.1), where the randomness comes from picking a copy  $S$  of  $F$  in  $H_n$ . This expectation can be expressed using the language of flag algebras.

It turned out, that for the proof of Theorem 1.3, it is enough to consider the inequalities for the pairs  $(F_2, F_3^0)$  and  $(F_2, F_3^1)$ , where  $F_2$  is the only 2-vertex 3-uniform hypergraph,  $F_3^0$  is the empty 3-vertex hypergraph, and  $F_3^1$  is the 3-vertex hypergraph with one edge.

Recall that our aim is to show that edge density of the sequence given in  $(\star)$  tends to zero. Unfortunately, we do not know how to apply flag algebras directly to derive this claim. Instead, we use them only to show that induced density of  $F_5$  tends to zero, where  $F_5$  is the 5-vertex hypergraph in Figure 2.1.



**Figure 2.1.** The hypergraph  $F_5$ .

To cope with this, we *sparsify* each hypergraph  $H_n$  given in  $(\star)$ , *i.e.*, we remove each edge of  $H_n$  independently with a sufficiently small but fixed positive probability, say  $\varepsilon/2$ . Therefore, we obtain a new sequence of

hypergraphs  $(I_n)$  such that  $d(I_n, 1/n) \geq 1/4$  almost surely. Hence, the induced density of  $F_5$  in the sequence  $(I_n)$  tends to zero. However, since the hypergraphs  $I_n$  were obtained from  $H_n$  by a random sparsification, the non-induced density of  $F_5$  in  $H_n$  has to tend to zero. But that implies, that almost every vertex in  $H_n$  has degree at most  $n$ . Therefore, the number of edges in  $H_n$  is  $O(n^2)$ , which finishes the proof of Theorem 1.3.

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# An analogue of the Erdős-Ko-Rado theorem for multisets

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**Abstract.** We verify a conjecture of Meagher and Purdy [4] by proving that if  $1 \leq t \leq k$ ,  $2k - t \leq n$  and  $\mathcal{F}$  is a family of  $t$ -intersecting  $k$ -multisets of  $[n]$ , then

$$|\mathcal{F}| \leq AK(n + k - 1, k, t),$$

where  $AK(n, k, t) := \max_i |\mathcal{A}_{n,k,t,i}|$  with  $\mathcal{A}_{n,k,t,i} := \{A : A \subseteq [n], |A| = k, |A \cap [t + 2i]| \geq t + i\}$ .

## 1 Introduction

### 1.1 Definitions, notation

Let  $n$  and  $l$  be positive integers and let

$$M(n, l) = \{(i, j) : 1 \leq i \leq n, 1 \leq j \leq l\}$$

be an  $n \times l$  rectangle. We call  $A \subseteq M(n, l)$  a  $k$ -multiset if the cardinality of  $A$  is  $k$  and  $(i, j) \in A$  implies  $(i, j') \in A$  for all  $j' \leq j$ . We think of multisets as sets with multiplicities, but it helps finding short and precise notation if we identify them with these special subsets of the rectangle. We denote the multiplicity of  $i$  in  $F$  by  $m(i, F)$ , i.e.  $m(i, F) := \max\{s : (i, s) \in F\}$  ( $m(i, F) \leq l$  by definition).

Let  $\mathcal{F}$  be a family of  $k$ -multisets of  $M(n, l)$ . We call  $\mathcal{F}$   $t$ -intersecting if  $t \leq |F_1 \cap F_2|$  for all  $F_1, F_2 \in \mathcal{F}$ . Let

$$\mathcal{M}(n, l, k, t) = \{\mathcal{F} : \mathcal{F} \text{ is } t\text{-intersecting set of } k\text{-multisets of } M(n, l)\},$$

i.e. the class of  $t$ -intersecting families of  $k$ -multisets.

Let  $\mathcal{F} \in \mathcal{M}(n, l, k, t)$ . We call  $T \subseteq M(n, l)$  a  $t$ -kernel for  $\mathcal{F}$  if  $|F_1 \cap F_2 \cap T| \geq t$  for all  $F_1, F_2 \in \mathcal{F}$ .

---

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## 1.2 History

Let us briefly summarize some results using our notation.

**Theorem 1.1 (Erdős, Ko, Rado, [3]).** *If  $n \geq 2k$  and  $\mathcal{F} \in \mathcal{M}(n, 1, k, 1)$  then*

$$|\mathcal{F}| \leq \binom{n-1}{k-1}.$$

*If  $n > 2k$ , then equality holds if and only if all members of  $\mathcal{F}$  contain a fixed element of  $[n]$ .*

They also proved that if  $n$  is large enough, every member of the largest  $t$ -intersecting family of sets contains a fixed  $t$ -element set, but did not give the optimal threshold. Frankl [5] showed for  $t \geq 15$  and Wilson [6] for every  $t$  that the optimal threshold is  $n = (k - t + 1)(t + 1)$ . Finally, Ahlswede and Khachatryan [1] determined the maximum families for all values of  $n$ .

**Theorem 1.2 (Ahlswede, Khachatryan [1]).** *Let  $t \leq k \leq n$  and  $\mathcal{A}_{n,k,t,i} = \{A : A \subseteq [n], |A| = k, |A \cap [t + 2i]| \geq t + i\}$ . If  $\mathcal{F} \in \mathcal{M}(n, 1, k, t)$ , then*

$$|\mathcal{F}| \leq \max_i |\mathcal{A}_{n,k,t,i}| = AK(n, k, t).$$

**Theorem 1.3 (Meagher, Purdy [4]).** *If  $n \geq k + 1$  and  $\mathcal{F} \in \mathcal{M}(n, k, k, 1)$ , then*

$$|\mathcal{F}| \leq \binom{n+k-2}{k-1}.$$

*If  $n > k + 1$ , then equality holds if and only if all members of  $\mathcal{F}$  contain a fixed element of  $M(n, k)$ .*

## 1.3 Conjectures

Brockman and Kay stated the following conjecture [2]:

**Conjecture 1.4 ([2], Conjecture 5.2).** *There is  $n_0(k, t)$  such that if  $n \geq n_0(k, t)$  and  $\mathcal{F} \in \mathcal{M}(n, k, k, t)$ , then*

$$|\mathcal{F}| \leq \binom{n+k-t-1}{k-t}.$$

Furthermore, equality is achieved if and only if each member of  $\mathcal{F}$  contains a fixed  $t$ -multiset of  $M(n, k)$ .

Meagher and Purdy also gave a possible candidate for the threshold  $n_0(k, t)$ .

**Conjecture 1.5 ([4], Conjecture 4.1).** Let  $k$ ,  $n$  and  $t$  be positive integers with  $t \leq k$ ,  $t(k-t) + 2 \leq n$  and  $\mathcal{F} \in \mathcal{M}(n, k, k, t)$ , then

$$|\mathcal{F}| \leq \binom{n+k-t-1}{k-t}.$$

Moreover, if  $n > t(k-t) + 2$ , then equality holds if and only if all members of  $\mathcal{F}$  contain a fixed  $t$ -multiset of  $M(n, k)$ .

Note that if  $n < t(k-t) + 2$ , then the family consisting of all multisets which contain a fixed  $t$ -multiset of  $M(n, k)$  still has cardinality  $\binom{n+k-t-1}{k-t}$ , but cannot be the largest. Indeed, if we fix a  $t+2$ -multiset  $T$  and consider the family of the multisets  $F$  with  $|F \cap T| \geq t+1$ , we get a larger family.

## 1.4 Results

The main idea of our proof is the following: instead of the well-known *left-compression* operation, which is a usual method in the theory of intersecting families, we define (in two different ways) an operation on  $\mathcal{M}(n, l, k, t)$  which can be called a kind of *down-compression*.

**Theorem 1.6.** Let  $1 \leq t \leq k$ ,  $2k-t \leq n$  and  $l$  be arbitrary. There exists

$$f : \mathcal{M}(n, l, k, t) \rightarrow \mathcal{M}(n, l, k, t)$$

satisfying the following properties:

- (i)  $|\mathcal{F}| = |f(\mathcal{F})|$  for all  $\mathcal{F} \in \mathcal{M}(n, l, k, t)$ ;
- (ii)  $M(n, 1)$  is a  $t$ -kernel for  $f(\mathcal{F})$ .

Using Theorem 1.6 we prove the following theorem which not only verifies Conjecture 1.5, but also gives the maximum cardinality of  $t$ -intersecting families of multisets in the case  $2k-t \leq n \leq t(k-t) + 2$ .

**Theorem 1.7.** Let  $1 \leq t \leq k$  and  $2k-t \leq n$ . If  $\mathcal{F} \in \mathcal{M}(n, k, k, t)$  then

$$|\mathcal{F}| \leq AK(n+k-1, k, t).$$

## 2 Concluding remarks

Note that for a family  $\mathcal{A}_{n+k-1, k, t, i}$  we can define a  $t$ -intersecting family of  $k$ -multisets in  $M(n, k)$ , hence the bound given in Theorem 1.7 is sharp. However, we do not know any nontrivial bounds in case  $n < 2k-t$ .

Another interesting problem is the case  $l < k$ . Theorem 1.6 gives us a small  $t$ -kernel, but the proof of Theorem 1.7 does not work in this case.

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# Polynomial gap extensions of the Erdős–Pósa theorem

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**Abstract.** Given a graph  $H$ , we denote by  $\mathcal{M}(H)$  all graphs that can be contracted to  $H$ . The following extension of the Erdős–Pósa Theorem holds: for every  $h$ -vertex planar graph  $H$ , there exists a function  $f_H: \mathbb{N} \rightarrow \mathbb{N}$  such that every graph  $G$ , either contains  $k$  disjoint copies of graphs in  $\mathcal{M}(H)$ , or contains a set of  $f_H(k)$  vertices meeting every subgraph of  $G$  that belongs in  $\mathcal{M}(H)$ . In this paper we prove that  $f_H$  can be polynomially (upper) bounded for every graph  $H$  of pathwidth at most 2 and, in particular, that  $f_H(k) = 2^{O(h^2)} \cdot k^2 \cdot \log k$ . As a main ingredient of the proof of our result, we show that for every graph  $H$  on  $h$  vertices and pathwidth at most 2, either  $G$  contains  $k$  disjoint copies of  $H$  as a minor or the treewidth of  $G$  is upper-bounded by  $2^{O(h^2)} \cdot k^2 \cdot \log k$ . We finally prove that the exponential dependence on  $h$  in these bounds can be avoided if  $H = K_{2,r}$ . In particular, we show that  $f_{K_{2,r}} = O(r^2 \cdot k^2)$ .

## 1 Introduction

In 1965, Paul Erdős and Lajos Pósa proved that every graph that does not contain  $k$  disjoint cycles, contains a set of  $O(k \log k)$  vertices meeting all its cycles [6]. Moreover, they gave a construction asserting that this bound is tight. This classic result can be seen as a “loose” min-max relation between covering and packing of combinatorial objects. Various extensions of this result, referring to different notions of packing and covering, attracted the attention of many researchers in modern Graph Theory (see, e.g. [1, 11]).

Given a graph  $H$ , we denote by  $\mathcal{M}(H)$  the set of all graphs that can be contracted to  $H$  (i.e. if  $H' \in \mathcal{M}(H)$ , then  $H$  can be obtained from  $H'$  after contracting edges). We call the members of  $\mathcal{M}(H)$  *models* of

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$H$ . Then the notions of covering and packing can be extended as follows: we denote by  $\mathbf{cover}_H(G)$  the minimum number of vertices that meet every model of  $H$  in  $G$  and by  $\mathbf{pack}_H(G)$  the maximum number of mutually disjoint models of  $H$  in  $G$ . We say that a graph  $H$  has the *Erdős–Pósa Property* if there exists a function  $f_H: \mathbb{N} \rightarrow \mathbb{N}$  such that for every graph  $G$ ,

$$\text{if } k = \mathbf{pack}_H(G), \text{ then } k \leq \mathbf{cover}_H(G) \leq f_H(k) \quad (1.1)$$

We will refer to  $f_H$  as the *gap* of the Erdős–Pósa Property. Clearly, if  $H = K_3$ , then (1.1) holds for  $f_{K_3} = O(k \log k)$  and the general question is to find, for each instantiation of  $H$ , the best possible estimation of the gap  $f_H$ , if it exists.

It turns out that  $H$  has the Erdős–Pósa Property if and only if  $H$  is a planar graph. This beautiful result appeared as a byproduct of the Graph Minors series of Robertson and Seymour. In particular, it is a consequence of the grid-exclusion theorem, proved in [14] (see also [3]).

**Proposition 1.1.** *There is a function  $g: \mathbb{N} \rightarrow \mathbb{N}$  such that if a graph excludes an  $r$ -vertex planar graph  $R$  as a minor, then its treewidth is bounded by  $g(r)$ .*

In [14] Robertson, Seymour, and Thomas conjectured that  $g$  is a low degree polynomial function. Currently, the best known bound for  $g$  is  $g(k) = 2^{O(k \log k)}$  and follows from [4] and [13] (see also [12, 14] for previous proofs and improvements). As the function  $g$  is strongly used in the construction of the function  $f_H$  in (1.1), the best, so far, estimation for  $f_H$  is far from being exponential in general. This initiated a quest for detecting instantiations of  $H$  where a polynomial gap  $f_H$  can be proved.

The first result in the direction of proving polynomial gaps for the Erdős–Pósa Property appeared in [9] where  $H$  is the graph  $\theta_c$  consisting of two vertices connected by  $c$  multiple edges (also called *c-pumpkin graph*). In particular, in [9] it was proved that  $f_{\theta_c}(k) = O(c^2 k^2)$ . More recently Fiorini, Joret, and Sau optimally improved this bound by proving that  $f_{\theta_c}(k) \leq c_t \cdot k \cdot \log k$  for some computable constant  $c_t$  depending on  $c$  [8]. In [15] Fiorini, Joret, and Wood proved that if  $T$  is a tree, then  $f_T(k) \leq c_T \cdot k$  where  $c_T$  is some computable constant depending on  $T$ . Finally, very recently, Fiorini [7] proved that  $f_{K_4} = O(k \log k)$ .

Our main result is a polynomial bound on  $f_H$  for a broad family of planar graphs, namely those of pathwidth at most 2. We prove the following:

**Theorem 1.2.** *If  $H$  is an  $h$ -vertex graph of pathwidth at most 2 and  $h > 5$ , then (1.1) holds for  $f_H(k) = 2^{O(h^2)} \cdot k^2 \cdot \log k$ .*

Note that the contribution of  $h$  in  $f_H$  is exponential. However, such a dependence can be waived when we restrict  $H$  to be  $K_{2,r}$ . Our second result is the following:

**Theorem 1.3.** *If  $H = K_{2,r}$ , then (1.1) holds for  $f_H(k) = O(r^2 \cdot k^2)$ .*

Both results above are based on a proof of Proposition 1.1, with polynomial  $g$ , for the cases where  $R$  consists of  $k$  disjoint copies of  $H$  and  $H$  is either a graph of pathwidth at most 2 or  $H = K_{2,3}$  (Theorems 2.1 and 2.2 respectively). For this, we follow an approach that makes strong use of the  $k$ -mesh structure introduced by Diestel *et al.* [4] in their proof of Proposition 1.1. Our proof indicates that, when excluding copies of some graph of pathwidth at most 2, the entangled machinery of [4] can be partially modified so that polynomial bounds on treewidth are possible. Finally, these bounds are then “translated” to polynomial bounds for the Erdős–Pósa gap using a technique developed in [10] (see also [9]).

*Definitions and preliminaries.* All graphs in this paper are simple, finite and undirected and logarithms are binary. We use standard notation in Graph Theory. We define  $k \cdot H$  as the graph obtained if we take  $k$  disjoint copies of  $H$ .

**Treewidth.** A *tree decomposition* of a graph  $G$  is a tree  $T$  whose vertices are some subsets of  $V(G)$  such that: (i)  $\bigcup_{X \in V(T)} X = V(G)$ , (ii) for every edge  $e$  of  $G$  there is a vertex of  $T$  containing both end of  $e$ , and (iii) for all  $v \in V(G)$ , the subgraph of  $T$  induced by  $\{X \in V(T), v \in X\}$  is connected. The *width* of a tree decomposition  $T$  is defined as equal to  $\max_{X \in V(T)} |X| - 1$ . The *treewidth* of  $G$ , written  $\mathbf{tw}(G)$ , is the minimum width of any of its tree decompositions. The *pathwidth* of  $G$ , written  $\mathbf{pw}(G)$ , is defined as the treewidth if we consider paths instead of trees.

## 2 Excluding packings of planar graphs

Theorems 1.2 and 1.3 follow combining the two following results with the machinery introduced in [10] (see also [9]). They have independent interest as they detect cases of Theorem 1.1 where  $g$  depends polynomially on  $k$ . In this extended abstract we only sketch the proof of Theorem 2.1.

**Theorem 2.1.** *Let  $H$  be a graph of pathwidth at most 2 on  $r > 5$  vertices. If  $G$  does not contain  $k$  disjoint copies of  $H$  as minor then  $\mathbf{tw}(G) \leq 2^{O(r^2)} \cdot k^2 \cdot \log 2k$ .*

**Theorem 2.2.** *For every positive integer  $r$ , if  $G$  does not contain  $k$  disjoint copies of  $K_{2,r}$  as minor then  $\mathbf{tw}(G) = O(r^2k^2)$ .*

*Proof of theorem 2.1. (sketch)* We prove the contrapositive. Let  $k$  be a integer,  $H$  a graph on  $r > 5$  vertices and of pathwidth at most 2 and  $G$  a graph. It can easily be proved that  $H \leq_m \Xi_r$  where  $\Xi_r$  is the graph obtained by a  $(r \times 2)$  if we subdivide once the ‘‘horizontal’’ edges. If we show that  $G$  contains  $k$  disjoint copies of  $\Xi_r$  as minors then we are done. Let  $g: \mathbb{N} \rightarrow \mathbb{N}$  such that  $g(k, r) = k^2 \log 2k \left( 180 \cdot 2^{r(r-2)} - 24 \cdot 2^{\frac{1}{2}r(r-2)} \right) + 6 \cdot 2^{\frac{1}{2}r(r-2)} - 1$ . We prove that for all graph  $G$ ,  $\mathbf{tw}(G) \geq g(k, r)$  implies that  $G \geq_m k \cdot \Xi_r$ . Let  $k$  and  $r > 5$  be two positive integers and assume that  $\mathbf{tw}(G) \geq g(k, r)$ . We examine below the case where  $\delta_c(G) < c \cdot 3rk\sqrt{\log 3rk}$  (the case  $\delta_c(G) \geq c \cdot 3rk\sqrt{\log 3rk}$  is omitted). Observe that  $c \cdot 3rk\sqrt{\log 3rk} < c \cdot 3r\sqrt{\log 6r} \cdot k\sqrt{\log 2k}$ . Let  $k_0 = k\sqrt{\log 2k}$  and  $r_0 = 3 \cdot 2^{\frac{r(r-2)}{2}}$ , and remark that  $k_0 \geq k$  and,  $r_0 \geq c \cdot 3r\sqrt{\log 6r}$  (remember that  $c \leq 648$  and  $r > 5$ ). With these notations, we have  $\delta_c(G) < 2k_0r_0$ . We will show that  $G \geq_m k_0 \cdot \Xi_r$  from which yields that  $G \geq_m k \cdot \Xi_r$ . By assumption,  $\mathbf{tw}(G) \geq g(k, r)$ . Using the results of [4], we can prove that  $G$  contains  $2k_0$  subsets  $X_1, \dots, X_{2k_0}$  of  $V(G)$  and a set  $\mathcal{P}$  of  $k_0r_0 = 3k_0 \cdot 2^{\frac{r(r-2)}{2}}$  disjoint paths of length at least 2 in  $G$  such that (i)  $\forall i \in \llbracket 1, 2k_0 \rrbracket$ ,  $X_i$  is of size  $r_0 = 3 \cdot 2^{\frac{r(r-2)}{2}}$  and is connected in  $G$  by a tree  $T_i$  using the elements of some set  $A \subseteq V(G)$ , (ii) any path in  $\mathcal{P}$  has one of its ends in some  $X_i$  with  $i \in \llbracket 1, k_0 \rrbracket$ , the other end in  $X_{2i}$  and its internal vertices are in none of the  $X_l$ , for all  $l \in \llbracket 1, 2k_0 \rrbracket$ , nor in  $A$ , and (iii)  $\forall i, j \in \llbracket 1, 2k_0 \rrbracket$ ,  $i \neq j \Rightarrow T_i \cap T_j = \emptyset$ .

We assume that for all  $i \in \llbracket 1, 2k_0 \rrbracket$ ,  $X_i = \{v \in V(T_i), \deg_{T_i}(v) \leq 2\}$ . It is easy to come down to this case by considering the minor of  $G$  obtained after deleting in  $T_i$  the leaves that are not in  $X_i$  and contracting one edge meeting a vertex of degree 2 which is not in  $X$  while such a vertex exists. As  $T_i$  is a ternary tree, one can easily prove that for all  $i \in \llbracket 1, 2k_0 \rrbracket$ ,  $T_i$  contains a path containing  $2 \log \frac{2}{3} |X_i| = (r-1)^2 + 1$  vertices of  $X_i$ . Let us call  $P_i$  such a path whose two ends are in  $X_i$ . Let us consider now the paths  $\{P_i\}_{i \in \llbracket 1, 2k_0 \rrbracket}$  and the paths that link the elements of different  $P_i$ 's. For each path  $i \in \llbracket 1, 2k_0 \rrbracket$ , we choose in  $P_i$  one end vertex (remember that both are in  $X_i$ ) that we name  $p_{i,0}$ . We follow  $P_i$  from this vertex and we denote the other vertices of  $P_i \cap X_i$  by  $p_{i,1}, p_{i,1}, \dots, p_{i,(r-1)^2}$  in this order. The *corresponding vertex* of some vertex  $p_{i,j}$  of  $P_i \cap X_i$  (for  $i \in \llbracket 1, k_0 \rrbracket$ ) is defined as the vertex of  $P_{2i} \cap X_{2i}$  to which  $p_{i,j}$  is linked to by a path of  $\mathcal{P}$ . As said before, the sets  $\{P_i \cap X_i\}_{i \in \llbracket 1, 2k_0 \rrbracket}$  are of size  $(r-1)^2 + 1$ . According to [5], one can find for all  $i \in \llbracket 1, k_0 \rrbracket$  a subsequence of length  $r$  in  $p_{i,0}, p_{i,1}, \dots, p_{i,(r-1)^2}$ , such that



the corresponding vertices in  $X_{2i}$  of this sequence are either in the same order (with respect to the subscripts of the names of the vertices), or in reverse order. For all  $i \in \llbracket 1, k_0 \rrbracket$ , this subsequence, its corresponding vertices and the vertices of the paths that link them together forms a  $\Xi_r$  model. We have thus  $k_0$  models of  $\Xi_r$  in  $G$ , that gives us  $k$  disjoint models of  $\Xi_r$  in  $G$  (since  $k \leq k_0$ ). We showed that for all  $k$  and  $r > 5$  positive integers, if a graph  $G$  has  $\text{tw}(G) \geq g(k, r)$ , then  $G \geq_m k \cdot \Xi_r$ . Consequently, if  $G$  has treewidth at least  $g(k, r)$ , then  $G$  contains  $k$  disjoint copies of  $H$  and we are done.  $\square$

*Postscript.* Very recently, the general open problem of estimating  $f_H(k)$  when  $H$  is a general planar graph has been tackled in [2]. Moreover, very recently, using the results of [13] we were able to improve both Theorems 2.1 and 1.3 by proving polynomial (on both  $k$  and  $|V(H)|$ ) bounds for more general instantiations of  $H$ .

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# The Erdős-Pósa property for long circuits

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**Abstract.** For an integer  $\ell$  at least 3, we prove that if  $G$  is a graph containing no two vertex-disjoint circuits of length at least  $\ell$ , then there is a set  $X$  of at most  $\frac{5}{3}\ell + \frac{29}{2}$  vertices that intersects all circuits of length at least  $\ell$ . Our result improves the bound  $2\ell + 3$  due to Birmel , Bondy, and Reed (The Erdős-Pósa property for long circuits, *Combinatorica* 27 (2007), 135-145) who conjecture that  $\ell$  vertices always suffice.

## 1 Introduction

A family  $\mathcal{F}$  of graphs is said to have the *Erdős-Pósa property* if there is a function  $f_{\mathcal{F}} : \mathbb{N} \rightarrow \mathbb{N}$  such that for every graph  $G$  and every  $k \in \mathbb{N}$ , either  $G$  contains  $k$  vertex-disjoint subgraphs that belong to  $\mathcal{F}$  or there is a set  $X$  of at most  $f_{\mathcal{F}}(k)$  vertices of  $G$  such that  $G - X$  has no subgraph that belongs to  $\mathcal{F}$ . The origin of this notion is [3] where Erdős and Pósa prove that the family of all circuits has this property.

Let  $\ell$  be an integer at least 3. Let  $\mathcal{F}_{\ell}$  denote the family of circuits of length at least  $\ell$ . In [2] Birmel , Bondy, and Reed show that  $\mathcal{F}_{\ell}$  has the Erdős-Pósa property with

$$f_{\mathcal{F}_{\ell}}(k) \leq 13\ell(k-1)(k-2) + (2\ell+3)(k-1), \quad (1.1)$$

which improves an earlier doubly exponential bound on  $f_{\mathcal{F}_{\ell}}(k)$  obtained by Thomassen [5]. The main contribution of Birmel , Bondy, and Reed [2] is to prove (1.1) for  $k=2$ , that is, to show

$$f_{\mathcal{F}_{\ell}}(2) \leq 2\ell + 3. \quad (1.2)$$

For  $k \geq 3$ , an inductive argument allows to deduce (1.1) from (1.2).

Birmel , Bondy, and Reed [2] conjecture that

$$f_{\mathcal{F}_{\ell}}(2) \leq \ell, \quad (1.3)$$

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that is, for every graph  $G$  containing no two vertex-disjoint circuits of length at least  $\ell$ , there is a set  $X$  of at most  $\ell$  vertices such that  $G - X$  has no circuit of length at least  $\ell$ . In view of the complete graph of order  $2\ell - 1$ , (1.3) would be best possible. For  $\ell = 3$ , (1.3) was shown by Lovász [4] and for  $\ell \in \{4, 5\}$ , (1.3) was shown by Birmelé [1].

Our contribution in the present paper is the following result.

**Theorem 1.1.** *Let  $\ell$  be an integer at least 3. Let  $G$  be a graph containing no two vertex-disjoint circuits of length at least  $\ell$ .*

*There is a set  $X$  of at most  $\frac{5}{3}\ell + \frac{29}{2}$  vertices that intersects all circuits of length at least  $\ell$ , that is,  $f_{\mathcal{F}_\ell}(2) \leq \frac{5}{3}\ell + \frac{29}{2}$ .*

While Theorem 1.1 is a nice improvement of (1.2), for  $k \geq 3$ , the above-mentioned inductive argument still leads to an estimate of the form  $f_{\mathcal{F}_\ell}(k) = O(\ell k^2)$ .

The rest of this paper is devoted to the proof of Theorem 1.1.

## 2 Proof of Theorem 1.1

With respect to notation and terminology we follow [2] and recall some specific notions. All graphs are finite, simple, and undirected. We abbreviate *vertex-disjoint* as *disjoint*. If  $A$  and  $B$  are sets of vertices of a graph  $G$ , then an  $(A, B)$ -*path* is a path  $P$  in  $G$  between a vertex in  $A$  and a vertex in  $B$  such that no internal vertex of  $P$  belongs to  $A \cup B$ . If  $P$  is a path and  $x$  and  $y$  are vertices of  $P$ , then  $P[x, y]$  denotes the subpath of  $P$  between  $x$  and  $y$ . Similarly, if  $C$  is a circuit endowed with an orientation and  $x$  and  $y$  are vertices of  $C$ , then  $C[x, y]$  denotes the segment of  $C$  from  $x$  to  $y$  following the orientation of  $C$ . In all figures of circuits the orientations will be counterclockwise.

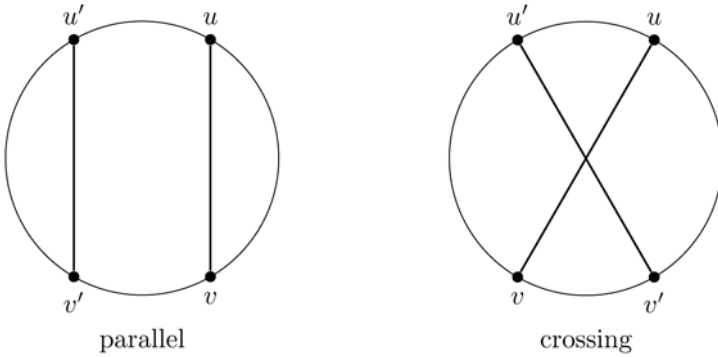
We fix an integer  $\ell$  at least 3 and call a circuit of length at least  $\ell$  *long*.

If  $C$  is a circuit and  $P$  and  $P'$  are disjoint  $(V(C), V(C))$ -paths such that  $P$  is between  $u$  and  $v$  and  $P'$  is between  $u'$  and  $v'$ , then

- $P$  and  $P'$  are called *parallel* (with respect to  $C$ ) if  $u, u', v', v$  appear in the given cyclic order on  $C$  and
- $P$  and  $P'$  are called *crossing* (with respect to  $C$ ) if  $u, u', v, v'$  appear in the given cyclic order on  $C$ .

See Figure 2.1.

In the proof of Theorem 1.1 below we consider three cases according to the length  $L$  of a shortest long circuit. If  $L$  is less than  $3\ell/2$ , the result is trivial. For  $L$  between  $3\ell/2$  and  $2\ell$  the following lemma implies the desired bound. Finally, for  $L$  larger than  $2\ell$ , Lemma 2.2 implies the desired bound.



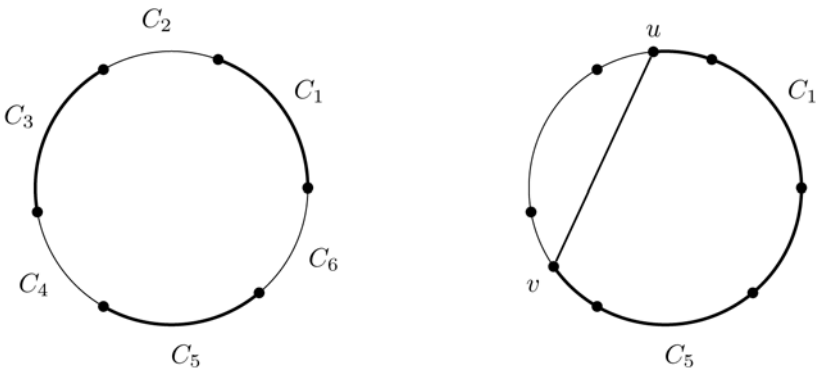
**Figure 2.1.** Parallel and crossing pairs of paths.

**Lemma 2.1.** *Let  $G$  be a graph containing no two disjoint long circuits.*

*If the shortest long circuit of  $G$  has length  $L$  with  $L \geq 3(\lceil \frac{1}{2}\ell \rceil - 2)$ , then there is a set  $X$  of at most  $\frac{1}{3}L + \ell + \frac{14}{3}$  vertices that intersects all long circuits.*

*Proof.* Let  $C$  be a shortest long circuit of  $G$ . We endow  $C$  with an orientation. We decompose  $C$  into 6 cyclically consecutive and internally disjoint segments  $C_1, \dots, C_6$  such that  $C_1, C_3$ , and  $C_5$  have length  $\lceil \frac{1}{2}\ell \rceil - 2$  and  $C_2, C_4$ , and  $C_6$  have lengths between  $\lfloor \frac{1}{3}L - (\lceil \frac{1}{2}\ell \rceil - 2) \rfloor$  and  $\lceil \frac{1}{3}L - (\lceil \frac{1}{2}\ell \rceil - 2) \rceil$ , that is, the six segments cover all of  $C$  and  $C_i$  and  $C_{i+1}$  overlap in exactly one vertex for every  $i \in [6]$  where we identify indices modulo 6.

Let  $X_1 = V(C_1) \cup V(C_3) \cup V(C_5)$ . See the left part of Figure 2.2.

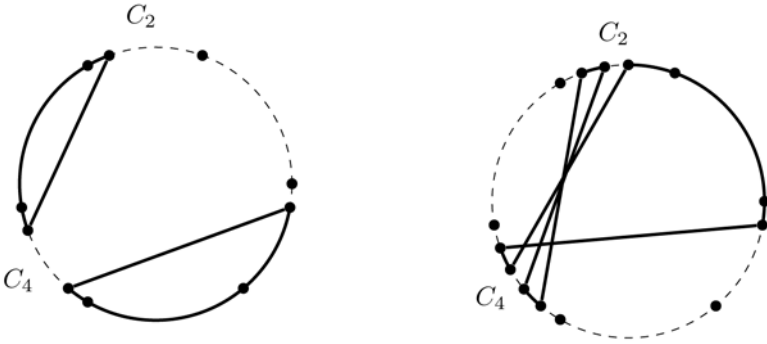


**Figure 2.2.** On the left the six segments of  $C$  and the set  $X$  in bold. On the right a long circuit formed by a  $(V(C_2), V(C_4))$ -path between  $u$  and  $v$  in  $G - (X_1 \cup V(C_6))$ .

Let  $i \in [6]$  be even. Let  $\mathcal{P}_i$  denote the set of  $(V(C_i), V(C_{i+2}))$ -paths in  $G - (X_1 \cup V(C_{i+4}))$ . The choice of  $C$  implies that every path  $P$  in  $\mathcal{P}_i$

has length at least  $\frac{1}{2}\ell$ ; otherwise  $P$  together with a segment of  $C$  avoiding  $V(C_{i+1})$  forms a long circuit that is shorter than  $C$ . See the right part of Figure 2.2. This implies that for every path  $P$  in  $\mathcal{P}_i$ ,  $P$  together with a segment of  $C$  containing  $V(C_{i+1})$  forms a long circuit.

Let  $\mathcal{P} = \mathcal{P}_2 \cup \mathcal{P}_4 \cup \mathcal{P}_6$ . Since  $G$  has no two disjoint long circuits, it follows that  $\mathcal{P}$  contains no two disjoint parallel paths and no four disjoint crossing paths. See Figure 2.3.



**Figure 2.3.** Two disjoint long circuits formed by two disjoint parallel paths in  $\mathcal{P}$  or by four disjoint crossing paths in  $\mathcal{P}$ .

Let  $X_2$  be a smallest set of vertices separating  $V(C_2)$  and  $V(C_4) \cup V(C_6)$  in  $G - X_1$ . Let  $X_3$  be a smallest set of vertices separating  $V(C_4)$  and  $V(C_6)$  in  $G - (X_1 \cup X_2)$ . By the above observations and Menger’s theorem,  $|X_2| \leq 3$  and  $|X_3| \leq 3$ .

There is some even  $j \in [6]$  such that in  $G - (X_1 \cup X_2 \cup X_3)$ , all long circuits intersect  $C$  only in  $V(C_j)$ ; otherwise there is a  $(V(C_i), V(C_{i+2}))$ -path in  $G - (X_1 \cup X_2 \cup X_3)$  for some even  $i \in [6]$ . This implies that  $X_1 \cup X_2 \cup X_3 \cup V(C_j)$  intersects all long circuits of  $G$ . Since

$$\begin{aligned}
 |X_1 \cup X_2 \cup X_3 \cup V(C_j)| &\leq 3 \left( \left\lceil \frac{1}{2}\ell \right\rceil - 2 \right) + 3 + 3 \\
 &\quad + \left\lceil \frac{1}{3}L - \left( \left\lceil \frac{1}{2}\ell \right\rceil - 2 \right) \right\rceil + 1 \\
 &\leq \frac{1}{3}L + \ell + \frac{14}{3}
 \end{aligned}$$

we obtain the desired result. □

**Lemma 2.2.** *Let  $G$  be a graph containing no two disjoint long circuits.*

*If the shortest long circuit of  $G$  has length at least  $2\ell - 3$ , then there is a set  $X$  of at most  $\frac{3}{2}\ell + \frac{29}{2}$  vertices that intersects all long circuits.*

*Sketch of the proof.* We only sketch the proof. Specifically, we do not give the proofs of the claims. Let  $C$  be shortest long circuit of  $G$ . Let  $L$  denote the length of  $C$ . We endow  $C$  with an orientation.

As in [2], a path between two vertices  $x$  and  $y$  of  $C$  that is internally disjoint from  $C$  is called *long*, if the segments  $C[x, y]$  and  $C[y, x]$  both have length at least  $\frac{1}{2}\ell$ .

**Claim A.** *Every long path has length at least  $\ell - 1$ .*

Choose a long circuit  $D$  of  $G$  distinct from  $C$  and a segment  $C[x, y]$  of  $C$  such that  $C[x, y]$  contains  $V(C) \cap V(D)$  and has minimum possible length. Note that  $x, y \in V(C) \cap V(D)$ .

We consider the two cases  $x \neq y$  and  $x = y$ . Here we only sketch the more general case  $x \neq y$ . Let  $X_1$  denote the set of  $\lceil \frac{1}{2}\ell \rceil - 1$  vertices immediately preceding  $x$  and let  $X_2$  denote the set of  $\lceil \frac{1}{2}\ell \rceil - 1$  vertices immediately following  $y$ . Let  $A = V(C) \setminus (X_1 \cup X_2 \cup V(C[x, y]))$  and  $B = V(C[x, y])$ . In  $G - (X_1 \cup X_2)$ , there are no two disjoint parallel  $(A, B)$ -paths and no four disjoint crossing  $(A, B)$ -paths; otherwise there would be two disjoint long circuits. Hence, by Menger's theorem, there is a set  $X_3$  of at most 3 vertices separating  $A$  and  $B$  in  $G - (X_1 \cup X_2)$ . The circuit  $D$  uniquely decomposes into a set  $\mathcal{P}$  of at least two  $(B, B)$ -paths of length at least 1.

**Claim B.** *If  $\mathcal{P}$  contains a path  $P$  between  $x$  and  $y$ , then in  $G - (X_1 \cup X_2 \cup X_3)$ , there are at most  $\lceil \frac{1}{2}\ell \rceil + 1$  disjoint  $(A, V(P))$ -paths.*

**Claim C.** *If  $\mathcal{P}$  contains two paths, say  $P$  and  $P'$ , between  $x$  and  $y$ , then in  $G - (X_1 \cup X_2 \cup X_3)$ , there are at most  $\lceil \frac{1}{2}\ell \rceil + 1$  disjoint  $(A, V(D))$ -paths.*

**Claim D.** *If  $P$  is a path in  $\mathcal{P}$  that is not a path between  $x$  and  $y$ , then in  $G - (X_1 \cup X_2 \cup X_3)$ , there are at most 3 disjoint  $(A, V(P))$ -paths.*

**Claim E.** *If  $P_1, \dots, P_4$  are four distinct paths in  $\mathcal{P}$  that are no paths between  $x$  and  $y$ , then in  $G - (X_1 \cup X_2 \cup X_3)$ , there are no four disjoint paths  $Q_1, \dots, Q_4$  such that  $Q_i$  is a  $(A, V(P_i))$ -path for  $i \in [4]$ .*

Let  $V_1$  denote the set of vertices  $r$  of  $D$  such that  $\mathcal{P}$  contains a path between  $x$  and  $y$  that contains  $r$  and let  $V_2$  denote the set of vertices  $s$  of  $D$  such that  $\mathcal{P}$  contains a path not between  $x$  and  $y$  that contains  $s$ . Clearly,  $V_1 \cup V_2 = V(D)$ . By Claims B and C and Menger's theorem, there is a set  $X_4$  of at most  $\lceil \frac{1}{2}\ell \rceil + 1$  vertices separating  $A$  and  $V_1$  in  $G - (X_1 \cup X_2 \cup X_3)$ . By Claims D and E and Menger's theorem, there is a set  $X_5$  of at most 9 vertices separating  $A$  and  $V_2$  in  $G - (X_1 \cup X_2 \cup X_3)$ . Let  $X = \{x, y\} \cup X_1 \cup X_2 \cup X_3 \cup X_4 \cup X_5$ .

If  $G - X$  contains a long circuit, say  $D'$ , then  $D'$  intersects  $A$ . Since  $D$  and  $D'$  intersect, there is an  $(A, V(D))$ -path  $P$  in  $G - X$ . In view of

$X_3$ ,  $P$  cannot end in  $B$ ; in view of  $X_4$ ,  $P$  cannot end in  $V_1$ ; and, in view of  $X_5$ ,  $P$  cannot end in  $V_2$ , which is a contradiction. Hence  $X$  intersects all long circuits. Since

$$|X| \leq 2 + |X_1| + |X_2| + |X_3| + |X_4| + |X_5| \leq \frac{3}{2}\ell + \frac{29}{2},$$

this completes the proof in the case  $x \neq y$ .  $\square$

*Proof of Theorem 1.1.* Let  $C$  be shortest long circuit of  $G$ . Let  $L$  denote the length of  $C$ .

If  $L$  is at most  $\frac{5}{3}\ell + \frac{29}{2}$ , then let  $X = V(C)$ . If  $L$  is larger than  $\frac{5}{3}\ell + \frac{29}{2}$  but less than  $2\ell - 4$ , then Lemma 2.1 implies the existence of a set  $X$  with the desired properties. If  $L$  is at least  $2\ell - 3$ , then Lemma 2.2 implies the existence of a set  $X$  with the desired properties.  $\square$

Our main interest was to improve the factor of  $\ell$  in the bound in Theorem 1.1 and not the additive constant, which can easily be improved slightly.

The main open problem remains the conjectured inequality (1.3). We believe that further ideas are needed for its proof. Furthermore, it is unclear whether the quadratic dependence on  $k$  in (1.1) is best possible. For  $\ell = 3$ , that is, the classical case considered by Erdős and Pósa [3], it is known that  $f_{\mathcal{F}_3}(k) = O(k \log k)$ .

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## Hypergraphs

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# A hypergraph Turán theorem via Lagrangians of intersecting families

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**Abstract.** Let  $\mathcal{K}_{3,3}^3$  be the 3-graph with 15 vertices  $\{x_i, y_i : 1 \leq i \leq 3\}$  and  $\{z_{ij} : 1 \leq i, j \leq 3\}$ , and 11 edges  $\{x_1, x_2, x_3\}$ ,  $\{y_1, y_2, y_3\}$  and  $\{x_i, y_j, z_{ij} : 1 \leq i, j \leq 3\}$ . We show that for large  $n$ , the unique largest  $\mathcal{K}_{3,3}^3$ -free 3-graph on  $n$  vertices is a balanced blow-up of the complete 3-graph on 5 vertices. Our proof uses the stability method and a result on Lagrangians of intersecting families that has independent interest.

## 1 Introduction

The *Turán number*  $ex(n, F)$  is the maximum number of edges in an  $F$ -free  $r$ -graph on  $n$  vertices. It is a long-standing open problem in Extremal Combinatorics to develop some understanding of these numbers for general  $r$ -graphs  $F$ . For ordinary graphs ( $r = 2$ ) the picture is fairly complete, but for  $r \geq 3$  there are very few known results. Turán [5] posed the natural question of determining  $ex(n, F)$  when  $F = K_t^r$  is a complete  $r$ -graph on  $t$  vertices. To date, no case with  $t > r > 2$  of this question has been solved, even asymptotically. For a summary of progress on hypergraph Turán problems we refer the reader to the survey [2].

In this paper, we determine the Turán number of the 3-graph  $\mathcal{K}_{3,3}^3$  with vertices  $\{x_i, y_i : 1 \leq i \leq 3\}$  and  $\{z_{ij} : 1 \leq i, j \leq 3\}$ , and edges  $\{x_1, x_2, x_3\}$ ,  $\{y_1, y_2, y_3\}$  and  $\{x_i, y_j, z_{ij} : 1 \leq i, j \leq 3\}$ . For an integer  $n \geq 5$ , let  $T_5^3(n)$  denote the balanced blow-up of  $K_5^3$  on  $n$  vertices, that is, we partition the vertices into 5 parts of sizes  $\lfloor n/5 \rfloor$  or  $\lceil n/5 \rceil$ , and take

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as edges all triples in which the vertices belong to 3 distinct parts. Write  $t_5^3(n) := e(T_5^3(n))$ . Our main result is as follows.

**Theorem 1.1.**  *$ex(n, \mathcal{K}_{3,3}^3) = t_5^3(n)$  for sufficiently large  $n$ . Moreover, if  $n$  is sufficiently large and  $G$  is a  $\mathcal{K}_{3,3}^3$ -free 3-graph with  $n$  vertices and  $t_5^3(n)$  edges, then  $G \cong T_5^3(n)$ .*

We prove Theorem 1.1 by the stability method and Lagrangians. Given an  $r$ -graph  $G$  on  $[n] = \{1, \dots, n\}$ , we define a polynomial in the variables  $x = (x_1, \dots, x_n)$  by

$$p_G(x) := \sum_{e \in E(G)} \prod_{i \in e} x_i.$$

The *Lagrangian* of  $G$  is

$$\lambda(G) = \max\{p_G(x) : x_i \geq 0 \text{ for } 1 \leq i \leq n \text{ and } \sum_{i=1}^n x_i = 1\}.$$

A key tool in the proof will be the following result that determines the maximum possible Lagrangian among all intersecting 3-graphs: it is uniquely achieved by  $K_5^3$ , which has  $\lambda(K_5^3) = \binom{5}{3}(1/5)^3 = 2/25$ .

**Theorem 1.2.** *Let  $G$  be an intersecting 3-graph. If  $G \neq K_5^3$ , then  $\lambda(G) \leq \lambda(K_5^3) - 10^{-3}$ .*

## 2 Lagrangians of intersecting 3-graphs

In this section we provide a rough sketch of the proof of Theorem 1.2. Our proof consists of the following two steps:

- (1) Prove that it suffices to bound from above the Lagrangian of a small number of specific 3-graphs.
- (2) Bound from above the Lagrangian of the 3-graphs determined in step (1).

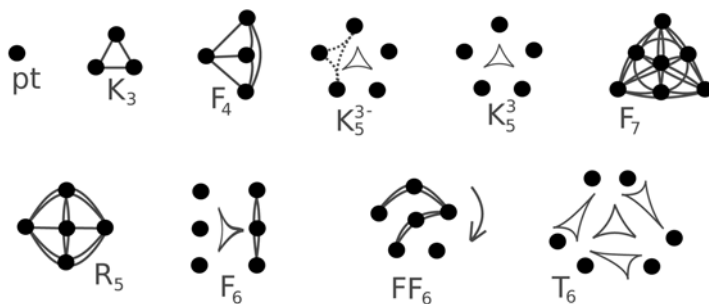
Step (1) consists of the following sub-steps:

- (a) Prove that it suffices to consider intersecting 3-graphs which satisfy certain desirable properties (e.g. the set of hyperedges is maximal with respect to inclusion and covers pairs of vertices whose weight in the assignment under consideration is strictly positive).
- (b) Apply a *down shifting operation* (or just shifting for brevity) to the set of hyperedges of the 3-graphs considered and prove that there is just a small number of resulting hypergraphs and that these have some desirable properties (all of these are depicted in Figure 2.1; note that

every such hypergraph has at most 7 vertices). Given an intersecting 3-graph  $\mathcal{F}$  with vertex set  $[n]$ , a *shift* of  $\mathcal{F}$  is any hypergraph obtained by applying the following rule: as long as there exist  $i \in A \in \mathcal{F}$  such that  $\mathcal{F}' = (\mathcal{F} \setminus \{A\}) \cup \{A \setminus \{i\}\}$  is intersecting, replace  $\mathcal{F}$  by  $\mathcal{F}'$  and repeat.

- (c) For every 3-graph  $\mathcal{F}$  as in (a) let  $S(\mathcal{F})$  be the hypergraph obtained from it by shifting in (b). We define a 3-graph  $G = \text{Gen}(n, S(\mathcal{F}))$  such that  $|V(G)| = n$  and  $G \supseteq \mathcal{F}$ . The family  $\mathcal{L}$  of all such  $G$  is the family of 3-graphs we will consider in step (2).

In step (2) we go through all 3-graphs in  $\mathcal{L}$  and prove a suitable upper bound on the Lagrangian of each of its elements. The details vary from 3-graph to 3-graph but the general idea is to use the method of lagrange multipliers while making use of a combinatorial weight shifting argument.



**Figure 2.1.** The family of hypergraphs resulting from down shifting.

### 3 An application to a hypergraph Turán problem

In this section we apply Theorem 1.2 to prove our main theorem on the Turán number of  $\mathcal{K}_{3,3}^3$ , namely that for large  $n$ , the unique extremal example is  $T_5^3(n)$ , i.e. the balanced blow-up of  $K_5^3$ . First we note some simple facts about  $T_5^3(n)$ . It is  $\mathcal{K}_{3,3}^3$ -free, as for any attempted embedding of  $\mathcal{K}_{3,3}^3$  in  $T_5^3(n)$ , there must be some  $1 \leq i, j \leq 3$  such that  $x_i$  and  $y_j$  lie in the same part, but then  $x_i y_j z_{ij}$  cannot be an edge. The number of edges satisfies

$$t_5^3(n) = \sum_{0 \leq i < j < k \leq 4} \left\lfloor \frac{n+i}{5} \right\rfloor \cdot \left\lfloor \frac{n+j}{5} \right\rfloor \cdot \left\lfloor \frac{n+k}{5} \right\rfloor = \frac{2}{25}n^3 + O(n^2).$$

Also, the minimum degree satisfies

$$\delta_5^3(n) = t_5^3(n) - t_5^3(n-1) = \sum_{0 \leq i < j \leq 3} \left\lfloor \frac{n+i}{5} \right\rfloor \cdot \left\lfloor \frac{n+j}{5} \right\rfloor = \frac{6}{25}n^2 + O(n).$$

We start by showing that the asymptotic result follows quickly from Theorem 1.2. First we need some definitions. Suppose  $F$  and  $G$  are  $r$ -graphs. The *Turán density* of  $F$  is  $\pi(F) = \lim_{n \rightarrow \infty} \binom{n}{r}^{-1} \text{ex}(n, F)$ . Given  $r$ -graphs  $F$  and  $G$  we say  $f : V(F) \rightarrow V(G)$  is a homomorphism if it preserves edges, i.e.  $f(e) \in E(G)$  for all  $e \in E(F)$ . We say that  $G$  is  $F$ -*hom-free* if there is no homomorphism from  $F$  to  $G$ . The *blow-up density* is  $b(G) = r! \lambda(G)$ . We say  $G$  is *dense* if every proper subgraph  $G'$  satisfies  $b(G') < b(G)$ . We also need the following two standard facts (see e.g. [2, Section 3]):

- (i)  $\pi(F)$  is the supremum of  $b(G)$  over  $F$ -hom-free dense  $G$ ,
- (ii) dense  $r$ -graphs cover pairs.

**Theorem 3.1.**  $\text{ex}(n, \mathcal{K}_{3,3}^3) = \frac{2}{25}n^3 + o(n^3)$ .

*Proof.* An equivalent formulation is that  $\pi(\mathcal{K}_{3,3}^3) = \frac{12}{25}$ . The lower bound is given by the construction  $T_5^3(n)$ . For the upper bound, by fact (i) above, it suffices to show that  $b(G) \leq 12/25$  for any  $\mathcal{K}_{3,3}^3$ -hom-free dense  $G$ . Suppose for a contradiction that  $G$  is  $\mathcal{K}_{3,3}^3$ -hom-free, dense, and has  $\lambda(G) > 2/25$ . By Theorem 1.2,  $G$  is not intersecting, so we can choose disjoint edges  $\{x_1, x_2, x_3\}$  and  $\{y_1, y_2, y_3\}$ . Then by fact (ii),  $G$  covers pairs, so for every  $1 \leq i, j \leq 3$  there exists an edge  $\{x_i, y_j, z_{ij}\}$ . However, this defines a homomorphism from  $\mathcal{K}_{3,3}^3$  to  $G$ , which contradicts  $G$  being  $\mathcal{K}_{3,3}^3$ -hom-free.  $\square$

### 3.1 Stability

In order to prove Theorem 1.1, we will first prove the following stability result.

**Theorem 3.2.** *For any  $\varepsilon > 0$  there exist  $\delta > 0$  and an integer  $n_0$  such that if  $\mathcal{F}$  is a  $\mathcal{K}_{3,3}^3$ -free 3-graph with  $n \geq n_0$  vertices and at least  $(\frac{2}{25} - \delta)n^3$  edges, then there exists a partition  $V(\mathcal{F}) = A_1 \cup \dots \cup A_5$  of the vertex set of  $\mathcal{F}$  such that  $\sum_{1 \leq i < j \leq 5} e(A_i \cup A_j) < \varepsilon n^3$ .*

Part of our proof follows the main ideas of [3]. The proof consists of two main stages. In the first stage we gradually change  $\mathcal{F}$  (as well as some other related structure) by iterating a process which is called *Symmetrization*. This process consists of two parts: *Cleaning* and *Merging*.

We refer to the basic object with which we operate as a *pointed partitioned 3-graph*; by this we mean a triple  $(\mathcal{G}, \mathcal{P}, U)$  where  $\mathcal{G} = (V, E)$  is a 3-graph,  $\mathcal{P} = \{P_u : u \in V\}$  is a partition of  $V$  such that  $u \in P_u$  for every  $u \in V$ , and  $U \subseteq V$  is a transversal of  $\mathcal{P}$ . The ultimate goal of this process is to end up with a pointed partitioned 3-graph  $(\mathcal{F}_{sym}, \mathcal{P}, U)$  such that  $\mathcal{F}_{sym}[U] \cong K_5^3$  and  $(1/5 - \beta)n \leq |A| \leq (1/5 + \beta)n$  for every part  $A \in \mathcal{P}$ , where  $\beta > 0$  is a sufficiently small real number. Note that Theorem 1.2 plays a crucial role in achieving this goal.

The second stage is to show that  $\mathcal{F}[V(\mathcal{F}_{sym})]$  is a subgraph of a blow-up of  $K_5^3$ . To do so, we will reverse the Merging steps performed during Symmetrization (this process was called Splitting in [3]). Since the first stage ensures that  $|V(\mathcal{F})| - |V(\mathcal{F}_{sym})|$  is very small, this will complete the proof of Theorem 3.2.

Using Theorem 3.2 we can now prove Theorem 1.1. Let  $\mathcal{F} = (V, E)$  be a maximum size  $\mathcal{K}_{3,3}^3$ -free 3-graph on  $n$  vertices, where  $n$  is sufficiently large. Clearly  $|E| \geq t_5^3(n)$ . The main steps of the proof are as follows:

- (1) Show that it suffices to prove Theorem 1.1 under the additional assumption that the minimum degree of  $\mathcal{F}$  is at least  $\delta_5^3(n)$ .
- (2) Let  $V = A_1 \cup \dots \cup A_5$  be a partition of the vertex set of  $\mathcal{F}$  which minimizes

$$\Sigma := \sum_{1 \leq i < j \leq 5} e(A_i \cup A_j) - 2 \sum_{i=1}^5 e(A_i).$$

Using Theorem 3.2 we can assume that  $\Sigma < \varepsilon n^3$ .

- (3) Observe that  $||A_i| - n/5| \leq \delta n$  for every  $1 \leq i \leq 5$  and some small real number  $\delta > 0$  as otherwise we would have the contradiction  $|E| \leq 2n^3/25 - \delta^2 n^3/40 + \varepsilon n^3 < t_5^3(n)$ .
- (4) Using (1), the fact that  $\mathcal{F}$  is  $\mathcal{K}_{3,3}^3$ -free and the minimality of the partition in (2), prove that for every  $u \in V$  there are at most  $\alpha n^2$  edges  $e \in E$  such that  $u \in e$  and  $|e \cap A_i| > 1$  holds for some  $1 \leq i \leq 5$ , where  $\alpha > 0$  is a small real number.
- (5) Using (1), the fact that  $\mathcal{F}$  is  $\mathcal{K}_{3,3}^3$ -free and the minimality of the partition in (2), prove that  $|e \cap A_i| \leq 1$  holds for every  $1 \leq i \leq 5$  and every  $e \in E$ .

## 4 Concluding remarks and open problems

The natural open problem is to extend our results from 3-graphs to general  $r$ -graphs. We would like to determine the Turán number of the  $r$ -graph  $\mathcal{K}_{r,r}^r$  with vertex set

$$V(\mathcal{K}_{r,r}^r) = \{x_i, y_i : 1 \leq i \leq r\} \cup \{z_{ijk} : 1 \leq i, j \leq r, 1 \leq k \leq r-2\}$$

and edge set

$$E(\mathcal{K}_{r,r}^r) = \{\{x_1, \dots, x_r\}, \{y_1, \dots, y_r\}\} \cup \{\{x_i, y_j, z_{ij1}, \dots, z_{ij(r-2)}\} : 1 \leq i, j \leq r\}.$$

The main difficulty seems to be in obtaining the analogue of Theorem 1.2, i.e. determining the maximum Lagrangian of an intersecting  $r$ -graph. At first, one might think that  $K_{2r-1}^r$  should be optimal, since this is the case when  $r = 3$ . However, this has Lagrangian  $\binom{2r-1}{r} \left(\frac{1}{2r-1}\right)^r$ , whereas stars (in which edges consist of all  $r$ -tuples containing some fixed vertex) give Lagrangians that approach  $\frac{1}{r!} \left(1 - \frac{1}{r}\right)^{r-1}$ , which is better for  $r \geq 4$ . We conjecture that stars are optimal for  $r \geq 4$  and that their blow-ups are extremal. Namely, we conjecture that the following hypergraph Turán result holds. Let  $S^r(n)$  be the  $r$ -graph on  $n$  vertices with parts  $A$  and  $B$ , where the edges consist of all  $r$ -tuples with 1 vertex in  $A$  and  $r-1$  vertices in  $B$ , and the sizes of  $A$  and  $B$  are chosen to maximise the number of edges (so  $|A| \sim n/r$ ). Write  $s^r(n) = e(S^r(n))$ .

**Conjecture 4.1.**  $ex(n, \mathcal{K}_{r,r}^r) = s^r(n)$  for  $r \geq 4$  and sufficiently large  $n > n_0(r)$ . Moreover, if  $n$  is sufficiently large and  $G$  is a  $\mathcal{K}_{r,r}^r$ -free  $r$ -graph with  $n$  vertices and  $s^r(n)$  edges, then  $G \cong S^r(n)$ .

More generally, our work suggests a direction of investigation in Extremal Combinatorics, namely to determine the maximum Lagrangian for any specified property of  $r$ -graphs. For this paper, the property was that of being intersecting. This direction was already started by Frankl and Füredi [1], who considered the question of maximising the Lagrangian of an  $r$ -graph with a specified number of edges. They conjectured that initial segments of the colexicographic order are extremal. Many cases of this have been proved by Talbot [4], but the full conjecture remains open.

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# Tight minimum degree conditions forcing perfect matchings in uniform hypergraphs

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**Abstract.** Given positive integers  $k$  and  $\ell$  where  $k/2 \leq \ell \leq k - 1$ , we give a minimum  $\ell$ -degree condition that ensures a perfect matching in a  $k$ -uniform hypergraph. This condition is best possible and improves on work of Pikhurko [12] who gave an asymptotically exact result, and extends work of Rödl, Ruciński and Szemerédi [15] who determined the threshold for  $\ell = k - 1$ . Our approach makes use of the absorbing method.

## 1 Introduction

A central question in graph theory is to establish conditions that ensure a (hyper)graph  $H$  contains some spanning (hyper)graph  $F$ . Of course, it is desirable to fully characterize those (hyper)graphs  $H$  that contain a spanning copy of a given (hyper)graph  $F$ . Tutte's theorem [18] characterizes those graphs with a perfect matching. (A *perfect matching* in a (hyper)graph  $H$  is a collection of vertex-disjoint edges of  $H$  which cover the vertex set  $V(H)$  of  $H$ .) However, for some (hyper)graphs  $F$  it is unlikely that such a characterization exists. Indeed, for many (hyper)graphs  $F$  the decision problem of whether a (hyper)graph  $H$  contains  $F$  is NP-complete. For example, in contrast to the graph case, the decision problem whether a  $k$ -uniform hypergraph contains a perfect matching is NP-complete for  $k \geq 3$  (see [4, 6]). Thus, it is desirable to find sufficient conditions that ensure a perfect matching in a  $k$ -uniform hypergraph.

Given a  $k$ -uniform hypergraph  $H$  with an  $\ell$ -element vertex set  $S$  (where  $0 \leq \ell \leq k - 1$ ) we define  $d_H(S)$  to be the number of edges containing  $S$ . The *minimum  $\ell$ -degree*  $\delta_\ell(H)$  of  $H$  is the minimum of  $d_H(S)$  over all

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$\ell$ -element sets  $S$  of vertices in  $H$ . Clearly  $\delta_0(H)$  is the number of edges in  $H$ . We also refer to  $\delta_1(H)$  as the *minimum vertex degree* of  $H$  and  $\delta_{k-1}(H)$  the *minimum codegree* of  $H$ .

Over the last few years there has been a strong focus in establishing minimum  $\ell$ -degree thresholds that force a perfect matching in a  $k$ -uniform hypergraph. See [13] for a survey on matchings (and Hamilton cycles) in hypergraphs. In particular, Rödl, Ruciński and Szemerédi [15] determined the minimum codegree threshold that ensures a perfect matching in a  $k$ -uniform hypergraph on  $n$  vertices for all  $k \geq 3$ . The threshold is  $n/2 - k + C$ , where  $C \in \{3/2, 2, 5/2, 3\}$  depends on the values of  $n$  and  $k$ . This improved bounds given in [9, 14].

Less is known about minimum vertex degree thresholds that force a perfect matching. One of the earliest results on perfect matchings was given by Daykin and Häggkvist [3], who showed that a  $k$ -uniform hypergraph  $H$  on  $n$  vertices contains a perfect matching provided that  $\delta_1(H) \geq (1 - 1/k) \binom{n-1}{k-1}$ . Hàn, Person and Schacht [5] determined, asymptotically, the minimum vertex degree that forces a perfect matching in a 3-uniform hypergraph. Kühn, Osthus and Treglown [10] and independently Khan [7] made this result exact. Khan [8] has also determined the exact minimum vertex degree threshold for 4-uniform hypergraphs. For  $k \geq 5$ , the precise minimum vertex degree threshold that ensures a perfect matching in a  $k$ -uniform hypergraph is not known.

The situation for  $\ell$ -degrees where  $1 < \ell < k - 1$  is also still open. Hàn, Person and Schacht [5] provided conditions on  $\delta_\ell(H)$  that ensure a perfect matching in the case when  $1 \leq \ell < k/2$ . These bounds were subsequently lowered by Markström and Ruciński [11]. Alon et al. [1] gave a connection between the minimum  $\ell$ -degree that forces a perfect matching in a  $k$ -uniform hypergraph and the minimum  $\ell$ -degree that forces a *perfect fractional matching*. As a consequence of this result they determined, asymptotically, the minimum  $\ell$ -degree which forces a perfect matching in a  $k$ -uniform hypergraph for the following values of  $(k, \ell)$ :  $(4, 1)$ ,  $(5, 1)$ ,  $(5, 2)$ ,  $(6, 2)$ , and  $(7, 3)$ .

Pikhurko [12] showed that if  $\ell \geq k/2$  and  $H$  is a  $k$ -uniform hypergraph whose order  $n$  is divisible by  $k$  then  $H$  has a perfect matching provided that  $\delta_\ell(H) \geq (1/2 + o(1)) \binom{n}{k-\ell}$ . This result is best possible up to the  $o(1)$ -term (see the constructions in  $\mathcal{H}_{\text{ext}}(n, k)$  below).

In [16, 17] we make Pikhurko's result exact. In order to state this result, we need some more definitions. Fix a set  $V$  of  $n$  vertices. Given a partition of  $V$  into non-empty sets  $A, B$ , let  $E_{\text{odd}}^k(A, B)$  ( $E_{\text{even}}^k(A, B)$ ) denote the family of all  $k$ -element subsets of  $V$  that intersect  $A$  in an odd (even) number of vertices. (Notice that the ordering of the vertex classes  $A, B$  is important.) Define  $\mathcal{B}_{n,k}(A, B)$  to be the  $k$ -uniform hypergraph with ver-

text set  $V = A \cup B$  and edge set  $E_{\text{odd}}^k(A, B)$ . Note that the complement  $\overline{\mathcal{B}}_{n,k}(A, B)$  of  $\mathcal{B}_{n,k}(A, B)$  has edge set  $E_{\text{even}}^k(A, B)$ .

Suppose  $n, k \in \mathbb{N}$  such that  $k$  divides  $n$ . Define  $\mathcal{H}_{\text{ext}}(n, k)$  to be the collection of the following hypergraphs:  $\mathcal{H}_{\text{ext}}(n, k)$  contains all hypergraphs  $\overline{\mathcal{B}}_{n,k}(A, B)$  where  $|A|$  is odd. Further, if  $n/k$  is odd then  $\mathcal{H}_{\text{ext}}(n, k)$  also contains all hypergraphs  $\mathcal{B}_{n,k}(A, B)$  where  $|A|$  is even; if  $n/k$  is even then  $\mathcal{H}_{\text{ext}}(n, k)$  also contains all hypergraphs  $\mathcal{B}_{n,k}(A, B)$  where  $|A|$  is odd.

It is easy to see that no hypergraph in  $\mathcal{H}_{\text{ext}}(n, k)$  contains a perfect matching. Indeed, first assume that  $|A|$  is even and  $n/k$  is odd. Since every edge of  $\mathcal{B}_{n,k}(A, B)$  intersects  $A$  in an odd number of vertices, one cannot cover  $A$  with an odd number of disjoint odd sets. Similarly  $\mathcal{B}_{n,k}(A, B)$  does not contain a perfect matching if  $|A|$  is odd and  $n/k$  is even. Finally, if  $|A|$  is odd then since every edge of  $\overline{\mathcal{B}}_{n,k}(A, B)$  intersects  $A$  in an even number of vertices,  $\overline{\mathcal{B}}_{n,k}(A, B)$  does not contain a perfect matching.

Given  $\ell \in \mathbb{N}$  such that  $k/2 \leq \ell \leq k-1$  define  $\delta(n, k, \ell)$  to be the maximum of the minimum  $\ell$ -degrees among all the hypergraphs in  $\mathcal{H}_{\text{ext}}(n, k)$ . For example, it is not hard to see that

$$\delta(n, k, k-1) = \begin{cases} n/2 - k + 2 & \text{if } k/2 \text{ is even and } n/k \text{ is odd} \\ n/2 - k + 3/2 & \text{if } k \text{ is odd and } (n-1)/2 \text{ is odd} \\ n/2 - k + 1/2 & \text{if } k \text{ is odd and } (n-1)/2 \text{ is even} \\ n/2 - k + 1 & \text{otherwise.} \end{cases} \quad (1.1)$$

In [16, 17] we prove the following exact version of Pikhurko's result.

**Theorem 1.1.** *Let  $k, \ell \in \mathbb{N}$  such that  $k \geq 3$  and  $k/2 \leq \ell \leq k-1$ . Then there exists an  $n_0 \in \mathbb{N}$  such that the following holds. Suppose  $H$  is a  $k$ -uniform hypergraph on  $n \geq n_0$  vertices where  $k$  divides  $n$ . If*

$$\delta_\ell(H) > \delta(n, k, \ell)$$

*then  $H$  contains a perfect matching.*

In [16] we prove Theorem 1.1 for  $k$  divisible by 4 and then in [17] we extend this result to all values of  $k$ . Independent to our work, Czygrinow and Kamat [2] have proven Theorem 1.1 in the case when  $k = 4$  and  $\ell = 2$ .

As explained before, the minimum  $\ell$ -degree condition in Theorem 1.1 is best possible. Theorem 1.1 and (1.1) together give the aforementioned result of Rödl, Ruciński and Szemerédi [15].

In general, the precise value of  $\delta(n, k, \ell)$  is unknown because it is not known what value of  $|A|$  maximizes the minimum  $\ell$ -degree of  $\mathcal{B}_{n,k}(A, B)$  (or  $\overline{\mathcal{B}}_{n,k}(A, B)$ ). (See [16] for a discussion on this.) However, in [16] we gave a tight upper bound on  $\delta(n, 4, 2)$ .

## 2 Overview of the proof of Theorem 1.1

The proof of Theorem 1.1 follows the so-called *stability approach*. We first prove that

- (i)  $H$  has a perfect matching or;
- (ii)  $H$  is ‘close’ to one of the hypergraphs  $\mathcal{B}_{n,k}(A, B)$  or  $\overline{\mathcal{B}}_{n,k}(A, B)$  in  $\mathcal{H}_{\text{ext}}(n, k)$ .

The extremal situation (ii) is then dealt with separately. For example, suppose  $H$  is ‘close’ to an element  $\mathcal{B}_{n,k}(A, B)$  from  $\mathcal{H}_{\text{ext}}(n, k)$ . (So we can view  $A, B$  as a partition of  $V(H)$ .) The minimum  $\ell$ -degree condition on  $H$  ensures that  $H$  contains an edge  $e$  that intersects  $A$  in an even number of vertices. Recall that no such edge exists in  $\mathcal{B}_{n,k}(A, B)$ ; this is the ‘reason’ why  $\mathcal{B}_{n,k}(A, B)$  does not have a perfect matching. Thus,  $e$  acts as a ‘parity breaking’ edge and can be used to form part of a perfect matching in  $H$ .

### Almost perfect matchings

To show that (i) or (ii) holds, we apply the following result of Markström and Ruciński [11] to ensure an ‘almost’ perfect matching in  $H$ .

**Theorem 2.1 (Lemma 2 in [11]).** *For each integer  $k \geq 3$ , every  $1 \leq \ell \leq k - 2$  and every  $\gamma > 0$  there exists an  $n_0 \in \mathbb{N}$  such that the following holds. Suppose that  $H$  is a  $k$ -uniform hypergraph on  $n \geq n_0$  vertices such that*

$$\delta_\ell(H) \geq \left( \frac{k - \ell}{k} - \frac{1}{k^{(k-\ell)}} + \gamma \right) \binom{n - \ell}{k - \ell}.$$

*Then  $H$  contains a matching covering all but at most  $\sqrt{n}$  vertices.*

(In [11], Markström and Ruciński only stated Theorem 2.1 for  $1 \leq \ell < k/2$ . In fact, their proof works for all values of  $\ell$  such that  $1 \leq \ell \leq k - 2$ .) In the case when  $\ell = k - 1$ , we need a result of Rödl, Ruciński and Szemerédi [15, Fact 2.1]: Suppose  $H$  is a  $k$ -uniform hypergraph on  $n$  vertices. If  $\delta_{k-1}(H) \geq n/k$ , then  $H$  contains a matching covering all but at most  $k^2$  vertices in  $H$ . Note that this minimum codegree condition is substantially smaller than the corresponding condition in Theorem 1.1. Further, if  $k/2 \leq \ell < k - 1$  then the minimum  $\ell$ -degree condition in Theorem 2.1 is also substantially smaller than the minimum  $\ell$ -degree in Theorem 1.1.

### Absorbing sets

Given a  $k$ -uniform hypergraph  $H$ , a set  $S \subseteq V(H)$  is called an *absorbing set* for  $Q \subseteq V(H)$ , if both  $H[S]$  and  $H[S \cup Q]$  contain perfect matchings.

If the hypergraph  $H$  in Theorem 1.1 contains a ‘small’ set  $S$  which is an absorbing set for any set  $Q \subseteq V(H)$  where  $|Q| \leq \sqrt{n}$  is divisible by  $k$ , then it is easy to find a perfect matching in  $H$ . Indeed, in this case the minimum  $\ell$ -degree of  $H - S$  satisfies the hypothesis of Theorem 2.1 (or the hypothesis of Fact 2.1 in [15] if  $\ell = k - 1$ ). Thus,  $H - S$  contains a matching  $M$  covering all but a set  $Q$  of at most  $\sqrt{n}$  vertices. Then since  $H[S \cup Q]$  contains a perfect matching  $M'$ ,  $M \cup M'$  is a perfect matching in  $H$ .

We give two conditions that ensure such an absorbing set  $S$  exists in  $H$  (and thus guarantee a perfect matching in  $H$ ). Roughly speaking, the first condition asserts that  $V(H)$  contains ‘many’  $\ell$ -tuples whose  $\ell$ -degree is ‘significantly’ larger than  $\delta(n, k, \ell)$ . The second condition concerns a certain ‘common neighbourhood’ property. (Fixing  $r := \lceil k/2 \rceil$ , this condition roughly asserts that for any  $r$ -tuple  $P \in \binom{V(H)}{r}$ , more than half of the  $r$ -tuples  $P'$  in  $\binom{V(H)}{r}$  are such that  $P$  and  $P'$  have a common neighbourhood which is not ‘too small’.) We will refer to these properties as  $(\alpha)$  and  $(\beta)$  respectively.

### The auxiliary graph $G$

We then show that if  $H$  does not satisfy  $(\alpha)$  and  $(\beta)$ , then (ii) must be satisfied. That is,  $H$  is ‘close’ to one of the hypergraphs  $\mathcal{B}_{n,k}(A, B)$  or  $\overline{\mathcal{B}}_{n,k}(A, B)$  in  $\mathcal{H}_{\text{ext}}(n, k)$ . For this, we consider an auxiliary bipartite graph  $G$  defined as follows: Set  $r := \lceil k/2 \rceil$ ,  $r' := \lfloor k/2 \rfloor$ ,  $X^r := \binom{V(H)}{r}$  and  $Y^{r'} := \binom{V(H)}{r'}$ . Further, let  $N := \binom{n}{r}$  and  $N' := \binom{n}{r'}$ .  $G$  has vertex classes  $X^r$  and  $Y^{r'}$ . Two vertices  $x_1 \dots x_r \in X^r$  and  $y_1 \dots y_{r'} \in Y^{r'}$  are adjacent in  $G$  if and only if  $x_1 \dots x_r y_1 \dots y_{r'} \in E(H)$ .

We show that, since  $H$  fails to satisfy  $(\alpha)$  and  $(\beta)$ ,  $G$  is ‘close’ to the disjoint union of two copies of  $K_{N/2, N'/2}$ . Once we have this information, we give direct arguments on  $G$  to show that this implies that (ii) is indeed satisfied.

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# Fractional and integer matchings in uniform hypergraphs

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**Abstract.** A conjecture of Erdős from 1965 suggests the minimum number of edges in a  $k$ -uniform hypergraph on  $n$  vertices which forces a matching of size  $t$ , where  $t \leq n/k$ . Our main result verifies this conjecture asymptotically, for all  $t < 0.48n/k$ . This gives an approximate answer to a question of Huang, Loh and Sudakov, who proved the conjecture for  $t \leq n/3k^2$ . As a consequence of our result, we extend bounds of Bollobás, Daykin and Erdős by asymptotically determining the minimum vertex degree which forces a matching of size  $t < 0.48n/(k-1)$  in a  $k$ -uniform hypergraph on  $n$  vertices. We also obtain further results on  $d$ -degrees which force large matchings. In addition we improve bounds of Markström and Ruciński on the minimum  $d$ -degree which forces a perfect matching in a  $k$ -uniform hypergraph on  $n$  vertices. Our approach is to inductively prove fractional versions of the above results and then translate these into integer versions.

## Large matchings in hypergraphs with many edges

A  $k$ -uniform hypergraph is a pair  $G = (V, E)$  where  $V$  is a finite set of vertices and the edge set  $E$  consists of unordered  $k$ -tuples of elements of  $V$ . A matching (or integer matching)  $M$  in  $G$  is a set of disjoint edges of  $G$ . The size of  $M$  is the number of edges in  $M$ .  $M$  is perfect if it has size  $|V|/k$ .

A classical theorem of Erdős and Gallai [6] determines the number of edges in a graph which forces a matching of a given size. In 1965, Erdős [5] made a conjecture which would generalize this to  $k$ -uniform hypergraphs.

**Conjecture 1.** Let  $n, k \geq 2$  and  $1 \leq s \leq n/k$  be integers. The minimum number of edges in a  $k$ -uniform hypergraph on  $n$  vertices which forces a matching of size  $s$  is

$$\max \left\{ \binom{ks-1}{k}, \binom{n}{k} - \binom{n-s+1}{k} \right\} + 1.$$

It is easy to see that the conjecture would be best possible: the first expression in the lower bound is obtained by considering the  $k$ -uniform

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clique  $K_{ks-1}^{(k)}$  (complemented by  $n - ks + 1$  isolated vertices); the second expression in the lower bound is obtained as follows. Let  $H(s)$  be a  $k$ -uniform hypergraph on  $n$  vertices with edge set consisting of all  $k$ -element subsets of  $V(H(s))$  intersecting a given subset of  $V(H(s))$  of size  $s - 1$ , that is  $H(s) = K_n^{(k)} - K_{n-s+1}^{(k)}$ .

The case  $s = 2$  of Conjecture 1 corresponds to the Erdős-Ko-Rado Theorem on intersecting families [7]. The conjecture also has applications to the Manickam-Mikós-Singhi conjecture in number theory (for details see *e.g.* [2]). Despite its seeming simplicity Conjecture 1 is still wide open in general. For the cases  $k \leq 4$ , it was verified asymptotically by Alon, Frankl, Huang, Rödl, Ruciński and Sudakov [1]. For  $k = 3$ , it was recently proved by Frankl [8]. Bollobás, Daykin and Erdős [3] proved Conjecture 1 for general  $k$  whenever  $s < n/(2k^3)$ , which extended earlier results of Erdős [5]. Huang, Loh and Sudakov [10] proved it for  $s < n/(3k^2)$ . The main result in this paper verifies Conjecture 1 asymptotically for matchings of any size up to almost half the size of a perfect matching. This gives an asymptotic answer to a question in [10].

**Theorem 2.** [15] *Let  $n, k \geq 2$  and  $0 \leq a < 0.48/k$  be such that  $n, k, an \in \mathbb{N}$ . The minimum number of edges in a  $k$ -uniform hypergraph on  $n$  vertices which forces a matching of size  $an$  is*

$$(1 - (1 - a)^k + o(1)) \binom{n}{k}.$$

### Large matchings in hypergraphs with large degrees

It is also natural to consider degree conditions that force matchings in uniform hypergraphs. Given a  $k$ -uniform hypergraph  $G = (V, E)$  and  $S \in \binom{V}{d}$ , where  $0 \leq d \leq k - 1$ , let  $\deg_G(S) = |\{e \in E : S \subseteq e\}|$  be the *degree* of  $S$  in  $G$ . Let  $\delta_d(G) = \min_{S \in \binom{V}{d}} \{\deg_G(S)\}$  be the *minimum  $d$ -degree* of  $G$ . When  $d = 1$ , we refer to  $\delta_1(G)$  as the *minimum vertex degree* of  $G$ . Note that  $\delta_0(G) = |E|$ .

For integers  $n, k, d, s$  satisfying  $0 \leq d \leq k - 1$  and  $0 \leq s \leq n/k$ , we let  $m_d^s(k, n)$  denote the minimum integer  $m$  such that every  $k$ -uniform hypergraph  $G$  on  $n$  vertices with  $\delta_d(G) \geq m$  has a matching of size  $s$ . So the results discussed in the previous section correspond to the case  $d = 0$ . The following degree condition for forcing perfect matchings has been conjectured in [9, 14] and also received much attention recently.

**Conjecture 3.** Let  $n$  and  $1 \leq d \leq k - 1$  be such that  $n, d, k, n/k \in \mathbb{N}$ . Then

$$m_d^{n/k}(k, n) = \left( \max \left\{ \frac{1}{2}, 1 - \left( \frac{k-1}{k} \right)^{k-d} \right\} + o(1) \right) \binom{n-d}{k-d}.$$

The lower bound here is given by the hypergraph  $H(n/k)$  defined after Conjecture 1 and the following parity-based construction from [13]. For any integers  $n, k$ , let  $H'$  be a  $k$ -uniform hypergraph on  $n$  vertices with vertex partition  $A \cup B = V(H')$ , such that  $||A| - |B|| \leq 2$  and  $|A|$  and  $n/k$  have different parity. Let  $H'$  have edge set consisting of all  $k$ -element subsets of  $V(H')$  that intersect  $A$  in an odd number of vertices. Observe that  $H'$  has no perfect matching, and that for every  $1 \leq d \leq k - 1$  we have that  $\delta_d(H') = (1/2 + o(1))\binom{n-d}{k-d}$ .

For  $d = k - 1$ ,  $m_{k-1}^{n/k}(k, n)$  was determined exactly by Rödl, Ruciński and Szemerédi [19]. This was generalized by Treglown and Zhao [20], who determined the extremal families for all  $d \geq k/2$ . The extremal constructions are similar to the parity based one of  $H'$  above. For  $d < k/2$  less is known. In [1] Conjecture 3 was proved for  $k - 4 \leq d \leq k - 1$ , by reducing it to a probabilistic conjecture of Samuels. In particular, this implies Conjecture 3 for  $k \leq 5$ . Khan [11], and independently Kühn, Osthus and Treglown [16], determined  $m_1^{n/k}(k, n)$  exactly for  $k = 3$ . Khan [12] also determined  $m_1^{n/k}(k, n)$  exactly for  $k = 4$ . As a consequence of these results,  $m_1^s(k, n)$  is determined exactly whenever  $s \leq n/k$  and  $k \leq 4$  (for details see the concluding remarks in [16]). More generally, we propose the following version of Conjecture 3 for non-perfect matchings.

**Conjecture 4.** For all  $\varepsilon > 0$  and all integers  $n, d, k, s$  with  $1 \leq d \leq k - 1$  and  $0 \leq s \leq (1 - \varepsilon)n/k$  we have

$$m_d^s(k, n) = \left(1 - \left(1 - \frac{s}{n}\right)^{k-d} + o(1)\right) \binom{n-d}{k-d}.$$

In fact it may be that the bound holds for all  $s \leq n - C$ , for some  $C$  depending only on  $d$  and  $k$ . The lower bound here is given by  $H(s)$ . The case  $d = k - 1$  of Conjecture 4 follows easily from the determination of  $m_{k-1}^s(k, n)$  for  $s$  close to  $n/k$  in [19]. Bollobás, Daykin and Erdős [3] determined  $m_1^s(k, n)$  for small  $s$ , *i.e.* whenever  $s < n/2k^3$ . As a consequence of our main result, for  $1 \leq d \leq k - 2$  we are able to determine  $m_d^s(k, n)$  asymptotically for non-perfect matchings of any size less than  $0.48n/(k - d)$ . Note that this proves Conjecture 4 in the case  $0.53k \leq d \leq k - 2$ , say.

**Theorem 5.** [15] *Let  $\varepsilon > 0$  and let  $n, k, d$  be integers with  $1 \leq d \leq k - 2$ , and let  $0 \leq a < \min\{0.48/(k - d), (1 - \varepsilon)/k\}$  be such that  $an \in \mathbb{N}$ . Then*

$$m_d^{an}(k, n) = \left(1 - (1 - a)^{k-d} + o(1)\right) \binom{n-d}{k-d}.$$



We now focus again on the case  $s = n/k$ , *i.e.* perfect matchings. It was shown by Hàn, Person and Schacht [9] that for  $k \geq 3$ ,  $1 \leq d < k/2$  we have  $m_d^{n/k}(k, n) \leq ((k-d)/k + o(1)) \binom{n-d}{k-d}$ . (The case  $d = 1$  of this is already due to Daykin and Häggkvist [4].) These bounds were slightly improved by Markström and Ruciński [17], using similar techniques, to

$$m_d^{n/k}(k, n) \leq \left( \frac{k-d}{k} - \frac{1}{k^{k-d}} + o(1) \right) \binom{n-d}{k-d}.$$

Using similar methods to those developed to prove Theorem 5, we are also able to slightly improve on this bound.

**Theorem 6 ([15]).** *Let  $n$  and  $1 \leq d < k/2$  be such that  $n, k, d, n/k \in \mathbb{N}$ . Then*

$$m_d^{n/k}(k, n) \leq \left( \frac{k-d}{k} - \frac{k-d-1}{k^{k-d}} + o(1) \right) \binom{n-d}{k-d}.$$

### Large fractional matchings

Our approach to proving these results uses the concepts of fractional matchings and fractional vertex covers. A *fractional matching* in a  $k$ -uniform hypergraph  $G = (V, E)$  is a function  $w : E \rightarrow [0, 1]$  of weights of edges, such that for each  $v \in V$  we have  $\sum_{e \in E: v \in e} w(e) \leq 1$ . The *size* of  $w$  is  $\sum_{e \in E} w(e)$ .  $w$  is *perfect* if it has size  $|V|/k$ . A *fractional vertex cover* in  $G$  is a function  $w : V \rightarrow [0, 1]$  of weights of vertices, such that for each  $e \in E$  we have  $\sum_{v \in e} w(v) \geq 1$ . The *size* of  $w$  is  $\sum_{v \in V} w(v)$ .

A key idea (already used *e.g.* in [1, 18]) is that we can switch between considering the largest fractional matching and the smallest fractional vertex cover of a hypergraph. The determination of these quantities are dual linear programming problems, and hence by the Duality Theorem they have the same size.

For  $s \in \mathbb{R}$  we let  $f_d^s(k, n)$  denote the minimum integer  $m$  such that every  $k$ -uniform hypergraph  $G$  on  $n$  vertices with  $\delta_d(G) \geq m$  has a fractional matching of size  $s$ . It was shown in [18] that  $f_{k-1}^{n/k}(k, n) = \lceil n/k \rceil$ . Similarly to [1], we now formulate the fractional version of Conjecture 1.

**Conjecture 7.** For all integers  $n, k, s$  with  $k \geq 2$  and  $1 \leq s \leq n/k$  we have

$$f_0^s(k, n) = \max \left\{ \binom{ks-1}{k}, \binom{n}{k} - \binom{n-s+1}{k} \right\} + 1.$$

As discussed in [1], this conjecture has applications to a problem on information storage and retrieval. To prove Theorems 2 and 5, we first

prove Conjecture 7 asymptotically for fractional matchings of any size up to  $0.48n/k$ .

**Theorem 8 ([15]).** *Let  $n, k \geq 2$  be integers and let  $0 \leq a \leq 0.48/k$ . Then*

$$f_0^{an}(k, n) = (1 - (1 - a)^k + o(1)) \binom{n}{k}.$$

We use Theorem 8, along with methods similar to those developed in [1], to convert our edge-density conditions for the existence of fractional matchings into corresponding minimum degree conditions. For  $1 \leq d \leq k - 2$  the following theorem asymptotically determines  $f_d^s(k, n)$  for fractional matchings of any size up to  $0.48n/(k - d)$ . Note that this determines  $f_d^s(k, n)$  asymptotically for all  $s \in (0, n/k)$  whenever  $d \geq 0.52k$ .

**Theorem 9 ([15]).** *Let  $n, k \geq 3$ , and  $1 \leq d \leq k - 2$  be integers and let  $0 \leq a \leq \min\{0.48/(k - d), 1/k\}$ . Then*

$$f_d^{an}(k, n) = (1 - (1 - a)^{k-d} + o(1)) \binom{n-d}{k-d}.$$

We then use Theorem 8 and a variant of Theorem 9, along with the Weak Hypergraph Regularity Lemma, to prove Theorems 2 and 5 respectively, by converting our fractional matchings into integer ones. We prove Theorem 6 in a similar fashion, via the following two theorems.

**Theorem 10 ([15]).** *Let  $n, k \geq 2, d \geq 1$  be integers. Then*

$$f_0^{n/(k+d)}(k, n) \leq \left( \frac{k}{k+d} - \frac{k-1}{(k+d)^k} + o(1) \right) \binom{n}{k}.$$

**Theorem 11 ([15]).** *Let  $n, k \geq 3, 1 \leq d \leq k - 2$  be integers. Then*

$$f_d^{n/k}(k, n) \leq \left( \frac{k-d}{k} - \frac{k-d-1}{k^{k-d}} + o(1) \right) \binom{n-d}{k-d}.$$

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## Cubic graphs

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# On cubic bridgeless graphs whose edge-set cannot be covered by four perfect matchings

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**Abstract.** The problem of establishing the number of perfect matchings necessary to cover the edge-set of a cubic bridgeless graph is strongly related to a famous conjecture of Berge and Fulkerson. In this paper we show that deciding whether this number is at most 4 for a given cubic bridgeless graph is NP-complete. Our proof makes heavy use of small cuts, so an interesting problem is to construct large *snarks* (cyclically 4-edge-connected cubic graphs of girth at least five and chromatic index four) whose edge-set cannot be covered by 4 perfect matchings. A well-known example is the Petersen graph and the unique other known examples were recently found by Häggglund using a computer program. In this paper we construct an infinite family  $\mathcal{F}$  of snarks whose edge-set cannot be covered by 4 perfect matchings. It turns out that the family  $\mathcal{F}$  also has interesting properties with respect to the shortest cycle cover problem. The Petersen graph and one of the graphs constructed by Häggglund are the only known snarks with  $m$  edges and no cycle cover of length  $\frac{4}{3}m$  (indeed their shortest cycle covers have length  $\frac{4}{3}m + 1$ ). We show that all the members of  $\mathcal{F}$  satisfy the former property, and we construct a snark with no cycle cover of length less than  $\frac{4}{3}m + 2$ .

## 1 Introduction

Throughout this paper, a graph  $G$  always means a simple connected finite graph (without loops and parallel edges). A *perfect matching* of  $G$  is a 1-regular spanning subgraph of  $G$ . In this context, a *cover*, or a  $k$ -*cover* of  $G$  is a set of  $k$  perfect matchings of  $G$  such that each edge of  $G$  belongs to at least one of the perfect matchings. Following the terminology introduced in [2], the *excessive index* of  $G$ , denoted by  $\chi'_e(G)$ , is the least integer  $k$  such that  $G$  has a  $k$ -cover.

A famous conjecture of Berge and Fulkerson [6] states that the edge-set of every cubic bridgeless graph can be covered by 6 perfect matchings,

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such that each edge is covered precisely twice. The second author recently proved that this conjecture is equivalent to another conjecture of Berge stating that every cubic bridgeless graph has excessive index at most five [10].

Note that a cubic bridgeless graph has excessive index 3 if and only if it is 3-edge-colorable, and deciding the latter is NP-complete. Hägglund [7, Problem 3] asked if it is possible to give a characterization of all cubic graphs with excessive index 5. We show that a characterization of all cubic graphs with excessive index at least 5 is unlikely unless P=NP.

**Theorem 1.1.** *Determining whether a cubic bridgeless graph  $G$  satisfies  $\chi'_e(G) \leq 4$  (resp.  $\chi'_e(G) = 4$ ) is an NP-complete problem.*

The gadgets used in the proof of NP-completeness have many 2-edge-cuts, so our first result does not say much about 3-edge-connected cubic graphs. A *snark* is a non 3-edge-colorable cubic graph with girth (length of a shortest cycle) at least five that is cyclically 4-edge-connected. A question raised by Fouquet and Vanherpe [5] is whether the Petersen graph is the unique snark with excessive index at least five. This question was answered by the negative by Hägglund using a computer program [7]. He proved that the smallest snark distinct from the Petersen graph having excessive index at least five is a graph  $\mathring{H}$  on 34 vertices. We show that the graph  $\mathring{H}$  found by Hägglund is a special member of an infinite family  $\mathcal{F}$  of snarks with excessive index precisely five.

We also show that our family  $\mathcal{F}$  has interesting properties with respect to shortest cycle covers. A *cycle cover* of a graph  $G$  is a covering of the edge-set of  $G$  by cycles (connected subgraphs with all degrees even), such that each edge is in at most one cycle. The *length* of a cycle cover is the sum of the number of edges in all cycles of the cover. The *Shortest Cycle Cover Conjecture* of Alon and Tarsi [1] states that every bridgeless graph  $G$  has a cycle cover of length at most  $\frac{7}{3}|E(G)|$ . Note that it was recently proved by Steffen [11] and Hou, Lai, and Zhang [8] that it is enough to prove the Cycle Double Cover conjecture (a famous conjecture due to Seymour and Szekeres, and implied by the Shortest Cycle Cover Conjecture) for snarks with excessive index at least five. For a cubic bridgeless graph  $G$  there is a trivial lower bound of  $\frac{4}{3}|E(G)|$  on the length of a cycle cover. On the other hand, the best known upper bound,  $\frac{34}{21}|E(G)|$ , was obtained by Kaiser, Král', Lidický, Nejedlý, and Šámal [9] in 2010.

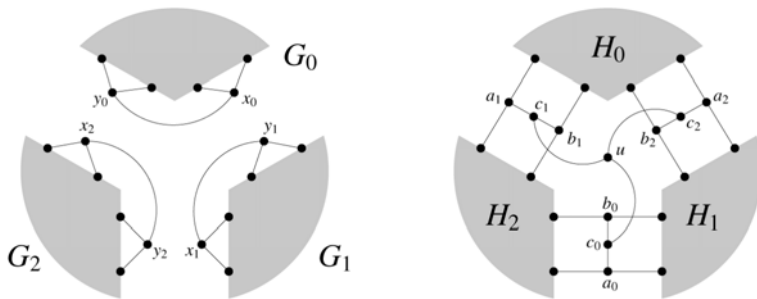
We show that no graph  $G$  in our infinite family  $\mathcal{F}$  has a cycle cover of length  $\frac{4}{3}|E(G)|$ . We also find the first known snark with no cycle cover of length less than  $\frac{4}{3}|E(G)| + 2$  (it has 106 vertices).

The reader is referred to the full version of the paper [4] for more results and proofs.

## 2 An infinite family of snarks with excessive index 5

In this section we show how to construct a snark  $G$  with excessive index at least 5 from three snarks  $G_0, G_1, G_2$  each having excessive index at least 5. Taking  $G_0 = G_1 = G_2$  to be the Petersen graph, we obtain the graph  $\mathring{H}$  found by Hägglund using a computer program [7], and for which no combinatorial proof showing that its excessive index is 5 was known. Our proof holds for any graph obtained using this construction, thus we exhibit an infinite family  $\mathcal{F}$  of snarks with excessive index 5. This answers a question of Fouquet and Vanherpe [5] in a very strong sense.

*The construction* For  $i = 0, 1, 2$ , consider a snark  $G_i$  with an edge  $x_i y_i$ . Let  $x_i^0$  and  $x_i^1$  (resp.  $y_i^0$  and  $y_i^1$ ) be the neighbors of  $x_i$  (resp.  $y_i$ ) in  $G_i$  different from  $y_i$  (resp.  $x_i$ ). For  $i = 0, 1, 2$ , let  $H_i$  be the graph obtained from  $G_i$  by removing vertices  $x_i$  and  $y_i$ . We construct a new graph  $G$  from the disjoint union of  $H_0, H_1, H_2$  and a new vertex  $u$  as follows. For  $i = 0, 1, 2$ , we introduce a set  $A_i = \{a_i, b_i, c_i\}$  of vertices such that  $a_i$  is adjacent to  $x_{i+1}^0$  and  $y_{i-1}^0$ ,  $b_i$  is adjacent to  $x_{i+1}^1$  and  $y_{i-1}^1$ , and  $c_i$  is adjacent to  $a_i, b_i$  and  $u$  (here and in the following all indices  $i$  are taken modulo 3). This construction is depicted in Figure 2.1.



**Figure 2.1.** The construction of  $G$  (right) from  $G_0, G_1$ , and  $G_2$ .

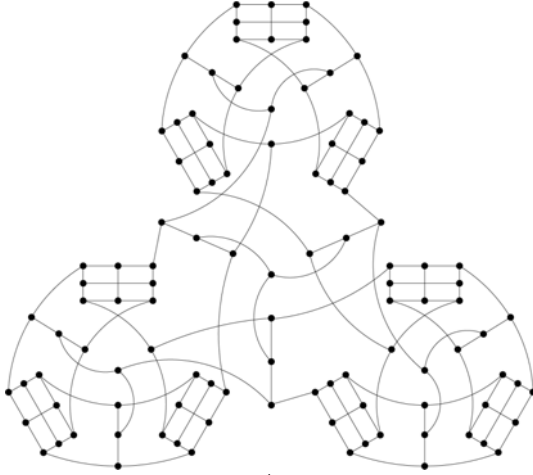
**Theorem 2.1.** *Let  $G_0, G_1, G_2$  be snarks such that  $\chi'_e(G_i) \geq 5$  for  $i = 0, 1, 2$ . Then the graph  $G$  obtained from  $G_0, G_1, G_2$  by the construction above is a snark with  $\chi'_e(G) \geq 5$ .*

## 3 Shortest cycle cover

The length of a shortest cycle cover of a bridgeless graph  $G$  is denoted by  $\text{scc}(G)$ . Note that for any cubic graph  $G$ ,  $\text{scc}(G) \geq \frac{4}{3}|E(G)|$ . We show

that for any graph  $G$  in the family  $\mathcal{F}$ ,  $\text{scc}(G) > \frac{4}{3}|E(G)|$ .

**Theorem 3.1.** *Let  $G_0, G_1, G_2$  be snarks such that  $\text{scc}(G_i) > \frac{4}{3}|E(G_i)|$  for  $i = 0, 1, 2$ . Then the graph  $G$  obtained from  $G_0, G_1, G_2$  by the construction of Section 2 is a snark with  $\text{scc}(G) > \frac{4}{3}|E(G)|$ .*



**Figure 3.1.** A snark  $G$  with  $\text{scc}(G) \geq \frac{4}{3}|E(G)| + 2$ .

Furthermore, we exhibit the first known snark  $G$  satisfying  $\text{scc}(G) > \frac{4}{3}|E(G)| + 1$ .

**Theorem 3.2.** *Any cycle cover of the graph  $G$  depicted in Figure 3.1 has length at least  $\frac{4}{3}|E(G)| + 2$ .*

## 4 Open problems

Häggglund proposed the following two problems (Problems 3 and 4 in [7]):

- Is it possible to give a simple characterization of cubic graphs  $G$  with  $\chi'_e(G) = 5$ ?
- Are there any cyclically 5-edge-connected snarks  $G$  with excessive index at least five distinct from the Petersen graph?

If Berge-Fulkerson conjecture is true, then the first problem has a negative answer by Theorem 1.1 (unless  $P=NP$ ). However, even assuming the correctness of Berge-Fulkerson conjecture, the second problem is still open, since each element of the infinite family  $\mathcal{F}$  contains cyclic 4-edge-cuts. Furthermore, the gadgets we use in the proof of Theorem 1.1



have many 2-edge-cuts. Hence we leave open the problem of establishing whether it is possible to give a simple characterization of 3-edge-connected or cyclically 4-edge-connected cubic graphs with excessive index (at least) 5.

Theorem 3.2 proves the existence of a snark  $G \in \mathcal{F}$  with no cycle cover of length less than  $\frac{4}{3}|E(G)| + 2$ . We believe that there exist snarks in  $\mathcal{F}$  for which the constant 2 can be replaced by an arbitrarily large number. On the other hand, Brinkmann, Goedgebeur, Hägglund, and Markström [3] conjectured that every snark  $G$  has a cycle cover of size at most  $(\frac{4}{3} + o(1))|E(G)|$ .

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# Relating ordinary and total domination in cubic graphs of large girth

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**Abstract.** For a cubic graph  $G$  of order  $n$ , girth at least  $g$ , and domination number  $(\frac{1}{4} + \epsilon)n$  for some  $\epsilon \geq 0$ , we show that the total domination number of  $G$  is at most  $\frac{13}{32}n + O(\frac{n}{g}) + O(\epsilon n)$ , which implies  $\frac{\gamma_t(G)}{\gamma(G)} \leq 1.92472 + O(\frac{n}{g})$ .

For a finite, simple, and undirected graph  $G$ , a set  $D$  of vertices of  $G$  is a *dominating set* of  $G$  if every vertex in  $V(G) \setminus D$  has a neighbor in  $D$ . Similarly, a set  $T$  of vertices of  $G$  is a *total dominating set* of  $G$  if every vertex in  $V(G)$  has a neighbor in  $T$ . Note that a graph has a total dominating set exactly if it has no isolated vertex. The minimum cardinalities of a dominating and a total dominating set of  $G$  are known as the *domination number*  $\gamma(G)$  of  $G$  and the *total domination number*  $\gamma_t(G)$  of  $G$ , respectively. These two parameters are among the most fundamental and well studied parameters in graph theory [3, 4, 6]. In view of their computational hardness especially upper bounds were investigated in great detail.

The two parameters are related by some very simple inequalities. Let  $G$  be a graph without isolated vertices. Since every total dominating set of  $G$  is also a dominating set of  $G$ , we immediately obtain

$$\gamma_t(G) \geq \gamma(G). \tag{1}$$

Similarly, if  $D$  is a dominating set of  $G$ , then adding, for every isolated vertex  $u$  of the subgraph  $G[D]$  of  $G$  induced by  $D$ , a neighbor of  $u$  in  $G$

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to the set  $D$ , results in a total dominating set of  $G$ , which implies

$$\gamma_t(G) \leq 2\gamma(G). \quad (2)$$

The complete bipartite graph  $K_{n/2, n/2}$  and the complete graph  $K_n$  show that (1) and (2) are sharp, respectively. In [2, 5] the trees that satisfy (1) or (2) with equality are characterized constructively.

While numerous very deep results concerning bounds on the domination number and the total domination number under various conditions have been obtained, the relation of these two parameters is not really well understood. The characterization of the extremal graphs for (1) and (2) and/or improvements of (1) and (2) even under strong additional assumptions appear to be very difficult. If the graph  $G$  arises, for instance, by subdividing every edge of the complete graph  $K_n$  with  $n \geq 3$  twice, then  $\gamma(G) = n$  and  $\gamma_t(G) = 2n - 1$ , that is, forbidding cycles of length up to 8 does not allow to improve (2) by much. Similarly, if the graph  $G$  has vertex set  $\bigcup_{i \in [k]} (A_i \cup B_i \cup C_i)$ , where

- the sets  $A_i$ ,  $B_i$ , and  $C_i$  for all  $i \in [k]$  are disjoint,
- $|A_i| = a$ ,  $|B_i| = a + 1$ , and  $|C_i| = ka$  for every  $i \in [k]$ ,
- the closed neighborhood  $N_G[u]$  of a vertex  $u$  in  $A_j$  for  $j \in [k]$  is  $B_j \cup \bigcup_{i \in [k]} A_i$ ,
- the closed neighborhood  $N_G[v]$  of a vertex  $v$  in  $B_j$  for  $j \in [k]$  is  $A_j \cup \{v\} \cup C_j$ , and
- the closed neighborhood  $N_G[w]$  of a vertex  $w$  in  $C_j$  for  $j \in [k]$  is  $B_j \cup C_j$ ,

then  $G$  is regular of degree  $(k + 1)a$ , has connectivity  $a$ , diameter 5,  $\gamma(G) = k + 1$ , and  $\gamma_t(G) = 2k$ , that is, a large minimum degree, a large degree of regularity, a large connectivity, a small diameter, and a large value of the domination number do not force any serious improvement of (2).

In the present extended abstract we consider the relation of the domination number and the total domination number for cubic graphs of large girth.

Let  $G$  be a cubic graph of order  $n$  and girth at least  $g$ , that is,  $G$  has no cycles of length less than  $g$ . Clearly,  $\gamma(G) \geq \frac{1}{4}n$  and  $\gamma_t(G) \geq \frac{1}{3}n$ . The best known upper bound on the domination number of  $G$ , improving earlier results from [10, 11], is due to Král *et al.* [9], who show

$$\gamma(G) \leq 0.299871n + O\left(\frac{n}{g}\right). \quad (3)$$

Combining the above bounds, we immediately obtain the following improvement of (1).

**Corollary 1.** *If  $G$  is a cubic graph of order  $n$  and girth at least  $g$ , then*

$$\frac{\gamma_t(G)}{\gamma(G)} \geq 1.111589 - O\left(\frac{1}{g}\right).$$

For a graph  $G$  of order  $n$ , minimum degree at least 2, and girth at least  $g$ , Henning and Yeo [7, 8] show  $\gamma_t(G) \leq \frac{1}{2}n + O\left(\frac{n}{g}\right)$ . Applying a trick from [10], this result leads to the following corollary.

**Corollary 2.** *If  $G$  is a cubic graph of order  $n$  and girth at least  $g$ , then*

$$\gamma_t(G) \leq \frac{121}{248}n + O\left(\frac{n}{g}\right) \leq 0.488n + O\left(\frac{n}{g}\right). \quad (4)$$

*Proof.* Let  $G$  be as in the statement. In view of the desired bound, we may assume that  $g$  is sufficiently large. Since the 5th power of the line graph of  $G$  is neither an odd cycle nor complete, has order  $\frac{3}{2}n$ , and maximum degree 124, the theorem of Brooks implies that there is a set  $M$  of at least  $\frac{3}{248}n$  edges of  $G$  at pairwise distance at least 5 in  $G$ . Let  $T_0$  denote the set of  $2|M|$  vertices incident with the edges in  $M$  and let  $G_1 = G \setminus N_G[T_0]$ . By construction, the graph  $G_1$  has order  $n - 6|M|$ , minimum degree at least 2, and girth at least  $g$ . By the above result of Henning and Yeo, the graph  $G_1$  has a total dominating set  $T_1$  of order at most  $\frac{1}{2}(n - 6|M|) + O\left(\frac{n}{g}\right)$ . Since  $T_0 \cup T_1$  is a total dominating set of  $G$ , we obtain

$$\begin{aligned} \gamma_t(G) &\leq 2|M| + \frac{1}{2}(n - 6|M|) + O\left(\frac{n}{g}\right) \\ &\leq \frac{1}{2}n - |M| + O\left(\frac{n}{g}\right) \\ &\leq \frac{1}{2}n - \frac{3}{248}n + O\left(\frac{n}{g}\right) \\ &= \frac{121}{248}n + O\left(\frac{n}{g}\right), \end{aligned}$$

which completes the proof.  $\square$

Combining the above bounds, we immediately obtain the following improvement of (2).

**Corollary 3.** *If  $G$  is a cubic graph of order  $n$  and girth at least  $g$ , then*

$$\frac{\gamma_t(G)}{\gamma(G)} \leq \frac{121}{62} + O\left(\frac{1}{g}\right) \leq 1.952 + O\left(\frac{1}{g}\right).$$

Note that Corollary 1 can only be close to the truth if the total domination number is close to  $\frac{1}{3}n$ . Similarly, Corollary 3 can only be close to the truth if the domination number is close to  $\frac{1}{4}n$ . Our main result shows that, for a cubic graph  $G$  of order  $n$  and girth at least  $g$ , for which the domination number is close to  $\frac{1}{4}n$ , the total domination number is smaller than guaranteed by (4). Specifically, we prove the following result.

**Theorem 4.** *If  $G$  is a cubic graph of order  $n$ , girth at least  $g$ , and domination number  $(\frac{1}{4} + \epsilon)n$  for some  $\epsilon \geq 0$ , then*

$$\gamma_t(G) \leq \frac{13}{32}n + \frac{15}{4(g-1)}n + \frac{187}{8}\epsilon n \leq 0.40625n + O\left(\frac{n}{g}\right) + O(\epsilon n).$$

Because of the space restriction, we omit the proof of Theorem 4 in the present extended abstract.

This result allows to improve Corollary 3 as follows.

**Corollary 5.** *If  $G$  is a cubic graph of order  $n$  and girth at least  $g$ , then*

$$\frac{\gamma_t(G)}{\gamma(G)} \leq \frac{701437}{364436} + O\left(\frac{n}{g}\right) \leq 1.92472 + O\left(\frac{n}{g}\right).$$

*Proof.* Let  $G$  be as in the statement and let  $\gamma(G) = (\frac{1}{4} + \epsilon)n$  for some  $\epsilon \geq 0$ . By Corollary 2 and Theorem 4, we obtain

$$\frac{\gamma_t(G)}{\gamma(G)} \leq \frac{\min\left\{\frac{13}{32} + \frac{187}{8}\epsilon, \frac{121}{248}n\right\}}{\frac{1}{4} + \epsilon} + O\left(\frac{n}{g}\right).$$

Since  $(\frac{13}{32} + \frac{187}{8}\epsilon) / (\frac{1}{4} + \epsilon)$  is increasing as a function of  $\epsilon \geq 0$  and  $\frac{13}{32} + \frac{187}{8}\epsilon = \frac{121}{248}n$  for  $\epsilon = \frac{81}{23188}$ , the desired result follows.  $\square$

While the constants in our results improve previous estimates, they are clearly far from the truth and should be improved. Suitably modifying the proof strategy of Theorem 4, it is possible to show an upper bound on the domination number of a cubic graph  $G$  of order  $n$  and girth at least  $g$ , for which the total domination number is close to  $\frac{1}{3}n$ . Unfortunately, this bound is weaker than the result of Král *et al.* [9].

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# Snarks with large oddness and small number of vertices

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**Abstract.** We estimate the minimum number of vertices of a cubic graph with given oddness and cyclic connectivity. We show that a 2-connected cubic graph  $G$  with oddness  $\omega(G)$  different from the Petersen graph has order at least  $5.41 \omega(G)$ , and for any integer  $k$  with  $2 \leq k \leq 6$  we construct an infinite family of cubic graphs with cyclic connectivity  $k$  and small oddness ratio  $|V(G)|/\omega(G)$ . For cyclic connectivity 2, 4, 5, and 6 we improve the upper bounds on the oddness ratio of snarks to 7.5, 13, 25, and 99 from the known values 9, 15, 76, and 118, respectively. We also construct a cyclically 4-connected snark of girth 5 with oddness 4 and order 44, improving the best previous value of 46.

## 1 Introduction

Snarks are connected bridgeless cubic graphs with chromatic index 4, sometimes required to satisfy additional conditions, such as cyclic 4-edge-connectivity and girth at least five, to avoid triviality. There are several important conjectures in graph theory where snarks are the principal obstacle: if true for snarks, they would hold for all graphs. Some of the conjectures have been verified for snarks that are close to 3-edge-colourable graphs. For example, the 5-flow conjecture is known to hold for snarks with oddness at most 2, and the cycle double cover conjecture has been verified for snarks with oddness at most 4 (see [4,5]). However, snarks with large oddness remain potential counterexamples to these and other conjectures, and therefore merit further investigation.

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The *oddness* of a cubic graph  $G$ , denoted by  $\omega(G)$ , is the minimum number of odd circuits in a 2-factor of  $G$ . A cubic graph is 3-edge-colourable if and only if its oddness is 0, so oddness provides a natural measure of uncolourability of a cubic graph. Another common measure of uncolourability is the *resistance* of  $G$ ,  $\rho(G)$ , the minimum number of edges whose removal from  $G$  yields a 3-edge-colourable graph. Since  $\rho(G) \leq \omega(G)$ , resistance provides a practical lower bound for oddness [8].

Our aim to provide bounds on the the ratio  $|V(G)|/\omega(G)$  for a snark  $G$  within the class of cyclically  $k$ -connected snarks. So far, only trivial lower bounds for oddness ratio have been known; as regards the upper bounds, there are various constructions, probably not optimal. Since the oddness ratio of the Petersen graph is 5, it is meaningless to attempt improving this absolute lower bound. In Section 22 we therefore take an asymptotic approach similar to that found in [3] and [8]. We summarise the known results and our improvements in Table 11; we only consider cyclic connectivity  $k \leq 6$  since no cyclically 7-connected snarks are known. In fact, they are believed not to exist.

connectivity $k$	LB	current UB	previous UB
2	5.41	7.5	9 (Steffen [8])
3	5.52	9	9 (Steffen [8])
4	5.52	13	15 (Hägglund [3])
5	5.83	25	76 (Steffen [8])
6	7	99	118 (Kochol [6])

**Table 1.** Upper (UB) and lower (LB) bounds on oddnes ratio  $|V|/\omega$ .

Besides general bounds, we are also interested in identifying the smallest snarks with oddness 4, addressing a long-standing open problem restated as Problem 4 in [1]. Our best results in this direction are shown in Figure 4.1 and described in Section 44. For more details and full proofs see [7].

## 2 Oddness and resistance ratios

The *oddness ratio* of a snark  $G$  is the quantity  $|V(G)|/\omega(G)$ , and its *resistance ratio* is  $|V(G)|/\rho(G)$ . We also examine asymptotic quantities

$$A_\omega = \liminf_{|V(G)| \rightarrow \infty} \frac{|V(G)|}{\omega(G)} \quad \text{and} \quad A_\rho = \liminf_{|V(G)| \rightarrow \infty} \frac{|V(G)|}{\rho(G)}.$$

Since the oddness ratio of a graph is at least as large as its resistance ratio, we have  $A_\omega \leq A_\rho$ . The oddness and resistance ratios heavily depend



on the cyclic connectivity of a graph in question. This suggests to study analogous values  $A_\omega^k$  and  $A_\rho^k$  obtained under the assumption that the class of snarks is restricted to those with cyclic connectivity at least  $k$ . Note that  $A_\omega^2 = A_\omega$ ,  $A_\rho^2 = A_\rho$ , and  $A_\omega^k \leq A_\rho^k$  for every  $k \geq 2$ . Similar ideas were pursued by Steffen [8] who proved that  $8 \leq A_\rho \leq 9$  and therefore  $A_\omega \leq A_\rho \leq 9$ . Since snarks constructed in [8] are cyclically 3-connected, we also have  $A_\omega = A_\omega^2 \leq A_\omega^3 \leq A_\rho^3 \leq 9$ .

### 3 Lower bounds on oddness ratio

A snark with girth at least 5 has oddness ratio at least 5. This bound is best possible because of the Petersen graph, but it can be improved for any other graph. Our approach is based on the following key observation.

**Proposition 3.1.** *Let  $\mathcal{C}$  be a set of 5-circuits of a bridgeless cubic graph  $G$ . Then  $G$  has a 2-factor that contains at most  $1/6$  of 5-circuits from  $\mathcal{C}$ .*

*Proof.* Let  $\mathcal{P}$  be the perfect matching polytope of  $G$ . For a vector  $\mathbf{x} \in \mathbb{R}^{|E(G)|}$  let  $\mathbf{x}(e)$  denote the entry corresponding to an edge  $e \in E(G)$ , and let  $\delta(U)$  be the set of all edges with precisely one end in the subgraph  $U$  of  $G$ . Since  $G$  is cubic and bridgeless, we have  $\mathcal{P} \neq \emptyset$ . Note that the vector  $\mathbf{t} = (1/3, 1/3, \dots, 1/3)$  always belongs to  $\mathcal{P}$ . Consider the function

$$f(\mathbf{x}) = \sum_{C \in \mathcal{C}} \sum_{e \in \delta(C)} \mathbf{x}(e)$$

defined for each  $\mathbf{x} \in \mathcal{P}$ . Since  $f$  is linear, there is a vertex of  $\mathcal{P}$  where  $f$  reaches its minimum. Since  $\mathbf{t} \in \mathcal{P}$ , we have  $f(\mathbf{x}_0) \leq 5/3 \cdot |\mathcal{C}|$ .

Let  $M$  be the perfect matching corresponding to  $\mathbf{x}_0$  and let  $F$  be the 2-factor complementary to  $M$ . Assume that  $F$  contains  $k$  circuits from  $\mathcal{C}$ . If a 5-circuit  $C \in \mathcal{C}$  belongs to  $F$ , it adds 5 to the sum in  $f(\mathbf{x}_0)$ . If  $C$  does not belong to  $F$ , it adds at least 1. Altogether  $f(\mathbf{x}_0) \geq 5k + (|\mathcal{C}| - k) = |\mathcal{C}| + 4k$ . Since  $f(\mathbf{x}_0) \leq 5/3 \cdot |\mathcal{C}|$ , we obtain  $k \leq |\mathcal{C}|/6$ .  $\square$

Proposition 3.1 has an interesting corollary: If  $G$  is a snark different from the Petersen graph, then for every vertex  $v$  of  $G$  there exists a 2-factor  $F$  of  $G$  such that every 5-circuit of  $F$  misses  $v$ . This gives a much shorter alternative way of proving the result of DeVos [2] that the Petersen graph is the only cubic graph having each 2-factor composed only of 5-cycles.

The following lemma provides the main tool for bounding oddness from above.

**Lemma 3.2.** *Let  $G$  be a snark of order  $n$  with girth at least 4. If  $G$  has  $q$  circuits of length 5, then  $\omega(G) \leq 3n + q/21$ .*

## 4 Constructions

**Cyclic connectivity 2.** Figure 4.1 (top) displays the smallest snarks of oddness 4. Their order is 28 and cyclic connectivity is 2 and 3, respectively. The proof of minimality is computer assisted. We also create an infinite family of snarks with oddness ratio approaching 7.5 from below and conjecture that every snark with oddness  $\omega$  has at least  $7.5\omega - 5$  vertices.

**Cyclic connectivity 4.** Let  $P_4^v$  and  $P_4^e$  be the Petersen graph with either two adjacent vertices removed or two non-adjacent edges disconnected, respectively, and the dangling edges retained. There are two pairs of dangling edges in both. By the Parity Lemma, in  $P_4^v$  the dangling edges of each pair must have the same colour for every 3-edge-colouring, while in  $P_4^e$  they must have different colours. If we join a pair of dangling edges from  $P_4^v$  to a pair of dangling edges from  $P_4^e$ , we get an uncolourable 4-pole  $N_1$  with 18 vertices. The 4-pole  $N_2$  with 26 vertices arises from  $P_4^e$  and two distinct copies of  $P_4^v$  by joining each pair of dangling edges of  $P_4^e$  to a pair of edges in a different copy of  $P_4^v$ . The 4-pole  $N_2$  is uncolourable even after the removal of a vertex  $w$ . This is clearly true if  $w$  belongs to a copy of  $P_4^v$ . If it was false for some  $w$  from a copy of  $P_4^e$ , then  $P_4^e$  would have a 3-edge-colouring where the edges in both pairs of dangling edges have the same colour. This would yield a 3-edge-colouring of the Petersen graph minus a vertex, but no such colouring exists.

To construct a cyclically 4-connected snark with arbitrarily large oddness we take a number of copies of  $N_1$  and a number of  $N_2$ , arrange them into a circuit, and join one pair of dangling edges from each copy to a pair of dangling edges of its predecessor and another pair of dangling edges to a pair of dangling edges of its successor. The way in which copies of  $N_1$  and  $N_2$  are arranged is not unique, therefore we may get several non-isomorphic graphs even if we only use copies of one of  $N_1$  and  $N_2$ .

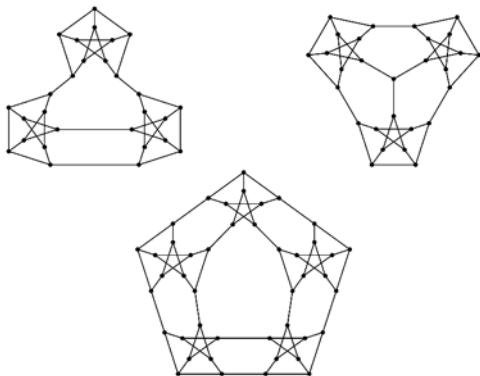
In this construction, each copy of  $N_1$  adds 1 and each copy of  $N_2$  adds 2 to the resistance of the resulting graph. Thus if we take  $r$  copies of  $N_2$ , we get a cyclically 4-connected snark of order  $26r$  with resistance  $2r$  and oddness at least  $2r$ . If we take  $r$  copies of  $N_2$  and one copy of  $N_1$  we get a cyclically 4-connected snark of order  $26r + 18$  with resistance  $2r + 1$  and oddness at least  $2r + 2$ . This shows, in particular, that  $A_\omega^4 \leq A_\rho^4 \leq 13$ .

For  $r = 1$  we obtain a cyclically 4-connected snark of order 44 with resistance 3 and oddness 4 shown in Figure 4.1, currently the smallest known non-trivial snark of oddness 4, improving the previous value of 46 [3].

**Cyclic connectivity 6.** Let  $P_3$  be the Petersen graph with one vertex removed and the dangling edges retained. Take  $r$  copies  $Q_1, Q_2, \dots, Q_r$  of  $P_3$ . Arrange them into a circuit and, for each  $i \in \{1, 2, \dots, r\}$ , join one dangling edge of  $Q_i$  to a dangling edge of  $Q_{i-1}$  and do the same for  $Q_{i+1}$  (indices reduced modulo  $r$ ). There are  $r/2$  pairs of oppositely positioned copies of  $P_3$ ; join the remaining dangling edges for each such pair. The result of is a cubic graph  $L_r$  of order  $9r$  and resistance  $r$ .

To obtain a snark with cyclic connectivity 6 we apply superposition to  $L_r$  (see [6] for details). Nontrivial supervertices will be copies of the 7-pole  $X$  having a vertex incident with three edges belonging to different connectors, plus two more edges joining the first two connectors of size 3. Nontrivial superedges will be copies of a 6-pole  $Y$  with 18 vertices created from the flower snark  $J_5$  by removing two nonadjacent vertices, one from the single 5-circuit of  $J_5$ . We choose a circuit  $C$  in  $L_r$  which passes through five vertices of each copy of  $P_3$  and finish the superposition by replacing each vertex on  $C$  with a copy of  $X$  and each edge on  $C$  with a copy of  $Y$ ; we use trivial supervertices and superedges everywhere outside  $C$ . The resulting graph has order  $99r$ . Its resistance is at least  $r$  due to the following proposition.

**Proposition 4.1.** *Let  $\tilde{G}$  be a snark resulting from a proper superposition of a snark  $G$ . Then  $\rho(\tilde{G}) \geq \rho(G)$ .*



**Figure 4.1.** Smallest snarks of oddness 4 with cyclic connectivity 2, 3, 4.

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# Non-trivial snarks with given circular chromatic index

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**Abstract.** We introduce a new framework designed for constructing graphs with given circular chromatic index. This framework allows construction of graphs with arbitrary maximum degree and with additional properties, *e.g.* high connectivity or large girth. We utilize this framework to construct a cyclically 4-edge-connected cubic graph with girth 5 and circular chromatic index  $r$  for any rational  $r \in (3, 3 + 1/4.5)$ .

## 1 Introduction

Edge-colourings of graphs emerged more than a century ago among the first topics in graph theory. They are especially important for 3-regular graphs because of their deep connections to various other branches of graph theory. According to the Vizing theorem, every cubic graph has a 4-edge-colouring; most of them have a 3-edge-colouring. Bridgeless cubic graphs with chromatic index 4 are called *snarks*.

It transpired that certain snarks are in many settings trivial when compared to others. The most interesting snarks are those that are cyclically 4-edge-connected and have girth at least 5. Such snarks are called *non-trivial*. A discussion of several aspects of triviality of snarks can be found in [6].

For a real number  $r \geq 2$ , a circular  $r$ -edge-colouring of a given graph  $G$  is a mapping  $c : E(G) \rightarrow [0, r)$  such that  $1 \leq |c(e) - c(f)| \leq r - 1$

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for any two adjacent edges  $e$  and  $f$  of  $G$ . The *circular chromatic index* of  $G$  is the infimum of all  $r$  such that  $G$  has a circular  $r$ -edge-colouring. In fact, this infimum is always attained and the value of circular chromatic index is always rational for a finite graph  $G$ . A proof of this and many other properties of circular colourings can be found in the surveys [7, 8]. It is known that if a graph  $G$  has a circular  $r$ -edge-colouring and  $r$  is an integer, then  $G$  also has an  $r$ -edge-colouring with colours being integers from the set  $\{0, 1, \dots, r - 1\}$ . Thus we may omit the word “circular” when referring to an  $r$ -edge-colouring.

Afshani *et al.* [1] proved that the circular chromatic index of a 2-edge-connected cubic graph lies in the interval  $[3, 11/3]$ . The only known bridgeless cubic graph with circular chromatic index greater than  $7/2$  is the Petersen graph with index  $11/3$ . It was conjectured that there are no other such graphs; this folklore conjecture was proved for bridgeless cubic graphs with girth at least six [3]. For any rational  $r \in (3, 10/3)$ , there are infinitely many cubic graphs with circular chromatic index  $r$ ; the only known construction giving an interval of realized values of  $\chi'_c$  is described in [4]. All new snarks constructed in [4] contain 2-edge-cuts and are thus trivial; the purpose of the present paper is to improve the construction method and then use it to produce non-trivial snarks with given circular chromatic index.

In the original construction, one has to construct the desired graph with circular chromatic index  $r$  together with a feasible  $r$ -edge-colouring. Our first improvement separates the construction of the graph from the construction of the colouring; in fact, the colouring is obtained from the modified construction without any additional effort. The second improvement consists in simplifying the construction to its bare essentials. The concept of balanced schemes hides many of the technical details and helps both authors and prospective readers. Another promising aspect of balanced schemes is that they do not depend on degrees of vertices of the graph being constructed. We describe the framework in Section 2.

Although we are dealing only with cubic graphs throughout the paper, our framework can be extended to regular graphs with larger degree in a straightforward way. In fact, only the building blocks would be different—the rest of the machinery works with no change. Thus only the current lack of suitable blocks separates us from a construction of regular graphs with given circular chromatic index. We have also managed to construct snarks of large girth with given circular chromatic index  $r$  from some interval  $(3, 3 + \alpha)$ , however, the relation between girth and  $\alpha$  is yet to be determined.

## 2 Balanced schemes

Our new construction is based on the concept of monochromatic networks devised in [4]. We refer the reader to this article for all the details on monochromatic networks not covered here.

A  $p$ -line monochromatic network  $N$  is a subcubic graph with  $2p$  vertices of degree one called *terminals*. The terminals of the network  $N$  are paired into  $p$  disjoint pairs in such a way that in every proper 3-edge-colouring of  $N$ , the edges incident to paired terminals have the same colour; we will denote this pairing by  $P(N)$ . The *terminal edge* incident to a terminal  $v$  of a network  $N$  is denoted by  $e(v)$ .

Two monochromatic networks can be naturally *joined* by identifying terminal edges. A *monochromatic line* of a network  $N$  is a set of edges having the same colour in every 3-edge-colouring of  $N$  and containing the two edges adjacent to the vertices in a pair of terminals from  $P(N)$ . By joining monochromatic networks we also merge monochromatic lines; in fact, we are only interested in *primary* monochromatic lines which are created from a starting terminal edge (which is a monochromatic line in itself) by repeatedly extending it by joining other monochromatic networks. Primary monochromatic lines are a crucial concept of our construction.

For  $\varepsilon \in [0, 1]$ , the  $\varepsilon$ -changeability of a network  $N$  is the maximal possible value of

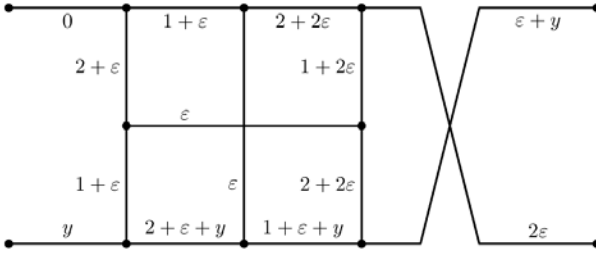
$$\sum_{(x,y) \in P(N)} |c(e(x)) - c(e(y))|_{3+\varepsilon}$$

over all  $(3 + \varepsilon)$ -edge-colourings  $c$  of  $N$ , where  $|q|_{3+\varepsilon} = \min\{|q|, |3 + \varepsilon - q|\}$ . It is clear that if we join two monochromatic networks  $M$  and  $N$ , the  $\varepsilon$ -changeability of the resulting monochromatic network does not exceed the sum of  $\varepsilon$ -changeabilities of  $M$  and  $N$ .

Monochromatic networks with one or two lines can be easily constructed from any cubic graph by cutting an edge of any cubic graph or by removing two adjacent vertices and turning the dangling edges into terminal edges. This construction yields a 2-line monochromatic network  $B$  from the Petersen graph (Figure 2.1).

Our aim is to simplify the view of monochromatic networks composed of many smaller networks. We will represent such monochromatic networks by schemes. An  $r$ -balanced scheme is a pair  $\mathcal{S} = (G, \varphi)$  such that

- $G$  is a multigraph that may contain parallel edges and semiedges; we denote the end of an edge  $e$  incident with a vertex  $v$  by  $e^v$ ;
- $\varphi$  is a labeling by rational numbers that assigns a value to both ends of every edge and one value to each semiedge;



**Figure 2.1.** The monochromatic network  $B$  with a  $(3 + \varepsilon)$ -edge-colouring.

- the sum of values at every vertex  $v$  of  $G$  (one value for every incident edge or semiedge) is  $r$ .

The set of vertices of the underlying multigraph  $G$  of a scheme  $S$  is denoted by  $V(S)$ . The vertices of  $S$  represent primary monochromatic lines of the network corresponding to the scheme  $S$  and the edges (semiedges) of  $S$  represent basic 2-line (1-line) monochromatic networks used in the construction.

We say that an edge  $e = uv$  of  $S$  is *representable* if there exists a 2-line monochromatic network  $N_2$  such that  $P(N_2) = \{(u_1, u_2), (v_1, v_2)\}$  and

- the  $(1/r)$ -changeability of  $N_2$  is  $\varphi(e^u) + \varphi(e^v)$ ;
- there exists a circular  $(3 + 1/r)$ -edge-colouring  $c$  of  $N_2$  such that  $|c(e(u_1)) - c(e(u_2))|_{3+\varepsilon} = \varphi(e^u)$ ,  $|c(e(v_1)) - c(e(v_2))|_{3+\varepsilon} = \varphi(e^v)$ , and the colours of the edges incident to  $u_1, u_2, v_1, v_2$  are all rational.

A network  $N_2$  satisfying the above definition is called a *representative* of the edge  $e$ . The representability of a semiedge of  $S$  can be defined in a similar way; we use a 1-line monochromatic network as the representative. An  $r$ -balanced scheme is *representable* if all its edges and semiedges are representable.

We will describe how to construct a subcubic graph  $G$  with circular chromatic index  $3 + 1/r$  from a representable  $r$ -balanced scheme  $S$ . Let  $p$  be the number of vertices of  $S$ . We start with a  $p$ -line monochromatic network  $N_0$  isomorphic to a matching containing  $p$  edges. Let  $\ell$  be an arbitrary bijection between  $V(S)$  and the monochromatic lines of  $N_0$ . For each edge of  $N_0$ , we fix one of its endvertices; these fixed vertices will remain untouched until the very last step of the construction.

For each edge  $uv$  (and also for each semiedge) of  $S$  we do the following: According to our definitions, the edge  $uv$  has a representative  $N$ . We join  $N$  to the monochromatic lines  $\ell(u)$  and  $\ell(v)$ . The resulting network has  $(1/r)$ -changeability at least  $p$ . Hence if we identify the paired



vertices of the resulting network, we get a graph with circular chromatic index at least  $3 + 1/r$ .

To produce a graph with circular chromatic index exactly  $3 + 1/r$ , we do a “copying trick”. Let  $d$  be the least common denominator of  $1/r$  and all the denominators in the fractions of the labeling  $\varphi$ . We start with  $d$  primary monochromatic lines for each vertex of  $\mathcal{S}$ . Instead of joining  $N$  to the lines  $\ell(u)$  and  $\ell(v)$ , we join  $d$  copies of  $N$  to all the lines  $\ell(u)$  and  $\ell(v)$  (note that both  $\ell(u)$  and  $\ell(v)$  contain  $d$  monochromatic lines) so that each line will be used exactly once. Apparently, the resulting network has  $(1/r)$ -changeability at least  $dp$ , thus if we identify endvertices paired according to the primary monochromatic lines, we get a graph with circular chromatic index at least  $3 + 1/r$ .

We will produce a  $(3 + 1/r)$ -colouring of our network alongside our construction. The idea is to guarantee that both before and after joining  $d$  copies of each network  $N$  for each vertex  $v \in \mathcal{S}$  all the rational numbers  $i/d \in [0, 3 + 1/r)$  are used as colours on the edges incident with non-fixed ends of the primary lines of  $N$ . The second property in the definition of a representable network guarantees that the total change of colour on each primary monochromatic line will be exactly 1. Therefore at the last step we may identify endvertices of primary monochromatic lines. The resulting graph will have the circular chromatic index equal to  $3 + 1/r$ .

We are now ready to sketch the proof of main result.

**Theorem 2.1.** *For each rational  $t \in (3, 3 + 1/4.5)$ , there is a non-trivial snark with circular chromatic index  $t$ .*

*Proof.* For each rational  $r \in (4.5, \infty)$ , we construct a representable  $r$ -balanced scheme that uses only the network  $B$  as a representative. After carrying out the construction described above we get a subcubic graph  $G$  with desired circular chromatic index. If  $G$  is not connected, we can take an arbitrary component of  $G$ . To make this graph cubic we just take two copies of it and connect the corresponding vertices of degree 2. If we use the same edge-colouring on both of the copies, then the circular chromatic index does not increase (note that the difference of colours incident to vertices of degree 2 is exactly 1 and therefore we can colour the third edge). By performing the construction a bit more carefully we can guarantee the resulting graph to be cyclically 4-edge connected.  $\square$

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## Graphs



# The graph formulation of the union-closed sets conjecture

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**Abstract.** In 1979 Frankl conjectured that in a finite non-trivial union-closed collection of sets there has to be an element that belongs to at least half the sets. We show that this is equivalent to the conjecture that in a finite non-trivial graph there are two adjacent vertices each belonging to at most half of the maximal stable sets. In this graph formulation other special cases become natural. The conjecture is trivially true for non-bipartite graphs and we show that it holds also for the class of chordal bipartite graphs and the class of bipartitioned circular interval graphs.

A set  $\mathcal{X}$  of sets is *union-closed* if  $X, Y \in \mathcal{X}$  implies  $X \cup Y \in \mathcal{X}$ . The following conjecture was formulated by Peter Frankl in 1979 [4].

**Union-closed sets conjecture.** *Let  $\mathcal{X}$  be a finite union-closed set of sets with  $\mathcal{X} \neq \{\emptyset\}$ . Then there is a  $x \in \bigcup_{X \in \mathcal{X}} X$  that lies in at least half of the members of  $\mathcal{X}$ .*

In spite of a great number of papers, see e.g. the good bibliography of Marković [7] for papers up to 2007, this conjecture is still wide open, although several special cases are known to hold. Various equivalent formulations are known, in particular by Poonen [9] who among other things translates the conjecture into the language of lattice theory.

In this paper we give a formulation of the conjecture in the language of graph theory. A set of vertices in a graph is *stable* if no two vertices of the set are adjacent. A stable set is *maximal* if it is maximal under inclusion, that is, every vertex outside has a neighbour in the stable set.

**Conjecture 1.** Let  $G$  be a finite graph with at least one edge. Then there will be two adjacent vertices each belonging to at most half of the maximal stable sets.

Note that Conjecture 1 is true for non-bipartite graphs. Indeed, if vertices  $u$  and  $v$  are adjacent there is no stable set containing them both and

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so one of them must belong to at most half of the maximal stable sets. An odd cycle will therefore imply the existence of two adjacent vertices each belonging to at most half of the maximal stable sets. The conjecture is for this reason open only for bipartite graphs. Moreover, in a connected bipartite graph, for any two vertices  $u$  and  $v$  in different bipartition classes we have a path from  $u$  to  $v$  containing an odd number of edges, so that if  $u$  and  $v$  each belongs to at most half the maximal stable sets there will be two adjacent vertices each belonging to at most half the maximal stable sets. Conjecture 1 is therefore equivalent to the following.

**Conjecture 2.** *Let  $G$  be a finite bipartite graph with at least one edge. Then each of the two bipartition classes contains a vertex belonging to at most half of the maximal stable sets.*

In this paper we show that Conjectures 1 and 2 are equivalent to the union-closed sets conjecture. The merit of this graph formulation is that other special cases become natural, in particular subclasses of bipartite graphs. We show that the conjecture holds for the class of chordal bipartite graphs and the class of bipartitioned circular interval graphs. Moreover, the reformulation allows to test Frankl's conjecture in a probabilistic sense: Bruhn and Schaudt [1] show that almost every random bipartite graph satisfies Conjecture 2 up to any given  $\varepsilon > 0$ , that is, almost every such graph contains in each bipartition class a vertex for which the number of maximal stable sets containing it is at most  $\frac{1}{2} + \varepsilon$  times the total number of maximal stable sets.

Stable sets are also called independent sets, with the maximal stable sets being exactly the independent dominating sets. The set of all maximal stable sets of a bipartite graph was studied by Prisner [10] who gave upper bounds on the size of this set, also when excluding certain subgraphs. More recently, Duffus, Frankl and Rödl [3] and Ilinca and Kahn [6] investigate the number of maximal stable sets in certain regular and biregular bipartite graphs. In work related to the graph parameter boolean-width, Rabinovich, Vatshelle and Telle [11] study balanced bipartitions of a graph that bound the number of maximal stable sets.

For a subset  $S$  of vertices of a graph we denote by  $N(S)$  the set of vertices adjacent to a vertex in  $S$ . We need two easy lemmas. The proof of the first is trivial.

**Lemma 3.** *Let  $G$  be a bipartite graph with bipartition  $U, W$ , and let  $S$  be a maximal stable set. Then  $S = (U \cap S) \cup (W \setminus N(U \cap S))$ .*

**Lemma 4.** *Let  $G$  be a bipartite graph with bipartition  $U, W$ , and let  $S$  and  $T$  be maximal stable sets. Then  $(U \cap S \cap T) \cup (W \setminus N(S \cap T))$  is a maximal stable set.*

*Proof.* Clearly,  $R = (U \cap S \cap T) \cup (W \setminus N(S \cap T))$  is stable. Trivially, any vertex in  $W \setminus R$  has a neighbour in  $R$ . A vertex  $u$  in  $U \setminus R$  does not lie in  $S$  or not in  $T$  (perhaps, it is not contained in either), let us say that  $u \notin T$ . As  $T$  is maximal,  $u$  has a neighbour  $w \in W \cap T$ . This neighbour  $w$  cannot be adjacent to any vertex in  $U \cap S \cap T$  as  $T$  is stable. So,  $w$  belongs to  $R$  as well, which shows that  $R$  is a maximal stable set.  $\square$

For a fixed graph  $G$  let us denote by  $\mathcal{A}$  the set of all maximal stable sets, and for any vertex  $v$  let us write  $\mathcal{A}_v$  for the sets of  $\mathcal{A}$  that contain  $v$  and  $\mathcal{A}_{\bar{v}}$  for the sets of  $\mathcal{A}$  that do not contain  $v$ . Let us call a vertex  $v$  *rare* if  $|\mathcal{A}_v| \leq \frac{1}{2}|\mathcal{A}|$ .

**Theorem 5.** *Conjecture 2 is equivalent to the union-closed sets conjecture.*

*Proof.* Let us consider first a union-closed set  $\mathcal{X} \neq \{\emptyset\}$ , which, without restricting generality, we may assume to include  $\emptyset$  as a member. We put  $U = \bigcup_{X \in \mathcal{X}} X$  and we define a bipartite graph  $G$  with vertex set  $U \cup \mathcal{X}$ , where we make  $X \in \mathcal{X}$  adjacent with all  $u \in X$ .

Now we claim that  $\tau : S \mapsto U \setminus S$  is a bijection between  $\mathcal{A}$  and  $\mathcal{X}$ . First note that indeed  $\tau(S) \in \mathcal{X}$  for every maximal stable set: Set  $A = U \cap S$  and  $B = \mathcal{X} \cap S$ . If  $U \subseteq S$  then  $U \setminus S = \emptyset \in \mathcal{X}$ , by assumption. So, assume  $U \not\subseteq S$ , which implies  $B \neq \emptyset$ . As  $S$  is a maximal stable set, it follows that  $U \setminus S = U \setminus A = N(B)$ . On the other hand,  $N(B)$  is just the union of the  $X \in S \cap \mathcal{X} = B$ , which is by the union-closed property equal to a set  $X' \in \mathcal{X}$ . To see that  $\tau$  is injective note that, by Lemma 3,  $S$  is determined by  $U \cap S$ , which in turn determines  $U \setminus S$ . For surjectivity, consider  $X \in \mathcal{X}$ . We set  $A = U \setminus N(X)$  and observe that  $S = A \cup (\mathcal{X} \setminus N(A))$  is a stable set. Moreover, as  $X \in \mathcal{X} \setminus N(A)$  every vertex in  $U \setminus A$  is a neighbour of  $X \in S$ , which means that  $S$  is maximal.

Now, assuming that Conjecture 2 is true, there is a rare  $u \in U$ , that is, it holds that  $|\mathcal{A}_u| \leq \frac{1}{2}|\mathcal{A}|$ . Clearly  $\mathcal{A}$  is the disjoint union of  $\mathcal{A}_u$  and of  $\mathcal{A}_{\bar{u}}$ , so that

$$|\tau(\mathcal{A}_{\bar{u}})| = |\mathcal{A}_{\bar{u}}| \geq \frac{1}{2}|\mathcal{A}| = \frac{1}{2}|\mathcal{X}|.$$

As  $u \in \tau(S) \in \mathcal{X}$  for every  $S \in \mathcal{A}_{\bar{u}}$ , the union-closed sets conjecture follows.

For the other direction, consider a bipartite graph with bipartition  $U, W$  and at least one edge. Define  $\mathcal{X} := \{U \setminus S : S \in \mathcal{A}\}$ , and note that  $\mathcal{X} \neq \{\emptyset\}$  as  $G$  has at least two distinct maximal stable sets. By Lemma 3, there is a bijection between  $\mathcal{X}$  and  $\mathcal{A}$ . Moreover, it is a direct consequence of Lemma 4 that  $\mathcal{X}$  is union-closed. From this, it is straightforward that Conjecture 2 follows from the union-closed sets conjecture.  $\square$

Let us say that a bipartite graph *satisfies the union-closed sets conjecture* if each of its bipartition classes contains a rare vertex. For a set  $X$  of vertices we define  $\mathcal{A}_X$  to be the set of maximal stable sets containing all of  $X$ . As before, we abbreviate  $\mathcal{A}_{\{x\}}$  to  $\mathcal{A}_x$ .

**Lemma 6.** *Let  $x$  be a vertex of a bipartite graph  $G$ . Then there is an injection  $\mathcal{A}_{N(x)} \rightarrow \mathcal{A}_x$ .*

*Proof.* We define

$$i : \mathcal{A}_{N(x)} \rightarrow \mathcal{A}_x, S \mapsto S \setminus L_1 \cup \{x\} \cup (L_2 \setminus N(S \cap L_3)),$$

where  $L_i$  denotes the set of vertices at distance  $i$  to  $x$ . That  $i(S)$  is stable and maximal is a direct consequence of the definition. Moreover,  $i(S) = i(T)$  for  $S, T \in \mathcal{A}_{N(x)}$  implies that  $S$  and  $T$  are identical outside  $L_1 \cup L_2$ . Moreover,  $S$  and  $T$  are also identical on  $L_1 \cup L_2$ : First,  $L_1 = N(x)$  shows that  $L_1$  lies in both  $S$  and  $T$ . Second, since every vertex in  $L_2$  is a neighbour of one in  $L_1 \subseteq S \cap T$ , no vertex of  $L_2$  can lie in either of  $S$  or  $T$ . Thus,  $S = T$ , and we see that  $i$  is an injection.  $\square$

We denote by  $N^2(x) = N(N(x))$  the second neighbourhood of  $x$ .

**Lemma 7.** *Let  $x, y$  be two adjacent vertices in a bipartite graph  $G$  with  $N^2(x) \subseteq N(y)$ . Then  $y$  is rare.*

*Proof.* From  $N^2(x) \subseteq N(y)$  it follows that every maximal stable set containing  $y$  must contain all of  $N(x)$ . Thus,  $\mathcal{A}_y = \mathcal{A}_{N(x)}$ , which means by Lemma 6 that  $|\mathcal{A}_y| \leq |\mathcal{A}_x|$  and as  $|\mathcal{A}_y| + |\mathcal{A}_x| \leq |\mathcal{A}|$  the lemma is proved.  $\square$

We now apply the lemma to the class of *chordal bipartite* graphs. This is the class of bipartite graphs in which every cycle with length at least six has a chord. A vertex  $v$  in a bipartite graph is *weakly simplicial* if the neighbourhoods of its neighbours form a chain under inclusion. Hammer, Maffray and Preissmann [5], and also Pelsmajer, Tokaz and West [8] prove the following:

**Theorem 8.** *A bipartite graph with at least one edge is chordal bipartite if and only if every induced subgraph has a weakly simplicial vertex. Moreover, such a vertex can be found in each of the two bipartition classes.*

**Theorem 9.** *Any chordal bipartite graph with at least one edge satisfies the union-closed sets conjecture.*

*Proof.* For a given bipartition class, let  $x$  be a weakly simplicial vertex in it. Among the neighbours of  $x$  denote by  $y$  the one whose neighbourhood includes the neighbourhoods of all other neighbours of  $x$ . Then  $y$  is rare, by Lemma 7.  $\square$

For two vertices  $u, v$  let us denote by  $\mathcal{A}_{uv}$  the set of  $S \in \mathcal{A}$  containing both of  $u$  and  $v$ , by  $\mathcal{A}_{u\bar{v}}$  the set of  $S \in \mathcal{A}$  containing  $u$  but not  $v$ , and by  $\mathcal{A}_{\bar{u}\bar{v}}$  the set of  $S \in \mathcal{A}$  containing neither of  $u$  and  $v$ .

**Lemma 10.** *Let  $G$  be a bipartite graph. Let  $y$  and  $z$  be two neighbours of a vertex  $x$  so that  $N^2(x) \subseteq N(y) \cup N(z)$ . Then one of  $y$  and  $z$  is rare.*

*Proof.* We may assume that  $|\mathcal{A}_{y\bar{z}}| \leq |\mathcal{A}_{\bar{y}z}|$ . Now, from  $N^2(x) \subseteq N(y) \cup N(z)$  we deduce that  $\mathcal{A}_{yz} = \mathcal{A}_{N(x)}$ . Thus, by Lemma 6, we obtain  $|\mathcal{A}_{yz}| \leq |\mathcal{A}_x|$ . Since  $\mathcal{A}_x \subseteq \mathcal{A}_{\bar{y}z}$  it follows that  $|\mathcal{A}_y| = |\mathcal{A}_{y\bar{z}}| + |\mathcal{A}_{yz}| \leq |\mathcal{A}_{\bar{y}z}| + |\mathcal{A}_{\bar{y}\bar{z}}| = |\mathcal{A}_{\bar{y}}|$ . As  $|\mathcal{A}| = |\mathcal{A}_y| + |\mathcal{A}_{\bar{y}}|$ , we see that  $y$  is rare.  $\square$

We give an application of Lemma 10. The class of circular interval graphs plays a fundamental role in the structure theorem of claw-free graphs of Chudnovsky and Seymour [2] and are defined as follows: Let a finite subset of a circle be the vertex set, and for a given set of subintervals of the circle consider two vertices to be adjacent if there is an interval containing them both. We may obtain a rich class of bipartite graphs from circular interval graphs: For any circular interval graph, partition its vertex set and delete every edge with both its endvertices in the same class. We call any graph arising in this manner a *bipartitioned circular interval graph*. The proof that these graphs satisfy the union-closed sets conjecture has been left out of this extended abstract.

**Theorem 11.** *Every bipartitioned circular interval graph with at least one edge satisfies the union-closed sets conjecture.*

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# The union-closed sets conjecture almost holds for almost all random bipartite graphs

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**Abstract.** Frankl's union-closed sets conjecture states that in every finite union-closed set of sets, there is an element that is contained in at least half of the member-sets (provided there are at least two members). The conjecture has an equivalent formulation in terms of graphs: In every bipartite graph with least one edge, both colour classes contain a vertex belonging to at most half of the maximal stable sets.

We prove that, for every fixed edge-probability, almost every random bipartite graph almost satisfies Frankl's conjecture.

## 1 Introduction

A full paper version of this extended abstract is available at the arXiv [4]. One of the most basic conjectures in extremal set theory is Frankl's conjecture on union-closed set systems. A set  $\mathcal{X}$  of sets is *union-closed* if  $X \cup Y \in \mathcal{X}$  for all  $X, Y \in \mathcal{X}$ .

**Union-closed sets conjecture.** *Let  $\mathcal{X} \neq \{\emptyset\}$  be a finite union-closed set of sets. Then there is a  $x \in \bigcup_{X \in \mathcal{X}} X$  that lies in at least half of the members of  $\mathcal{X}$ .*

While Frankl [6] dates the conjecture to 1979, it apparently did not appear in print before 1985, when it was mentioned as an open problem in Rival [8]. Despite being widely known, there is only little substantial progress on the conjecture. For a survey on the conjecture, see [3].

Recently, Bruhn, Charbit, Schaudt and Telle [2] gave an equivalent formulation in terms of graphs. For this, let us say that a vertex set  $S$  in a graph is *stable* if no two of its vertices are adjacent, and that it is *maximally stable* if, in addition, every vertex outside  $S$  has a neighbour in  $S$ .

**Conjecture 1.1 (Bruhn, Charbit, Schaudt and Telle [2]).** *Let  $G$  be a bipartite graph with at least one edge. Then each of the two biparti-*

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tion classes contains a vertex belonging to at most half of the maximal stable sets.

So far, the graph formulation is only verified for chordal-bipartite graphs, bipartite subcubic graphs, bipartite series-parallel graphs and bipartitioned circular interval graphs [2].

We prove a slight weakening of Conjecture 1.1 for random bipartite graphs. For  $\delta > 0$ , we say that a bipartite graph *satisfies the union-closed sets conjecture up to  $\delta$*  if each of its two bipartition classes has a vertex for which the number of maximal stable sets containing it is at most  $\frac{1}{2} + \delta$  times the total number of maximal stable sets. A *random bipartite graph* is a graph on bipartition classes of cardinalities  $m$  and  $n$ , where any two vertices from different classes are independently joined by an edge with probability  $p$ . We say that *almost every* random bipartite graph has property  $P$  if for every  $\varepsilon > 0$  there is an  $N$  such that, whenever  $m + n \geq N$ , the probability that a random bipartite graph on  $m + n$  vertices has  $P$  is at least  $1 - \varepsilon$ .

Our main result is the following.

**Theorem 1.2.** *Let  $p \in (0, 1)$  be a fixed edge-probability. For every  $\delta > 0$ , almost every random bipartite graph satisfies the union-closed sets conjecture up to  $\delta$ .*

## 2 Discussion of averaging

Many of the partial results on Frankl's conjecture are based on a technique called *averaging*. It consists in taking the average of the number of member sets containing a given element, where the average ranges over the set  $U = \bigcup_{X \in \mathcal{X}} X$  of all elements. If that average is at least  $\frac{1}{2}|U|$  then clearly  $\mathcal{X}$  satisfies the conjecture. Averaging was used successfully by Balla, Bollobás and Eccles [1] when  $n \geq \lceil \frac{1}{3}2^{m+1} \rceil$ . Reimer [7] showed that the average is always at least  $\log_2(|U|)$ .

Averaging does not always work. It is easy to construct union-closed set systems in which the average is too low. Czédli, Maróti and Schmidt [5] even found such set systems of size  $|\mathcal{X}| = \lfloor 2^{|U|+1}/3 \rfloor$ . Nevertheless, we see that, in the graph formulation, averaging almost always allows us to conclude that the union-closed sets conjecture is satisfied (up to any  $\delta > 0$ ).

To describe the averaging technique for bipartite graphs, let us write  $\mathcal{A}(G)$  for the set of maximal stable sets of a bipartite graph  $G$ . Conjecture 1.1 is satisfied if  $G$  contains a *rare* vertex in both bipartition classes, that is, a vertex that lies in at most half of the maximal stable sets.

We consider a bipartite graph  $G$  to have a fixed bipartition, which we denote by  $(L(G), R(G))$ . When discussing the bipartition classes, we

often refer to  $L(G)$  as the *left side* and to  $R(G)$  as the *right side* of the graph. Let us consider a fixed edge probability  $p \in (0, 1)$  and put  $q = 1 - p$ . We denote by  $\mathcal{B}(m, n; p)$  the probability space whose elements are the random bipartite graphs  $G$  with  $|L(G)| = m$  and  $|R(G)| = n$ . We note first that exchanging the sides turns a random bipartite graph  $G \in \mathcal{B}(m, n; p)$  into a member of  $\mathcal{B}(n, m; p)$ , which means that it suffices to show the existence of a rare vertex in  $L(G)$ . All the discussion that follows focuses on the left side  $L(G)$ .

That a vertex  $v$  is rare means that  $|\mathcal{A}_v(G)|$ , the number of maximal stable sets containing  $v$ , is at most  $\frac{1}{2}|\mathcal{A}(G)|$ . Thus, if for the average

$$\sum_{v \in L(G)} \frac{|\mathcal{A}_v(G)|}{|\mathcal{A}(G)|} \leq \frac{1}{2}|L(G)|$$

then  $L(G)$  contains a rare vertex. Double-counting shows that the above average is equal to

$$\text{left-avg}(G) := \sum_{A \in \mathcal{A}(G)} \frac{|A \cap L(G)|}{|\mathcal{A}(G)|},$$

and thus our aim is to show that when  $m + n$  is very large, it follows with high probability that  $\text{left-avg}(G) \leq \frac{m}{2}$  for any  $G \in \mathcal{B}(m, n; p)$ .

Unfortunately, we do not reach this aim. While we show for large parts of the parameter space  $(m, n)$  that the average is, with high probability, small enough, we also see that when  $n$  is roughly  $q^{-\frac{m}{2}}$  the average becomes very close to  $\frac{m}{2}$ , so close that our tools are not sharp enough to separate the average from slightly above  $\frac{m}{2}$ . Therefore, we provide for a bit more space by settling on bounding the average away from  $(\frac{1}{2} + \delta)m$  for any positive  $\delta$ , which then only allows us to deduce the existence of a vertex  $v \in L(G)$  that is *almost rare*, in the sense that  $v$  lies in at most  $(\frac{1}{2} + \delta)|\mathcal{A}(G)|$  maximal stable sets.

Much of the previous discussion is subsumed in the following lemma.

**Lemma 2.1.** *Let  $G$  be a bipartite graph, and let  $\delta \geq 0$ . If*

$$\text{left-avg}(G) \leq \left(\frac{1}{2} + \delta\right) |L(G)|$$

*then there exists a vertex in  $L(G)$  that lies in at most  $(\frac{1}{2} + \delta)|\mathcal{A}(G)|$  maximal stable sets.*

The following result is the heart of our main result, Theorem 1.2:

**Theorem 2.2.** *For all  $\delta > 0$  and all  $\varepsilon > 0$  there is an integer  $N$  so that for  $G \in \mathcal{B}(m, n; p)$*

$$\Pr \left[ \text{left-avg}(G) \leq \left( \frac{1}{2} + \delta \right) m \right] \geq 1 - \varepsilon$$

for all  $m, n$  with  $m + n \geq N$  and  $n \geq \max\{20, (\lceil 3 \log_{1/q}(2) \rceil + 2)^2\} + 1$ .

In order to show how Theorem 1.2 follows from Theorem 2.2, we need to deal with the special case when one side is of constant size while the other becomes ever larger. Indeed, in this case averaging might fail—for a trivial reason. If we fix a constant right side  $R(G)$ , while  $L(G)$  becomes ever larger, then  $L(G)$  contains many isolated vertices. Since the isolated vertices lie in every maximal stable set they may push up  $\text{left-avg}(G)$  to above  $\frac{m}{2}$ .

However, isolated vertices are never a threat to Frankl’s conjecture: A bipartite graph satisfies the union-closed sets conjecture if and only if it satisfies the conjecture with all isolated vertices deleted. More generally, it turns out that the special case of a constant right side is easily taken care of:

**Lemma 2.3.** *Let  $c$  be a positive integer, and let  $\varepsilon > 0$ . Then there is an  $N$  so that for  $G \in \mathcal{B}(m, n; p)$*

$$\Pr [L(G) \text{ contains a rare vertex}] \geq 1 - \varepsilon,$$

for all  $m, n$  with  $m \geq N$  and  $n \leq c$ .

*Proof of Theorem 1.2.* Let  $\delta > 0$  be given. By symmetry, it is enough to show that the left side  $L(G)$  of almost every random bipartite graph  $G$  in  $\mathcal{B}(m, n; p)$  contains a vertex that lies in at most  $(\frac{1}{2} + \delta)|\mathcal{A}|$  maximal stable sets. For this, consider a  $\varepsilon > 0$ , and let  $N$  be the maximum of the  $N$  given by Theorem 2.2 and Lemma 2.3 with  $c = \max\{20, (\lceil 3 \log_{1/q}(2) \rceil + 2)^2\}$ . Consider a pair  $m, n$  of positive integers with  $m + n \geq N$ . If  $n \leq \max\{20, (\lceil 3 \log_{1/q}(2) \rceil + 2)^2\}$  then Lemma 2.3 yields a rare vertex in  $L(G)$  with probability at least  $1 - \varepsilon$ . If, on the other hand,  $n \geq \max\{20, (\lceil 3 \log_{1/q}(2) \rceil + 2)^2\} + 1$ , Theorem 2.2 becomes applicable, which is to say that with probability at least  $1 - \varepsilon$  we have  $\text{left-avg}(G) \leq (\frac{1}{2} + \delta)m$ . Now, Lemma 2.1 yields the desired vertex in  $L(G)$ .  $\square$

### 3 Sketch of the proof of Theorem 2.2

In order to prove Theorem 2.2, we distinguish several cases, depending on the relative sizes,  $m$  and  $n$ , of the two sides of the random bipartite graph  $G$ . In each of the cases we need a different method.

Our general strategy follows the observation that if there are many more maximal stable sets with small left side than with large left side, then the average over the left sides is small, too:

**Lemma 3.1.** *Let  $\nu > 0$  and  $\delta \geq 0$ , and let  $G$  be a bipartite graph with  $|L(G)| = m$ . Let  $\mathcal{L}$  be the maximal stable sets  $A$  of  $G$  with  $|A \cap L(G)| \geq (\frac{1}{2} + \delta)m$ , and let  $\mathcal{S}$  be those maximal stable sets  $B$  with  $|B \cap L(G)| \leq (1 - \nu)\frac{m}{2}$ . If  $|\mathcal{S}| \geq \frac{1}{\nu}|\mathcal{L}|$  then*

$$\text{left-avg}(G) \leq (\frac{1}{2} + \delta)m.$$

We bound the number of maximal stable sets with large left side, usually counted by a random variable  $\mathcal{L}_G$ , and at the same time we show that there are many maximal stable sets with a small left side; those we count with  $\mathcal{S}_G$ .

Up to  $n < q^{-\frac{m}{2}}$  we are able to use the same bound for the number  $\mathcal{L}_G$  of maximal stable sets whose left sides are of size at least  $\frac{m}{2}$ : We prove that with high probability  $\mathcal{L}_G$  is bounded by a polynomial in  $n$ . For right sides that are much larger than the left side, i.e.  $m \gg n$ , we even extend such a bound to maximal stable sets with left side  $\geq \frac{m}{3}$ .

For the maximal stable sets with small left side, counted by  $\mathcal{S}_G$ , we need to distinguish several cases. When the left side of the graph is much larger than the right side, namely  $m \geq q^{-\sqrt[5]{n}}$ , we find with high probability a large induced matching in  $G$ . This in turn implies that the total number of maximal stable sets is high, and thus clearly also the number of those with small left side.

When the sides of the graph do not differ too much in size,  $m \leq q^{-\sqrt[5]{n}}$  and  $n \leq q^{-\sqrt[5]{m}}$ , the variance of the number of maximal stable sets with small left side is moderate enough to apply Chebychev's inequality. Since the expectation of  $\mathcal{S}_G$  is high, we again can use Lemma 3.1 to deduce Theorem 2.2.

However, when the left side of the graph becomes much larger than the right side, we cannot control the variance of  $\mathcal{S}_G$  anymore. Instead, for  $q^{-\sqrt[5]{m}} \leq n \leq q^{-\frac{m}{16}}$ , we cut the right side into many pieces each of large size and apply Hoeffding's inequality to each of the pieces together with the left side. The inequality ensures that we find on at least one of the pieces a large number of maximal stable sets of small left side. Surpassing  $n \geq q^{-\frac{m}{16}}$ , we have to refine our estimations but we can still use this strategy up to slightly below  $n = q^{-\frac{m}{2}}$ .

In the interval  $q^{-\frac{m}{2}} \leq n \leq q^{-m^3}$ , we encounter a serious obstacle. There, we have to cope with an average that is very close to  $\frac{m}{2}$ . It is precisely for this reason that, overall, we only prove that  $\text{left-avg}(G) \leq$

$(\frac{1}{2} + \delta)m$  instead of  $\text{left-avg}(G) \leq \frac{m}{2}$ . To keep below the slightly higher average, we only need to bound the number of maximal stable sets with left side  $> (\frac{1}{2} + \delta)m$ . This number we almost trivially bound by  $2^{\lambda m}$ , with some  $\lambda < 1$ . On the other hand, we see that the number  $\mathcal{S}_G$  of maximal stable sets of small left side is  $2^{\lambda' m}$  with a  $\lambda'$  as close to 1 as we want.

In the remaining case, we are dealing with an enormous right side:  $n \geq q^{-m^3}$ . Then, it is easy to see that with high probability there is an induced matching that covers all of the left side, which implies that every subset of  $L(G)$  is the left side of a maximal stable set. This immediately gives us  $\text{left-avg}(G) = \frac{m}{2}$ .

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# The robust component structure of dense regular graphs

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**Abstract.** We study the large-scale structure of dense regular graphs. This involves the notion of robust expansion, a recent concept which has already been used successfully to settle several longstanding problems. Roughly speaking, a graph is robustly expanding if it still expands after the deletion of a small fraction of its vertices and edges. Our main result states that every dense regular graph can be partitioned into ‘robust components’, each of which is a robust expander or a bipartite robust expander. We apply our result to obtain the following.

- (i) We prove that whenever  $\varepsilon > 0$ , every sufficiently large 3-connected  $D$ -regular graph on  $n$  vertices with  $D \geq (1/4 + \varepsilon)n$  is Hamiltonian. This asymptotically confirms the only remaining case of a conjecture raised independently by Bollobás and Häggkvist in the 1970s.
- (ii) We prove an asymptotically best possible result on the circumference of dense regular graphs of given connectivity. The 2-connected case of this was conjectured by Bondy and already proved by Wei.

## 1 Introduction

Our main result states that any dense regular graph  $G$  is the vertex-disjoint union of boundedly many ‘robust components’. Each such component has a strong expansion property that is highly ‘resilient’ and almost all edges of  $G$  lie inside these robust components. In other words, the result implies that the large scale structure of dense regular graphs is remarkably simple. This can be applied *e.g.* to Hamiltonicity problems in dense regular graphs. Note that the structural information obtained in this way is quite different from that given by Szemerédi’s regularity lemma.

The crucial notion in our partition is that of robust expansion. This is a structural property which has close connections to Hamiltonicity. Given a graph  $G$  on  $n$  vertices,  $S \subseteq V(G)$  and  $0 < \nu \leq \tau < 1$ , we define the  $\nu$ -robust neighbourhood  $RN_{\nu,G}(S)$  of  $S$  to be the set of all those vertices of  $G$  with at least  $\nu n$  neighbours in  $S$ . We say  $G$  is a *robust*  $(\nu, \tau)$ -expander if, for every  $S \subseteq V(G)$  with  $\tau n \leq |S| \leq (1 - \tau)n$ , we have that  $|RN_{\nu,G}(S)| \geq |S| + \nu n$ .

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There is an analogous notion of robust outexpansion for digraphs. Kühn, Osthus and Treglown showed in [13] that every sufficiently large dense robust outexpander contains a directed Hamilton cycle. The notion has been crucial in several other recent papers.

Let  $G$  be a bipartite graph with vertex classes  $A$  and  $B$ . Then clearly  $G$  is not a robust expander. However, we can obtain a bipartite analogue of robust expansion by only considering sets  $S \subseteq A$  with  $\tau|A| \leq |S| \leq (1 - \tau)|A|$ . This notion extends in a natural way to graphs which are ‘close to bipartite’.

Our main result implies that for fixed  $r \in \mathbb{N}$ ,  $\varepsilon > 0$  and  $n$  sufficiently large, any  $D$ -regular graph on  $n$  vertices with  $D \geq (\frac{1}{r+1} + \varepsilon)n$  has a partition into at most  $r$  (bipartite) robust expander components, so that the number of edges between these is very small. We now give a more formal statement of this.

**Definition 1.1.** (*Robust partitions*) Let  $n, D, k, \ell \in \mathbb{N}$  and let  $0 < \rho \leq \nu \leq \tau < 1$ . Suppose that  $G$  is a  $D$ -regular graph on  $n$  vertices. We say that  $\mathcal{V}$  is a *robust partition of  $G$  with parameters  $\rho, \nu, \tau, k, \ell$*  if the following hold:

- $\mathcal{V} = \{V_1, \dots, V_k, W_1, \dots, W_\ell\}$  is a partition of  $V(G)$ ;
- for all  $X \in \mathcal{V}$  we have that  $|X| \geq D - \rho n$  and  $e(X, V(G) \setminus X) \leq \rho n^2$ ;
- for all  $1 \leq j \leq \ell$ ,  $W_j$  has bipartition  $A_j, B_j$  such that  $G[W_j]$  can be made into a balanced bipartite graph with respect to this bipartition by removing at most  $\rho n$  vertices and at most  $\rho n^2$  edges;
- for all  $1 \leq i \leq k$ ,  $G[V_i]$  is a robust  $(\nu, \tau)$ -expander and for all  $1 \leq j \leq \ell$ ,  $G[W_j]$  is a bipartite robust  $(\nu, \tau)$ -expander with vertex classes  $A_j, B_j$ ;
- for all  $X, X' \in \mathcal{V}$  and all  $x \in X$  we have  $d_X(x) \geq d_{X'}(x)$ . In particular,  $d_X(x) \geq D/(k + \ell)$ ;
- for all  $1 \leq j \leq \ell$  and for all  $u \in A_j$  we have  $d_{B_j}(u) \geq d_{A_j}(u)$ , and analogously for  $v \in B_j$ ;
- for all  $X \in \mathcal{V}$  and all but at most  $\rho n$  vertices  $x \in X$  we have  $d_X(x) \geq D - \rho n$ ;
- $k + 2\ell \leq \lfloor (1 + \rho^{1/3})n/D \rfloor$ .

Note that the last property implies there are only a small number of possible choices for  $k$  and  $\ell$  when  $D$  is large. Our main result is the following.

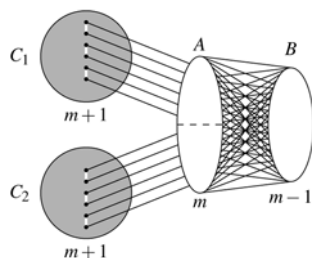
**Theorem 1.2 ([12]).** *For all  $\alpha, \tau > 0$  and every non-decreasing function  $f : (0, 1) \rightarrow (0, 1)$ , there exists  $n_0 \in \mathbb{N}$  such that the following holds. For all  $D$ -regular graphs  $G$  on  $n \geq n_0$  vertices where  $D \geq \alpha n$ , there exist  $k, \ell \in \mathbb{N}$  and  $\rho, \nu$  with  $1/n_0 \leq \rho \leq \nu \leq \tau$ ;  $\rho \leq f(\nu)$  and  $1/n_0 \leq f(\rho)$  such that  $G$  has a robust partition  $\mathcal{V}$  with parameters  $\rho, \nu, \tau, k, \ell$ .*

In the special case of dense vertex-transitive graphs (which are always regular) a related partition result was obtained by Christofides, Hladký and Máthé [6], who resolved the dense case of a question of Lovász [16] on Hamilton paths (and cycles) in vertex-transitive graphs. It would be interesting to obtain such robust partition results for further classes of graphs. In particular, it might be possible to generalise Theorem 1.2 to sparser graphs.

## 2 Two applications to longest cycles in regular graphs

Consider the classical result of Dirac that every graph on  $n \geq 3$  vertices with minimum degree at least  $n/2$  contains a Hamilton cycle. Suppose we wish to strengthen this by reducing the degree threshold at the expense of introducing some other condition(s). The two extremal examples for Dirac's theorem (*i.e.* the disjoint union of two cliques and the almost balanced complete bipartite graph) make it natural to consider regular graphs with some connectivity property, see *e.g.* the recent survey of Li [14] and handbook article of Bondy [3].

In particular, Szekeres (see [9]) asked for which  $D$  every 2-connected  $D$ -regular graph  $G$  on  $n$  vertices is Hamiltonian. Jackson [9] showed that  $D \geq n/3$  suffices. This improved earlier results of Nash-Williams [17], Erdős and Hobbs [7] and Bollobás and Hobbs [2]. Hilbig [8] improved the degree condition to  $n/3 - 1$ , unless  $G$  is the Petersen graph or another exceptional graph. Bollobás [1] as well as Häggkvist (see [9]) independently made the natural and far more general conjecture that any  $t$ -connected regular graph on  $n$  vertices with degree at least  $n/(t + 1)$  is Hamiltonian. However, the following counterexample (see Figure 2.1), due to Jung [11] and independently Jackson, Li and Zhu [10], disproves this conjecture for  $t > 3$ .



**Figure 2.1.** An extremal example for Conjecture 2.1.

For  $m$  divisible by four, construct  $G$  as follows. Let  $C_1, C_2$  be two disjoint copies of  $K_{m+1}$  and let  $A, B$  be two disjoint independent sets of orders  $m, m - 1$  respectively. Add every edge between  $A$  and  $B$ . Add a set of  $m/2$  independent edges from each of  $C_1$  and  $C_2$  to  $A$  so that together

these edges form a matching of size  $m$ . Delete  $m/4$  independent edges in each of  $C_1, C_2$  so that  $G$  is  $m$ -regular. Then  $G$  has  $4m + 1$  vertices and is  $m/2$ -connected. However  $G$  is not Hamiltonian since  $G \setminus A$  has  $|A| + 1$  components (in other words,  $G$  is not 1-tough). Note that  $G$  has a robust partition  $C_1, C_2, A \cup B$  (so  $(k, \ell) = (2, 1)$  in the definition of robust partitions in Section 1).

Jackson, Li and Zhu believe that the conjecture of Bollobás and Häggkvist is true in the remaining open case when  $t = 3$ .

**Conjecture 2.1 ([10]).** Let  $G$  be a 3-connected  $D$ -regular graph on  $n \geq 13$  vertices such that  $D \geq n/4$ . Then  $G$  contains a Hamilton cycle.

The above example shows that the degree condition in Conjecture 2.1 cannot be reduced. One can also show that none of the other conditions can be relaxed.

There have been several partial results towards this conjecture. For instance, Li and Zhu [15] proved it in the case when  $D \geq 7n/22$  and Broersma, van den Heuvel, Jackson and Veldman [5] proved it for  $D \geq 2(n + 7)/7$ . In [10] it is proved that, if  $G$  satisfies the conditions of the conjecture, any longest cycle in  $G$  is dominating provided that  $n$  is not too small. (Here, a subgraph  $H$  of a graph  $G$  is *dominating* if  $G \setminus V(H)$  is an independent set.) By considering robust partitions, we are able to prove an approximate version of the conjecture. We hope that our methods can be used to obtain the exact bound  $D \geq n/4$  for large  $n$ . This is work in progress.

**Theorem 2.2 ([12]).** For all  $\varepsilon > 0$ , there exists  $n_0 \in \mathbb{N}$  such that every 3-connected  $D$ -regular graph on  $n \geq n_0$  vertices with  $D \geq (1/4 + \varepsilon)n$  is Hamiltonian.

More generally, one can consider the circumference of dense regular graphs of given connectivity. Bondy [4] conjectured that, for  $r \geq 3$ , every sufficiently large 2-connected  $D$ -regular graph  $G$  on  $n$  vertices with  $D \geq n/r$  has circumference  $c(G) \geq 2n/(r - 1)$ . (Here the circumference  $c(G)$  of  $G$  is the length of the longest cycle in  $G$ .) This was confirmed by Wei [18], who proved the conjecture for all  $n$  and in fact showed that  $c(G) \geq 2n/(r - 1) + 2(r - 3)/(r - 1)$ , which is best possible. We are able to extend this (asymptotically) to  $t$ -connected dense regular graphs.

**Theorem 2.3.** Let  $t, r \in \mathbb{N}$ . For all  $\varepsilon > 0$  there exists  $n_0 \in \mathbb{N}$  such that the following holds. Whenever  $G$  is a  $t$ -connected  $D$ -regular graph on  $n \geq n_0$  vertices where  $D \geq (1/r + \varepsilon)n$ , the circumference of  $G$  is at least  $\min\{t/(r - 1), 1 - \varepsilon\}n$ .

This is asymptotically best possible. Moreover, as discussed above, the extremal example in Figure 2.1 shows that in general  $\min\{t/(r - 1), 1 -$

$\varepsilon\}n$  cannot be replaced by  $\min\{t/(r-1), 1\}n$ . Theorem 2.3 shows that the conjecture of Bollobás and Häggkvist is in fact close to being true after all – any  $t$ -connected regular graph with degree slightly higher than  $n/(t+1)$  contains an almost spanning cycle.

We are also confident that our robust partition result (Theorem 1.2) will have applications to other problems.

### 3 Sketch of the proof of Theorems 1.2 and 2.2

#### 3.1 Sketch proof of Theorem 1.2

The basic proof strategy is to successively refine an appropriate partition of  $G$ . So let  $G$  be a  $D$ -regular graph on  $n$  vertices, where  $D$  is linear in  $n$ . Suppose that  $G$  is not a (bipartite) robust expander. Then  $V(G)$  contains a set  $S$  such that  $N$  is not much larger than  $S$ , where  $N := RN_{v,G}(S)$  for appropriate  $v$ . Consider a minimal  $S$  with this property. Since  $G$  is regular,  $N$  cannot be significantly smaller than  $S$ . One can use this to show that there are very few edges between  $S \cup N$  and  $X := V(G) \setminus (S \cup N)$ . Moreover, one can show that  $S$  and  $N$  are either almost identical or almost disjoint. In the former case,  $G[S \cup N]$  is a robust expander and in the latter  $G[S \cup N]$  is close to a bipartite robust expander. So in both cases,  $S \cup N$  is a (bipartite) robust expander component. Similarly, if  $X$  is non-empty, it is either a (bipartite) robust expander component or we can partition it further along the above lines. In this way, we eventually arrive at the desired partition.

#### 3.2 Sketch proof of Theorem 2.2

Let  $\varepsilon > 0$  and let  $G$  be a 3-connected  $D$ -regular graph on  $n$  vertices, where  $D \geq (1/4 + \varepsilon)n$ . Theorem 1.2 gives us a robust partition  $\mathcal{V}$  of  $G$  containing exactly  $k$  robust expander components and  $\ell$  bipartite robust expander components where  $k + 2\ell \leq 3$ , so there are only five possible choices of  $(k, \ell)$ . Assume for simplicity that  $\mathcal{V}$  consists of three robust expander components  $G_1, G_2, G_3$ . So  $(k, \ell) = (3, 0)$ . (The cases when  $\ell > 0$  are harder.) The result of [13] mentioned in Section 1 implies that  $G_i$  contains a Hamilton cycle for  $i = 1, 2, 3$ . In fact, it can be used to show that  $G_i$  is Hamilton  $p$ -linked for each bounded  $p$ . (Here a graph  $G$  is *Hamilton  $p$ -linked* if, whenever  $x_1, y_1, \dots, x_p, y_p$  are distinct vertices, there exist vertex-disjoint paths  $P_1, \dots, P_p$  such that  $P_j$  connects  $x_j$  to  $y_j$ , and such that together these paths cover all vertices of  $G$ .) This means that the problem of finding a Hamilton cycle in  $G$  can be reduced to finding only a suitable set of *external edges*, where an edge is external if it has endpoints in different  $G_i$ . We use the assumption of 3-connectivity to find these external edges.

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# The $(\Delta, D)$ and $(\Delta, N)$ problems in double-step digraphs with unilateral diameter

Cristina Dalfo<sup>1</sup> and Miquel Àngel Fiol<sup>2</sup>

## 1 Preliminaries

We study the  $(\Delta, D)$  and  $(\Delta, N)$  problems for double-step digraphs considering the unilateral distance, which is the minimum between the distance in the digraph and the distance in its converse digraph, obtained by changing the directions of all the arcs.

The first problem consists of maximizing the number of vertices  $N$  of a digraph, given the maximum degree  $\Delta$  and the unilateral diameter  $D^*$ , whereas the second one consists of minimizing the unilateral diameter given the maximum degree and the number of vertices. We solve the first problem for every value of the unilateral diameter and the second one for some infinitely many values of the number of vertices.

Miller and Sirán [4] wrote a comprehensive survey about  $(\Delta, D)$  and  $(\Delta, N)$  problems. In particular, for the double-step graphs considering the standard diameter, the first problem was solved by Fiol, Yebra, Alegre and Valero [3], whereas Bermond, Iliades and Peyrat [2], and also Beivide, Herrada, Balcázar and Arruabarrena [1] solved the  $(\Delta, N)$  problem. In the case of the double-step digraphs, also with the standard diameter, Morillo, Fiol and Fàbrega [5] solved the  $(\Delta, D)$  problem and provided some infinite families of digraphs which solve the  $(\Delta, N)$  problem for their corresponding numbers of vertices.

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### 1.1 Double-step digraphs

A *double-step digraph*  $G(N; a, b)$  has set of vertices  $\mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z}$  and arcs from every vertex  $i$  to vertices  $i + a \pmod N$  and  $i + b \pmod N$ , for  $0 \leq i \leq N - 1$ , where  $a, b$  are some integers called *steps* such that  $1 \leq a < b \leq N - 1$ . Because of the automorphisms  $i \mapsto i + \alpha$  for  $1 \leq \alpha \leq N - 1$ , the double-step digraphs are vertex-transitive. Moreover, they are strongly connected if and only if  $\gcd(N, a, b) = 1$ . It is known that the maximum order  $N$  of a double-step digraph with diameter  $k$  is upper bounded by the Moore-like bound  $N \leq M_{DSD}(2, k) = \binom{k+2}{2}$ , where the equality would hold if all the numbers  $ma + nb$  were different modulo  $N$ , with  $m, n \geq 0$  and  $m + n \leq k$ . In fact, this bound cannot be attained for  $k > 1$ .

Every double-step digraph has an L-shaped form associated, which tessellates the plane. If one of the steps, say  $a$ , equals 1, we can choose an L-shaped tile with dimensions  $\ell = b, h$  being the quotient obtained dividing  $N$  by  $\ell, w = \ell - s$  with  $s$  being the remainder of such a division, and  $y = 1$ . Then,  $N = \ell h + s$  with  $0 \leq s < \ell$  (see Figure 1.1).

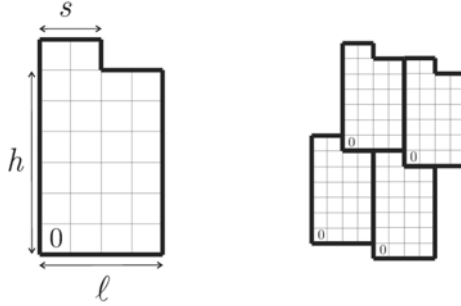


Figure 1.1. An L-shaped form and its tessellation.

### 1.2 Unilateral distance

Given a digraph  $G = (V, A)$ , the *unilateral distance* between two vertices  $u, v \in V$  is defined as

$$\text{dist}_G^*(u, v) = \min\{\text{dist}_G(u, v), \text{dist}_G(v, u)\} = \min\{\text{dist}_G(u, v), \text{dist}_{\overline{G}}(u, v)\},$$

where  $\text{dist}_G$  is the standard distance in digraph  $G$  and  $\text{dist}_{\overline{G}}$  is the distance in its *converse* digraph  $\overline{G}$ , that is, the digraph obtained by changing the directions of all the arcs of  $G$ . From this concept, we can define the *unilateral eccentricity*  $\text{ecc}^*$  from vertex  $u$ , the *unilateral radius*  $r^*$  of  $G$ , and the *unilateral diameter*  $D^*$  of  $G$  as follows:

$$\text{ecc}^*(u) = \max_{v \in V} \{\text{dist}_G^*(u, v)\}, \quad r^* = \min_{u \in V} \{\text{ecc}^*(u)\}, \quad \text{and} \quad D^* = \max_{u \in V} \{\text{ecc}^*(u)\}.$$

As an example, if we have  $G = C_N$ , the directed cycle on  $N$  vertices, then  $D^* = \lfloor N/2 \rfloor$ .

Note that, obviously, these parameters have as lower bounds the ones corresponding to the underlying graph, obtained from digraph  $G$  by changing the arcs for edges without direction.

## 2 The unilateral diameter of double-step digraphs with step $a=1$

In this section we study the unilateral diameter of the double-step digraphs with  $a = 1$  having ‘small’  $b$ . Although we have not been able to prove that the optimal results can be obtained always by taking such values of the steps, computational experiments seem to support this claim. In fact, as we see in the next section, this approach allows us to solve the  $(\Delta, D^*)$  problem for every value of  $D^*$ , and also to solve the  $(\Delta, N)^*$  problem for infinitely many values of  $N$ .

As we have already seen, a double-step digraph  $G(N; 1, b)$  with  $N = \ell h + s$  and  $0 \leq s < \ell$ , can be described by an L-shaped form with dimensions  $\ell = b$ ,  $h = \lfloor N/\ell \rfloor$ ,  $y = 1$ , and  $w = \ell - s$ . See again Figure 1.1. In this context we have the following result for the unilateral diameter  $D^*$ .

**Proposition 2.1.** *For  $N = \ell h + s$ , where  $1 < \ell \leq \lceil \sqrt{N} \rceil$  and  $0 \leq s \leq \ell - 1$ , a double-step digraph  $G(N; a, b)$  with  $a = 1$  and  $b = \ell$  has unilateral diameter*

$$D^* = \begin{cases} \left\lfloor \frac{\ell + h + s - 1}{2} \right\rfloor & \text{if } 0 \leq s \leq \ell - 2, \\ \left\lfloor \frac{\ell + h - 1}{2} \right\rfloor & \text{if } s = \ell - 1. \end{cases} \tag{2.1}$$

If there is not restriction for the value of  $\ell$ , then the values in Eq. (2.1) are an upper bound for the unilateral diameter. For instance, is for  $N = 430$ , the unilateral diameter is 22 and the upper bound given by Equation (2.1) is 23.

## 3 The $(\Delta, D^*)$ and $(\Delta, N)^*$ problems for double-step digraphs with unilateral diameter

### 3.1 The $(\Delta, D^*)$ problem

In our context, the  $(\Delta, D^*)$  problem consists of finding the double-step digraph  $G(N; a, b)$  with maximum number of vertices given a unilateral diameter  $D^*$  and the maximum degree  $\Delta = 2$ , that is, to find the steps



that maximize the number of vertices for such a unilateral diameter. To get a Moore-like bound (see Miller and Sirán [4]), notice that at distance  $k = 1, 2, \dots, D^*$  from vertex 0 there are at most  $2(k + 1)$  vertices ( $k + 1$  of them going forward and the other  $k + 1$  going backwards). Then, this gives

$$N \leq M(2, D^*) = 2(1 + 2 + \dots + D^* + 1) - 1 = (D^*)^2 + 3D^* + 1. \quad (3.1)$$

Moreover, if the maximum is attained, we get an ‘optimal’ X-shaped tile which tessellates the plane, and this allows us to solve the  $(\Delta, D^*)$  problem, as shown in the following result.

**Proposition 3.1.** *For each integer value  $k \geq 0$ , the double-step digraph  $G(N; 1, b)$ , with  $N = M(2, k) = k^2 + 3k + 1$  and  $b = k + 1$  has unilateral diameter  $D^* = k$ .*

### 3.2 The $(\Delta, N)^*$ problem

In our context, the  $(\Delta, N)^*$  problem consists of finding the minimum unilateral diameter  $D^*$  in double-step digraphs given a number of vertices  $N$  and their maximum degree  $\Delta = 2$ , that is, to find the steps that minimize the unilateral diameter for such a number of vertices. We begin with the following general upper bound for the unilateral diameter.

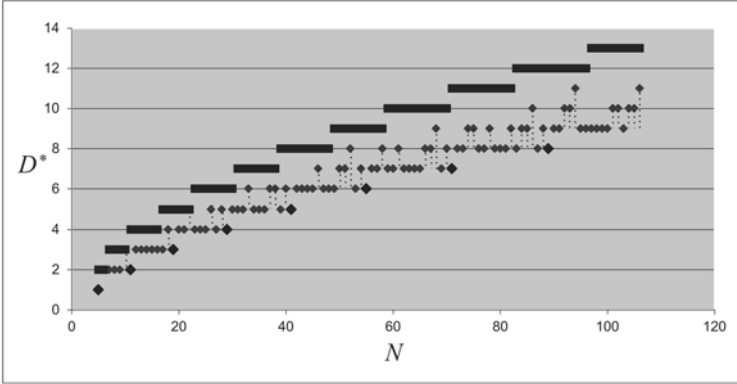
**Proposition 3.2.** *Given any number of vertices  $N \geq 5$ , there exists a double-step digraph with unilateral diameter  $D^*$  satisfying*

$$D^* \leq \left\lceil \sqrt{2(N + 2)} \right\rceil - 2.$$

To solve the  $(\Delta, N)^*$  problem for double-step digraphs with minimum unilateral diameter we consider the case  $s = \ell - 1$  of Proposition 2.1. Moreover, to keep track of the excluded vertices from the maximum  $M(2, k)$ , we define  $r$  as the subindex of the triangular number  $T_r = 1 + 2 + \dots + r = \binom{r+1}{2}$ .

### Proposition 3.3.

- (a) *If  $0 \leq r < \frac{1}{2}(\sqrt{8k + 9} - 1)$ , the double-step digraph  $G(N; a, b)$ , with number of vertices  $N = k^2 + 3k + 1 - r(r + 1)$  and steps  $a = 1$  and  $b = \ell = k - r + 1$ , has minimum unilateral diameter  $D^* = k$ .*
- (b) *If  $0 \leq r < \sqrt{k + 1}$ , the double-step digraph  $G(N; a, b)$ , with number of vertices  $N = k^2 + 2k - r^2$  and steps  $a = 1$  and  $b = \ell = k - r + 1$ , has minimum unilateral diameter  $D^* = k$ .*



**Figure 3.1.** The minimum unilateral diameter  $D^*$  with respect to the number of vertices  $N$ , for  $5 \leq N \leq 106$ . (The largest points correspond to the  $(\Delta, D^*)$  problem, and the thick lines to the upper bound given in Proposition 3.2.)

**Table 1.** Some results of the  $(\Delta, D^*)$  and  $(\Delta, N)^*$  problems solved with Proposition 3.3.

Problem	$\ell + h$	$r$	$\ell$	$h$	$N = \ell h + \ell - 1$	$D^*$
$(\Delta, D^*)$	even	0	$k + 1$	$k + 2$	$k^2 + 3k + 1$	$k$
$(\Delta, N)^*$	even	1	$k$	$k + 3$	$k^2 + 3k - 1$	$k$
$(\Delta, N)^*$	even	2	$k - 1$	$k + 4$	$k^2 + 3k - 5$	$k$
$(\Delta, N)^*$	even	3	$k - 2$	$k + 5$	$k^2 + 3k - 11$	$k$
...	...	...	...	...	...	...
$(\Delta, N)^*$	odd	0	$k + 1$	$k + 1$	$k^2 + 2k$	$k$
$(\Delta, N)^*$	odd	1	$k$	$k + 2$	$k^2 + 2k - 1$	$k$
$(\Delta, N)^*$	odd	2	$k - 1$	$k + 3$	$k^2 + 2k - 4$	$k$
$(\Delta, N)^*$	odd	3	$k - 2$	$k + 4$	$k^2 + 2k - 9$	$k$
...	...	...	...	...	...	...

As shown in Figure 3.1, the unilateral diameter  $D^*$  does not increase monotonously with the number of vertices  $N$ .

Note that if we fix  $r$  for any  $k$  large enough, we get an infinite family of digraphs with minimum unilateral diameter for each number of vertices. See some examples of the cases of Proposition 3.3 in Table 1.

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# Critical groups of generalized de Bruijn and Kautz graphs and circulant matrices over finite fields

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**Abstract.** We determine the critical groups of the generalized de Bruijn graphs  $DB(n, d)$  and generalized Kautz graphs  $Kautz(n, d)$ , thus extending and completing earlier results for the classical de Bruijn and Kautz graphs. Moreover, for a prime  $p$  the critical groups of  $DB(n, p)$  are shown to be in close correspondence with groups of  $n \times n$  circulant matrices over  $\mathbb{F}_p$ , which explains numerical data in [11] and suggests the possibility to construct normal bases in  $\mathbb{F}_{p^n}$  from spanning trees in  $DB(n, p)$ .

## 1 Introduction

The *critical group* of a directed graph  $G$  is an abelian group obtained from the Laplacian matrix  $\Delta$  of  $G$ ; it determines and is determined by the Smith Normal Form (SNF) of  $\Delta$ . (For precise definitions of these and other terms, we refer to the next section.) The *sandpile group*  $S(G, v)$  of  $G$  at a vertex  $v$  is an abelian group obtained from the reduced Laplacian  $\Delta_v$  of  $G$ ; its order is equal to the *complexity*  $\kappa(G)$  of  $G$ , the number of directed trees rooted at  $v$ , a fact that is related to the Matrix Tree Theorem, see for example [8] and its references. If  $G$  is Eulerian, then  $S(G, v)$  does not depend on  $v$ , and is then simply written as  $S(G)$ ; in that case, it is equal to the critical group of  $G$ . The critical group has been studied in other contexts under several other names, such as group of components, Picard or Jacobian group, and Smith group. For more details and background, see, *e.g.*, [6].

Critical groups have been determined for a large number of graph families. For some examples, see the references in [1]. Here, we determine the critical group of the generalized de Bruijn graphs  $DB(n, d)$  and generalized Kautz graphs  $Kautz(n, d)$ , thus extending and completing the results from [8] for the binary de Bruijn graphs  $DB(2^\ell, 2)$  and Kautz graphs

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Kautz $((p-1)p^{\ell-1}, p)$  (with  $p$  prime), and [3] for the classical de Bruijn graphs  $\text{DB}(d^\ell, d)$  and Kautz graphs  $\text{Kautz}((d-1)d^{\ell-1}, d)$ . Unlike the classical case, the generalized versions are not necessarily iterated line graphs, so to obtain their critical groups, different techniques have to be applied.

Our original motivation for studying these groups stems from their relations to some algebraic objects, such as the groups  $C(n, p)$  of invertible  $n \times n$ -circulant matrices over  $\mathbb{F}_p$  (mysterious numerical coincidences were noted in the OIES entry A027362 [11] by the third author, computed with the help of [12,15]), and *normal bases* (cf. e.g. [9]) of the finite fields  $\mathbb{F}_{p^n}$ . The latter were noted to be closely related to circulant matrices and to *necklaces* by Reutenauer [13, Sect. 7.6.2], see also [5], and the related numeric data collected in [2]. Here, we show that  $C(n, p)/(\mathbb{Z}_{p-1} \times \mathbb{Z}_n)$  is isomorphic to the critical group of  $\text{DB}(n, p)$ . Although we were not able to construct an explicit bijection between the former and the latter, we could speculate that potentially one might be able to design a new deterministic way to construct normal bases of  $\mathbb{F}_{p^n}$ .

## 2 Preliminaries

Let  $M$  be an  $m \times n$  integer matrix of rank  $r$ . For a ring  $F$ , we write  $R_F(M) = M^\top F^n$ , the  $F$ -module generated by the rows of  $M$ . The *Smith group* [14] of  $M$  is defined as  $\Gamma(M) = \mathbb{Z}^n / R_{\mathbb{Z}}(M)$ . The submodule  $\bar{\Gamma}(M) = \mathbb{Z}^n / R_{\mathbb{Q}}(M) \cap \mathbb{Z}^n$  of  $\Gamma(M)$  is a finite abelian group called the *finite part* of  $\Gamma(M)$ . Indeed, if  $M$  has rank  $r$ , then  $\Gamma(M) = \mathbb{Z}^{n-r} \oplus \bar{\Gamma}(M)$  with  $\bar{\Gamma}(M) = \bigoplus_{i=1}^r \mathbb{Z}_{d_i}$ , where  $d_1, \dots, d_r$  are the nonzero invariant factors of  $M$ , so that  $d_i | d_{i+1}$  for  $i = 1, \dots, r-1$ . For invariant factors and the Smith Normal Form, we refer to [10]. See [14] for further details and proofs.

Let  $G = (V, E)$  be a directed graph on  $n = |V|$  vertices. The indegree  $d^-(v)$  and outdegree  $d^+(v)$  is the number of edges ending or starting in  $v \in V$ , respectively. The *adjacency matrix* of  $G$  is the  $n \times n$  matrix  $A = (A_{v,w})$ , with rows and columns indexed by  $V$ , where  $A_{v,w}$  is the number of edges from  $v$  to  $w$ . The *Laplacian* of  $G$  is the matrix  $\Delta = D - A$ , where  $D$  is diagonal with  $D_{v,v} = d_v^-$ . The *critical group*  $K(G)$  of  $G$  is the finite part of the Smith group of the Laplacian  $\Delta$  of  $G$ . The *sandpile group*  $S(G, v)$  of  $G$  at a  $v \in V$  is the finite part of the Smith group of the  $(n-1) \times (n-1)$  *reduced Laplacian*  $\Delta_v$ , obtained from  $\Delta$  by deleting the row and the column of  $\Delta$  indexed by  $v$ . Note that by the Matrix Tree Theorem for directed graphs, the order of  $S(G, v)$  equals the number of directed spanning trees rooted at  $v$ .

## 2.1 Generalized de Bruijn and Kautz graphs

Generalized de Bruijn graphs and generalized Kautz graphs [4] are known to have a relatively small diameter and attractive connectivity properties, and have been studied intensively due to their applications in interconnection networks. The generalized Kautz graphs were first investigated in [7], and are also known as *Imase-Itoh digraphs*. Both classes of graphs are Eulerian.

We will determine the critical group, or, equivalently, the sandpile group, of a generalized de Bruijn or Kautz graph on  $n$  vertices by embedding this group as a subgroup of index  $n$  in a group that we will refer to as the *sand dune* group of the corresponding digraph. Let us now turn to the details.

The *generalized de Bruijn graph*  $DB(n, d)$  has vertex set  $\mathbb{Z}_n$ , the set of integers modulo  $n$ , and (directed) edges  $v \rightarrow dv + i$  for  $i = 0, \dots, d - 1$  and all  $v \in \mathbb{Z}_n$ . The *generalized Kautz graph*  $Kautz(n, d)$  has vertex set  $\mathbb{Z}_n$  and directed edges  $v \rightarrow -d(v + 1) + i$  for  $i = 0, \dots, d - 1$  and all  $v \in \mathbb{Z}_n$ . Note that both  $DB(n, d)$  and  $Kautz(n, d)$  are Eulerian. In what follows, we will focus on the generalized de Bruijn graph; the generalized Kautz graph can be handled in a similar way, essentially by replacing  $d$  by  $-d$  in certain places.

Let  $\mathcal{Z}_n = \{a(x) \in \mathbb{Z}[x] \bmod x^n - 1 \mid a(1) = 0\}$ . With each vertex  $v \in \mathbb{Z}_n$ , we associate the polynomial  $f_v(x) = dx^v - x^{dv} \sum_{i=0}^{d-1} x^i \in \mathcal{Z}_n$ . Since  $f_v(x)$  is the associated polynomial of the  $v$ th row of the Laplacian  $\Delta^{(n,d)}$  of the generalized de Bruijn graph  $DB(n, d)$ , the Smith group  $\Gamma(\Delta^{(n,d)})$  of the Laplacian of  $DB(n, d)$  is the quotient of  $\mathbb{Z}[x] \bmod x^n - 1$  by the  $\mathbb{Z}_n$ -span  $\langle f_v(x) \mid v \in \mathbb{Z}_n \rangle_{\mathbb{Z}_n}$  of the polynomials  $f_v(x)$ . Now note that  $\mathbb{Z}[x] \bmod x^n - 1 \cong \mathbb{Z} \oplus \mathcal{Z}_n$ , so since  $\sum_{v \in \mathbb{Z}_n} f_v(x) = 0$ , we have that

$$\begin{aligned} \Gamma(\Delta^{(n,d)}) &= (\mathbb{Z}[x] \bmod x^n - 1) / \langle f_v(x) \mid v \in \mathbb{Z}_n \rangle_{\mathbb{Z}_n} \\ &\cong \mathbb{Z} \oplus \mathcal{Z}_n / \langle f_v(x) \mid v \in \mathbb{Z}'_n \rangle_{\mathbb{Z}_n} \end{aligned} \quad (2.1)$$

where  $\mathbb{Z}'_n = \mathbb{Z}_n \setminus \{0\}$ . It is easily checked that the polynomials  $f_v(x)$  with  $v \in \mathbb{Z}'_n$  are independent over  $\mathbb{Q}$ , hence they constitute a basis for  $\mathcal{Z}_n$  over  $\mathbb{Q}$ . As a consequence, each element in the quotient group

$$S(n, d) = S_{DB}(n, d) = \mathcal{Z}_n / \langle f_v(x) \mid v \in \mathbb{Z}'_n \rangle_{\mathbb{Z}_n} \quad (2.2)$$

has finite order, and so  $S(n, d)$  is the critical group, or, equivalently, the sandpile group, of the generalized de Bruijn graph  $DB(n, d)$ . We define the *sand dune group*  $\Sigma(n, d) = \Sigma_{DB}(n, d)$  of  $DB(n, d)$  as  $\Sigma(n, d) = \mathcal{Z}_n / \langle g_v(x) \mid v \in \mathbb{Z}'_n \rangle_{\mathbb{Z}_n}$ , where  $g_v(x) = (x - 1)f_v(x) = dx^v(x - 1) - x^{dv}(x^d - 1)$ . Now let  $e_v = x^v - 1$ ; we have that  $e_0 = 0$ , and  $\mathcal{Z}_n =$

$\langle e_v \mid v \in \mathbb{Z}'_n \rangle_{\mathbb{Z}}$ , the  $\mathbb{Z}$ -span of the polynomials  $e_v$ . Furthermore, let  $\epsilon_v = de_v - e_{dv}$ . The span in  $\mathcal{Q}_n = \{a(x) \in \mathbb{Q}[x] \bmod x^n - 1 \mid a(1) = 0\}$  of the polynomials  $g_v(x)$  with  $v \in \mathbb{Z}_n$  is the set of polynomials of the form  $dc(x) - c(x^d)$  with  $c(1) = 0$ ; since  $\epsilon_v = g_0(x) + \dots + g_{v-1}(x)$  for all  $v \in \mathbb{Z}_n$ , we conclude that

$$\Sigma(n, d) = \mathcal{Z}_n / \mathcal{E}_{n,d}, \tag{2.3}$$

where  $\mathcal{Z}_n = \langle e_v \mid v \in \mathbb{Z}'_n \rangle_{\mathbb{Z}}$  and  $\mathcal{E}_{n,d} = \langle \epsilon_v \mid v \in \mathbb{Z}'_n \rangle_{\mathbb{Z}}$  is the  $\mathbb{Z}$ -submodule of  $\mathcal{Z}_n$  generated by the polynomials  $\epsilon_v = de_v - e_{dv}$ . The next result is crucial: it identifies the elements of the sand dune group  $\Sigma(n, d)$  that are actually contained in the sandpile group  $S(n, d)$ . (Due to lack of space, we omit the not too difficult proofs in the remainder of this section.)

**Theorem 2.1.** *If  $a \in \Sigma(n, d)$  with  $a = \sum_v a_v e_v$ , then  $a \in S(n, d)$  if and only if  $\sum_v v a_v \equiv 0 \pmod n$ .*

**Corollary 2.2.** *We have  $\Sigma(n, d) / S(n, d) = \mathbb{Z}_n$  and so  $|\Sigma(n, d)| = n|S(n, d)|$ .*

The above descriptions of the sandpile group  $S(n, d)$  and sand dune group  $\Sigma(n, d)$ , and the embedding of  $S(n, d)$  as a subgroup of  $\Sigma(n, d)$  are very suitable for the determination of these groups. In the process, repeatedly information is required about the order of various group elements. The following two results provide that information.

**Lemma 2.3.** *Let  $a = \sum_v a_v \epsilon_v \in \Sigma(n, d)$ . Then the order of  $a$  in  $\Sigma(n, d)$  is the smallest positive integer  $m$  for which  $ma_v \in \mathbb{Z}$  for each  $v$ .*

We say that  $v \in \mathbb{Z}_n$  has  $d$ -type  $(f, e)$  in  $\mathbb{Z}_n$  if  $v, dv, \dots, d^{e+f-1}v$  are all distinct, with  $d^{e+f}v = d^f v$ . Now, by expressing  $e_v$  in terms of the  $\epsilon_v$ , we can determine the order of  $e_v$ . The result is as follows.

**Lemma 2.4.** *Supposing  $v$  has  $d$ -type  $(f, e)$ , then  $e_v = \sum_{i=0}^{f-1} d^{-i-1} \epsilon_{d^i v} + \sum_{j=0}^{e-1} d^{j-f} (d^e - 1)^{-1} \epsilon_{d^{f+j} v}$  in  $\mathcal{Z}_n$ , and hence  $e_v$  has order  $d^f (d^e - 1)$  in  $\Sigma(n, d)$ .*

## 2.2 Invertible circulant matrices

Let  $Q_n$  be the  $n \times n$  permutation matrix over a field  $F$  corresponding to the cyclic permutation  $(1, 2, \dots, n)$ . An  $n \times n$  circulant matrix over  $F$  is a matrix that can be written as  $a_1 Q_n + a_2 Q_n^2 + \dots + a_n Q_n^n$  with  $a_i \in F$  for  $1 \leq i \leq n$ . All the invertible circulant matrices form a commutative group (w.r.t. matrix multiplication), namely, the centralizer

of  $Q_n$  in  $GL_n(F)$ . In the case  $F = \mathbb{F}_p$  we consider here we denote this commutative group by  $C(n, p)$ . Note that  $C(n, p)$  contains a subgroup isomorphic to  $\mathbb{Z}_{p-1} \oplus \mathbb{Z}_n$ , namely the direct product of the group of scalar matrices  $F_p^* I := \{\lambda I \mid \lambda \in \mathbb{F}_p^*\}$  and the cyclic subgroup generated by  $Q_n$ . Each circulant matrix has all-ones vector  $\mathbf{1} := (1, \dots, 1)^\top$  as an eigenvector. Thus  $C'(n, p) := \{g \in C(n, p) \mid g\mathbf{1} = \mathbf{1}\}$  is a subgroup of  $C(n, p)$ , and we have the following formula.

$$C(n, p) = C'(n, p) \times F_p^* I. \tag{2.4}$$

### 3 Main results

Let  $n, d > 0$  be fixed integers. The description of the sandpile group  $S(n, d)$  and the sand-dune group  $\Sigma(n, d)$  of the generalized the Bruin graph  $DB(n, d)$  involves a sequence of numbers defined as follows. Put  $n_0 = n$ , and for  $i = 1, 2, \dots$ , define  $g_i = \gcd(n_i, d)$  and  $n_{i+1} = n_i/g_i$ . We have  $n_0 > \dots > n_k = n_{k+1}$ , where  $k$  is the smallest integer for which  $g_k = 1$ . We will refer to the sequence  $n_0 > \dots > n_k = n_{k+1}$  as the  $d$ -sequence of  $n$ . In what follows, we will write  $m = n_k$  and  $g = g_0 \cdots g_{k-1}$ . Note that  $n = gm$  with  $\gcd(m, d) = 1$ .

Since  $\gcd(m, d) = 1$ , the map  $x \rightarrow dx$  partitions  $\mathbb{Z}_m$  into orbits of the form  $O(v) = (v, dv, \dots, d^{o(v)-1}v)$ . We will refer to  $o(v) = |O(v)|$  as the order of  $v$ .

For every prime  $p|m$ , we define  $\pi_p(m)$  to be the largest power of  $p$  dividing  $m$ . Let  $V$  be a complete set of representatives of the orbits  $O(v)$  different from  $\{0\}$ , where we ensure that for every divisor  $p$  of  $m$ , all integers of the form  $m/p^j$  are contained in  $V$ .

**Theorem 3.1.** *With the above definitions and notation, we have that*

$$\Sigma(n, d) = \left[ \bigoplus_{i=0}^{k-1} \mathbb{Z}_{d^{i+1}}^{n_i - 2n_{i+1} + n_{i+2}} \right] \oplus \left[ \bigoplus_{v \in V} \mathbb{Z}_{d^{o(v)} - 1} \right], \tag{3.1}$$

and

$$S(n, d) = \left[ \bigoplus_{i=0}^{k-1} \mathbb{Z}_{d^{i+1}/g_i} \oplus \mathbb{Z}_{d^{i+1}}^{n_i - 2n_{i+1} + n_{i+2} - 1} \right] \oplus \left[ \bigoplus_{v \in V} \mathbb{Z}_{(d^{o(v)} - 1)/c(v)} \right], \tag{3.2}$$

where  $c(v) = 1$  except in the following cases. For any  $p|m$ ,

$$c(m/\pi_p(m)) = \begin{cases} \pi_p(m), & \text{if } p \neq 2 \text{ or } d \equiv 1 \pmod 4 \text{ or } 4 \nmid m; \\ \pi_2(m)/2, & \text{if } p = 2 \text{ and } d \equiv 3 \pmod 4 \text{ and } 4|m, \end{cases}$$

and if  $4|m$  and  $d \equiv 3 \pmod 4$ , then  $c(m/2) = 2$ .



For the generalized Kautz graph, a similar result holds. For  $v \in \mathbb{Z}_m$ , we let  $O'(v)$  denote the orbit of  $v$  under the map  $x \rightarrow -dv$ , and we define  $o'(v) = |O'(v)|$ . Now take  $V'$  to be a complete set of representatives of the orbits on  $\mathbb{Z}'_m$ . Finally, define  $c'(v)$  similar to  $c(v)$ , except that now  $d$  is replaced by  $-d$  (so the special case now involves  $d \equiv 1 \pmod{4}$ ). Then we have the following.

**Theorem 3.2.** *The sandpile group  $S_{\text{Kautz}}(n, d)$  of the generalized Kautz graph  $\text{Kautz}(n, d)$  is obtained from  $S(n, d)$  by replacing  $V$  by  $V'$ ,  $o(v)$  by  $o'(v)$ , and  $c(v)$  by  $c'(v)$  in (3.2).*

The above results can be proved in a number of steps. In what follows, we outline the method for the generalized de Bruijn graphs; for the generalized Kautz graphs, a similar approach can be used. Furthermore, we note that many of the steps below repeatedly use Theorem 2.1 and Lemma 2.4. First, we investigate the “multiplication-by- $d$ ” map  $d : x \rightarrow dx$  on the sandpile and sand-dune group. Let  $\Sigma_0(n, d)$  and  $S_0(n, d)$  denote the kernel of the map  $d^k$  on  $\Sigma(n, d)$  and  $S(n, d)$ , respectively. It is not difficult to see that  $\Sigma(n, d) \cong \Sigma_0(n, d) \oplus \Sigma(m, d)$  and  $S(n, d) \cong S_0(n, d) \oplus S(m, d)$ . Then, we use the map  $d$  to determine  $\Sigma_0(n, d)$  and  $S_0(n, d)$ . It is easy to see that for any  $n$ , we have  $d\Sigma(n, d) \cong \Sigma(n/(n, d), d)$  and  $dS(n, d) \cong S(n/(n, d), d)$ . With much more effort, it can be shown that the kernel of the map  $d$  on  $\Sigma(n, d)$  and  $S(n, d)$  is isomorphic to  $\mathbb{Z}_d^{n-n/(n,d)}$  and  $\mathbb{Z}_{d/(n,d)} \oplus \mathbb{Z}_d^{n-1-n/(n,d)}$ , respectively. Then we use induction over the length  $k + 1$  of the  $d$ -sequence of  $n$  to show that  $\Sigma_0(n, d)$  and  $S_0(n, d)$  have the form of the left part of the right hand side in (3.1) and (3.2), respectively. This part of the proof, although much more complicated, resembles the method used by [8] and [3].

Now it remains to handle the parts  $\Sigma(m, d)$  and  $S(m, d)$  with  $\gcd(m, d) = 1$ . For the “helper” group  $\Sigma(m, d)$  that embeds  $S(m, d)$ , this is trivial: it is easily seen that  $\Sigma(m, d) = \bigoplus_{v \in V} \langle e_v \rangle$ , and the order of  $e_v$  is equal to the size  $o(v)$  of its orbit  $O(v)$  under the map  $d$ , so (3.1) follows immediately. The  $e_v$  are not contained in  $S(m, d)$ , but we can try to modify them slightly to obtain a similar decomposition for  $S(m, d)$ . The idea is to replace  $e_v$  by a modified version  $\tilde{e}_v = e_v - \sum_{p|m} \lambda_p(v) e_{\pi_p(v)m/\pi_p(m)}$ , where the numbers  $\lambda_p(v)$  are chosen such that  $\tilde{e}_v \in S(m, d)$ , or by a suitable multiple of  $e_v$ , in some exceptional cases (these are cases where  $c(v) > 1$ ). It turns out that this is indeed possible, and in this way the proof of Theorem 3.1 can be completed.

Finally, with the notation from Subsect. 2.2, we have the following isomorphisms, connecting critical groups and circulant matrices.

**Theorem 3.3.** *Let  $d$  be a prime. Then*

$$S(n, d) \cong C'(n, d)/\langle Q_n \rangle, \quad \text{and} \quad \Sigma(n, d) \cong C'(n, d).$$

The proof of Theorem 3.3 is by reducing to the case  $\gcd(n, p) = 1$  by an explicit construction, and then by diagonalizing  $C(n, p)$  over an appropriate extension of  $\mathbb{F}_p$ . Essentially, as soon as  $\gcd(n, p) = 1$ , one can read off a decomposition of  $C(n, p)$  into cyclic factors from the irreducible factors of the polynomial  $x^n - 1$  over  $\mathbb{F}_p$ .

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# Two notions of unit distance graphs

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**Abstract.** A *complete (unit) distance graph* in  $\mathbb{R}^d$  is a graph whose set of vertices is a finite subset of the  $d$ -dimensional Euclidean space, and two vertices are adjacent if and only if the Euclidean distance between them is exactly 1. A *(unit) distance graph* in  $\mathbb{R}^d$  is any subgraph of such a graph. We study various properties of both types of distance graphs. We show that for any fixed  $d$  the number of complete distance graphs in  $\mathbb{R}^d$  on  $n$  labelled vertices is  $2^{(1+o(1))dn \log_2 n}$ , while the number of distance graphs in  $\mathbb{R}^d$  on  $n$  labelled vertices is  $2^{(1-1/\lfloor d/2 \rfloor + o(1))n^2/2}$ . This is used to study a Ramsey type question involving these graphs. Finally, we discuss the following problem: what is the minimum number of edges a graph must have so that it is not realizable as a complete distance graph in  $\mathbb{R}^d$ ?

## 1 Introduction

This paper is devoted to the notion of a (unit) distance graph. There are two well-known definitions:

**Definition 1.1.** A graph  $G = (V, E)$  is a *(unit) distance graph in  $\mathbb{R}^d$*  if  $V \subset \mathbb{R}^d$  and  $E \subseteq \{(x, y), x, y \in \mathbb{R}^d, |x - y| = 1\}$ , where  $|x - y|$  denotes the Euclidean distance between  $x$  and  $y$ .

**Definition 1.2.** A graph  $G = (V, E)$  is a *complete (unit) distance graph in  $\mathbb{R}^d$*  if  $V \subset \mathbb{R}^d$  and  $E = \{(x, y), x, y \in \mathbb{R}^d, |x - y| = 1\}$ .

Distance graphs arise naturally in the study of two well-known problems of combinatorial geometry. First one, posed by Erdős [3], is the fol-

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lowing: determine the maximum number  $f_2(n)$  of unit distances among  $n$  points on the plane. Second is the following question, posed by Nelson (see [2]): what is the minimum number  $\chi(\mathbb{R}^2)$  of colours needed to color the points of the plane so that no two points at unit distance apart receive the same colour?

Let  $\mathcal{D}(d)$  denote the set of all labeled distance graphs in  $\mathbb{R}^d$ , and let  $\mathcal{D}_n(d)$  denote the set of all those of order  $n$ . Similarly, denote by  $\mathcal{CD}(d)$  the set of all labeled complete distance graphs in  $\mathbb{R}^d$ , and let  $\mathcal{CD}_n(d)$  denote the set of those of order  $n$ .

We reformulate the stated above questions in terms of distance graphs:

$$f_2(n) = \max_{G \in \mathcal{D}_n(2)} |E(G)| = \max_{G \in \mathcal{CD}_n(2)} |E(G)|.$$

$$\chi(\mathbb{R}^d) = \max_{G \in \mathcal{D}(d)} \chi(G) = \max_{G \in \mathcal{CD}(d)} \chi(G),$$

The second series of equalities follows from the well-known Erdős–de Bruijn theorem, which states that the chromatic number of the space  $\mathbb{R}^d$  is equal to the chromatic number of some finite distance graph in  $\mathbb{R}^d$ . It is easy to see that it does not matter which definition of distance graph we use in the study of these two problems. However, sets  $\mathcal{D}(d)$  and  $\mathcal{CD}(d)$  differ greatly, and we discuss it in the next section.

## 2 Results

The first theorem shows that in any dimension  $d$  the number of distance graphs is far bigger than the number of complete distance graphs.

### Theorem 2.1.

1. For any  $d \in \mathbb{N}$ , we have  $|\mathcal{CD}_n(d)| = 2^{(1+o(1))dn \log_2 n}$ .
2. For any  $d \in \mathbb{N}$  we have  $|\mathcal{D}_n(d)| = 2^{\left(1 - \frac{1}{|d/2|} + o(1)\right) \frac{n^2}{2}}$ .
3. If  $d = d(n) = o(n)$ , then we have  $|\mathcal{CD}_n(d)| = 2^{o(n^2)}$ .
4. If  $d = d(n) \geq c \frac{n}{\log_2 n}$ , where  $c > 4$ , then  $|\mathcal{D}_n(d)| = (1 + o(1))2^{\frac{n(n-1)}{2}}$ .

In other words, almost every graph on  $n$  vertices can be realized as a distance graph in  $\mathbb{R}^d$ .

*Sketch of the proof.* Let  $P_1, \dots, P_m$  be  $m$  real polynomials in  $l$  real variables. For a point  $x \in \mathbb{R}^l$  the zero pattern of the  $P_j$ 's at  $x$  is the tuple  $(\varepsilon_1, \dots, \varepsilon_m) \in \{0, 1\}^m$ , where  $\varepsilon_j = 0$ , if  $P_j(x) = 0$  and  $\varepsilon_j = 1$  if  $P_j(x) \neq 0$ . Denote by  $z(P_1, \dots, P_m)$  the total number of zero patterns of polynomials  $P_1, \dots, P_m$ . Upper bounds from point 1 and 3 of the theorem are proved using the following proposition from real algebraic geometry ([7], Theorem 1.3 and Corollary 1.5):

**Proposition 2.2 ([7]).** *Let  $P_1, \dots, P_m$  be  $m$  real polynomials in  $l$  real variables, and suppose the degree of each  $P_j$  does not exceed  $k$ . Then  $z(P_1, \dots, P_m) \leq \binom{km}{\ell} \leq (ekm/l)^l$ .*

Denote by  $(v_1^i, \dots, v_d^i)$  the coordinates of the vertex  $v_i$  in the distance graph. For each unordered pair  $\{i, j\}$  of vertices of the graph define the following polynomial  $P_{ij}$ :

$$P_{ij} = -1 + \sum_{r=1}^d (v_r^i - v_r^j)^2.$$

It is easy to see that each labeled distance graph in  $\mathbb{R}^d$  corresponds to a zero pattern of the polynomials  $P_{12}, \dots, P_{n-1n}$ .

Lower bound in part 1 of the theorem follows from the fact that certain bipartite graphs with maximum degree  $d$  in one part are realizable as complete distance graphs in  $\mathbb{R}^d$ .

Lower bound in part 1 of the theorem is a corollary of the fact that any  $[d/2]$ -partite graph is realizable as a distance graph in  $\mathbb{R}^d$ . Upper bound has several proofs, with the simplest one based on the regularity lemma. It rests on the fact that complete  $([d/2] + 1)$ -partite graph with three vertices in each part is not realizable as distance graph in  $\mathbb{R}^d$ .

The fourth part of the theorem is an easy consequence of the famous theorem by Bollobás [1] concerning the chromatic number of the random graph  $G(n, 1/2)$ .  $\square$

Next, we study the following two Ramsey-type quantities.

**Definition 2.3.** The (complete) distance Ramsey number  $R_D(s, t, d)$  ( $R_{CD}(s, t, d)$ ) is the minimum natural  $m$  such that for any graph  $G$  on  $m$  vertices the following holds: either  $G$  contains an induced  $s$ -vertex subgraph isomorphic to a (complete) distance graph in  $\mathbb{R}^d$  or its complement  $\bar{G}$  contains an induced  $t$ -vertex subgraph isomorphic to a (complete) distance graph in  $\mathbb{R}^d$ .

The quantity  $R_D(s, s, d)$  was introduced in [6], and studied in several follow-up papers, while the quantity  $R_{CD}(s, s, d)$  was not studied so far. The following theorem was proved in [4]:

**Theorem 2.4 ([4]).**

1. For every fixed  $d \geq 2$  we have

$$R_D(s, s, d) \geq 2^{\left(\frac{1}{2[d/2]} + o(1)\right)s}.$$

2. For any  $d = d(s)$ ,  $2 \leq d \leq s/2$  we have

$$R_D(s, s, d) \leq d \cdot R\left(\left\lceil \frac{s}{\lfloor d/2 \rfloor} \right\rceil, \left\lceil \frac{s}{\lfloor d/2 \rfloor} \right\rceil\right),$$

where  $R(k, \ell)$  is the classical Ramsey number: the minimum number  $n$  so that any graph on  $n$  vertices contains either a clique of size  $k$  or an independent set of size  $\ell$ .

By the last theorem the bounds for  $R_D(s, s, d)$  are roughly the same as for the classical Ramsey number  $R\left(\left\lceil \frac{s}{\lfloor d/2 \rfloor} \right\rceil, \left\lceil \frac{s}{\lfloor d/2 \rfloor} \right\rceil\right)$ :

$$\frac{s}{2\lfloor d/2 \rfloor}(1 + o(1)) \leq \log R_D(s, s, d) \leq \frac{2s}{\lfloor d/2 \rfloor}(1 + o(1)),$$

where the  $o(1)$ -terms tend to zero as  $s$  tends to infinity.

Using Theorem 2.1 we can show that  $R_{CD}(s, s, d)$  is far larger than  $R_D(s, s, d)$ .

**Theorem 2.5.**

1. For any  $d = d(s) = o(s)$  we have  $R_{CD}(s, s, d) \geq 2^{(1+o(1))s/2}$ .
2. For  $d = d(s) \leq cs$ , where  $c < 1/2$  and  $H(c) < 1/2$ , there exists a constant  $\alpha = \alpha(c) > 0$  such that  $R_{CD}(s, s, d) \geq 2^{(1+o(1))\alpha s}$ .

This theorem is proved via standard probabilistic approach used to obtain lower bounds on the Ramsey numbers.

The last (possible) difference between  $\mathcal{D}(d)$  and  $\mathcal{CD}(d)$  we point out is the following. Fix an  $l \in \mathbb{N}$ . The following theorem was proved in [5].

**Theorem 2.6 ([5]).** For any  $g \in \mathbb{N}$  there exists a sequence of distance graphs in  $\mathbb{R}^d$ ,  $d = 1, 2, \dots$ , with girth greater than  $g$  such that the chromatic number of the graphs in the sequence grows exponentially with  $d$ .

Unfortunately, we cannot prove a similar theorem for complete distance graphs. All we can prove is the following

**Proposition 2.7.** For any  $g \in \mathbb{N}$  there exists a sequence of complete distance graphs in  $\mathbb{R}^d$ ,  $d = 1, 2, \dots$ , with girth greater than  $g$  such that the chromatic number of the graphs in the sequence grows as  $\Omega\left(\frac{d}{\log d}\right)$ .

Every bipartite graph is realizable as a distance graph in  $\mathbb{R}^4$ . However, for any fixed  $d$  there exists a bipartite graph that is not realizable as a complete distance graph in  $\mathbb{R}^d$ . In general, it seems difficult even for a

bipartite graph  $G$  to decide whether  $G$  is realizable as a complete distance graph in  $\mathbb{R}^d$  or not. In particular, we introduce the quantity  $g_2(d)$ , which is equal to the minimum possible number of edges in a bipartite graph  $K$  that is not realizable as a complete distance graph in  $\mathbb{R}^d$ . We obtained the following theorem.

**Theorem 2.8.** *For any  $d \geq 4$  we have  $\binom{d+2}{2} \leq g_2(d) \leq \binom{d+3}{2} - 6$ .*

The lower bound in this theorem states that if a bipartite graph is not realizable as a complete distance graph in  $\mathbb{R}^d$ ,  $d \geq 4$ , then it must have at least as many edges as the complete graph  $K_{d+2}$  on  $d+2$  vertices, which is an obvious example of a graph that is not realizable as a distance graph in  $\mathbb{R}^d$ . This is not the case for  $d=3$ , since the graph  $K_{3,3}$  is not realizable as a distance graph in  $\mathbb{R}^3$  and it has  $9 < \binom{5}{2}$  edges. It is interesting to determine, whether an arbitrary graph  $G$  that is not realizable as a complete distance graph in  $\mathbb{R}^d$ ,  $d \geq 4$ , must have at least  $\binom{d+2}{2}$  edges, and to study the similar question for distance graphs.

*Sketch of the proof of theorem 2.8.* Both upper and lower bounds are based on linear-algebraic considerations, mostly dealing with the notion of affine dependence.

To obtain the upper bound, we prove that the bipartite graph  $K''$  with the parts  $A = \{a_1, \dots, a_d\}$ ,  $B = \{b_1, \dots, b_d\}$  and with the set of edges  $E = \{(a_i, b_j) : i > j\} \cup \{(a_i, b_j) : i \leq 3\}$  is not realizable as a complete distance graph in  $\mathbb{R}^d$ .

To prove the lower bound we provide sufficient conditions for a bipartite graph to be realizable as a complete distance graph in  $\mathbb{R}^d$ . The construction of the realization is algorithmic.  $\square$

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# An interlacing approach for bounding the sum of Laplacian eigenvalues of graphs

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**Abstract.** We apply interlacing techniques for obtaining lower and upper bounds for the sums of Laplacian eigenvalues of graphs. Mainly, we generalize two theorems of Grone and Grone & Merris, providing tight bounds and studying the cases of equality. As a consequence, some results on well-known parameters of a graph, such as the maximum and minimum cuts, the edge-connectivity, the (almost) dominating number, and the edge isoperimetric number, are derived.

## 1 Introduction

Throughout this paper,  $G = (V, E)$  is a finite simple graph with  $n = |V|$  vertices and  $e = |E|$  edges. Recall that the Laplacian matrix of  $G$  is  $L = D - A$  where  $D$  is the diagonal matrix of the vertex degrees and  $A$  is the adjacency matrix of  $G$ . Let us also recall the following result about interlacing (see [2]).

**Theorem 1.1.** *Let  $A$  be a real symmetric  $n \times n$  matrix with eigenvalues  $\lambda_1 \geq \dots \geq \lambda_n$ . For some  $m < n$ , let  $S$  be a real  $n \times m$  matrix with orthonormal columns,  $S^\top S = I$ , and consider the matrix  $B = S^\top A S$ , with eigenvalues  $\mu_1 \geq \dots \geq \mu_m$ . Then,*

(a) *The eigenvalues of  $B$  interlace those of  $A$ , that is,*

$$\lambda_i \geq \mu_i \geq \lambda_{n-m+i}, \quad i = 1, \dots, m. \quad (1.1)$$

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- (b) *If the interlacing is tight, that is, for some  $0 \leq k \leq m$ ,  $\lambda_i = \mu_i$ ,  $i = 1, \dots, k$ , and  $\mu_i = \lambda_{n-m+i}$ ,  $i = k + 1, \dots, m$ , then  $\mathbf{SB} = \mathbf{AS}$ .*

Two interesting particular cases are when  $\mathbf{B}$  is a principal submatrix of  $\mathbf{A}$ ; and when  $\mathbf{B}$  is the so-called *quotient matrix* of  $\mathbf{A}$  with respect to a given partition  $\mathcal{P} = \{I_1, \dots, I_m\}$  of  $\{1, \dots, n\}$ .

The first case gives useful conditions for an induced subgraph  $G'$  of a graph  $G$ , because the adjacency matrix of  $G'$  is a principal submatrix of the adjacency matrix of  $G$ . However, the Laplacian matrix  $\mathbf{L}'$  of  $G'$  is in general not a principal submatrix of the Laplacian matrix  $\mathbf{L}$  of  $G$ . But  $\mathbf{L}' + \mathbf{D}'$  is a principal submatrix of  $\mathbf{L}$  for some nonnegative diagonal matrix  $\mathbf{D}'$ . Therefore the left hand inequalities in (1.1) still hold for the Laplacian eigenvalues, because adding the positive semi-definite matrix  $\mathbf{D}'$  decreases no eigenvalue.

In the case that  $\mathbf{B}$  is a quotient matrix of  $\mathbf{A}$ , the element  $b_{ij}$  of  $\mathbf{B}$  is the average row sum of the block of  $\mathbf{A}$  with rows and columns indexed by  $I_i$  and  $I_j$ , respectively. Actually, the quotient matrix  $\mathbf{B}$  does not need to be symmetric or equal to  $\mathbf{S}^\top \mathbf{AS}$ , but in this case  $\mathbf{B}$  is similar to (and therefore has the same spectrum as)  $\mathbf{S}^\top \mathbf{AS}$ .

If the interlacing is tight, then (b) of Theorem 1.1 reflects that  $\mathcal{P}$  is a *regular* (or *equitable*) partition of  $\mathbf{A}$ , that is, each block of the partition has constant row and column sums. Moreover, if  $\mathbf{A}$  is the adjacency matrix of a graph  $G$ , then the bipartite induced subgraphs  $G[I_i, I_j]$  are biregular, and  $G[I_i]$  is regular (then we say that  $\mathcal{P}$  is a *regular* partition of  $G$ .) Alternatively, if the interlacing is tight for the quotient matrix of the Laplacian matrix of  $G$ , then the first condition also holds, but not necessarily the second (now we speak about an *almost regular* partition of  $G$ .)

Assuming that  $G$  has  $n$  vertices, with degrees  $d_1 \geq d_2 \geq \dots \geq d_n$ , and Laplacian matrix  $\mathbf{L}$  with eigenvalues  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n (= 0)$ , it is known that, for  $1 \leq m \leq n$ ,

$$\sum_{i=1}^m \lambda_i \geq \sum_{i=1}^m d_i. \tag{1.2}$$

This is a consequence of Schur's theorem [6] stating that the spectrum of any symmetric, positive definite matrix majorizes its main diagonal. In particular, note that if  $m = n$  we have equality in (1.2), because both terms correspond to the trace of  $\mathbf{L}$ . To prove (1.2) by using interlacing, let  $\mathbf{B}$  be a principal  $m \times m$  submatrix of  $\mathbf{L}$  indexed by the subindexes corresponding to the  $m$  largest degrees, with eigenvalues  $\mu_1 \geq \mu_2 \geq \dots \geq \mu_m$ . Then,  $\text{tr } \mathbf{B} = \sum_{i=1}^m d_i = \sum_{i=1}^m \mu_i$ , and, by interlacing,

$\lambda_{n-m+i} \leq \mu_i \leq \lambda_i$  for  $i = 1, \dots, m$ , whence (1.2) follows. Similarly, reasoning with the principal submatrix  $\mathbf{B}$  (of  $\mathbf{L}$ ) indexed by the  $m$  vertices with minimum degrees we get:

$$\sum_{i=1}^m \lambda_{n-m+i} \leq \sum_{i=1}^m d_{n-m+i}. \quad (1.3)$$

The next result, which is an improvement of (1.2), is due to Grone [4], who proved that if  $G$  is connected and  $m < n$  then,

$$\sum_{i=1}^m \lambda_i \geq \sum_{i=1}^m d_i + 1. \quad (1.4)$$

In this paper, we give a generalization of this result by considering the degrees of vertices in a given subset. Note that if we take  $m = 1$  in (1.4), we get  $\lambda_1 \geq d_1 + 1$ . Guo [3] conjectured another generalization looking at individual eigenvalues, which was proved by Brouwer and Haemers [1]. They showed that if  $\lambda_i$  is the  $i$ -th largest Laplacian eigenvalue, and  $d_i$  is the  $i$ -th largest degree of a connected graph  $G$  on  $n$  vertices (in fact, it is enough to assume that  $G \neq K_m \cup (n - m)K_1$ ), then  $\lambda_i \geq d_i - i + 2$ ,  $1 \leq i \leq n - 1$ .

## 2 New results

In this section we present the main results of the paper. In particular, we use interlacing for generalizing two results of Grone [4] and Grone and Merris [5].

### 2.1 A generalization of Grone's result

We begin with a basic result from where most of our bounds derive. Given a graph  $G$  with a vertex subset  $U \subset V$ , let  $\partial U$  be the *vertex-boundary* of  $U$ , that is, the set of vertices in  $\bar{U} = V \setminus U$  which have some adjacent vertex in  $U$ . Also, let  $\partial(U, \bar{U})$  denote the *edge-boundary* of  $U$ , which is the set of edges which connect vertices in  $U$  with vertices in  $\bar{U} = V \setminus U$ , with cardinality  $e(U, \bar{U}) = |\partial(U, \bar{U})|$ . For every vertex  $v \in V$ , let  $d_v = d(v)$  stands for its degree.

**Proposition 2.1.** *Let  $G$  be a graph on  $n = |V|$  vertices, having Laplacian matrix  $\mathbf{L} = (l_{uv})$  with eigenvalues  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n (= 0)$ . For any given vertex subset  $U = \{u_1, \dots, u_m\}$  with  $0 < m < n$ , we have*

$$\sum_{i=1}^m \lambda_{n-i} \leq \sum_{u \in U} d_u + \frac{e(U, \bar{U})}{|\bar{U}|} \leq \sum_{i=1}^m \lambda_i. \quad (2.1)$$

The equality on either side of (2.1) implies that the interlacing is tight, and therefore that the partition of  $G$  is almost equitable. In other words, in case of equality every vertex  $x \in U$  is adjacent to either all or 0 vertices in  $\bar{U}$ , whereas each vertex  $x \in \bar{U}$  has precisely  $b = e(U, \bar{U})/|\bar{U}|$  neighbors in  $U$ . Using this, it is straightforward to construct nontrivial examples with equality. (If  $n = m + 1$ , equality holds trivially on both sides.)

In this context, observe that there is no graph (with  $n > 2$ ) satisfying (1.4) for every  $0 < m < n$ . However the complete graph  $K_n$  provides an example for which the inequalities in Proposition 2.1 are equalities for all  $0 < m < n$ . Indeed, in this case we have  $\lambda_i = n$  for any  $1 \leq i < n$  and for any set  $U$  of size  $m$ ,  $e(U, \bar{U})/|\bar{U}| = m$ , thus giving for any  $0 < m < n$ ,

$$\sum_{i=1}^m \lambda_i = mn = m(n - 1) + m = \sum_{u \in U} d_u + \frac{e(U, \bar{U})}{|\bar{U}|},$$

If the vertex degrees of  $G$  are  $d_1 \geq d_2 \geq \dots \geq d_n$ , we can choose conveniently the  $m$  vertices of  $U$  (that is, those with maximum or minimum degrees) to obtain the best inequalities in (2.1). Namely,

$$\sum_{i=1}^m \lambda_i \geq \sum_{i=1}^m d_i + \frac{e(U, \bar{U})}{|\bar{U}|}, \tag{2.2}$$

and

$$\sum_{i=1}^m \lambda_{n-i} \leq \sum_{i=1}^m d_{n-i+1} + \frac{e(U, \bar{U})}{|\bar{U}|}. \tag{2.3}$$

Note that, since  $e(U, \bar{U}) \geq 1$ , (2.2) is a slight improvement of (1.2). Moreover, (2.3), together with (1.3) for  $m + 1$ , yields

$$\sum_{i=1}^m \lambda_{n-m+i} = \sum_{i=1}^m \lambda_{n-i} \leq \sum_{i=1}^m d_{n-i+1} + \min \left\{ d_{n-m}, \frac{e(U, \bar{U})}{|\bar{U}|} \right\}. \tag{2.4}$$

If we have more information on the structure of the graph, we can improve the above results by either bounding  $e(U, \bar{U})$  or ‘optimizing’ the ratio  $b = e(U, \bar{U})/|\bar{U}|$ . In fact, the right inequality in (2.1) (and, hence, (2.2)) can be improved when  $\bar{U} \neq \partial U$ . Simply first delete the vertices (and corresponding edges) of  $\bar{U} \setminus \partial U$ , and then apply the inequality. Then  $d_1, \dots, d_m$  remain the same and  $\lambda_1, \dots, \lambda_m$  do not increase. Thus we obtain:

**Theorem 2.2.** *Let  $G$  be a connected graph on  $n = |V|$  vertices, with Laplacian eigenvalues  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n (= 0)$ . For any given vertex subset  $U = \{u_1, \dots, u_m\}$  with  $0 < m < n$ , we have*

$$\sum_{i=1}^m \lambda_i \geq \sum_{u \in U} d_u + \frac{e(U, \bar{U})}{|\partial U|}. \tag{2.5}$$

Notice that, as a corollary, we get Grone’s result [4] since  $e(U, \bar{U}) \geq |\partial U|$ .

### 2.2 A generalization of a bound by Grone and Merris

In [5], Grone and Merris gave another lower bound for the sum of the Laplacian eigenvalues, in the case when there is an induced subgraph consisting of isolated vertices and edges. If the induced subgraph of a subset  $U \subset V$  with  $|U| = m$  consists of  $r$  pairwise disjoint edges and  $m - 2r$  isolated vertices, then

$$\sum_{i=1}^m \lambda_i \geq \sum_{u \in U} d_u + m - r. \tag{2.6}$$

An improvement of this result was given by Brouwer and Haemers in [2]. Let  $G$  be a (not necessarily connected) graph with a vertex subset  $U$ , with  $m = |U|$ , and let  $h$  be the number of connected components of  $G[U]$  that are not connected components of  $G$ . Then,

$$\sum_{i=1}^m \lambda_i \geq \sum_{u \in U} d_u + h. \tag{2.7}$$

Using interlacing, the bound (2.6) can also be generalized as follows:

**Theorem 2.3.** *Let  $G$  be a connected graph of order  $n > 2$  with Laplacian eigenvalues  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ . Given a vertex subset  $U \subset V$ , with  $m = |U| < n$ , let  $G[U] = (U, E[U])$  and  $G[\bar{U}]$  be the corresponding induced subgraphs. Let  $\theta_1$  be the largest Laplacian eigenvalue of  $G[\bar{U}]$ . Then,*

$$\sum_{i=1}^{m+1} \lambda_i \geq \sum_{u \in U} d_u + m - |E[U]| + \theta_1. \tag{2.8}$$

The previous bounds on the sum of Laplacian eigenvalues are used to provide meaningful results for the size of the maximum cut of a graph, the edge-connectivity of the graph, the minimum size of a dominating set and the (edge-)isoperimetric number. Given a graph  $G$  on  $n$  vertices, the *isoperimetric number* is defined as  $i(G) = \min_{U \subset V} \{e(U, \bar{U})/|U| : 0 < |U| \leq n/2\}$ .

**Proposition 2.4.** *Let  $G$  be a graph on  $n$  vertices, with vertex degrees  $d_1 \geq d_2 \geq \dots \geq d_n$ , and Laplacian eigenvalues  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n (= 0)$ . Let  $m$  such that  $\frac{n}{2} \leq m < n$ . Then,*

$$i(G) \leq \sum_{i=1}^m (\lambda_i - d_i). \quad (2.9)$$

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# On the structure of the group of balanced labelings on graphs

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## 1 Abstract

Let  $G = (V, E)$  be an undirected graph with possible multiple edges and loops (a multigraph). Let  $A$  be an Abelian group. In this work we study the following topics:

1) A function  $f : E \rightarrow A$  is called balanced if the sum of its values along every closed truncated trail of  $G$  is zero. By a truncated trail we mean a trail without the last vertex. The set  $H(E, A)$  of all the balanced functions  $f : E \rightarrow A$  is a subgroup of the free Abelian group  $A^E$  of all functions from  $E$  to  $A$ . We give a full description of the structure of the group  $H(E, A)$ , and provide an  $O(|E|)$ -time algorithm to construct a set of the generators of its cyclic direct summands.

2) A function  $g : V \rightarrow A$  is called balanceable if there exists some  $f : E \rightarrow A$  such that the sum of all the values of  $g$  and  $f$  along every closed truncated trail of  $G$  is zero. The set  $B(V, A)$  of all balanceable functions  $g : V \rightarrow A$  is a subgroup of the free Abelian group  $A^V$  of all the functions from  $V$  to  $A$ . We give a full description of the structure of the group  $B(V, A)$ .

3) A function  $h : V \cup E \rightarrow A$  taking values on vertices and edges is called balanced if the sum of its values along every closed truncated trail of  $G$  is zero. The set  $W(V \cup E, A)$  of all balanced functions  $h : V \cup E \rightarrow A$  is a subgroup of the free Abelian group  $A^{V \cup E}$  of all functions from  $V \cup E$  to  $A$ . The group  $H(E, A)$  is naturally isomorphic to the subgroup of  $W(V \cup E, A)$  consisting of all functions taking every vertex to 0. So we, abusing the notations, treat  $H(E, A)$  as that subgroup of  $W(V \cup E, A)$ .

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There is a natural epimorphism from  $W(V \cup E, A)$  onto  $B(V, A)$ , which “forgets” the values of  $h$  on the edges. The kernel of this epimorphism is precisely  $H(E, A)$ . Thus,  $W(V \cup E, A)/H(E, A) \cong B(V, A)$ . We use this fact to give a full description of the group  $W(V \cup E, A)$ .

## 2 Introduction

For directed graphs, the study of integer-valued functions that vanish on all cycles of a graph, and the notion of the cycle space are classical in Graph Theory. For undirected graphs these questions were first addressed in [1]. In that work such functions are called cycle-vanishing edge valuations. It turns out that in the undirected case the dimensions of the cycle space and, dually, of the space of cycle vanishing edge valuations are closely related to 3-edge connectivity. In a very recent article [4] the balanced functions from the union of the set of vertices and the set of edges of a graph to a finite Abelian group  $A$  are considered. In that work, which is closely related to [1], the number of such functions is calculated.

In [6] they study the triples  $(\Gamma, g, G)$ , where  $\Gamma$  is a graph,  $G$  is a group, and  $g$  is a function from the set of edges of  $\Gamma$  to  $G$ . The edges of  $\Gamma$  are thought of as having arbitrary, but fixed, orientation and the equality  $g(-e) = (g(e))^{-1}$  holds for every edge  $e$ , where  $-e$  is  $e$  with the inverted orientation. Such triples are called gain or voltage graphs. A gain graph is called balanced if the product of values of  $g$  along every cycle equals the identity element. For the voltage graphs, being balanced is equivalent to satisfying Kirchhoff’s voltage law. In [6] several strong criteria for the gain graphs to be balanced are obtained. Voltage graphs are discussed in [3]. For the survey on signed graphs, gain graphs, and related topics see [8].

## 3 Results

A truncated trail (ttrail)  $p$  from a vertex  $x$  to a vertex  $y$  is an alternating sequence  $v_1, e_1, v_2, e_2, \dots, v_n, e_n$  of vertices and edges such that  $v_1 = x$ , each  $e_j$ , for  $j = 1, \dots, n-1$ , connects  $v_j$  and  $v_{j+1}$ ,  $e_n$  connects  $v_n$  and  $y$ , and  $e_i \neq e_j$  for  $i \neq j$ . Let  $p = v_1, e_1, v_2, e_2, \dots, v_n, e_n$  be a ttrail from a vertex  $x$  to a vertex  $y$  and  $p' = v'_1, e'_1, v'_2, e'_2, \dots, v'_{n'}, e'_{n'}$  be a ttrail from  $y$  to a vertex  $w$ . Then, if  $p$  and  $p'$  have no common edges, the ttrail  $p + p'$  from  $x$  to  $w$  is defined as  $v_1, e_1, v_2, e_2, \dots, v_n, e_n, v'_1, e'_1, v'_2, e'_2, \dots, v'_{n'}, e'_{n'}$ . Let  $p = v_1, e_1, v_2, e_2, \dots, v_n, e_n$  be a ttrail from a vertex  $x$  to a vertex  $y$ . Then the ttrail  $p^{-1}$  from  $y$  to  $x$  is defined as  $v'_1, e'_1, v'_2, e'_2, \dots, v'_n, e'_n$  where  $v'_1 = y$ ,  $v'_i = v_{n+2-i}$  for  $i = 2, \dots, n$ , and  $e'_j = e_{n+1-j}$  for  $j = 1, \dots, n$ . A ttrail  $p$  from a vertex  $x$  to itself is called a closed ttrail. A closed ttrail is called a cycle if it contains every vertex only one time.

The subgroup of all elements of  $A$  of order 2 is denoted  $A_2$ . The image  $2A$  of the doubling self-map  $a \mapsto 2a$  of  $A$  is a subgroup of  $A$ .

**Definition 3.1.** Vertices  $v$  and  $w$  of  $G$  are called  $k$ -edge-connected if there exist  $k$  different ttrails between  $v$  and  $w$  such that no two of these ttrails have any common edges. By definition, we say that a vertex  $v$  of  $G$  is  $k$ -edge-connected to itself for all  $k$ .

The edge version of the famous Menger's Theorem, which asserts that two distinct vertices  $v$  and  $w$  of  $G$  are  $k$ -edge-connected if and only if no deletion of any  $k-1$  edges from  $G$  disconnects  $v$  and  $w$ , plays crucial role in our work. Its corollary is that  $k$ -edge-connectivity is an equivalence relation on the vertices of  $G$ .

We denote the number of equivalence classes of  $k$ -edge-connected vertices of  $G$  by  $con_k(G)$ . Note that  $con_1(G) = con(G)$  is just the number of the connected components of  $G$ . We denote the equivalence class of  $k$ -edge-connected vertices of  $G$ , which contains a vertex  $v$ , by  $Con_k(v)$ .

**Definition 3.2.** A ttrail between two  $k$ -edge-connected vertices is called a  $k$ -weakly closed ttrail of  $G$ .

Clearly, every closed ttrail of  $G$  is also a  $k$ -weakly closed ttrail of  $G$  for every  $k$ .

In this work we mostly concentrate on the case when  $k = 3$ , since it plays a crucial role in the structure of groups  $H(E, A)$ ,  $B(V, A)$  and  $W(V \cup E, A)$ . It is shown in [7] that there exists a simple linear-time algorithm for finding all 3-edge-connected components of an undirected graph. Further, we introduce short ttrails, and obtain several related structural results, like Lemma 3.6 and Theorem 3.7. Then, we obtain the dual results, like Lemma 3.13 and Theorem 3.12. Finally, we use these results to obtain the group structure of  $H(E, A)$ ,  $B(V, A)$ , and  $W(V \cup E, A)$ .

**Definition 3.3.** The boundary linear map  $\delta : \mathbb{F}_2^E \rightarrow \mathbb{F}_2^V$  is defined by taking each edge to the sum of its two adjacent vertices.

**Definition 3.4.** The Cycle Space of  $G$  is defined as  $Ker(\delta)$ , and its elements are called the homological cycles of  $G$ .

**Definition 3.5.** A 3-weakly closed ttrail between vertices  $v$  and  $w$ , where  $v$  can be equal to  $w$ , is called short if no one of its inner vertices is 3-edge-connected to  $v$ .

**Lemma 3.6.** *Every 3-weakly closed ttrail is a unique sum of short 3-weakly closed ttrails.*

Let  $x, y, u, w$  be (not necessarily distinct) vertices belonging to the same 3-edge-connected component of  $G$ .

**Theorem 3.7.** *If two short 3-weakly closed ttrails, one from  $x$  to  $y$  and the other from  $u$  to  $w$ , have a common inner vertex, then  $x = u$ ,  $y = w$ , and their first and last edges coincide.*

**Definition 3.8.** Two short 3-weakly closed ttrails are called twins if their first and last edges coincide.

Twinship is an equivalence relation between short 3-weakly closed ttrails.

**Definition 3.9.** For two vertices  $v, w$  of  $G$ , a homological path  $p$  between  $v$  and  $w$  is a vector  $p \in \mathbb{F}_2^E$  such that  $\delta(p) = v + w$ .

**Definition 3.10.** A homological path between two (not necessarily distinct)  $k$ -edge-connected vertices of  $G$  is called a  $k$ -weak homological cycle of  $G$ .

**Definition 3.11.** The subspace of  $\mathbb{F}_2^E$ , spanned by all the  $k$ -weak homological cycles of  $G$ , is called the  $k$ -Weak Cycle Space of  $G$ .

The Cycle Space is a subspace of the  $k$ -Weak Cycle Space for each  $k$ .

Next, we obtain the following dual results on functions from  $G$  to  $A$ .

**Theorem 3.12.** *If a function  $f : E \rightarrow A$  takes the Cycle Space of  $G$  to 0 then  $f$  is balanced. If a function  $f : E \rightarrow A$  is balanced then it takes the 3-Weak Cycle Space of  $G$  into  $A_2$  and the Cycle Space of  $G$  to 0.*

**Lemma 3.13.** *A function  $g : V \rightarrow A$  such that  $g(V) \in 2A$  or that for some 3-edge-connected component  $W$  of  $V$  and any  $a \in A$ ,  $g(w) = a$ , if  $w \in W$ , and  $g(w) = 0$  otherwise, is balanceable. In the other direction, if a function  $g : V \rightarrow A$  is balanceable then for any 3-edge-connected vertices  $v$  and  $w$  we must have  $g(v) - g(w) \in 2A$ .*

Using the above findings, we construct certain appropriate bases for the Cycle Space of  $G$ , the 3-Weak Cycle Space of  $G$ ,  $\mathbb{F}_2^E$ ,  $\mathbb{F}_2^{\text{con}_3(G)}$ , and  $\mathbb{F}_2^V$ , which we need to prove our three concluding theorems.

**Theorem 3.14.** *The Abelian group  $H(E, A)$  of all the balanced functions  $f : E \rightarrow A$  is isomorphic to  $A^{\text{con}_3(G) - \text{con}(G)} \times A_2^{|V| - \text{con}_3(G)}$ .*

**Theorem 3.15.** *The Abelian group  $B(V, A)$  of all the balanceable functions  $g : V \rightarrow A$  is isomorphic to  $A^{\text{con}_3(G)} \times (2A)^{|V| - \text{con}_3(G)}$ .*

**Theorem 3.16.** *The Abelian group  $W(V \cup E, A)$  of all the balanced functions  $h : V \cup E \rightarrow A$  is isomorphic to  $A^{|V| + \text{con}_3(G) - \text{con}(G)}$ .*

Notice that Theorem 4 in [4] follows from our Theorem 3.16 for a finite  $A$ .

## 4 Conclusions

When every  $a \in A$  is of order 2, for example, if  $A = Z_2$ , then  $2A = 0$  and  $A_2 = A$ . In that case our results for  $H(E, A)$  coincide with the classical results for balanced gain graphs.

When  $A$  is the additive group of the real numbers then  $2A = A$  and  $A_2 = 0$ , and our findings on  $H(E, A)$  coincide with the results of [1].

When  $A$  is a finite group, Theorem 4 in [4] follows from our Theorem 3.16.

We have studied the group structure of the group  $W(V \cup E, A)$ , its subgroup  $H(E, A)$ , and its factor-group  $B(V, A)$ . The dual problem is to understand the structure of the subgroup  $B'(V, A)$  of all balanced functions on vertices and the factor-group  $H'(E, A)$  of all balanceable functions on edges. The elements of  $B'(V, A)$  are, by abuse of notation, such  $h \in W(V \cup E, A)$  that  $h(e) = 0$  for every edge  $e \in E$ . In the case when  $A = Z_2$  this is identical to describing the consistent marked (vertex-signed) graphs. These graphs were treated and characterized in [5].

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# The price of connectivity for feedback vertex set

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**Abstract.** Let  $\text{fvs}(G)$  and  $\text{cfvs}(G)$  denote the cardinalities of a minimum feedback vertex set and a minimum connected feedback vertex set of a graph  $G$ , respectively. In general graphs, the ratio  $\text{cfvs}(G)/\text{fvs}(G)$  can be arbitrarily large. We study the interdependence between  $\text{fvs}(G)$  and  $\text{cfvs}(G)$  in graph classes defined by excluding one induced subgraph  $H$ . We show that the ratio  $\text{cfvs}(G)/\text{fvs}(G)$  is bounded by a constant for every connected  $H$ -free graph  $G$  if and only if  $H$  is a linear forest. We also determine exactly those graphs  $H$  for which there exists a constant  $c_H$  such that  $\text{cfvs}(G) \leq \text{fvs}(G) + c_H$  for every connected  $H$ -free graph  $G$ , as well as exactly those graphs  $H$  for which we can take  $c_H = 0$ .

## 1 Introduction

Numerous important graph parameters are defined as the cardinality of a smallest subset of vertices satisfying a certain property. Well-known examples of such parameters include the cardinality of a minimum vertex cover, a minimum dominating set, or a minimum feedback vertex set in a graph. In many cases, requiring the subset of vertices to additionally induce a connected subgraph defines a natural variant of the original parameter. The cardinality of a minimum connected vertex cover or a minimum connected dominating set are just two examples of such parameters that have received considerable interest from both the algorithmic and structural graph theory communities. An interesting question is what effect the additional connectivity constraint has on the value of the graph parameter in question.

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One notable graph parameter that has been studied in this context is the vertex cover number  $\tau(G)$ , defined as the cardinality of a minimum vertex cover of a graph  $G$ . The connected variant of this parameter is the connected vertex cover number, denoted by  $\tau_c(G)$  and defined as the cardinality of a minimum connected vertex cover in  $G$ . The following observation on the interdependence between  $\tau(G)$  and  $\tau_c(G)$  for connected graphs  $G$  is due to Camby et al. [2].

**Observation 1.1 ([2]).** For every connected graph  $G$ , it holds that  $\tau_c(G) \leq 2 \cdot \tau(G) - 1$ .

Given a graph class  $\mathcal{G}$ , the worst-case ratio  $\tau_c(G)/\tau(G)$  over all connected graphs  $G$  in  $\mathcal{G}$  is defined to be the *price of connectivity* for vertex cover for the class  $\mathcal{G}$ . Observation 1.1 implies that for general graphs, the price of connectivity for vertex cover is upper bounded by 2, and the class of all paths shows that the bound of 2 is asymptotically sharp [2]. Cardinal and Levy [4], who coined the term “price of connectivity for vertex cover”, showed a stronger upper bound of  $2/(1 + \epsilon)$  for graphs with average degree  $\epsilon n$ . Camby et al. [2] provided forbidden induced subgraph characterizations of graph classes for which the price of connectivity for vertex cover is upper bounded by  $t$ , for  $t \in \{1, 4/3, 3/2\}$ .

The above idea applies to other graph parameters as well. The following observation, due to Duchet and Meyniel [5], shows the interdependence between the connected domination number  $\gamma_c(G)$  and the domination number  $\gamma(G)$  of a connected graph  $G$ .

**Observation 1.2 ([5]).** For every connected graph  $G$ , it holds that  $\gamma_c(G) \leq 3 \cdot \gamma(G) - 2$ .

Adapting the terminology used above for vertex cover, Observation 1.2 implies that the price of connectivity for dominating set on general graphs is upper bounded by 3. The class of all paths again shows that this bound is asymptotically sharp. Zverovich [9] proved that for any graph  $G$ , it holds that  $\gamma_c(H) = \gamma(H)$  for every induced subgraph  $H$  of  $G$  if and only if  $G$  is  $(P_5, C_5)$ -free, that is, if and only if  $G$  does not contain an induced subgraph isomorphic to  $P_5$  or  $C_5$ . This implies that the price of connectivity for dominating set is exactly 1 for the class of  $(P_5, C_5)$ -free graphs. Camby and Schaudt [3] proved that  $\gamma_c(G) \leq \gamma(G) + 1$  for every connected  $(P_6, C_6)$ -free graph  $G$ , and showed that this bound is best possible. They also obtained a sharp upper bound of 2 on the price of connectivity for dominating set for  $(P_8, C_8)$ -free graphs, and showed that the general upper bound of 3 is asymptotically sharp for  $(P_9, C_9)$ -free graphs.

A *feedback vertex set* of a graph  $G$  is a set  $F$  of vertices such that deleting  $F$  makes  $G$  acyclic, that is, the graph  $G - F$  is a forest. The

cardinalities of a minimum feedback vertex set and a minimum connected feedback vertex set of a graph  $G$  are denoted by  $\text{fvs}(G)$  and  $\text{cfvs}(G)$ , respectively. For any graph class  $\mathcal{G}$ , we define the price of connectivity for feedback vertex set to be the worst-case ratio  $\text{cfvs}(G)/\text{fvs}(G)$  over all connected graphs  $G$  in  $\mathcal{G}$ . In contrast to the aforementioned upper bounds of 2 and 3 on the price of connectivity for vertex cover and dominating set, respectively, the price of connectivity for feedback vertex is not upper bounded by a constant. Graphs consisting of two disjoint cycles that are connected to each other by an arbitrarily long path show that the price of connectivity for feedback vertex set is not even bounded by a constant for planar graphs. Interestingly, Grigoriev and Sitters [6] showed that for planar graphs of minimum degree at least 3, the price of connectivity for feedback vertex set is at most 11. This upper bound of 11 was later improved to 5 by Schweitzer and Schweitzer [8], who also showed that this bound is tight.

**Our Results.** We study the price of connectivity for feedback vertex set for graph classes characterized by one forbidden induced subgraph  $H$ . A graph is called  $H$ -free if it does not contain an induced subgraph isomorphic to  $H$ . We show that the price of connectivity for feedback vertex set is bounded by a constant on the class of  $H$ -free graphs if and only if  $H$  is a linear forest, that is, a forest of maximum degree at most 2. In fact, we obtain a more refined tetrachotomy result on the interdependence between  $\text{fvs}(G)$  and  $\text{cfvs}(G)$  for all connected  $H$ -free graphs  $G$ , depending on the structure of the graph  $H$ . In order to formally state our result, we need the following terminology. The *disjoint union*  $G + H$  of two vertex-disjoint graphs  $G$  and  $H$  is the graph with vertex set  $V(G) \cup V(H)$  and edge set  $E(G) \cup E(H)$ . We write  $sH$  to denote the disjoint union of  $s$  copies of  $H$ , and  $P_n$  to denote the path on  $n$  vertices. Let  $H$  be a graph and  $\mathcal{G}$  the class of  $H$ -free graphs. The class  $\mathcal{G}$  is called *identical*, *additive*, or *multiplicative* if for all connected graphs  $G$  in  $\mathcal{G}$ , it holds that  $\text{cfvs}(G) = \text{fvs}(G)$ ,  $\text{cfvs}(G) \leq \text{fvs}(G) + c_H$  for some constant  $c_H \geq 0$ , or  $\text{cfvs}(G) \leq d_H \cdot \text{fvs}(G)$  for some constant  $d_H \geq 0$ , respectively. Our result can now be formulated as follows.

**Theorem 1.3.** *Let  $H$  be a graph, and let  $\mathcal{G}$  be the class of  $H$ -free graphs. Then*

- (i)  $\mathcal{G}$  is multiplicative if and only if  $H$  is a linear forest;
- (ii)  $\mathcal{G}$  is additive if and only if  $H$  is an induced subgraph of  $P_5 + sP_1$  or  $sP_3$  for some  $s \geq 0$ ;
- (iii)  $\mathcal{G}$  is identical if and only if  $H$  is an induced subgraph of  $P_3$ .

## 2 The proof of Theorem 1.3

Statements (i), (ii) and (iii) in Theorem 1.3 follow from Lemmas 2.1, 2.3 and 2.4 below, respectively.

**Lemma 2.1.** *Let  $H$  be a graph. Then there is a constant  $d_H$  such that  $\text{cfvs}(G) \leq d_H \cdot \text{fvs}(G)$  for every connected  $H$ -free graph  $G$  if and only if  $H$  is a linear forest.*

*Proof.* If  $H$  is a linear forest, then the statement of the lemma follows from the fact that every connected  $H$ -free graph has bounded diameter.

Before proving the reverse direction, we first introduce a family of graphs that will be used later in the proof. Let  $C_n$  denote the cycle on  $n$  vertices. For any three integers  $i, j, k$ , we define  $B_{i,j,k}$  to be the graph obtained from  $C_i + C_j$  by choosing a vertex  $x$  in  $C_i$  and a vertex  $y$  in  $C_j$ , and adding a path of length  $k$  between  $x$  and  $y$ ; if  $k = 0$ , then we simply identify  $x$  and  $y$ . The graph  $B_{3,3,0}$  is called the *butterfly*.

Now suppose  $H$  is not a linear forest. We distinguish two cases. Suppose  $H$  contains a cycle, and let  $C$  be a shortest cycle in  $H$ ; in particular,  $C$  is an induced cycle. For any integer  $\ell$ , the graph  $B_\ell := B_{|V(C)|+1, |V(C)|+1, \ell}$  is  $C$ -free and therefore  $H$ -free. The observation that  $\text{cfvs}(B_\ell) = \text{fvs}(B_\ell) + \ell - 1$  for every  $\ell \geq 1$  shows that no constant  $d_H$  exists as described in the lemma. If  $H$  does not contain a cycle, then  $H$  is a forest. For any integer  $\ell$ , the graph  $B_{3,3,\ell}$  is claw-free. Since we assumed that  $H$  is not a linear forest and hence contains a claw,  $B_{3,3,\ell}$  is also  $H$ -free. The observation that  $\text{cfvs}(B_{3,3,\ell}) = \text{fvs}(B_{3,3,\ell}) + \ell - 1$  for every  $\ell \geq 1$  completes the proof of Lemma 2.1.  $\square$

Lemma 2.2 below exhibits a structural property of  $sP_3$ -free graphs that will be used in the proof of Lemma 2.3 below. The proof of Lemma 2.2 has been omitted due to page restrictions.

**Lemma 2.2.** *For every integer  $s$ , there is a constant  $c_s$  such that  $\text{cfvs}(G) \leq \text{fvs}(G) + c_s$  for every connected  $sP_3$ -free graph  $G$ .*

**Lemma 2.3.** *Let  $H$  be a graph. Then there is a constant  $c_H$  such that  $\text{cfvs}(G) \leq \text{fvs}(G) + c_H$  for every connected  $H$ -free graph  $G$  if and only if  $H$  is an induced subgraph of  $P_5 + sP_1$  or  $sP_3$  for some integer  $s$ .*

*Proof.* First suppose  $H$  is an induced subgraph of  $P_5$ . Let  $G$  be a connected  $H$ -free graph. In particular,  $G$  is  $P_5$ -free. Hence, due to a result by Bacsó and Tuza [1], there exists a dominating set  $D \subseteq V(G)$  such that  $D$  is a clique or  $D$  induces a  $P_3$  in  $G$ . Let  $F$  be a minimum feedback vertex set of  $G$ . Note that  $|D \setminus F| \leq 2$  if  $D$  is a clique and  $|D \setminus F| \leq 3$  if  $D$  induces a  $P_3$ . Since  $D$  is a connected dominating set in  $G$ , the set



$F \cup D$  is a connected feedback vertex set of  $G$  of size at most  $|F| + 3$ . Hence, we can take  $c_H = 3$ .

Now suppose  $H$  is an induced subgraph of  $P_5 + sP_1$  for some integer  $s$ . Let  $G$  be a connected  $H$ -free graph. If  $G$  is  $P_5$ -free, then we can take  $c_H = 3$  due to the above arguments. Suppose  $G$  contains an induced path  $P$  on 5 vertices. Let  $S$  be a maximal independent set in the graph obtained from  $G$  by deleting the five vertices of  $P$  as well as all their neighbors in  $G$ . Since  $G$  is  $P_5 + sP_1$ -free, we know that  $|S| \leq s - 1$ . Note that  $V(P) \cup S$  is a dominating set of  $G$ . Hence, by Observation 1.2, there is a connected dominating set  $D$  in  $G$  of size at most  $3(|V(P) \cup S|) - 2 \leq 3s + 10$ . Let  $F$  be a minimum feedback vertex set in  $G$ . Then  $F \cup D$  is a connected feedback vertex set in  $G$  of size at most  $|F| + 3s + 10$ . Hence, we can take  $c_H = 3s + 10$ .

If  $H$  is an induced subgraph of  $sP_3$  for some integer  $s$ , then the existence of a constant  $c_H$  as mentioned in Lemma 2.3 is guaranteed by Lemma 2.2.

It remains to show that if  $H$  is not an induced subgraph of  $P_5 + sP_1$  or  $sP_3$  for any integer  $s$ , then there is no constant  $c_H$  such that  $\text{cfvs}(G) - \text{fvs}(G) + c_H$  for every connected  $H$ -free graph  $G$ . Let  $H$  be a graph that is not an induced subgraph of  $P_5 + sP_1$  or  $sP_3$  for any integer  $s$ . First suppose  $H$  is not a linear forest. Then, by Lemma 2.1, there does not exist a constant  $c$  such that  $\text{cfvs}(G) \leq c \cdot \text{fvs}(G)$  for every connected  $H$ -free graph  $G$ . This implies that there cannot be a constant  $c_H$  such that  $\text{cfvs}(G) \leq \text{fvs}(G) + c_H$  for every connected  $H$ -free graph  $G$ . Finally, suppose  $H$  is a linear forest. Since  $H$  is not an induced subgraph of  $P_5 + sP_1$  or  $sP_3$  for any integer  $s$ , it contains  $P_6$  or  $P_4 + P_2$  as an induced subgraph. Consequently, the class of  $H$ -free graphs is a superclass of the class of  $\{P_6, P_4 + P_2\}$ -free graphs. Hence, in order to complete the proof of Lemma 2.3, it suffices to show that if  $\mathcal{G}$  is the class of  $\{P_6, P_4 + P_2\}$ -free graphs, then there exists no constant  $c_H$  such that  $\text{cfvs}(G) \leq \text{fvs}(G) + c_H$  for every connected  $G \in \mathcal{G}$ .

For every integer  $k \geq 1$ , let  $L_k$  be the graph obtained from  $k$  disjoint copies of the butterfly by adding a new vertex  $x$  that is made adjacent to all vertices of degree 2. For every  $k \geq 1$ , the unique minimum feedback vertex set in  $L_k$  is the set  $\{x, y_1, y_2, \dots, y_k\}$ , so  $\text{fvs}(L_k) = k + 1$ . Every minimum connected feedback vertex set in  $L_k$  contains the set  $\{x, y_1, y_2, \dots, y_k\}$ , as well as exactly one additional vertex for each of the vertices  $y_i$  to make this set connected. Hence,  $\text{cfvs}(L_k) = 2k + 1 = \text{fvs}(L_k) + k$ . The observation that  $L_k$  is  $\{P_6, P_4 + P_2\}$ -free for every  $k \geq 1$  implies that if  $\mathcal{G}$  is the class of  $\{P_6, P_4 + P_2\}$ -free graphs, then there exists no constant  $c$  such that  $\text{cfvs}(G) \leq \text{fvs}(G) + c$  for every connected  $G \in \mathcal{G}$ .  $\square$

The proof of the next lemma has been omitted due to page restrictions.

**Lemma 2.4.** *Let  $H$  be a graph. Then  $\text{cfvs}(G) = \text{fvs}(G)$  for every connected  $H$ -free graph  $G$  if and only if  $H$  is an induced subgraph of  $P_3$ .*

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# A local flow algorithm in bounded degree networks

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## 1 Introduction

We show a deterministic local algorithm for constructing an almost maximum flow and an almost minimum cut in multisource-multitarget networks with bounded degrees. Locality means that we decide about each edge or node depending only on its constant radius neighbourhood. We show two applications of the flow algorithm, one is about how the neighborhood distributions of arbitrary bounded degree graphs can be approximated by bounded size graphs, and the other one is related to the Aldous–Lyons conjecture.

In our case, network means a graph with three kind of nodes: sources, targets and regular nodes; and we have directed edges with capacities. A local flow algorithm means a function that gets an edge and a constant-radius neighbourhood of it, including the types of vertices and the capacities of the edges; and the function outputs the amount of flow on that edge. A local flow algorithm is correct if for each network  $N$ , the amounts produced by the local algorithm on all edges provide a flow.

For typical problems, we expect from local algorithms approximate solutions only. We measure this error compared to the sum of all capacities  $cap(N)$ . Namely, we say that we can find an almost maximum flow in multisource-multitarget networks if for each  $\varepsilon > 0$ , there exists a correct local flow algorithm that for each network  $N$ , outputs a flow with size at most  $\varepsilon \cdot cap(n)$  less than the size of the maximum flow.

Local algorithms are an equivalent description of constant-time dis-

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tributed algorithms. For more about local algorithms, see the survey paper by Suomela [5]. It turned out that this is a useful tool for property testing, as well. Property testing is an analogue of the Szemerédi regularity theory, but for bounded degree graphs. Namely, we want to describe very large graphs by bounded-size structures, such that this structure describes several properties and parameters of the entire graph. Nguyen and Onak [4] proved the estimability of several parameters using randomized local algorithms. For more about this topic, see the book by Lovász [3].

## 2 Model and results

There is an input network  $N = (G, c)$ , as follows.  $G = (S, R, T, \vec{E})$  is a graph with degrees bounded by  $d$ .  $d$  is a global constant throughout the paper. The vertices of  $G$  are separated into the disjoint union of the sets  $S$  (source),  $R$  (regular) and  $T$  (target).  $\vec{E}$  is the set of directed edges of  $G$ , which satisfies that  $(a, b) \in \vec{E} \Leftrightarrow (b, a) \in \vec{E}$ . We have a capacity function  $c : \vec{E} \rightarrow [0, \infty)$  of the directed edges. The total capacity on all edges is denoted by  $\text{cap}(N) = \sum_{e \in \vec{E}} c(e)$ . We will use the terms “graph”, “path” and “edge” in the directed sense. Let  $V = V(G) = V(N) = S \cup R \cup T$ ,  $|V| = n$ ,  $\text{out}(A) = \{(a, b) \in \vec{E} \mid a \in A, b \notin A\}$ , and  $\text{out}(v) = \text{out}(\{v\})$ , and for an edge  $e = (a, b)$ , let  $-e = (b, a)$  denote the edge in the opposite direction.

A function  $f : \vec{E} \rightarrow \mathbb{R}$  is called a *flow* if it satisfies that  $\forall e \in \vec{E}(G)$ :  $f(-e) = -f(e)$  and  $f(e) \leq c(e)$ , and  $\forall r \in R$ :  $\sum_{e \in \text{out}(r)} f(e) = 0$ . The value of a flow  $f$  is  $\|f\| = \sum_{e \in \text{out}(S)} f(e)$ . Denote a maximum flow by  $f^* = f^*(N)$ .

A set  $S \subseteq X \subseteq S \cup R$  is called a *cut*. The value of a cut is  $\|X\| = \|X\|_N = \sum_{e \in \text{out}(X)} c(e)$ . The Maximum Flow Minimum Cut Theorem [2] says that  $\min_{S \subseteq X \subseteq S \cup R} \|X\| = \|f^*\|$ .

The rooted  $r$ -neighborhood of a vertex  $v$  or edge  $e$ , denoted by  $h_r(v) = h_r(G, v)$  and  $h_r(e)$ , means the (vertex- or edge-)rooted induced subnetwork of the vertices at distance at most  $r$  from  $v$  or  $e$ , rooted at  $v$  or  $e$ , respectively. The set of all possible  $r$ -neighborhoods are denoted by  $\mathcal{B}(r)$  and  $\mathcal{B}^{(2)}(r)$ , respectively. A function  $F : \mathcal{B}^{(2)}(r) \rightarrow \mathbb{R}$  is called a *local flow algorithm*, and for each network  $N$ , we define the flow on  $N$  generated by  $F$  as  $F(N) = (e \rightarrow F(h_r(e)))$ . Similarly,  $C : \mathcal{B}(r) \rightarrow \{\text{true}, \text{false}\}$  is called a *local cut algorithm*, and for each network  $N$ , we define the cut on  $N$  generated by  $C$  as  $C(N) = \left\{ v \in V(G) \mid C(h_r(v)) \right\}$ .

**Theorem 2.1.** *For each  $\varepsilon > 0$ , there exists a local flow algorithm  $F$  that for each network  $N$ ,  $\|F(N)\| \geq \|f^*(N)\| - \varepsilon \cdot \text{cap}(n)$ .*

*Summary of proof.* We start from the empty flow, and we augment on all augmenting paths in increasing order of length, and in a random order within the same length. This provides an expectedly almost maximum flow. In order to keep the locality, we skip the augmentation in some specific low probability events. At each edge, this changes the flow with low probability. Finally, we average the flow on all random orderings.  $\square$

**Theorem 2.2.** *For each  $\varepsilon > 0$ , there exists a probability distribution  $\mathcal{D}$  of local cut algorithms such that for each network  $N$ ,  $F(N)$  is a flow, and*

$$\mathbb{E}_{C \in \mathcal{D}} \|C(N)\| \leq \|f^*(N)\| + \varepsilon \cdot \text{cap}(n).$$

*Summary of proof.* First, we construct the flow with the local algorithm. Then we construct a fractional cut, based on the length of the shortest path from each node to a target node, using edges with not too small free capacities. This fractional cut naturally provides a probability distribution on cuts.  $\square$

**Corollary 2.3.** *In the class of networks with capacities bounded by a constant,  $\|f^*(N)\|/n$  is estimable. In other words, for every  $\varepsilon > 0$ , there exist  $k, r \in \mathbb{N}$  and a function  $g : \mathcal{B}(r)^k \rightarrow \mathbb{R}$  such that if the vertices  $v_1, v_2, \dots, v_k$  are chosen independently with uniform distribution, then*

$$\mathbb{E} \left( \left| \frac{\|f^*(N)\|}{n} - g(h_r(v_1), h_r(v_2), \dots, h_r(v_k)) \right| \right) < \varepsilon.$$

*Summary of proof.* If we run the local flow algorithm on each neighborhood, then the sample proportion of source nodes, multiplied by the average flow starting from these nodes is a good approximation.  $\square$

### 3 Applications on neighborhood distributions

Let  $\mathcal{G}$  denote the set of all graphs with degrees bounded by  $d$ . Let  $H_r(G)$  denote the distribution of the  $r$ -neighborhood of a random vertex of a graph  $G \in \mathcal{G}$ . We call a family  $\mathcal{F}$  of graphs **nice** if  $G_1, G_2 \in \mathcal{F} \Rightarrow G_1 \cup G_2 \in \mathcal{F}$  and  $G_1 \subseteq G \in \mathcal{F} \Rightarrow G_1 \in \mathcal{F}$  – where  $\subseteq$  denotes nonempty induced subgraph – and  $\emptyset \notin \mathcal{F}$ . Let us denote the closure of the set of all  $r$ -neighborhood distributions in  $\mathcal{F}$  by  $D(\mathcal{F}, r) = \text{cl}\{H_r(G) \mid G \in \mathcal{F}\}$ . It is easy to see that  $D(\mathcal{F}, r)$  is a convex compact subset of  $\mathbb{R}^{\mathcal{B}(r)}$ . Therefore,  $D(\mathcal{F}, r)$  is determined by its dual, as follows.

Let us identify the natural base of  $\mathbb{R}^{\mathcal{B}(r)}$  by the elements of  $\mathcal{B}(r)$ , and let the linear extension of  $w : \mathcal{B}(r) \rightarrow \mathbb{R}$  defined as the function  $\tilde{w} : \mathbb{R}^{\mathcal{B}(r)} \rightarrow \mathbb{R}$ ,  $\tilde{w}\left(\sum_{b \in \mathcal{B}(r)} \lambda_b b\right) = \sum_{b \in \mathcal{B}(r)} \lambda_b w(b)$ . Let us define

$$m(\mathcal{F}, w) = \max_{P \in D(\mathcal{F}, r)} \sum_{b \in \mathcal{B}(r)} w(b) P(b) = \max_{P \in D(\mathcal{F}, r)} \tilde{w}(P) = \sup_{G \in \mathcal{F}} \tilde{w}(H_r(G)).$$

It is easy to see that, for a distribution  $P$  on  $\mathcal{B}(r)$ ,

$$P \in D(\mathcal{F}, r) \Leftrightarrow \forall w : \mathcal{B}(r) \rightarrow [0, 1], \tilde{w}(P) \leq m(\mathcal{F}, w).$$

The following theorem expresses that if a graph  $G$  is distinguishable with high probability from a nice family  $\mathcal{F}$ , then it is distinguishable based on the constant-radius neighborhood of only one random vertex, as well.

**Theorem 3.1.** *Assume that  $H_r(G_0) \notin D(\mathcal{F}, r)$  holds for a graph  $G_0 \in \mathcal{G}$  and a nice family  $\mathcal{F}$  of graphs; namely, there exists a  $w : \mathcal{B}(r) \rightarrow [0, 1]$  satisfying*

$$\tilde{w}(H_r(G_0)) - m(\mathcal{F}, w) \geq \delta > 0.$$

*Then for all  $\varepsilon > 0$ , with  $r' = r'(r, \varepsilon, \delta)$ , there exists a subset  $M \subset \mathcal{B}(r')$  and an induced subgraph  $G_1$  of  $G_0$  such that the following holds.*

$$\begin{aligned} \mathbb{P}(H_{r'}(G_1) \in M) &> 1 - \varepsilon, \\ \forall G \in \mathcal{F}: \quad \mathbb{P}(H_{r'}(G) \in M) &< \varepsilon. \end{aligned}$$

*Summary of proof.* For each graph  $G$  and vertex weights  $\mu_1 : V(G) \rightarrow [0, 1]$  and  $\alpha \in [0, 1]$ , we construct a network with bounded degrees and bounded edge capacities, in which a maximum flow defines a redistribution on the weights to  $\mu_2 : V(G) \rightarrow [0, \alpha]$  satisfying that, under some condition,  $\sum_{v \in V(G)} \mu_1(v) = \sum_{v \in V(G)} \mu_2(v)$ . If we use our local flow algorithm on this network, then it provides a local redistribution of the weights with a small loss. Using this tool two times, from  $w : \mathcal{B}(r) \rightarrow [0, 1]$ , we get another function  $w' : \mathcal{B}(r') \rightarrow [0, 1]$  such that  $\tilde{w}'(H_{r'}(G_1)) > 1 - \varepsilon'$  for some  $G_1 \in \mathcal{G}$ , but  $\tilde{w}'(H_{r'}(G)) < \varepsilon'$  for all  $G \in \mathcal{F}$ . Then the set  $M = \{B \in \mathcal{B}(r') : w'(B) > 1/2\}$  satisfies the requirements.  $\square$

Lovász [3] asked to find, for every radius  $r \in \mathbb{N}$  and error bound  $\varepsilon > 0$ , an explicit  $n = n(r, \varepsilon) \in \mathbb{N}$  such that the  $r$ -neighborhood distribution of each graph can be  $\varepsilon$ -approximated by a graph of size at most  $n$ . Formally,  $\forall G \in \mathcal{G} : \exists G' \in \mathcal{G} : |V(G')| \leq n$  such that  $\|H_r(G), H_r(G')\|_1 < \varepsilon$ .

Alon gave a simple proof of the existence of such a function, but the proof did not provide any explicit bound. It is still open whether there exists, say, a recursive function satisfying the requirement, and also how to compute the graph  $G'$  from  $G$ . See Lovász [3] for details.

As a corollary of Theorem 3.1, we can show that if there exists an arbitrary large error bound  $\lambda < 1$  such that, for all  $r$ , we can find an explicit upper bound on  $n(r, \lambda)$ , then it provides explicit upper bounds on  $n(r, \varepsilon)$  for all  $r \in \mathbb{N}$  and  $\varepsilon > 0$ , as well.

### 3.1 Connection with the Aldous–Lyons Conjecture

We only show here the definitions about the Aldous–Lyons conjecture. For details about this topic, we suggest reading [1] or [3].

Consider a probability distribution  $U$  of rooted graphs, including infinite graphs, with degrees bounded by  $d$ . Select a connected rooted graph from  $U$ , and then select a uniform random edge  $e$  from the root. We consider  $e$  as oriented away from the root. This way we get a probability distribution  $\sigma$  with an oriented “root edge”. Let  $R(\sigma)$  denote the distribution obtained by reversing the orientation of the root of a random element of  $\sigma$ . We say that  $U$  is a **unimodular** random graph if  $\sigma = R(\sigma)$ .

Let  $\mathcal{U}$  denote the set of unimodular random graphs. Let  $H_r(U)$  denote the distribution of the  $r$ -neighborhoods of the root of  $U \in \mathcal{U}$ . Let  $D(\mathcal{U}, r) = cl\{H_r(U) \mid U \in \mathcal{U}\}$ . It can be easily shown that every graph  $G$  with a uniform random root provides a  $U \in \mathcal{U}$  with  $H_r(G) = H_r(U)$ , therefore,  $D(\mathcal{G}, r) \subseteq D(\mathcal{U}, r)$ .

The Aldous–Lyons conjecture says that every unimodular distribution on rooted connected graphs with bounded degree is the “limit” of a bounded degree graph sequence. More precisely,

**Conjecture 3.2 (Aldous–Lyons).**  $\forall r \in \mathbb{N}: D(\mathcal{G}, r) = D(\mathcal{U}, r)$ .

About this conjecture, the same idea as what we used in Theorem 3.1 provides the following result.

**Theorem 3.3.** *If the Aldous–Lyons Conjecture is false, then there exists a unimodular random graph  $U$  that can be distinguished with high probability from any graph, based on the constant-radius neighborhood of only one random vertex. In other words, for all  $\varepsilon > 0$ , there exists an  $r \in \mathbb{N}$ , a subset  $M \subset \mathcal{B}(r)$  and a unimodular random graph  $U \in \mathcal{U}$  that for all  $G \in \mathcal{G}$ , the  $r$ -neighborhood of a random vertex of  $G$  is in  $M$  with probability at most  $\varepsilon$ , but the  $r$ -neighborhood of the root of  $U$  is in  $M$  with probability at least  $1 - \varepsilon$ .*

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# The maximum time of 2-neighbour bootstrap percolation: algorithmic aspects

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**Abstract.** In 2-neighbourhood bootstrap percolation on a graph  $G$ , an infection spreads according to the following deterministic rule: infected vertices of  $G$  remain infected forever and in consecutive rounds healthy vertices with at least 2 already infected neighbours become infected. Percolation occurs if eventually every vertex is infected. In this paper, we are interested in calculating the maximum time  $t(G)$  the process can take, in terms of the number of rounds needed to eventually infect the entire vertex set. We prove that the problem of deciding if  $t(G) \geq k$  is NP-complete for: (a) fixed  $k \geq 4$ ; (b) bipartite graphs with fixed  $k \geq 7$ ; and (c) planar bipartite graphs. Moreover, we obtain polynomial time algorithms for (a)  $k \leq 2$ , (b) chordal graphs and (c)  $(q, q - 4)$ -graphs, for every fixed  $q$ .

## 1 Introduction

Under  $r$ -neighbour bootstrap percolation on a graph  $G$ , the spreading rule is a threshold rule in which  $S_{i+1}$  is obtained from  $S_i$  by adding to it the vertices of  $G$  which have at least  $r$  neighbours in  $S_i$ . We say that a set  $S_0$  percolates  $G$  (or that percolation occurs) if eventually every vertex of  $G$  becomes infected, that is, there exists a  $t$  such that  $S_t = V(G)$ . In that case we define  $t_r(S)$  as the minimum  $t$  such that  $S_t = V(G)$ . And define, the *percolation time of  $G$*  as  $t_r(G) = \max\{t_r(S) : S \text{ percolates } G\}$ . In this paper, we shall focus on the case where  $r = 2$  and in such case we omit the subscript of the functions  $t_r(S)$  and  $t_r(G)$ .

Bootstrap percolation was introduced by Chalupa et al. [8] as a model for interacting particle systems in physics. Since then it has found applications in clustering phenomena, sandpiles, and many other areas of sta-

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tistical physics, as well as in neural networks and computer science [10].

There are two broad classes of questions one can ask about bootstrap percolation. The first, and the most extensively studied, is what happens when the initial configuration  $S_0$  is chosen randomly under some probability distribution? One would like to know how likely percolation is to occur, and if it does occur, how long it takes. The answer to the first of these questions is now well understood for various graphs [1, 2, 15].

The second broad class of questions is the one of extremal questions. For example, what is the smallest or largest size of a percolating set with a given property? Interesting cases are solved in [3–5, 7, 9, 17–19].

Here, we consider the decision version of the problem, as stated below.

#### PERCOLATION TIME PROBLEM

*Input:* A graph  $G$  and an integer  $k$ .

*Question:* Is  $t(G) \geq k$ ?

It is interesting to notice that infection problems appear in the literature under many different names and were studied by researchers of various fields. The particular case in which  $r = 2$  in  $r$ -neighbourhood bootstrap percolation is also a particular case of a infection problem related to convexities in graph, which are also of our interest.

A finite *convexity space* is a pair  $(V, \mathcal{C})$  consisting of a finite ground set  $V$  and a set  $\mathcal{C}$  of subsets of  $V$  satisfying  $\emptyset, V \in \mathcal{C}$  and  $\mathcal{C}$  is closed under intersection. The members of  $\mathcal{C}$  are called  *$\mathcal{C}$ -convex sets* and the *convex hull* of a set  $S$  is the minimum convex set  $H(S) \in \mathcal{C}$  containing  $S$ .

A convexity space  $(V, \mathcal{C})$  is an *interval convexity* [6] if there is a so-called *interval function*  $I : \frac{V}{2} \rightarrow 2^V$  such that a subset  $C$  of  $V$  belongs to  $\mathcal{C}$  if and only if  $I(\{x, y\}) \subseteq C$  for every two distinct elements  $x$  and  $y$  of  $C$ . With no risk of confusion, for any  $S \subset V$ , we also denote by  $I(S)$  the union of  $S$  with  $\bigcup_{x, y \in S} I(\{x, y\})$ . In interval convexities, the convex hull of a set  $S$  can be computed by exhaustively applying the corresponding interval function until obtaining a convex set.

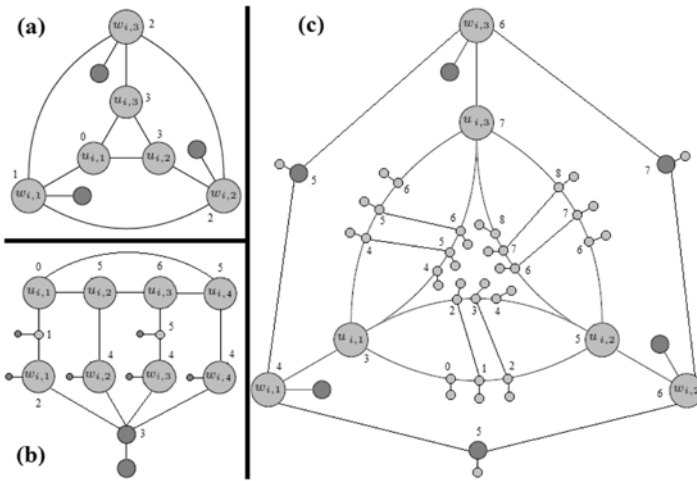
The most studied graph convexities defined by interval functions are those in which  $I(\{x, y\})$  is the union of paths between  $x$  and  $y$  with some particular property. Some common examples are the  $P_3$ -convexity [12], geodetic convexity [13] and monophonic convexity [11]. We observe that the spreading rule in 2-neighbours bootstrap percolation is equivalent to  $S_{i+1} = I(S_i)$  where  $I$  is the interval function which defines the  $P_3$ -convexity:  $I(S)$  contains  $S$  and every vertex belonging to some path of 3 vertices whose extreme vertices are in  $S$ . It will be convenient to denote  $S_i$  by  $I_i(S)$ , where  $I_i(S)$  is obtained by applying  $i$  times the operation  $I$ . Related to the geodetic convexity, there exists the *geodetic iteration number of a graph* [14] which is similar to our definition of  $t(G)$ .

## 2 Results

We first prove the following NP-hardness results.

**Theorem 2.1.** PERCOLATION TIME is NP-complete for any fixed  $k \geq 4$ . If the graph is bipartite, it is NP-Complete for any fixed  $k \geq 7$ . It is also NP-Complete for planar bipartite graphs.

We first make a reduction from 3-SAT. We construct a graph  $G$  as follows. For each clause  $C_i$  with literals  $\ell_{i,1}$ ,  $\ell_{i,2}$  and  $\ell_{i,3}$ , add to  $G$  a gadget as in Figure 2(a). For each pair of literals  $\ell_{i,a}$ ,  $\ell_{j,b}$  such that one is the negation of the other, add a vertex  $y_{(i,a),(j,b)}$  adjacent to  $w_{i,a}$  and  $w_{j,b}$ . Let  $Y$  be the set of all vertices created this way. Finally, add a vertex  $z$  adjacent to all vertices in  $Y$  and a pendant vertex  $z'$  adjacent to  $z$ . It is possible to prove that the formula is satisfiable if and only if  $t(G) \geq 4$ . In the case of bipartite graphs, the reduction is from 4-SAT and the construction is similar, but using the gadget in Figure 2(b). The case of planar bipartite graphs is more technical and is by a reduction from PLANAR 3-SAT using the gadget in Figure 2(c).



**Figure 2.1.** Gadget for each clause  $C_i$ .

Now we present our polynomial results. It is clear that  $t(G) \geq 1$  if and only if  $G$  has a vertex of degree  $\geq 2$ . The next result characterizes the graphs with  $t(G) \geq 2$ . The same question for  $t(G) \geq 3$  is still open.

**Theorem 2.2.** Let  $G$  be a graph. Then  $t(G) \geq 2$  if and only if there exist  $u \in V(G)$  and a neighbour  $v$  of  $u$  such that  $A \cup \{v\}$  is a hull set, where  $A \subset V(G)$  contains every vertex which is neither  $u$  nor a neighbour of  $u$ .

Now we show how to determine  $t(T)$  in linear time, for any tree  $T$ . Given two adjacent vertices  $u, v \in V(T)$ , denote by  $s(u, v)$  the maximum time that  $u$  enters in the convex hull of  $S$ , among all hull sets  $S$  of the subtree of  $T - v$  containing  $u$ ; and by  $t(u)$  the maximum time that  $u$  enters in the convex hull of  $S$ , among all hull sets  $S$  of  $T$ . Clearly,  $t(T)$  equals  $\max_{u \in V(T)} t(u)$  and the values  $s(u, v), t(u)$  are given by:

$$s(u, v) = \begin{cases} 0, & \text{if } |N(u)| \leq 2, \\ 1 + \text{the second value in the non-decreasing} \\ \text{ordering of the set } \{s(x, u) : x \in N(u) \setminus \{v\}\}, & \text{if } |N(u)| \geq 3. \end{cases}$$

$$t(u) = \begin{cases} 0, & \text{if } |N(u)| \leq 1, \\ 1 + \text{the second value in the non-decreasing} \\ \text{ordering of the set } \{s(x, u) : x \in N(u)\}, & \text{if } |N(u)| \geq 2. \end{cases} \quad (2.1)$$

In order to compute the values  $s(u, v)$  and  $s(v, u)$ , for each edge  $uv \in E(T)$ , we use a directed graph  $D$  with vertex set  $V(T)$  and edges  $\{(u, v), (v, u) \mid uv \in E(T)\}$ . Observe that  $s(u, v)$  can be computed only after all values  $s(x, y)$  are known, where  $(x, y)$  is an arc of  $D - v$  belonging to a directed path from some leaf of  $T$  to  $u$ . Thus, consider a partition of the arcs of  $D$  into sets  $S_0, \dots, S_{d-1}$ , where  $d$  is the diameter of  $T$ , and  $S_i$  contains the arcs  $(u, v)$  such that the longest directed path of  $D - v$  from some leaf to  $u$  has length  $i$ . In the beginning, we know  $s(u, v)$ , for all arc  $(u, v) \in S_0$ . Further, as long as  $s(x, y)$  is known for each arc  $(x, y) \in \bigcup_{j=0}^{i-1} S_j$ , we can compute  $s(u, v)$ , for every arc  $(u, v) \in S_i$ ,  $i \in [1, d-1]$ . Therefore, we can compute all values  $s(u, v)$  in linear time. With some modifications, we can adapt these ideas to obtain a polynomial time algorithm for chordal graphs.

**Theorem 2.3.** *If  $T$  is a tree, then  $t(T)$  can be computed in linear time. Let  $G$  be a chordal graph. If  $G$  is 2-connected, then  $t(G)$  can be computed in time  $O(n^2m)$ ; otherwise,  $t(G)$  can be computed in time  $O(n^2m^2)$ .*

Considering  $(q, q-4)$ -graphs,  $q$  fixed, we proved that the percolation time  $t(G)$  is bounded by  $q$ , and we give linear time algorithms to obtain  $t(G)$ . We mention that these graphs generalize the  $P_4$ -sparse graphs.

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# A multipartite Hajnal-Szemerédi theorem

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**Abstract.** The celebrated Hajnal-Szemerédi theorem gives the precise minimum degree threshold that forces a graph to contain a perfect  $K_k$ -packing. Fischer’s conjecture states that the analogous result holds for all multipartite graphs except for those formed by a single construction. Using recent results on perfect matchings in hypergraphs, we prove that (a generalisation of) this conjecture holds for any sufficiently large graph.

## 1 Introduction

The celebrated Hajnal-Szemerédi theorem [6] states that if  $k$  divides  $n$  then any graph  $G$  on  $n$  vertices with minimum degree  $\delta(G) \geq (k-1)n/k$  contains a perfect  $K_k$ -packing<sup>3</sup>. This theorem generalised a result of Corradi and Hajnal [3], who established the case  $k = 3$ , and is best-possible in the sense that the theorem would not hold assuming any weaker minimum degree condition. More recently, a series of papers [1,2,8,9] determined the minimum degree thresholds which force a perfect  $H$ -packing in a graph for non-complete graphs  $H$ , culminating in the work of Kühn and Osthus [11], who essentially settled the problem by giving the best-possible such condition (up to an additive constant) for any graph  $H$ , in terms of the so-called *critical chromatic number*.

In many applications it is natural to instead consider packings in a multipartite setting, in which the analogous problem seems to be considerably more difficult. More precisely, let  $V_1, \dots, V_k$  be pairwise-disjoint sets of  $n$  vertices each, and  $G$  be a  $k$ -partite graph with vertex classes  $V_1, \dots, V_k$  (so  $G$  has vertex set  $V_1 \cup \dots \cup V_k$  and each  $V_j$  is an inde-

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<sup>3</sup> A *perfect  $H$ -packing* in a graph  $G$  is a spanning collection of vertex-disjoint copies of  $H$  in  $G$ ; other sources have referred to the same notion as a *perfect  $H$ -tiling* or  *$H$ -factor*.

pendent set in  $G$ ). We define the *partite minimum degree* of  $G$ , denoted  $\delta^*(G)$ , to be the largest  $m$  such that every vertex has at least  $m$  neighbours in each part other than its own, so

$$\delta^*(G) := \min_{i \in [k]} \min_{v \in V_i} \min_{j \in [k] \setminus \{i\}} |N(v) \cap V_j|,$$

where  $N(v)$  denotes the neighbourhood of  $v$ .

Fischer [5] conjectured that the natural multipartite analogue of the Hajnal-Szemerédi theorem should hold. That is, he conjectured that if  $\delta^*(G) \geq (k-1)n/k$  then  $G$  must contain a perfect  $K_k$ -packing. This conjecture is straightforward for  $k=2$ , as it is not hard to see that any maximal matching must be perfect. However, Magyar and Martin [13] constructed a counterexample for  $k=3$ , and furthermore showed that their construction gives the only counterexample for large  $n$ . More precisely, they showed that if  $n$  is sufficiently large,  $G$  is a 3-partite graph with vertex classes each of size  $n$  and  $\delta^*(G) \geq 2n/3$ , then either  $G$  contains a perfect  $K_3$ -packing, or  $G$  is isomorphic to the graph  $\Gamma_{n,3,3}$  defined in Construction 1 for some odd  $n$  which is divisible by 3.

The implicit conjecture behind this result (stated explicitly by Kühn and Osthus [10]) is that the only counterexamples to Fischer's original conjecture are the constructions given by the graphs  $\Gamma_{n,k,k}$  defined in Construction 1 when  $n$  is odd and divisible by  $k$ . We refer to this as the modified Fischer conjecture. If  $k$  is even then  $n$  cannot be both odd and divisible by  $k$ , so the modified Fischer conjecture is the same as the original conjecture in this case. Martin and Szemerédi [15] proved that (the modified) Fischer's conjecture holds for  $k=4$ . Another partial result was obtained by Csaba and Mydlarz [4], who gave a function  $f(k)$  with  $f(k) \rightarrow 0$  as  $k \rightarrow \infty$  such that the conjecture holds for large  $n$  if one strengthens the degree assumption to  $\delta^*(G) \geq (k-1)n/k + f(k)n$ . However, for general  $k$  the validity of even an asymptotic version of Fischer's conjecture (*i.e.* assuming that  $\delta^*(H) \geq (k-1)n/k + o(n)$ ) was unknown until recently, when the results described below were obtained.

## 2 New results

Keevash and Mycroft [7] used new results on perfect matchings in  $k$ -uniform hypergraphs<sup>4</sup> to deduce the following asymptotic result (which

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<sup>4</sup> A *hypergraph*  $H$  consists of a vertex set  $V$  and an edge set  $E$ , where each edge  $e \in E$  is a subset of  $V$ . The edges are not required to be the same size; if they are then we say that  $H$  is a  *$k$ -uniform hypergraph*, or  *$k$ -graph*, where  $k$  is the common size of the edges.

was also proved independently and simultaneously by Lo and Markström [12] using the ‘absorbing’ method.)

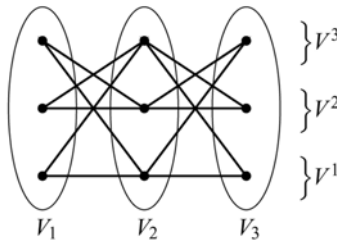
**Theorem 2.1.** *For any  $k$  and  $\varepsilon > 0$  there exists  $n_0$  such that any  $k$ -partite graph  $G$  whose vertex classes each have size  $n \geq n_0$  with  $\delta^*(G) \geq (k - 1)n/k + \varepsilon n$  contains a perfect  $K_k$ -packing.*

An  $r$ -partite graph can only contain a  $K_k$ -packing for  $r \geq k$ , since otherwise we do not have even a single copy of  $K_k$ . Fischer’s conjecture pertains to the case  $r = k$ , but it is natural to ask also for an analogous result for the case  $r > k$ . By a careful analysis of the extremal cases of Theorem 2.1, we can prove an exact result answering both Fischer’s conjecture and also this more general question for large  $n$ . This is the following theorem, the case  $r = k$  of which shows that (the modified) Fischer’s conjecture holds for any sufficiently large graph. (The graph  $\Gamma_{n,r,k}$  referred to in the statement is defined in Construction 1.)

**Theorem 2.2.** *For any  $r \geq k$  there exists  $n_0$  such that for any  $n \geq n_0$  with  $k \mid rn$  the following statement holds. Let  $G$  be a  $r$ -partite graph whose vertex classes each have size  $n$  such that  $\delta^*(G) \geq (k - 1)n/k$ . Then  $G$  contains a perfect  $K_k$ -packing, unless  $rn/k$  is odd,  $k \mid n$ , and  $G \cong \Gamma_{n,r,k}$ .*

We now give the generalised version of the construction of Magyar and Martin [13] showing Fischer’s original conjecture to be false.

**Construction 1.** *Suppose  $rn/k$  is odd and  $k$  divides  $n$ . Let  $V$  be a vertex set partitioned into parts  $V_1, \dots, V_r$  of size  $n$ . Partition each  $V_i, i \in [r]$  into subparts  $V_i^j, j \in [k]$  of size  $n/k$ . Define a graph  $\Gamma_{n,r,k}$ , where for each  $i, i' \in [r]$  with  $i \neq i'$  and  $j \in [k]$ , if  $j \geq 3$  then any vertex in  $V_i^j$  is adjacent to all vertices in  $V_{i'}^{j'}$  with  $j' \in [k] \setminus \{j\}$ , and if  $j = 1$  or  $j = 2$  then any vertex in  $V_i^j$  is adjacent to all vertices in  $V_{i'}^{j'}$  with  $j' \in [k] \setminus \{3 - j\}$ .*



**Figure 2.1.** Construction 1 for the case  $k = r = 3$ .



Figure 2.1 shows Construction 1 for the case  $k = r = 3$ . To avoid complicating the diagram, edges between  $V_1$  and  $V_3$  are not shown: these are analogous to those between  $V_1$  and  $V_2$  and between  $V_2$  and  $V_3$ . For  $n = k$  this is the exact graph of the construction; for larger  $n$  we ‘blow up’ the graph above, replacing each vertex by a set of size  $n/k$ , and each edge by a complete bipartite graph between the corresponding sets. In general, it is helpful to picture the construction as an  $r$  by  $k$  grid, with columns corresponding to parts  $V_i$ ,  $i \in [r]$  and rows  $V^j = \bigcup_{i \in [r]} V_i^j$ ,  $j \in [k]$  corresponding to subparts of the same superscript. Vertices have neighbours in other rows and columns to their own, except in rows  $V^1$  and  $V^2$ , where vertices have neighbours in other columns in their own row and other rows besides rows  $V^1$  and  $V^2$ . Thus  $\delta^*(G) = (k - 1)n/k$ . We claim that there is no perfect  $K_k$ -packing. For any  $K_k$  has at most one vertex in any  $V^j$  with  $j \geq 3$ , so at most  $k - 2$  vertices in  $\bigcup_{j \geq 3} V^j$ . Also  $|\bigcup_{j \geq 3} V^j| = (k - 2)rn/k$ , and there are  $rn/k$  copies of  $K_k$  in a perfect packing. Thus each  $K_k$  must have  $k - 2$  vertices in  $\bigcup_{j \geq 3} V^j$ , and so 2 vertices in  $V^1 \cup V^2$ , which must either both lie in  $V^1$  or both lie in  $V^2$ . However,  $|V^1| = rn/k$  is odd, so  $V^1$  cannot be perfectly covered by pairs. Thus  $G$  contains no perfect  $K_k$ -packing.

### 3 Rough outline of the proofs

As described above, Theorem 2.1, the asymptotic version of Fischer’s conjecture, is proved by a short deduction from results on perfect matchings in uniform hypergraphs proved in [7]. Indeed, the result used gives fairly general conditions on a  $k$ -graph  $H$  which guarantee that either

- (a)  $H$  contains a perfect matching, or
- (b)  $H$  is close to a ‘divisibility barrier’, one of a family of lattice-based constructions which do not contain a perfect matching.

Given a graph  $G$ , we define the *clique  $k$ -complex* of  $G$  to be the hypergraph  $J$  on  $V(G)$  whose edges are the cliques of size  $j$  in  $G$  for  $1 \leq j \leq k$ . Then a perfect  $K_k$ -packing in  $G$  is a perfect matching in the  $k$ -graph  $J_k$  consisting of all edges of  $J$  of size  $k$ . It is straightforward to show that if  $G$  meets the conditions of Theorem 2.1, then  $J_k$  satisfies the conditions necessary to apply the theorem from [7] described above. Furthermore, it is similarly not difficult to show that  $J_k$  is not close to a divisibility barrier, ruling out (b). So the theorem implies that (a) must hold, completing the proof of Theorem 2.1.

However, if we instead only assume that  $G$  satisfies the weaker conditions of Theorem 2.2, we can no longer deduce that  $J_k$  is not close to a

divisibility barrier. Indeed, the clique  $k$ -complex of the graph  $\Gamma_{n,r,k}$  constructed in Construction 1 is actually isomorphic to a divisibility barrier. On the other hand, if  $J_k$  is close to a divisibility barrier then we can obtain significant structural information regarding  $G$ . In fact, for  $k \geq 3$  we find that we may partition  $G$  into two ‘rows’. That is, we may find a subset  $U_i$  of each vertex class  $V_i$  of size  $pn/k$  for some  $1 \leq p \leq k - 1$  such that the bipartite graphs  $G[U_i, V_j \setminus U_j]$  for  $i \neq j$  are almost-complete. Except for a small number of ‘bad’ vertices, the rows  $G_1 := G[\bigcup U_i]$  and  $G_2 := G[\bigcup V_i \setminus U_i]$  satisfy a similar degree condition to  $G$ , but with  $p$  and  $k - p$  respectively in place of  $k$ . This suggests our approach: we argue inductively to find a perfect  $K_p$ -packing in  $G_1$  and a perfect  $K_{k-p}$ -packing in  $G_2$ . Using the fact that we have almost all edges between rows, we join each copy of  $K_p$  in the former packing to a copy of  $K_{k-p}$  in the latter packing to form a  $K_k$ -packing in  $G$ , as required.

However, for  $k = 2$  there is another possibility for  $G$  for which  $J_k$  is close to a divisibility barrier. This is that  $G$  is *pair-complete*, meaning that we may choose  $U_i \subseteq V_i$  of size  $n/2$  for each  $i$  so that  $G_1 := G[\bigcup U_i]$  and  $G_2 := G[\bigcup V_i \setminus U_i]$  are almost-complete  $r$ -partite graphs, and there are very few edges in the bipartite graphs  $G[U_i, V_j \setminus U_j]$ . If there are in fact no edges in these bipartite graphs, and  $r$  and  $n/2$  are both odd, then  $G$  cannot contain a perfect matching (*i.e.* perfect  $K_2$ -packing). This presents an obstacle to the proof strategy described above for  $k \geq 3$  (since our inductive argument may fail for this reason). It transpires that we can avoid this problem by initially deleting a well-chosen small  $K_k$ -packing in  $G$  except for when  $G$  is exactly isomorphic to the graph  $\Gamma_{n,r,k}$ , and the theorem follows from this.

## 4 Future directions

As described in the introduction, the Hajnal-Szemerédi theorem on perfect  $K_k$ -packings in a graph  $G$  was followed by a sequence of papers addressing the problem of finding an  $H$ -packing in  $G$  for an arbitrary graph  $H$ . Following Theorem 2.2, it seems natural to ask for multipartite analogues of these theorems as well. In this direction, Martin and Skokan [14] recently proved an approximate multipartite version of the Alon-Yuster theorem. That is, they proved that if  $H$  is a graph with  $\chi(H) \leq k$ , and  $G$  is a  $k$ -partite graph with vertex classes  $V_1, \dots, V_k$  of size  $n$  which satisfies  $\delta^*(G) \geq (k - 1)n/k + o(n)$ , then  $G$  contains a perfect  $H$ -packing. One natural question is whether this minimum degree bound can be improved to include only a constant error term. Moreover, this bound is not even asymptotically best possible for many graphs: to find the degree threshold which forces a perfect  $H$ -packing in a  $k$ -partite

graph for an arbitrary  $k$ -partite graph  $H$  an analogue of the critical chromatic number seems necessary.

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# Directed cycle double covers: hexagon graphs

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**Abstract.** Jaeger’s directed cycle double cover conjecture can be formulated as a problem of existence of special perfect matchings in a class of graphs that we call hexagon graphs. A hexagon graph can be associated with any cubic graph. We show that the hexagon graphs of cubic *bridgeless graphs* are braces that can be generated from the ladder on 8 vertices using two types of McCuaig’s augmentations.

The long-standing Jaeger’s directed cycle double cover conjecture [1] (DCDC conjecture in short) is broadly considered to be among the most important open problems in graph theory. A typical formulation asks whether every 2-connected graph admits a family of cycles such that one may prescribe an orientation on each cycle of the family in such a way that each edge  $e$  of the graph belongs to exactly two cycles and these cycles induce opposite orientations on  $e$ . In order to prove the DCDC conjecture, a wide variety of approaches have arisen [1, 4], including a topological approach. The topological formulation of the DCDC conjecture is as follows: every cubic bridgeless graph admits an embedding in a closed Riemann surface such that every edge belongs to exactly two distinct face boundaries defined by the embedding; that is, with no dual loop.

**Our results.** In this work, we formulate the DCDC conjecture as a problem of existence of special perfect matchings in a class of graphs that we call *hexagon graphs*.

The class of the hexagon graphs of the cubic bridgeless graphs turns out to be a subclass of braces. A brace is a simple (*i.e.* no loops and no multiple edges), connected, bipartite graph on at least six vertices such

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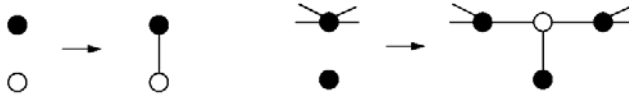
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that for every pair of non-adjacent edges, there is a perfect matching containing the pair of edges. The class of braces, along with bricks, are a fundamental class of graphs in matching theory, mainly because they are building blocks of a perfect matching decomposition procedure, namely of the tight cut decomposition procedure [2]. In [3], McCuaig introduced a method for generating all braces starting from a large base set of graphs and recursively making use of 4 distinct types of operations. Our main results are:

*A cubic graph  $G$  has a directed cycle double cover if and only if its hexagon graph  $H$  admits a safe perfect matching.*

*The hexagon graphs arising from cubic bridgeless graphs are braces that can be generated from the ladder on 8 vertices using 2 simple augmentations of McCuaig (see Figure 1).*



**Figure 1.** Simple augmentations.

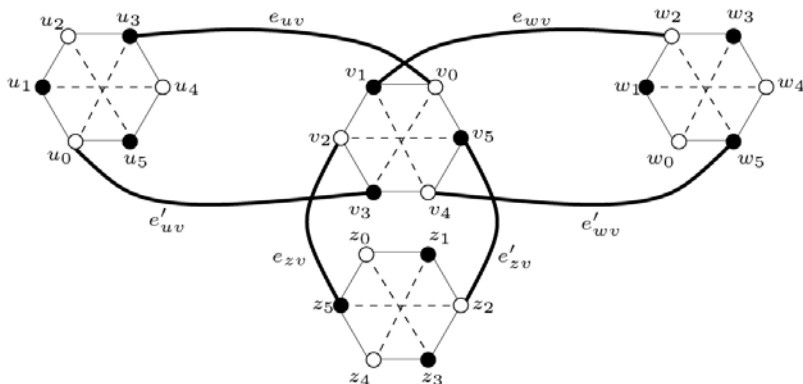
Next we define the hexagon graphs and explain our results.

**Hexagon graphs.** We refer to the complete bipartite graph  $K_{3,3}$  as a *hexagon*. For a graph  $G$  and a vertex  $v$  of  $G$ , let  $N_G(v)$  denote the set of neighbors of  $v$  in  $G$ .

Let  $G$  be a cubic graph with vertex set  $V$  and edge set  $E$ . A *hexagon graph* of  $G$  is a graph  $H$  obtained from  $G$  following the next rules:

1. We replace each vertex  $v$  in  $V$  by a hexagon  $h_v$  so that for every pair  $u, v \in V$ , if  $u \neq v$ , then  $h_u$  and  $h_v$  are vertex disjoint. Let  $V(H) = \{V(h_v) : v \in V\}$ .
2. For each vertex  $v \in V$ , let  $\{v_i : i \in \mathbb{Z}_6\}$  denote the vertex set of  $h_v$  and  $\{v_i v_{i+1}, v_i v_{i+3} : i \in \mathbb{Z}_6\}$  its edge set. With each neighbor  $u$  of  $v$  in  $G$ , we associate an index  $i_{v(u)}$  from the set  $\{0, 1, 2\} \subset \mathbb{Z}_6$  so that if  $N_G(v) = \{u, w, z\}$ , then  $i_{v(u)}, i_{v(w)}, i_{v(z)}$  are pairwise distinct.
3. (See Figure 2). Let  $X = \cup_{v \in V} \{v_{2i} : i \in \mathbb{Z}_6\}$  and  $Y = \cup_{v \in V} \{v_{2i+1} : i \in \mathbb{Z}_6\}$ . We replace each edge  $uv$  in  $E$  by two vertex disjoint edges  $e_{uv}, e'_{uv}$  so that if both  $v_{i_{v(u)}}, u_{i_{u(v)}}$  belong to either  $X$  or  $Y$ , then  $e_{uv} = v_{i_{v(u)}} u_{i_{u(v)}+3}, e'_{uv} = v_{i_{v(u)}+3} u_{i_{u(v)}}$ . Otherwise,  $e_{uv} = v_{i_{v(u)}} u_{i_{u(v)}}, e'_{uv} = v_{i_{v(u)}+3} u_{i_{u(v)}+3}$ . Let  $E(H) = \{E(h_v) : v \in V\} \cup \{e_{uv}, e'_{uv} : uv \in E\}$ .

We say that  $h_v$  is the hexagon of  $H$  associated with the vertex  $v$  of  $G$  and that  $\{h_v : v \in V\}$  is the *set of hexagons of  $H$* . We shall refer to the set of edges  $\cup_{v \in V} \{v_i v_{i+3} : i \in \mathbb{Z}_6\}$  as the *set of red edges of  $H$* , to the set



**Figure 2.** Red edges:= dashed lines, blue edges:= thin lines and white edges:= thick lines. Moreover,  $X :=$  white vertices,  $Y :=$  black vertices.

of edges  $\{e_{uv}, e'_{uv} : uv \in E\}$  as the set of *white edges* of  $H$ , and finally to the set of edges  $\bigcup_{v \in V} \{v_i v_{i+1} : i \in \mathbb{Z}_6\}$  as the set of *blue edges* of  $H$  (see Figure 2). Moreover, we shall say that a perfect matching of  $H$  containing only blue edges is a *blue perfect matching*.

Let  $G$  be a cubic graph and  $H$  be a hexagon graph of  $G$ . We observe two important properties: (i)  $H$  is bipartite; and (ii) if  $H'$  is another hexagon graph of  $G$ , then  $H$  and  $H'$  are isomorphic.

**Embeddings and hexagon graphs.** Let  $G$  be a cubic graph,  $H$  the hexagon graph of  $G$ , and  $W$  the set of white edges of  $H$ . Each blue perfect matching  $M$  of  $H$  encodes an embedding of  $G$  on a closed Riemann surface with set of face boundaries the set of subgraphs of  $G$  induced by the cycles in  $M \Delta W$ . The converse also holds. That is, each embedding of  $G$  on a closed Riemann surface defines a blue perfect matching  $M$  of  $H$ , where the set of subgraphs of  $G$  induced by all cycles in  $M \Delta W$  coincides with the set of face boundaries of the embedding.

Finally, the embedding of  $G$  encoded by  $M$  has a dual loop if and only if there is a cycle in  $M \Delta W$  that contains both end vertices of a red edge. We shall say that a blue perfect matching  $M$  is *safe* if no cycle of  $M \Delta W$  contains the end vertices of a red edge. This implies the first main result stated above.

**Braces and hexagon graphs.** The generation of the hexagon graphs of the cubic bridgeless graphs from the ladder on 8 vertices using 2 simple augmentations of McCuaig is a hard technical result and it is contained in the full version of this write-up.

The weak form is: Let  $G$  be a cubic graph. Then the hexagon graph  $H$  of  $G$  is a brace if and only if  $G$  is bridgeless. Below we sketch the proof.

Let  $B$ ,  $W$ , and  $R$  denote the set of blue, white, and red edges, respectively. Moreover, a blue edge is denoted by  $b$ , a white edge by  $w$ , and a red edge by  $r$ . Each pair of disjoint edges,  $\{b, b'\}$ ,  $\{r, r'\}$ , or  $\{b, r\}$ , can be simply extended to a perfect matching of  $H$ .

We note that each component of  $W \cup R$  is a cycle on four vertices, a *square*. Let  $w, w'$  be a pair of disjoint white edges. The edges  $w, w'$  belong to the same square of  $W \cup R$ , or to two different squares of  $W \cup R$ . In either case  $w, w'$  can be naturally extended to a perfect matching of  $H$ . Similarly, each edge of a pair  $w, r$  of disjoint white and red edge belongs to different square of  $W \cup R$ , and therefore it can be completed into a perfect matching of  $H$ .

Finally we consider a pair  $b, w$  of disjoint white and blue edge. If the hexagon with  $b$  does not contain an end vertex of  $w$ , then it is not difficult to extend  $b, w$  to a perfect matching of  $H$ . Hence, let  $h_u$  be the hexagon that contains  $b$  and an end vertex of  $w$ , and let  $h_v$  be the hexagon that contains the other end vertex of  $w$ . Let  $b = u_i u_{i+1}$ ,  $w = u_k v_j$ , where  $i, j, k \in \mathbb{Z}_6$ .

If  $k \notin \{i+3, i+4\}$ , then  $b, w$  can be completed into a perfect matching of  $H$  that contains the edges  $b, w$ , and  $u_{i+3} u_{i+4}$ .

Hence, without loss of generality we can assume that  $k = i+3$ . Let  $e_{uv} = u_i v_{j+3}$  and  $e_{uz} = u_{i+1} z_l$ , where  $z$  is the neighbor of  $v$  in  $G$  such that the white edge with an end vertex  $u_{i+1}$  has an end vertex in  $h_z$ , and  $l \in \mathbb{Z}_6$ . Given that in  $G$ , edges  $uv, uz$  have a common end vertex  $u$  represented by hexagon  $h_u$ , edge  $b = u_i u_{i+1}$  can be seen as *the transition* between  $uv, uz$ , while  $u_k u_{k+1}$  can be seen as this transition reversed.

Now let  $G$  be bridgeless. We observe that two adjacent edges in a cubic bridgeless graph belong to a common cycle. Let  $C$  be such a cycle for  $uv, uz$ . The two possible orientations of  $C$  correspond to two disjoint cycles  $C_b, C_w$  in  $H$ , where  $b \in C_b$  and  $w \in C_w$ ; they contain the transition and transition reversed (between  $uv, uz$ ), respectively. Let  $M_b$  be the perfect matching of  $C_b$  consisting of all blue edges and  $M_w$  be the perfect matching of  $C_w$  consisting of all white edges. In particular,  $b \in M_b$  and  $w \in M_w$ . Since each hexagon of  $H$  is intersected by  $C_b \cup C_w$  either in a pair of disjoint blue edges, or in the empty set,  $M_b \cup M_w$  can be extended to a perfect matching of  $H$ .

On the other hand, if  $G$  has a bridge  $e = \{u, v\}$ , then let  $V_1$  be the component of  $G - e$  containing  $u$ . Any perfect matching of  $G$  extending  $b, w$  must induce a perfect matching of  $\cup_{x \in V_1} h_x \setminus \{u_{i+3}\}$ , but this set consists of an odd number of vertices and thus no perfect matching containing  $b, w$  can exist.

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# Finding an Odd $K_{3,3}$

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Peter Whalen<sup>1</sup> and Robin Thomas<sup>1</sup>

**Abstract.** In their 1999 paper, “Permanents, Pfaffian orientations, and even directed circuits”, Neil Robertson, Paul Seymour, and Robin Thomas provided a good characterization for whether or not a bipartite graph has a Pfaffian orientation. Robin Thomas and Peter Whalen recently provided a shorter proof of the central result using elementary methods. The first step in this new proof is the finding of a subgraph of a brace isomorphic to a subdivision of  $K_{3,3}$  in which each edge has been subdivided an even number of times. Here, we extend this result to prove that any internally 4-connected non-planar bipartite graph contains a subgraph isomorphic to a subdivision of  $K_{3,3}$  in which each edge has been subdivided an even number of times.

## 1 Introduction

While a well-known algorithm of Edmonds [1] provides a polynomial time solution to the problem of finding a perfect matching in a graph, the problem of counting the number of perfect matchings is #P-Complete [5]. Work by Kasteleyn [2], however, has led to the notion of a Pfaffian orientation and an algorithm for computing the number of perfect matchings in graphs admitting a Pfaffian orientation.

**Definition 1.1.** Let  $G$  be a directed graph. Then  $G$  is *Pfaffian* if for every even cycle,  $C$  such that  $G - V(C)$  contains a perfect matching, the number of edges directed in either direction of the cycle is odd. If  $G$  is an undirected graph, we say that  $G$  is *Pfaffian* if there is an orientation of the edges such that the resulting directed graph is Pfaffian. Such an orientation is called a *Pfaffian orientation*.

The question of determining whether or not a graph has a Pfaffian orientation has been solved for several cases, though remains open in gen-

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eral. One solved case is that of bipartite graphs. The following elegant result of Little [3] provides a characterization for this problem:

**Theorem 1.2.** *Let  $G$  be a bipartite graph. Then  $G$  is Pfaffian if and only if it does not contain a subgraph  $H$  and a perfect matching  $M$  such that  $E(G) = E(H) \cup M$ ,  $E(H) \cap M = \emptyset$ , and  $H$  is isomorphic to a subdivision of  $K_{3,3}$  in which each edge is subdivided an even number of times.*

The disadvantage to this theorem is that it does not seem to immediately give rise to a polynomial time algorithm. Instead, a result of Robertson, Seymour, and Thomas [4] provides an algorithm for determining whether a brace has a Pfaffian orientation which relies on a theorem also independently proven by McCuaig.

Recently, Robin Thomas and Peter Whalen have provided a new proof for the central theorem behind this algorithm that relies on more elementary methods. The first step of this process is to find a subgraph of  $G$  isomorphic to a subdivision of  $K_{3,3}$  in which each edge is subdivided an even number of times. We refer to such a subdivision as an *odd  $K_{3,3}$*  since each branch of the  $K_{3,3}$  is a path of odd length. In the context of that result, it sufficed to find such a subgraph only when  $G$  is a brace.

The quality of having an odd  $K_{3,3}$ , however, seems more naturally tied to connectivity rather than to matching properties. So in this result, we extended our theorem to show that any internally 4-connected, non-planar bipartite graph contains a subgraph isomorphic to an odd  $K_{3,3}$ . This result seems natural and has practical application in the study of Pfaffian orientations.

## 2 Statement of the theorem

We first require several definitions

**Definition 2.1.** Let  $G$  be a 3-connected graph on at least 5 vertices such that for every pair of sets  $X, Y \subseteq E(G)$  with  $E(G) = X \cup Y$ ,  $X \cap Y = \emptyset$ , if three vertices are incident with an edge in  $A$  and an edge in  $B$  then  $|A| \leq 3$  or  $|B| \leq 4$ . Then we say that  $G$  is *internally 4-connected*.

**Definition 2.2.** Let  $G$  be a graph and  $H$  a subgraph of  $G$  isomorphic to a subdivision of  $K_{3,3}$ . Let  $v_1, v_2, \dots, v_6$  be the degree three vertices of  $H$  and  $P_1, P_2, \dots, P_9$  be the paths in  $H$  between  $v_i$ . We then refer to  $H$  as a *hex* or a *hex of  $G$* , the  $v_i$  as the *feet of  $H$* , and the  $P_i$  as the *segments of  $H$* .

**Definition 2.3.** Let  $G$  be a graph and  $H$  a hex of  $G$ . We refer to a segment  $P$  of  $H$  as *even* if it has even length and *odd* otherwise. We refer to  $H$  as even if all of its segments are even and similarly for odd.

The remainder of the paper deals with the proof of the following theorem:

**Theorem 2.4.** *Let  $G$  be an internally 4-connected non-planar bipartite graph. Then  $G$  has an odd hex.*

### 3 The proof technique

The sketch of the proof of the theorem is as follows.  $G$  contains a hex since  $G$  is 3-connected, non-planar, and not  $K_5$ . Choose the hex with the fewest number of even segments. If some segment of that hex is even, it has a vertex  $v$  with degree 2 with respect to the hex. Since  $G$  is 3-connected, we can use Menger's theorem to augment the segment of the hex containing  $v$  to find a path from  $v$  to elsewhere in the hex. With some analysis, we show that this (or a similar argument) will always produce a hex with more odd segments which completes the proof.

At each step in our proof, we consider a subgraph of  $G$ ,  $H$ , that consists of a hex plus potentially additional paths. Our main operation is to find some piece of  $H$  that is not internally 4-connected and try to find a new path that we might be able to use to find a hex with more odd segments than our original choice.

Let the bipartition of  $G$  be  $(A, B)$ . Suppose we have three internally-vertex disjoint paths  $P_1, P_2, P_3$  meeting at a single vertex  $v$ . Let the ends of  $P_1, P_2, P_3$  be  $a, b, c$  respectively and let  $v, a \in A$  and  $b, c \in B$ . Let  $X$  be a subset of the vertices of  $G$  disjoint from this structure. We would like a path with one end in the union of the interior of our paths and the other in  $X$ . The core lemma that we use describes possible outcomes of this situation. Note first that in the context of this subgraph,  $\{a, b, c\}$  represents a three separation, so we should be able to find another path out of this structure. By possibly rerouting the original paths, we find four outcomes, one of which is a straightforward jump from a  $B$  vertex along  $P_1$  to an  $A$  vertex in  $X$ . The other three are similarly explicit, but somewhat more complicated. The proof of this lemma follows from Menger's theorem and induction and carefully analyzing the resulting outcomes.

To prove the main theorem given the lemma, let  $H$  be the hex of  $G$  with the fewest even segments. If at most 3 of the segments are odd, say most of the feet are in  $A$ , then choose a  $B$ -vertex on one of the even segments,  $v$  and apply Menger's theorem to find a path to one of the other segments. Choosing  $v$  as one of the feet of our new hex, we can immediately find a hex with more odd segments. When  $H$  has at least 4 odd segments, we apply a similar argument. Suppose we look at a  $B$  vertex on an  $A - A$  segment and find that the path we get from Menger's theorem ends at another  $B$  vertex. Then we can apply the main lemma. In

the first outcome, we find a new path that ends instead at an  $A$ -vertex. In the other outcomes, we get a large amount of structure (including parity) in a neighborhood of  $B$ . We can generally take advantage of this structure to find a new hex with more odd segments than our original.

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# Zero-error source-channel coding with entanglement

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and Giannicola Scarpa<sup>1</sup>

## 1 Introduction

We study a problem from zero-error information theory—a topic well-known for its rich connections to combinatorics [1, 8, 10–12, 14]—in a setting where a sender and receiver may use quantum entanglement, one of the most striking features of quantum mechanics. The problem that we consider is the classical *source-channel coding problem*, where Alice and Bob are each given an input from a random source and get access to a noisy channel through which Alice can send messages to Bob. Their goal is to minimize the average number of channel uses per source input while allowing Bob to learn Alice’s inputs. Here we show that entanglement can allow for an unbounded decrease in the asymptotic rate of classical source-channel codes. We also consider the *source problem*, the case where Alice can send messages to Bob without noise. We prove a lower bound on the rate of source codes with entanglement in terms of a variant of the Lovász theta number [10, 13], a graph parameter given by a semidefinite program.

### 1.1 Classical source-channel coding

We briefly explain the three relevant problems from zero-error information theory and their well-known graph-theoretical characterizations.

A *discrete dual source*  $\mathcal{M} = (\mathbf{X}, \mathbf{U}, P)$  consists of a finite set  $\mathbf{X}$ , a (possibly infinite) set  $\mathbf{U}$  and a probability distribution  $P$  over  $\mathbf{X} \times \mathbf{U}$ . In a dual-source instance, with probability  $P(x, u)$ , Alice gets an input

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$x \in \mathbf{X}$  and Bob a  $u \in \mathbf{U}$ . Bob needs to learn Alice's input without error by having Alice send Bob as few bits as possible. Associated to  $\mathcal{M}$  is its *characteristic graph*  $G = (\mathbf{X}, E)$ , where  $\{x, y\} \in E$  if there exists a  $u \in \mathbf{U}$  such that  $P(x, u) > 0$  and  $P(y, u) > 0$ . As observed in [14], solving the zero-error source coding problem is equivalent to finding a proper coloring of  $G$  that uses the minimum number of colors and the *Witsenhausen rate*

$$R(G) = \lim_{m \rightarrow \infty} \frac{1}{m} \log \chi(G^{\boxtimes m}) \quad (1.1)$$

is the minimum asymptotic *cost rate* (i.e., the average number of bits Alice needs to send Bob per source input) of a zero-error code for  $\mathcal{M}$ . Here  $G^{\boxtimes m}$  is the  $m^{\text{th}}$  strong graph power [12] and  $\log$  is the logarithm in base 2.

A *discrete channel*  $\mathcal{N} = (\mathbf{S}, \mathbf{V}, Q)$  consists of a finite input set  $\mathbf{S}$ , a (possibly infinite) output set  $\mathbf{V}$  and a probability distribution  $Q(\cdot|s)$  over  $\mathbf{V}$  for each  $s \in \mathbf{S}$ . If Alice sends  $s \in \mathbf{S}$  through the channel, then Bob receives  $v \in \mathbf{V}$  with probability  $Q(v|s)$ . Associated to a channel is its *confusability graph*  $H = (\mathbf{S}, F)$ , where  $\{s, t\} \in F$  if there exists a  $v \in \mathbf{V}$  such that both  $Q(v|s) > 0$  and  $Q(v|t) > 0$ . As observed in [12],  $\alpha(H^{\boxtimes n})$  is the maximum number of distinct possible messages that Alice can send to Bob without error by using the channel  $n$  times. The *Shannon capacity*

$$c(H) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \alpha(H^{\boxtimes n}) \quad (1.2)$$

gives the maximum asymptotic rate of a zero-error channel code.

In the *source-channel coding problem* the parties get inputs from a dual source  $\mathcal{M}$  and get access to a channel  $\mathcal{N}$ . As observed in [11], if  $\mathcal{M}$  has characteristic graph  $G$  and  $\mathcal{N}$  has confusability graph  $H$ , then a zero-error coding scheme which encodes length  $m$  source-input-sequences into length  $n$  channel-input-sequences defines a homomorphism from  $G^{\boxtimes m}$  to  $H^{\boxtimes n}$ . The parameter

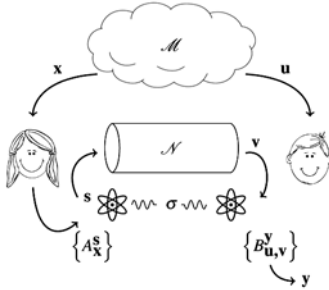
$$\eta(G, H) := \lim_{m \rightarrow \infty} \frac{1}{m} \min \left\{ n \in \mathbb{N} : G^{\boxtimes m} \xrightarrow{\exists \text{ homomorphism}} H^{\boxtimes n} \right\}$$

gives the minimum asymptotic cost rate (i.e., the minimum average number of channel uses per source input) of a zero-error code. Note that  $R(G) = \eta(G, \overline{K_2})$  and  $1/c(H) = \eta(K_2, H)$ . In general  $\eta(G, H) \leq R(G)/c(H)$  and in [11] it is shown that unbounded separations between  $\eta(G, H)$  and  $R(G)/c(H)$  can occur.

## 1.2 Source-channel coding with entanglement

We briefly introduce the model of entanglement-assisted coding, but a direct algebraic definition of the entanglement-assisted variant of  $\eta(G, H)$  is given in Definition 1.1. A *state* is a complex positive semidefinite matrix with trace 1. The possible states of a pair of  $d$ -dimensional quantum systems  $(\mathcal{A}, \mathcal{B})$  are the states in  $\mathbb{C}^{d \times d} \otimes \mathbb{C}^{d \times d}$ . Such a pair is *entangled* if its state is not a convex combination of states of the form  $\rho_A \otimes \rho_B$  with  $\rho_A \in \mathbb{C}^{d \times d}$  and  $\rho_B \in \mathbb{C}^{d \times d}$ . A  $t$ -*outcome measurement* is a collection  $\mathbf{A} = \{A^i \in \mathbb{C}^{d \times d} : i \in [t]\}$  of positive semidefinite matrices  $A^i$  that satisfy  $\sum_{i=1}^t A^i = I$ . A measurement describes an experiment that one may perform on a  $d$ -dimensional quantum system. If  $(\mathcal{A}, \mathcal{B})$  is in a state  $\sigma$ , and Alice performs a  $t$ -outcome measurement  $\mathbf{A}$  on  $\mathcal{A}$  and Bob performs an  $r$ -outcome measurement  $\mathbf{B}$  on  $\mathcal{B}$ , then they obtain outcomes  $i \in [t]$  and  $j \in [r]$ , respectively, with probability  $\text{Tr}((A^i \otimes B^j)\sigma)$ .

The *entanglement-assisted* protocol for solving the source-channel coding problem is as follows:



- (1) Alice and Bob get inputs  $\mathbf{x} \in \mathbf{X}^m$  and  $\mathbf{u} \in \mathbf{U}^m$ , respectively, from the source  $\mathcal{M}$ ;
- (2) Alice performs a measurement  $\{A_{\mathbf{x}}^{\mathbf{s}} \in \mathbb{C}^{d \times d} : \mathbf{s} \in \mathbf{S}^n\}$  on her system  $\mathcal{A}$  (the measurement may depend on  $\mathbf{x}$ );
- (3) Alice sends the outcome  $\mathbf{s}$  over  $\mathcal{N}$ ;
- (4) Bob receives an output  $\mathbf{v}$  from  $\mathcal{N}$ ;
- (5) Bob performs a measurement  $\{B_{\mathbf{u}, \mathbf{v}}^{\mathbf{y}} \in \mathbb{C}^{d \times d} : \mathbf{y} \in \mathbf{X}^m\}$  (which may depend on  $\mathbf{u}$  and  $\mathbf{v}$ ) on his system  $\mathcal{B}$ ;
- (6) Bob obtains a measurement outcome  $\mathbf{y} \in \mathbf{X}^n$ .

A zero-error entanglement-assisted coding scheme satisfies that Bob's measurement outcome  $\mathbf{y}$  is Alice's input  $\mathbf{x}$  with probability 1. With a similar technique as in [6], we can define the entanglement variant of  $\eta(G, H)$  as follows.

**Definition 1.1.** For graphs  $G, H$  and  $m \in \mathbb{N}$ , define  $\eta_m^*(G, H)$  as the minimum positive integer  $n$  such that there exist  $d \in \mathbb{N}$  and  $d \times d$  positive semidefinite matrices  $\{\rho_{\mathbf{x}}^{\mathbf{s}} : \mathbf{x} \in V(G^{\boxtimes m}), \mathbf{s} \in V(H^{\boxtimes n})\}$  and  $\rho$  such that  $\text{Tr}(\rho) = 1$ ,

$$\sum_{\mathbf{s} \in V(H^{\boxtimes n})} \rho_{\mathbf{x}}^{\mathbf{s}} = \rho \quad \forall \mathbf{x} \in V(G^{\boxtimes m}),$$

$$\rho_{\mathbf{x}}^{\mathbf{s}} \rho_{\mathbf{y}}^{\mathbf{t}} = 0 \quad \forall \{\mathbf{x}, \mathbf{y}\} \in E(G^{\boxtimes m}), \{\mathbf{s}, \mathbf{t}\} \in V(H^{\boxtimes n}) \cup E(H^{\boxtimes n}).$$

Define  $\eta^*(G, H) = \lim_{m \rightarrow \infty} \eta_m^*(G, H)/m$ .

We regain the parameter  $\eta(G, H)$  if we restrict the above matrices  $\rho$  and  $\rho_{\mathbf{x}}^{\mathbf{s}}$  to be  $\{0, 1\}$ -valued scalars. As in the classical setting, we obtain the entangled variants of the Witsenhausen rate  $R^*(G) = \eta^*(G, \overline{K_2})$  and Shannon capacity  $1/c^*(H) = \eta^*(K_2, H)$ . Alternatively these parameters can be defined analogously to (1.1) and (1.2) based on entangled variants of the chromatic and independence numbers  $\chi^*(G)$  and  $\alpha^*(H)$ , whose definitions are similar to Definition 1.1. The parameters  $\alpha^*(H)$  and  $c^*(H)$  were first defined in [6], where it was first shown that a separation  $\alpha < \alpha^*$  is possible. It was later shown in [5, 9] that even the zero-error capacity can be increased with entanglement (*i.e.*,  $c < c^*$ ). To the best of our knowledge, neither source nor source-channel coding has been considered in the context of shared entanglement before.

## 2 Our results

### 2.1 The entangled chromatic number and Szegedy’s number

Our first result gives a lower bound for the entangled chromatic number, which can be efficiently computed with semidefinite programming.

**Theorem 2.1.** *For every graph  $G$ ,  $\vartheta^+(G) \leq \chi^*(\overline{G})$  and  $\log \vartheta(G) \leq R^*(\overline{G})$ .*

Here  $\vartheta(G)$  is the celebrated theta number of Lovász [10] defined by

$$\vartheta(G) = \min\{\lambda : \exists Z \in \mathbb{R}_{\geq 0}^{V \times V}, Z_{u,u} = \lambda - 1 \text{ for } u \in V, Z_{u,v} = -1 \text{ for } \{u, v\} \notin E\},$$

where  $\mathbb{R}_{\geq 0}^{V \times V}$  is the space of positive semidefinite matrices, and  $\vartheta^+(G) \geq \vartheta(G)$  is the variant of Szegedy [13] obtained by adding the constraint  $Z_{u,v} \geq -1$  for  $\{u, v\} \in E$ .

Combining with results of [3, 7], we get the chain of inequalities

$$c(G) \leq c^*(G) \leq \log \vartheta(G) \leq R^*(\overline{G}) \leq R(\overline{G}).$$

As in the classical setting, the problem of giving stronger bounds on the entangled Witsenhausen rate and Shannon capacity is wide open.



## 2.2 Classical versus entangled source-channel coding rates

Our second result says that entanglement allows for an unbounded advantage in the asymptotic cost rate of a zero-error source-channel coding scheme. For this we use (as in [5]) the “quarter orthogonality graph”  $H_k$  (for odd  $k$ ), with vertices all vectors in  $\{-1, 1\}^k$  with an even number of “ $-1$ ” entries and with edges the pairs with inner product  $-1$ . We also use the result of [15] showing the existence of a Hadamard matrix (*i.e.*, a matrix  $A \in \{-1, 1\}^{N \times N}$  that satisfies  $AA^T = NI$ ) of size  $N = 4q^2$  if  $q$  is an odd prime power with  $q \equiv 1 \pmod{4}$ .

**Theorem 2.2.** *For every odd integer  $k \geq 5$ , we have*

$$\eta^*(H_k, H_k) \leq \frac{\log(k+1)}{(k-1) \left(1 - \frac{4 \log(k+1)}{k-3}\right)}. \quad (2.1)$$

Moreover, if  $p$  is an odd prime and  $\ell \in \mathbb{N}$  such that there exists a Hadamard matrix of size  $4p^\ell$  (which holds e.g. for  $p = 5$  and  $\ell$  even) and  $k = 4p^\ell - 1$ , then

$$\eta(H_k, H_k) > \frac{0.154k - 1}{k - 1 - \log(k+1)}. \quad (2.2)$$

The proof of the bound (2.1) uses the inequality  $\eta^*(H_k, H_k) \leq R^*(H_k)/c^*(H_k)$ . To show  $R^*(H_k) \leq \log(k+1)$ , we prove  $\chi^*(H_k) \leq k+1$  (by constructing feasible operators from a  $(k+1)$ -dimensional orthonormal representation of  $H_k$ ) and then conclude using the sub-multiplicativity of  $\chi^*$  under strong graph powers. To show  $c^*(H_k) \geq (k-1) \left(1 - \frac{4 \log(k+1)}{k-3}\right)$ , we use the celebrated *quantum teleportation* scheme of [4] to exhibit an explicit protocol that achieves such capacity on any channel with confusability graph  $H_k$ . This proof-technique appears to be new in the context of zero-error entanglement-assisted communication.

To show (2.2) we use properties of the fractional chromatic number and vertex transitivity of  $H_k$  by which we lower bound  $\eta(H_k, H_k)$  by lower bounding  $\alpha(\overline{H_k})$  and upper bounding  $\alpha(H_k)$ . The lower bound uses the existence of a Hadamard matrix of size  $k+1$  and the upper bound combines the linear algebra method of Alon [1] with the beautiful construction of certain polynomials in [2].

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## Ramsey Theory



# Ramsey numbers for bipartite graphs with small bandwidth

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**Abstract.** We determine asymptotically the two color Ramsey numbers for bipartite graphs with small bandwidth and constant maximum degree and the three color Ramsey numbers for *balanced* bipartite graphs with small bandwidth and constant maximum degree. In particular, we determine asymptotically the two and three color Ramsey numbers for grid graphs.

## 1 Introduction and Results

For graphs  $G_1, G_2, \dots, G_r$ , the Ramsey number  $R(G_1, G_2, \dots, G_r)$  is the smallest positive integer  $n$  such that if the edges of a complete graph  $K_n$  are partitioned into  $r$  disjoint color classes giving  $r$  graphs  $H_1, H_2, \dots, H_r$ , then at least one  $H_i$  ( $1 \leq i \leq r$ ) contains a subgraph isomorphic to  $G_i$ . The existence of such a positive integer is guaranteed by Ramsey's classical theorem. The number  $R(G_1, G_2, \dots, G_r)$  is called the Ramsey number for the graphs  $G_1, G_2, \dots, G_r$ . Determining  $R(G_1, G_2, \dots, G_r)$  for general graphs appears to be a difficult problem (see *e.g.* [13]). For  $r = 2$ , a well-known theorem of Gerencsér and Gyárfás [7] states that  $R(P_n, P_n) = \lfloor \frac{3n-2}{2} \rfloor$ , where  $P_n$  denotes the path with  $n \geq 2$  vertices.

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In [9] more general trees were considered. For a tree  $T$ , we write  $t_1$  and  $t_2$ ,  $t_2 \geq t_1$ , for the sizes of the vertex classes of  $T$  as a bipartite graph. Note that if  $2t_1 \geq t_2$ , then  $R(T, T) \geq 2t_1 + t_2 - 1$ , since the following edge-coloring of  $K_{2t_1+t_2-2}$  has no monochromatic copy of  $T$ . Partition the vertices into two classes  $V_1$  and  $V_2$  such that  $|V_1| = t_1 - 1$  and  $|V_2| = t_1 + t_2 - 1$ , then use color “red” for all edges inside the classes and use color “blue” for all edges between the classes. On the other hand, if  $2t_1 < t_2$ , a similar edge-coloring of  $K_{2t_2-2}$  with two classes both of size  $t_2 - 1$  shows that  $R(T, T) \geq 2t_2$ . Thus,  $R(T, T) \geq \max\{2t_1 + t_2, 2t_2\} - 1$ . Haxell, Łuczak and Tingley proved in [9] that for a tree  $T$  with maximum degree  $o(t_2)$ , this lower bound is the asymptotically correct value of  $R(T, T)$ .

We try to extend this to bipartite graphs with small bandwidth (although with a more restrictive maximum degree condition). A graph is said to have *bandwidth* at most  $b$ , if there exists a labelling of the vertices by numbers  $1, \dots, n$  such that for every edge  $\{i, j\}$  of the graph we have  $|i - j| \leq b$ . We will focus on the following class of bipartite graphs.

**Definition 1.1.** A bipartite graph  $H$  is called a  $(\beta, \Delta)$ -graph if it has bandwidth at most  $\beta|V(H)|$  and maximum degree at most  $\Delta$ . Furthermore, we say that  $H$  is a balanced  $(\beta, \Delta)$ -graph if it has a legal 2-coloring  $\chi : V(H) \rightarrow [2]$  such that  $1 - \beta \leq |\chi^{-1}(1)|/|\chi^{-1}(2)| \leq 1 + \beta$ .

For example, all bounded degree planar graphs  $G$  are  $(\beta, \Delta(G))$ -graphs for any  $\beta > 0$  [3]. Our first theorem is an analogue of the result in [9] for  $(\beta, \Delta)$ -graphs.

**Theorem 1.2.** *For every  $\gamma > 0$  and natural number  $\Delta$ , there exist a constant  $\beta > 0$  and natural number  $n_0$  such that for every  $(\beta, \Delta)$ -graph  $H$  on  $n \geq n_0$  vertices with a legal 2-coloring  $\chi : V(H) \rightarrow [2]$  where  $t_1 = |\chi^{-1}(1)|$  and  $t_2 = |\chi^{-1}(2)|$ ,  $t_1 \leq t_2$ , we have*

$$R(H, H) \leq (1 + \gamma) \max\{2t_1 + t_2, 2t_2\}.$$

For more recent results on the Ramsey number of graphs of higher chromatic number and sublinear bandwidth, we refer the reader to the recent paper by Allen, Brightwell and Skokan [1].

For  $r \geq 3$  less is known about Ramsey numbers. Proving a conjecture of Faudree and Schelp [5], it was shown in [8] that for sufficiently large  $n$   $R(P_n, P_n, P_n) = 2n - 1$ , for odd  $n$  and  $R(P_n, P_n, P_n) = 2n - 2$ , for even  $n$ . Asymptotically this was also proved independently by Figaj and Łuczak [6]. Benevides and Skokan proved [2] that  $R(C_n, C_n, C_n) = 2n$  for sufficiently large even  $n$ . In our second theorem we extend these results (asymptotically) to balanced  $(\beta, \Delta)$ -graphs.

**Theorem 1.3.** *For every  $\gamma > 0$  and natural number  $\Delta$ , there exist a constant  $\beta > 0$  and natural number  $n_0$  such that for every balanced  $(\beta, \Delta)$ -graph  $H$  on  $n \geq n_0$  vertices we have  $R(H, H, H) \leq (2 + \gamma)n$ .*

In particular, Theorems 1.2 and 1.3 determine asymptotically the two and three color Ramsey numbers for grid graphs.

We conclude this section with a few words about the proof method for our main theorems. The proof of Theorem 1.2 combines ideas from [9] and [4], while the proof of Theorem 1.3 follows a similar approach as in [6], again, together with the result in [4]. Since the strategies for both theorems are close to each other, we focus on the proof of Theorem 1.3, for which we present an outline in the next section. Details can be found in [12].

## 2 Sketch of the proof of Theorem 1.3

Here we will sketch the main ideas of our proof. The proof relies on the regularity method for graphs and we refer the reader to the survey [10] for related notation and definitions.

The first part of the proof follows the same pattern as the proof by Figaj and Łuczak [6] for the case where  $H$  is a path. Namely, we apply a multicolored variant of Szemerédi's Regularity Lemma [14] to the 3-colored complete graph  $K_N$  with  $N = (2 + \gamma)n$  and get a partition with a very dense reduced graph. The edges of the reduced graph inherit the majority color of the respective pair. Applying Lemma 8 from [6] gives us a monochromatic tree  $T$  in the reduced graph that contains a matching  $M$  covering almost half of the vertices.

Switching back from the reduced graph to the colored complete graph, we denote by  $G_T$  the subgraph of  $K_N$  whose vertices are contained in the clusters represented by the vertices of  $T$  and whose edges run inside the pairs represented by the edges of  $T$  and have the same color as the edges of  $T$ . Thus  $G_T$  is a monochromatic subgraph of  $K_N$  whose regular pairs are arranged in a structure mirroring that of  $T$  and all have density at least  $1/3$ . Finally, we localize almost spanning super-regular subgraphs in the pairs in  $G_T$  represented by edges in  $M$  and denote the subgraph formed by the union of these pairs by  $G_M \subset G_T$ .

To understand the motivation for the second part, recall that our overall goal is to embed  $H$  into  $G_T$ . Notice that  $G_M$  has in fact enough vertices to accommodate all of  $H$ . Indeed, most of the vertices of  $H$  will be mapped to  $G_M$ , and we will only have to use parts from  $G_T \setminus G_M$  because we may need to connect the various parts of  $H$  embedded into  $G_M$ . Let us explain this more precisely by assuming for the moment that  $H$  is just a path. Let

$m$  be an integer which is just a bit smaller than the size of the clusters in  $G_T$  (that we assume to be all of the same size). Applying the Blow-up Lemma by Komlós, Sárközy and Szemerédi [11], we then embed the first  $m$  vertices from each color class of  $H$  into the super-regular pair represented by the first matching edge in  $M$ .

To be able to ‘reach’ the next super-regular pair in  $G_M$ , where we can embed the next  $m+m$  vertices of  $H$ , we need to make use of the fact that the vertices representing these two pairs are connected by a path in  $T$ . This path translates into a sequence of regular pairs in  $G_T$ , into each of which we embed an intermediate edge of  $H$ , thereby ‘walking’ towards the next super-regular pair in  $G_M$ . In this way, we only use few edges of the regular pairs in  $G_T \setminus G_M$ , thus keeping them regular all the way through, and leaving a bit of space in the super-regular pairs in  $G_M$ , in case we need to walk through them later again.

The task for the second part of our proof is to restructure our balanced  $(\beta, \Delta)$ -graph  $H$  in such a way that it behaves like the path in the embedding approach described before. Here two major problems occur:

- Suppose for example that  $H$  is a graph consisting of a path whose vertices are labelled by  $1, \dots, n$ , with some additional edges between vertices whose label differ by at most  $\beta n$  and have different parity (because  $H$  is bipartite). For such a graph ‘making the connections’ as above is now more difficult. Suppose, for instance, that we have a chain of regular pairs  $(V_i, V_{i+1})$  in  $G_T$  for  $i = 1, \dots, 4$  and want to use it to ‘walk’ with  $H$  from  $V_1$  to  $V_5$ . We cannot simply assign vertex 1 to  $V_1$ , then 2 to  $V_2$  and so on up to 5 to  $V_5$ , because maybe  $\{2, 5\}$  forms an edge in  $H$  but  $(V_2, V_5)$  is not a regular pair in  $G_T$ .

The solution to this problem is to walk more slowly: assign vertex 1 to  $V_1$ , then with the vertices  $2, 3, \dots, \beta n + 1$  alternate between  $V_2$  and  $V_3$ , the next  $\beta n$  vertices continue the zig-zag pattern between  $V_3$  and  $V_4$ , and finally we send the last vertex to  $V_5$ . What does this buy us? Consider, *e.g.*, the final vertex  $y$  that got mapped to  $V_5$ . Due to the bandwidth condition, all its potential neighbours were embedded in  $V_3$  or  $V_4$ , and due to the parity condition, they must all lie in  $V_4$ . This is good, because we have a regular pair  $(V_4, V_5)$ .

- The second problem that we have to face is as follows. By definition,  $H$  has a 2-coloring of its vertices that uses both colors similarly often *in total*, but this does not have to be true *locally* – among the first  $m+m$  vertices of  $H$ , there could be far more vertices of color 1 than of color 2, which means that our approach to embed them into a super-regular pair with two classes of the same size would fail.

The solution to this problem is to re-balance  $H$ . We use an ordering

of  $H$  with bandwidth at most  $\beta n$  and cut  $H$  into small blocks of size  $\xi n$ , where  $\beta n \ll \xi n \ll m$ . Then it is not hard to see that we can obtain a new ordering of the vertices of  $H$  by changing the order in which the blocks appear, so that in every interval of blocks summing to roughly  $m$  consecutive vertices of  $H$  the two colors are balanced up to  $2\xi n$  vertices. We can now assign the blocks forming these intervals to super-regular pairs in  $G_M$  in such a way that they there represent a balanced 2-coloring and can therefore be embedded via the Blow-up Lemma into the super-regular pair.

Both these problems can appear at the same time, but one can combine these two solutions. Hence  $H$  can indeed be embedded into  $G_T$  similarly to the example of the path example given above, which finishes the proof.

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# Polynomial bounds on geometric Ramsey numbers of ladder graphs

Josef Cibulka<sup>1</sup>, Pu Gao<sup>2</sup>, Marek Krčál<sup>1</sup>, Tomáš Valla<sup>3</sup> and Pavel Valtr<sup>1</sup>

**Abstract.** We prove that the geometric Ramsey numbers of the ladder graph on  $2n$  vertices are bounded by  $O(n^3)$  and  $O(n^{10})$ , in the convex and general case, respectively. We also prove polynomial upper bounds of geometric Ramsey numbers of pathwidth-2 outerplanar triangulations in both convex and general cases.

## 1 Introduction and basic definitions

A finite set  $P \subset \mathbb{R}^2$  of points is in a *general position* if no three points of  $P$  are collinear. The *complete geometric graph on  $P$* , denoted by  $K_P$ , is the complete graph with vertex set  $P$ , whose edges are drawn as the straight-line segments between pairs of points of  $P$ .

The set of points  $P$  is in *convex position* if  $P$  is the set of vertices of a convex polygon. If  $P$  is in convex position, we say that  $K_P$  is a *convex complete geometric graph*.

Károlyi, Pach and Tóth [4] introduced the concept of Ramsey numbers for geometric graphs as follows. Given a graph  $G$ , the *geometric Ramsey number* of  $G$ , denoted by  $R_g(G)$ , is the smallest integer  $n$  such that every complete geometric graph  $K_P$  on  $n$  vertices with edges arbitrarily coloured by two colours contains a monochromatic non-crossing copy of  $G$ . The *convex geometric Ramsey number* of  $G$ ,  $R_c(G)$ , is defined the same way except that  $K_P$  is restricted to the convex complete geometric

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graph. A graph  $G$  is said to be *outerplanar* if  $G$  can be drawn in the plane without any edge crossing and with all vertices of  $G$  incident to the unbounded face. Apparently, the numbers  $R_g(G)$  and  $R_c(G)$  are finite only if  $G$  is outerplanar, and it follows immediately from the definitions that  $R_c(G) \leq R_g(G)$  for every outerplanar graph  $G$ .

The Ramsey numbers of outerplanar graphs, as well as of all planar graphs, are bounded by a function linear in the number of vertices by a result of Chen and Schelp [2]. However, the corresponding geometric Ramsey numbers can be larger and it remains open whether there is a general polynomial bound for all outerplanar graphs. By a simple constructive proof, it is easy to see that for any  $G$  on  $n$  vertices that contains a Hamilton cycle,  $R_c(G) \geq (n - 1)^2 + 1$ . It has also been proved that, for any cycle  $C_n$  on  $n$  vertices,  $R_g(C_n) \leq 2(n - 2)(n - 1) + 2$  [5], and this upper bound is only known to be tight when  $n \in \{3, 4\}$ . Károlyi *et al.* [5] found the exact value  $R_c(P_n) = 2n - 3$  and the upper bound  $R_g(P_n) \in O(n^{3/2})$ , where  $P_n$  is a path on  $n > 2$  vertices. The bounds  $2n - 3 \leq R_g(P_n) \leq O(n^{3/2})$  remain the best known bounds on the geometric Ramsey number of paths.

In this extended abstract, we contribute to this subject by showing polynomial upper bounds on the geometric Ramsey numbers of the ladder graphs, see Definition 2.1, and their generalisation. In Section 2, we show that the geometric Ramsey numbers of the ladder graph on  $2n$  vertices are bounded by  $O(n^3)$  and  $O(n^{10})$  in the convex and general case, respectively. In Section 3, we generalise the polynomial upper bounds to the class of all subgraphs of pathwidth-2 outerplanar triangulations, see Definition 3.1. These bounds are  $20n^7$  and  $O(n^{22})$  in the convex and general case, respectively

We note here that all colourings in this extended abstract, unless specified, refer to edge colourings. As a convention, in any 2-colouring, we assume that the colours used are blue and red.

We abbreviate the set  $\{1, 2, \dots, k\}$  with  $[k]$  and  $\{l, l + 1, \dots, k\}$  with  $[l, k]$ . We write  $(x_i)_{i=1}^k$  for the sequence  $x_1, x_2, \dots, x_k$ . The sequence of vertices  $(v_i)_{i=1}^{\ell+1}$  is a *path of colour  $c$*  and length  $\ell$  if every pair  $\{v_i v_{i+1}\}$ ,  $i \in [\ell]$  is an edge and has colour  $c$ . A sequence  $(A_i)_{i=1}^k$  is said to be a *partition* of  $A$  if  $A_i$  are pairwise disjoint and  $\cup_{i=1}^k A_i = A$ .

## 2 Ladder graphs

The ladder graphs are defined as follows.

**Definition 2.1.** For any integer  $n \geq 1$ , the ladder graph on  $2n$  vertices, denoted by  $L_{2n}$ , is the graph composed of two paths  $(u_i)_{i=1}^n$  and  $(v_i)_{i=1}^n$ , together with the set of edges  $\{u_i v_i : i \in [n]\}$ .

In the following two subsections, we prove upper bounds on  $R_c(L_{2n})$  and  $R_g(L_{2n})$ . Both proofs use the following lemma due to Gritzmann *et al.* [3].

**Lemma 2.2 (Gritzmann *et al.* 1991[3]).** *Let  $G$  be an outerplanar graph on  $n$  vertices and let  $P$  be a set of  $n$  points in general position. Then  $K_P$  contains a non-crossing copy of  $G$ .*

### 2.1 Convex position

In this section, let  $C$  denote a set of  $32n^3$  points in convex position. That is,  $C$  is the set of vertices of some convex polygon. We label the vertices  $v_1, v_2, \dots, v_{|C|}$  in the clockwise order starting at an arbitrarily chosen vertex  $v_1$ . We write  $v_i < v_j$  if and only if  $i < j$ . Let  $A, B \subset C$ . We say that  $A$  precedes  $B$  and write  $A < B$  if and only if for every  $u \in A$  and every  $v \in B$ ,  $u < v$ . Notice that if  $A < B$ , then the sets  $A$  and  $B$  can be separated by a line.

For a pair of disjoint vertex sets  $(L, R)$ ,  $L \subset C$ ,  $R \subset C$ , the complete bipartite graph on  $(L, R)$ , denoted by  $K_{L,R}$ , is the set of edges  $\{u, v\}$ , where  $u \in L$  and  $v \in R$ . A complete bipartite graph  $K_{L,R}$  is said to be *well-split* if  $L < R$  or  $R < L$ . A well-split  $K_{m,n}$  is a well-split  $K_{L,R}$ , for some  $L$  and  $R$  such that  $|L| = m$ ,  $|R| = n$ .

The following lemmas are used frequently throughout the proofs.

**Lemma 2.3.** *If a 2-colouring of  $K_C$  contains a monochromatic well-split  $K_{2n^2, 2n^2}$ , then it contains a monochromatic non-crossing copy of  $L_{2n}$ .*

**Lemma 2.4.** *Let  $N$  be a positive integer. Let  $G$  be the complete graph on a set  $A$  of points in general position and let  $(A_i)_{i=1}^n$  be a partition of  $A$  with  $|A_i| \geq N$  for every  $i \in [n]$ . Then for any 2-colouring of the edges of  $G$ , either there is a red path  $(u_i)_{i=1}^n$  with  $u_i \in A_i$  for each  $i \in [n]$  or for some  $i \in [n - 1]$  there exists a blue  $K_{B_i, B_{i+1}}$  with  $B_i \subseteq A_i$ ,  $B_{i+1} \subseteq A_{i+1}$  and  $\min\{|B_i|, |B_{i+1}|\} \geq N/2$ .*

Using Lemma 2.3 and 2.4, we are able to prove the following theorem.

**Theorem 2.5.** *For every  $n \geq 1$ ,  $R_c(L_{2n}) \leq 32n^3$ .*

### 2.2 General geometric position

**Definition 2.6.** Two sets of points  $A$  and  $B$  in the plane are *mutually avoiding* if  $|A|, |B| \geq 2$  and no line subtended by a pair of points in  $A$  intersects the convex hull of  $B$ , and vice versa.

A simple example of a pair of mutually avoiding sets are sets  $A$  and  $B$  such that  $A \cup B$  is in convex position and  $A$  and  $B$  can be separated by a straight line. Observe that for any mutually avoiding pair  $(A, B)$ , every point in  $A$  “sees” all the vertices in  $B$  in the same order and vice versa. That is, there are unique total orders  $u_1 \prec u_2 \prec \dots \prec u_{|A|}$  of the points in  $A$  and  $v_1 \prec v_2 \prec \dots \prec v_{|B|}$  of the the points in  $B$  such that every point in  $B$  “sees”  $u_1, \dots, u_{|A|}$  consecutively in a clockwise order before seeing any vertex in  $B$ , whereas every point in  $A$  “sees”  $v_1, \dots, v_{|B|}$  consecutively in a counterclockwise order before seeing any vertex in  $A$ . A path  $(p_i)_{i=1}^\ell$  in either  $A$  or  $B$  is an *increasing path* if  $p_1 \prec p_2 \prec \dots \prec p_\ell$ .

For any two sets of vertices  $A_1, A_2$  both contained in  $A$  (or  $B$ ), we write  $A_1 \prec A_2$  if and only if for every  $u \in A_1$  and  $v \in A_2, u \prec v$ . An embedding of the complete bipartite graph  $K_{m,n}$  is said to be *well-split* if the two sets of points representing the two vertex parts are mutually avoiding.

The following proposition follows from the definition of a pair of mutually avoiding sets.

**Proposition 2.7.** *Assume  $A$  and  $B$  are mutually avoiding. Let  $P_u = (x_i)_{i=1}^n$  be an increasing path in  $A$  and let  $P_v = (y_i)_{i=1}^n$  be an increasing path in  $B$ . Then the ladder graph composed of the paths  $P_u$  and  $P_v$  and edges  $\{\{x_i, y_i\} : i \in [n]\}$  is non-crossing.*

Given a set of points in general position, the following theorem guarantees the existence of two mutually avoiding subsets, each with reasonably large sizes.

**Theorem 2.8 (Aronov et al. 1994 [1]).** *Let  $A'$  and  $B'$  be two sets of points separated by a line, each of size  $6n^2$ . Then there exist mutually avoiding subsets  $A \subset A'$  and  $B \subset B'$  such that  $A$  and  $B$  are both of size  $n$ .*

By Lemma 2.2 and Proposition 2.7, we have the following generalisation of Lemma 2.3.

**Lemma 2.9.** *If a 2-colouring of  $K_P$  contains a monochromatic well-split  $K_{2n^2, 2n^2}$ , then it contains a monochromatic non-crossing  $L_{2n}$ .*

A complete geometric bipartite graph  $K_{L,R}$  is said to be *separable* if  $L$  and  $R$  can be separated by a line. Notice that if  $L \cup R$  is in convex position, then  $K_{L,R}$  is separable if and only if it is well-split. Obviously, every complete bipartite geometric graph  $K_{L,R}$  contains a separable complete

bipartite graph with parts of sizes  $|L|/2$  and  $|R|/2$ . However, all complete bipartite geometric graphs that we encounter in subsequent proofs are separable, so we state the following corollary of Theorem 2.8 and Lemma 2.9 for separable complete bipartite graphs only.

**Corollary 2.10.** *Every 2-colouring of  $K_P$  containing a monochromatic separable  $K_{24n^4, 24n^4}$  contains a monochromatic non-crossing copy of  $L_{2n}$ .*

We conclude by the resulting theorem.

**Theorem 2.11.** *The geometric Ramsey number of the ladder graph  $L_{2n}$  satisfies  $R_g(L_{2n}) = O(n^{10})$ .*

### 3 Generalisation to pathwidth-2 outerplanar triangulations

An *outerplanar triangulation*  $G$  is a planar graph that can be drawn in the plane in such a way that the outer face is incident with all the vertices of  $G$  and every other face is incident with exactly three vertices.

The pathwidth of a graph was first defined by Robertson and Seymour [6] as follows. A *path decomposition* of a graph  $G$  is a sequence  $(G_i)_{i=1}^m$  of subgraphs of  $G$  such that each edge of  $G$  is in at least one of  $G_i$  and for every vertex  $v$  of  $G$ , the set of graphs  $G_i$  containing  $v$  forms a contiguous subsequence of  $(G_i)_{i=1}^m$ . The *pathwidth* of a graph  $G$  is the smallest  $k$  such that  $G$  has a path decomposition in which every  $G_i$  has at most  $k + 1$  vertices. Let  $pw(G)$  denote the pathwidth of  $G$ .

We now give an equivalent definition of pathwidth-2 outerplanar triangulations that will be used in the later proofs.

**Definition 3.1.** Let  $PW_2(n)$  be the class of outerplanar triangulations  $G$  whose vertices can be decomposed into two disjoint sets  $V_u \cup V_v = V(G)$  such that the subgraphs induced by the two sets,  $P_u = G[V_u]$  and  $P_v = G[V_v]$ , are paths.

We then state the following theorem.

**Theorem 3.2.** *For any  $G \subseteq G' \in PW_2(n)$ ,  $R_c(G) \leq 20n^7$  and  $R_g(G) \leq O(n^{22})$ .*

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## Geometry and Surfaces





# Arrangements of pseudocircles and circles

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**Abstract.** An arrangement of pseudocircles is a finite collection of Jordan curves in the plane with the additional properties that (i) no three of the curves meet in a point; (ii) every two curves meet in at most two points; and (iii) if two curves meet in a point  $p$ , then they cross at  $p$ .

We say that two arrangements  $\mathcal{C} = (c_1, \dots, c_n)$ ,  $\mathcal{D} = (d_1, \dots, d_n)$  are equivalent if there is a homeomorphism  $\varphi$  of the plane onto itself such that  $\varphi[c_i] = d_i$  for all  $1 \leq i \leq n$ . Linhart and Ortner (2005) gave an example of an arrangement of five pseudocircles that is not equivalent to an arrangement of circles, and conjectured that every arrangement of at most four pseudocircles is equivalent to an arrangement of circles. We prove their conjecture.

We consider two related recognition problems. The first is the problem of deciding, given a pseudocircle arrangement, whether it is equivalent to an arrangement of circles. The second is deciding, given a pseudocircle arrangement, whether it is equivalent to an arrangement of convex pseudocircles. We prove that both problems are NP-hard, answering questions of Bultena, Grünbaum and Ruskey (1998) and of Linhart and Ortner (2008).

We also give an example of a collection of convex pseudocircles with the property that its intersection graph (*i.e.* the graph with one vertex for each pseudocircle and an edge between two vertices if and only if the corresponding pseudocircles intersect) cannot be realized as the intersection graph of a collection of circles. This disproves a folklore conjecture communicated to us by Pyatkin.

## 1 Introduction

An *arrangement of pseudocircles* is a finite list  $\mathcal{C} = (c_1, \dots, c_n)$  of Jordan curves in the plane satisfying the following three conditions:

1. no three curves intersect in a point;
2. every two curves intersect in at most two points; and
3. if two curves meet in a point  $p$ , then they cross at  $p$ .

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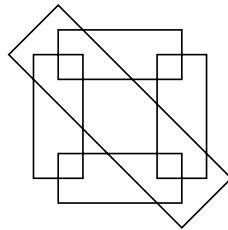
We will say that two arrangements  $\mathcal{C} = (c_1, \dots, c_n)$  and  $\mathcal{D} = (d_1, \dots, d_n)$  are *equivalent* if there exists a homeomorphism  $\varphi$  from the plane onto itself with the property that  $\varphi[c_i] = d_i$  for all  $1 \leq i \leq n$ .

Naturally, an arrangement of pseudocircles  $\mathcal{C} = (c_1, \dots, c_n)$  is called an *arrangement of circles* if each  $c_i$  is a circle. We will say that an arrangement of pseudocircles is *circleable* if it is equivalent to an arrangement of circles. In this work, we are interested in a number of problems which are related to the question of whether a given pseudocircle arrangement is circleable. Unfortunately, due to page limitations, we are unable to provide any full proofs for the results given below; however, we give indications of the proof methodology used.

ACKNOWLEDGEMENTS. We thank Artem Pyatkin for bringing to our attention the folklore conjecture described in Section 4 below as well as for helpful pointers to the literature.

## 2 A smallest circleable arrangement

A natural question is whether every arrangement of pseudocircles is circleable. This question was studied before by Linhart and Ortner [6], who showed that the pseudocircle arrangement in Figure 2.1 is not circleable.



**Figure 2.1.** An arrangement of five pseudocircles that is not circleable.

They also conjectured that this is a minimal example of a non-circleable pseudocircle arrangement. We confirm their conjecture.

**Theorem 2.1.** *Every arrangement of at most four pseudocircles is equivalent to an arrangement of circles.*

The proof of this theorem necessitates an involved case analysis which is thankfully shortened by the use of circle inversions in the extended complex plane. Theorem 2.1 provides a natural analogue of a celebrated result of Goodman and Pollack [5], who showed that every arrangement of up to eight pseudolines is equivalent to an arrangement of lines. Prior to this an example was known of an arrangement of nine pseudolines not equivalent to an arrangement of lines.

### 3 Circleability and convexibility complexity

In this section, we shall discuss the computation complexity of some pseudocircle recognition problems. Before we do so, we must appropriately define the input format.

Given an arbitrary list  $\mathcal{C} = (c_1, \dots, c_n)$  of Jordan curves in the plane, the *arrangement multigraph* for  $\mathcal{C}$  is defined as follows: the vertices are all the intersection points among the curves, while the edges are all the maximal curve segments that have no intersection points. This multigraph is already naturally endowed with an embedding in the plane, which we may specify combinatorially for instance by a rotation system (*i.e.* each vertex  $v$  is assigned a cyclic permutation of the edges incident with  $v$ ). Because of conditions **(i)** and **(iii)** of a pseudocircle arrangement, we can straightforwardly derive from the embedded multigraph the information of which edge belongs to which curve of  $\mathcal{C}$ . Therefore, we may assume that the input to the computational problems below are given as the rotation system of a multigraph.

Given that not all pseudocircle arrangements are circleable, one might wonder if there is an efficient characterization of circleable arrangements. We write CIRCLEABILITY for the computational problem of deciding, given a list  $\mathcal{C}$  of Jordan curves in the plane, whether  $\mathcal{C}$  can be realized as an oriented circle arrangement. We indeed consider a restricted version of this problem, CONVEX CIRCLEABILITY, on the input of arrangements of convex pseudocircles. By using a corollary of a deep result of Mněv [8] on the recognition of stretchable pseudoline arrangements (see also Shor [10]), we show the following.

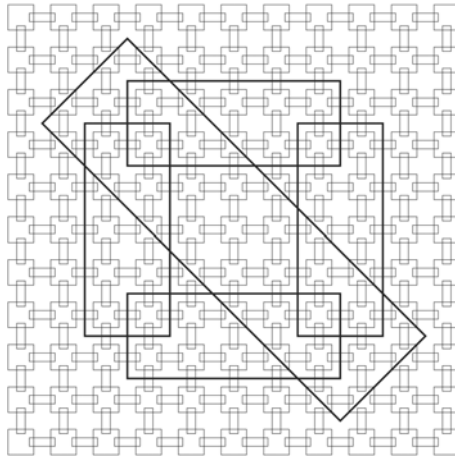
**Theorem 3.1.** *CONVEX CIRCLEABILITY is NP-hard.*

It is natural also to consider the recognition problem for arrangements of convex pseudocircles. We say that an arrangement of pseudocircles is *convexible* if it is equivalent to an arrangement of convex pseudocircles. We write PSEUDOCIRCLE CONVEXIBILITY for the problem of deciding, given a list  $\mathcal{C}$  of Jordan curves in the plane, whether  $\mathcal{C}$  can be realized as an arrangement of convex pseudocircles. Bultena, Grünbaum and Ruskey [1] have asked about the complexity of this computational problem. Later, Linhart and Ortner [7] asked the weaker question of whether there exists a pseudocircle arrangement that is not convexible. Again using the result of Mněv, we answer both of these questions.

**Theorem 3.2.** *PSEUDOCIRCLE CONVEXIBILITY is NP-hard.*

#### 4 A folklore conjecture

If  $\mathcal{A} = (A_1, \dots, A_n)$  is a list of sets, then the *intersection graph* of  $\mathcal{A}$  is the graph  $G = (V, E)$  with vertex set  $V = \{1, \dots, n\}$  and an edge  $ij \in E$  if and only if  $A_i \cap A_j \neq \emptyset$ . A folklore conjecture that was communicated to us by Artem Pyatkin [9] states that every intersection graph of a list of convex curves is also the intersection graph of a list of circles. (We do not use the word “arrangement” here because we do not necessarily want to impose the restrictions 1–3 above.) This conjecture was apparently inspired by the work of Dobrynin and Mel’nikov [2–4] on the chromatic number of “arrangement graphs” of Jordan curves in the plane (*i.e.* graphs whose vertices are the intersection points of the curves and whose edges are the curve segments between these intersection points). To get a feel for the conjecture observe for instance that, while all the pseudocircles of the arrangement in Figure 2.1 are convex curves and the arrangement is not equivalent to any arrangement of circles, one can easily construct a family of five circles in the plane with the same intersection graph. We are however able to produce a counterexample to the folklore conjecture by placing additional pseudocircles as in Figure 4.1.



**Figure 4.1.** A collection of convex curves such that no collection of circles defines the same intersection graph.

**Theorem 4.1.** *The intersection graph of the convex pseudocircles in Figure 4.1 cannot be realized as the intersection graph of a list of circles.*

The idea for this additional “gridding” of smaller pseudocircles is that it ensures that, if the resultant intersection graph were realizable as the intersection graph of a list of circles, then the original (Linhart-Ortner) arrangement of pseudocircles would be circleable, a contradiction.

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# Extended abstract for structure results for multiple tilings in 3D

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**Abstract.** We study multiple tilings of 3-dimensional Euclidean space by a convex body. In a multiple tiling, a convex body  $P$  is translated with a discrete multiset  $\Lambda$  in such a way that each point of  $\mathbb{R}^d$  gets covered exactly  $k$  times, except perhaps the translated copies of the boundary of  $P$ . It is known that all possible multiple tilers in  $\mathbb{R}^3$  are zonotopes. In  $\mathbb{R}^2$  it was known by the work of M. Kolountzakis [9] that, unless  $P$  is a parallelogram, the multiset of translation vectors  $\Lambda$  must be a finite union of translated lattices (also known as quasi periodic sets). In that work [9] the author asked whether the same quasi-periodic structure on the translation vectors would be true in  $\mathbb{R}^3$ . Here we prove that this conclusion is indeed true for  $\mathbb{R}^3$ .

Namely, we show that if  $P$  is a convex multiple tiler in  $\mathbb{R}^3$ , with a discrete multiset  $\Lambda$  of translation vectors, then  $\Lambda$  has to be a finite union of translated lattices, unless  $P$  belongs to a special class of zonotopes. This exceptional class consists of two-flat zonotopes  $P$ , defined by the Minkowski sum of two 2-dimensional symmetric polygons in  $\mathbb{R}^3$ , one of which may degenerate into a single line segment. It turns out that rational two-flat zonotopes admit a multiple tiling with an aperiodic (non-quasi-periodic) set of translation vectors  $\Lambda$ . We note that it may be quite difficult to offer a visualization of these 3-dimensional non-quasi-periodic tilings, and that we discovered them by using Fourier methods.

The study of multiple tilings of Euclidean space began in 1936, when the famous Minkowski facet-to-facet conjecture [15] for classical tilings was extended to the setting of  $k$ -tilings with the unit cube, by Furtwängler [3]. Minkowski's facet-to-facet conjecture states that for any *lattice* tiling of  $\mathbb{R}^d$  by translations of the unit cube, there exist at least two translated cubes that share a facet (face of co-dimension 1). The conjecture was strengthened by Furtwängler [3] who conjectured the same conclusion for any *multiple* lattice tiling.

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To define a multiple tiling, suppose we translate a convex body  $P$  with a discrete multiset  $\Lambda$ , in such a way that each point of  $\mathbb{R}^d$  gets covered exactly  $k$  times, except perhaps the translated copies of the boundary of  $P$ . We then call such a body a  $k$ -tiler, and such an action has been given the following names in the literature: a  $k$ -tiling, a **tiling at level  $k$** , a **tiling with multiplicity  $k$** , and sometimes simply a **multiple tiling**. We may use any of these synonyms here, and we immediately point out, for polytopes  $P$ , a trivial but useful algebraic equivalence for a tiling at level  $k$ :

$$\sum_{\lambda \in \Lambda} \mathbf{1}_{P+\lambda}(x) = k, \quad (1)$$

for almost all  $x \in \mathbb{R}^d$ , where  $\mathbf{1}_P$  is the indicator function of the polytope  $P$ .

Furtwängler's conjecture was disproved by Hajós [7] for dimension larger than 3 and for  $k \geq 9$  while Furtwängler himself [3] proved it for dimension at most 3. Hajós [8] also proved Minkowski's conjecture in all dimensions. The ideas of Furtwängler were subsequently extended (but still restricted to cubes) by the important work of Perron [16], Robinson [17], Szabó [21], Gordon [4] and Lagarias and Shor [11]. These authors showed that for some levels  $k$  and dimensions  $d$  and under the lattice assumption as well as not, a facet-to-facet conclusion for  $k$ -tilings is true in  $\mathbb{R}^d$ , while for most values of  $k$  and  $d$  it is false.

It was known to Bolle [2] that in  $\mathbb{R}^2$ , every  $k$ -tiling convex polytope has to be a centrally symmetric polygon, and using combinatorial methods Bolle [2] gave a characterization for all polygons in  $\mathbb{R}^2$  that admit a  $k$ -tiling with a lattice  $\Lambda$  of translation vectors. Kolountzakis [10] proved that if a convex polygon  $P$  tiles  $\mathbb{R}^2$  multiply with any discrete multiset  $\Lambda$ , then  $\Lambda$  must be a *finite union of two-dimensional lattices*. The ingredients of Kolountzakis' proof include the idempotent theorem for the Fourier transform of a measure. Roughly speaking, the idempotent theorem of Meyer [14] tells us that if the square of the Fourier transform of a measure is itself, then the support of the measure is contained in a finite union of lattices.

A multiple tiling is called **quasi-periodic** if its multiset of discrete translation vectors  $\Lambda$  is a finite union of translated lattices, not necessarily all of the same dimension.

**Theorem (Kolountzakis, 2002 [9]).** *Suppose that  $K$  is a symmetric convex polygon which is not a parallelogram. Then  $K$  admits only quasi-periodic multiple tilings if any.*

In this work, we extend this result to  $\mathbb{R}^3$ , and we also find a fascinating class of polytopes analogous to the parallelogram of the theorem above. To describe this class, we first recall the definition of a **zonotope**, which

is the Minkowski sum of a finite number of line segments. In other words, a zonotope equals a translate of  $[-\mathbf{v}_1, \mathbf{v}_1] + \cdots + [-\mathbf{v}_N, \mathbf{v}_N]$ , for some positive integer  $N$  and vectors  $\mathbf{v}_1, \dots, \mathbf{v}_N \in \mathbb{R}^d$ . A zonotope may equivalently be defined as the projection of some  $l$ -dimensional cube. A third equivalent condition is that for a  $d$ -dimensional zonotope, all of its  $k$ -dimensional faces are centrally symmetric, for  $1 \leq k \leq d$ . For example, the zonotopes in  $\mathbb{R}^2$  are the centrally symmetric polygons.

We shall say that a polytope  $P \subseteq \mathbb{R}^3$  is a **two-flat zonotope** if  $P$  is the Minkowski sum of  $n + m$  line segments which lie in the union of two different two-dimensional subspaces  $H_1$  and  $H_2$ . In other words,  $H_1$  contains  $n$  of the segments and  $H_2$  contains  $m$  of the segments (if one of the segments belongs to both  $H_1$  and  $H_2$  we list it twice, once for each plane). Equivalently,  $P$  may be thought of as the Minkowski sum of two 2-dimensional symmetric polygons one of which may degenerate into a single line segment.

It turns out that if  $P$  is a rational two-flat zonotope, then  $P$  admits a  $k$ -tiling with a non-quasi-periodic set of translation vectors  $\Lambda$ . For some of the classical study of 1-tilings, and their interesting connections to zonotopes, the reader may refer to the work of [12, 13, 19, 23], and [1]. Here we find it very useful to use the intuitive language of distributions [18, 20] in order to think – and indeed discover – facts about  $k$ -tilings. To that end we introduce the distribution (which is locally a measure)

$$\delta_\Lambda := \sum_{\lambda \in \Lambda} \delta_\lambda, \quad (2)$$

where  $\delta_\lambda$  is the Dirac delta function at  $\lambda \in \mathbb{R}^d$ . To develop some intuition, we may check formally that

$$\delta_\Lambda * \mathbf{1}_P = \sum_{\lambda \in \Lambda} \delta_\lambda * \mathbf{1}_P = \sum_{\lambda \in \Lambda} \mathbf{1}_{P+\lambda},$$

so that from the first definition (1) of  $k$ -tiling, we see that a polytope  $P$  is a  $k$ -tiler if and only if

$$\delta_\Lambda * \mathbf{1}_P = k. \quad (3)$$

Suppose the polytope  $P$  tiles multiply with the translates  $\Lambda \subseteq \mathbb{R}^d$ . We will need to understand some basic facts about how the  $\Lambda$  points are distributed.

For any symmetric polytope  $P$ , and any face  $F \subset P$ , we define  $F^-$  to be the face of  $P$  symmetric to  $F$  with respect to  $P$ 's center of symmetry. We call  $F^-$  the **opposite face** of  $F$ . We use the standard convention of boldfacing all vectors, to differentiate between  $\mathbf{v}$  and  $v$ , for example. We furthermore use the convention that  $[\mathbf{e}]$  denotes the 1-dimensional line



segment from 0 to the endpoint of the vector  $\mathbf{e}$ . Whenever it is clear from context, we will also write  $[e]$  to denote the same line segment - for example, in the case that  $e$  denotes an edge of a polytope.

Suppose  $P \in \mathbb{R}^3$  is a zonotope (symmetric polytope with symmetric facets). A collection of four edges of  $P$  is called a **4-legged-frame** if whenever  $e$  is one of the edges then there exist two vectors  $\boldsymbol{\tau}_1$  and  $\boldsymbol{\tau}_2$  such that the four edges are

$$[e], [e] + \boldsymbol{\tau}_1, [e] + \boldsymbol{\tau}_2 \text{ and } [e] + \boldsymbol{\tau}_1 + \boldsymbol{\tau}_2,$$

and such that the edges  $[e]$  and  $[e] + \boldsymbol{\tau}_1$  belong to the same face of  $P$  and the edges  $[e] + \boldsymbol{\tau}_2$  and  $[e] + \boldsymbol{\tau}_1 + \boldsymbol{\tau}_2$  belong to the opposite face.

With the notion of 4-legged frames, we can introduce the following so-called **intersection property**, which plays an important role in the proof of the main theorem.

Suppose  $P$  is a  $k$ -tiler with a discrete multiset  $\Lambda$ , in  $\mathbb{R}^3$ . We say that the intersection property holds, if

$$\bigcap_{e, \boldsymbol{\tau}_1, \boldsymbol{\tau}_2} (\mathbf{e}^\perp \cup \boldsymbol{\tau}_1^\perp \cup \boldsymbol{\tau}_2^\perp) = \{0\}, \tag{4}$$

where the intersection above is taken over all sets of 4-legged frames of  $P$ .

Recently, a structure theorem for convex  $k$ -tilers in  $\mathbb{R}^d$  was found, and is as follows.

**Theorem (Gravin, Robins, Shiryayev 2012 [6]).** *If a convex polytope  $k$ -tiles  $\mathbb{R}^d$  by translations, then it is centrally symmetric and its facets are centrally symmetric.*

This theorem generalizes the theorem for 1-tilers by Minkowski [15]. One-tiler case was extensively studied in the past, and the complete characterization for 1-tilers was given independently by Venkov [22] and McMullen [13].

It follows immediately from the latter theorem that a  $k$ -tiler  $P \subset \mathbb{R}^3$  is necessarily a zonotope. The following theorem extends the result of Kolountzakis [9] from  $\mathbb{R}^2$  to  $\mathbb{R}^3$ , providing a structure theorem for multiple tilings by polytopes in three dimensions.

The main result here is the following theorem [5]:

**Main Theorem.** *Suppose a polytope  $P$   $k$ -tiles  $\mathbb{R}^3$  with a discrete multiset  $\Lambda$ , and suppose that  $P$  is not a two-flat zonotope. Then  $\Lambda$  is a finite union of translated lattices.*

Although the proof is involved, one of the main ideas is to compute the Fourier transform of any 4-legged frame of a polytope, and show that

its zeros form a certain countable union of hyperplanes. Another main idea is to show that if the intersection property holds for  $P$ , then  $\delta_\Lambda$  has discrete support.

We also show that each rational two-flat zonotope admits a very peculiar non-quasi-periodic  $k$ -tiling. We note that it may be quite difficult to offer a visualization of these 3-dimensional non-quasi-periodic tilings, and that we discovered them by using Fourier methods.

## Open Questions

The proof of the main theorem does not directly generalize to dimensions higher than 3, since  $k$ -tilers in these dimensions are no longer all zonotopes. So it might require some new ideas and methods to deal with the higher dimension case, and we propose this as a primary direction for future work.

It would also be very helpful to generalize Bolle's characterization of 2-dimensional lattice  $k$ -tilers to higher dimensions.

Another important topic of future research would be to generalize the Venkov-McMullen characterization [13] from 1-tilers in  $\mathbb{R}^n$  to  $k$ -tilers. It is already established that any  $k$ -tiler in  $\mathbb{R}^n$  is centrally symmetric and has centrally symmetric facets, and it is reasonable to conjecture that it is enough to add one more condition on co-dimension 2 facets to get a complete characterization.

Finally, it would be interesting to prove or disprove the following conjecture: if a polytope has a quasi-periodic multiple tiling, then it also has a tiling with just one lattice.

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# On the nonexistence of $k$ -reptile simplices in $\mathbb{R}^3$ and $\mathbb{R}^4$

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**Abstract.** A  $d$ -dimensional simplex  $S$  is called a  $k$ -reptile (or a  $k$ -reptile simplex) if it can be tiled without overlaps by  $k$  simplices with disjoint interiors that are all mutually congruent and similar to  $S$ . For  $d = 2$ , triangular  $k$ -reptiles exist for many values of  $k$  and they have been completely characterized by Snover, Waiveris, and Williams. On the other hand, the only  $k$ -reptile simplices that are known for  $d \geq 3$ , have  $k = m^d$ , where  $m$  is a positive integer. We substantially simplify the proof by Matoušek and the second author that for  $d = 3$ ,  $k$ -reptile tetrahedra can exist only for  $k = m^3$ . We also prove a weaker analogue of this result for  $d = 4$  by showing that four-dimensional  $k$ -reptile simplices can exist only for  $k = m^2$ .

## 1 Introduction

A closed set  $X \subset \mathbb{R}^d$  with nonempty interior is called a  $k$ -reptile (or a  $k$ -reptile set) if there are sets  $X_1, X_2, \dots, X_k$  with disjoint interiors and with  $X = X_1 \cup X_2 \cup \dots \cup X_k$  that are all mutually congruent and similar to  $X$ . Such sets have been studied in connection with fractals and also with crystallography and tilings of  $\mathbb{R}^d$  [3, 6, 8].

It is easy to see that whenever  $S$  is a  $d$ -dimensional  $k$ -reptile simplex, then all of  $\mathbb{R}^d$  can be tiled by congruent copies of  $S$ : indeed, using the tiling of  $S$  by its smaller copies  $S_1, \dots, S_k$  as a pattern, one can inductively tile larger and larger similar copies of  $S$ . On the other hand, not all space-filling simplices must be  $k$ -reptiles for some  $k \geq 2$ .

Clearly, every triangle tiles  $\mathbb{R}^2$ . Moreover, every triangle  $T$  is a  $k$ -reptile for  $k = m^2$ , since  $T$  can be tiled in a regular way with  $m^2$  congruent tiles, each positively or negatively homothetic to  $T$ . See *e.g.* Snover *et al.* [16] for an illustration.

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The question of characterizing the tetrahedra that tile  $\mathbb{R}^3$  is still open and apparently rather difficult. The first systematic study of space-filling tetrahedra was made by Sommerville. Sommerville [17] discovered a list of exactly four tilings (up to isometry and rescaling), but he assumed that all tiles are *properly congruent* (that is, congruent by an orientation-preserving isometry) and meet face-to-face. Edmonds [5] noticed a gap in Sommerville's proof and by completing the analysis, he confirmed that Sommerville's classification of proper, face-to-face tilings is complete. In the non-proper and non face-to-face situations there are infinite families of non-similar tetrahedral tilers. Goldberg [7] described three such families, obtained by partitioning a triangular prism. In fact, Goldberg's first family was found by Sommerville [17] before, but he selected only special cases with a certain symmetry. Goldberg [7] noticed that even the general case admits a proper tiling of  $\mathbb{R}^3$ . Goldberg's first family also coincides with the family of simplices found by Hill [11], whose aim was to classify *rectifiable* simplices, that is, simplices that can be cut by straight cuts into finitely many pieces that can be rearranged to form a cube. The simplices in Goldberg's second and third families are obtained from the simplices in the first family by splitting into two congruent halves. According to Senechal's survey [14], no other space-filling tetrahedra than those described by Sommerville and Goldberg are known.

For general  $d$ , Debrunner [4] constructed  $\lfloor d/2 \rfloor + 2$  one-parameter families and a finite number of additional special types of  $d$ -dimensional simplices that tile  $\mathbb{R}^d$ . Smith [15] generalized Goldberg's construction and using Debrunner's ideas, he obtained  $(\lfloor d/2 \rfloor + 2)\phi(d)/2$  one-parameter families of space-filling  $d$ -dimensional simplices; here  $\phi(d)$  is the Euler's totient function. It is not known whether for some  $d$  there is an acute space-filling simplex or a two-parameter family of space-filling simplices [15].

In recent years the subject of tilings has received a certain impulse from computer graphics and other computer applications. In fact, our original motivation for studying simplices that are  $k$ -reptiles comes from a problem of probabilistic marking of Internet packets for IP traceback [1, 2]. See [12] for a brief summary of the ideas of this method. For this application, it would be interesting to find a  $d$ -dimensional simplex that is a  $k$ -reptile with  $k$  as small as possible.

For dimension 2 there are several possible types of  $k$ -reptile triangles, and they have been completely classified by Snover *et al.* [16]. In particular,  $k$ -reptile triangles exist for all  $k$  of the form  $a^2 + b^2$ ,  $a^2$  or  $3a^2$  for arbitrary integers  $a, b$ . In contrast, for  $d \geq 3$ , reptile simplices seem to be much more rare. The only known constructions of higher-dimensional  $k$ -reptile simplices have  $k = m^d$ . The best known examples are the *Hill sim-*

*plices* (or the *Hadwiger–Hill simplices*) [4, 9, 11]. A  $d$ -dimensional Hill simplex is the convex hull of vectors  $0, b_1, b_1 + b_2, \dots, b_1 + \dots + b_d$ , where  $b_1, b_2, \dots, b_d$  are vectors of equal length such that the angle between every two of them is the same and lies in the interval  $(0, \frac{\pi}{2} + \arcsin \frac{1}{d-1})$ .

Concerning nonexistence of  $k$ -reptile simplices in dimension  $d \geq 3$ , Hertel [10] proved that a 3-dimensional simplex is an  $m^3$ -reptile using a “standard” way of dissection (which we will not define here) if and only if it is a Hill simplex. He conjectured that Hill simplices are the only 3-dimensional reptile simplices. Herman Haverkort recently pointed us to an example of a  $k$ -reptile tetrahedron which is not Hill, which contradicts Hertel’s conjecture. In fact, except for the one-parameter family of Hill tetrahedra, three other space-filling tetrahedra described by Sommerville [17] and Goldberg [7] are also  $k$ -reptiles for every  $k = m^3$ . The simplices and their tiling are based on the barycentric subdivision of the cube. The construction can be naturally extended to find similar examples of  $d$ -dimensional  $k$ -reptile simplices for  $d \geq 4$  and  $k = m^d$ . Matoušek [12] showed that there are no 2-reptile simplices of dimension 3 or larger. For dimension  $d = 3$  Matoušek and the second author [13] proved the following theorem.

**Theorem 1.1 ([13]).** *In  $\mathbb{R}^3$ ,  $k$ -reptile simplices (tetrahedra) exist only for  $k$  of the form  $m^3$ , where  $m$  is a positive integer.*

We give a new, simple proof of Theorem 1.1 in Section 3.

Matoušek and the second author [13] conjectured that a  $d$ -dimensional  $k$ -reptile simplex can exist only for  $k$  of the form  $m^d$  for some positive integer  $m$ . We prove a weaker version of this conjecture for four-dimensional simplices.

**Theorem 1.2.** *Four-dimensional  $k$ -reptile simplices can exist only for  $k$  of the form  $m^2$ , where  $m$  is a positive integer.*

Four-dimensional Hill simplices are examples of  $k$ -reptile simplices for  $k = m^4$ . Whether there exists a four-dimensional  $m^2$ -reptile simplex for  $m$  non-square remains an open question.

## 2 Angles in simplices and Coxeter diagrams

Given a  $d$ -dimensional simplex  $S$  with vertices  $v_0, \dots, v_d$ , let  $F_i$  be the facet opposite to  $v_i$ . A *dihedral angle*  $\beta_{i,j}$  of  $S$  is the internal angle of the facets  $F_i$  and  $F_j$ , that meet at the  $(d - 2)$ -face  $F_i \cap F_j$ .

The *Coxeter diagram* of  $S$  is a graph  $c(S)$  with labeled edges such that the vertices of  $c(S)$  represent the facets of  $S$  and for every pair of facets  $F_i$  and  $F_j$ , there is an edge  $e_{i,j}$  labeled by the dihedral angle  $\beta_{i,j}$ .

The most important tool we use is Debrunner's lemma [4, Lemma 1], which connects the symmetries of a  $d$ -simplex with the symmetries of its Coxeter diagram (which represents the "arrangement" of the dihedral angles). This lemma allows us to substantially simplify the proof of Theorem 1.1 and enables us to step up by one dimension and prove Theorem 1.2, which seemed unmanageable before.

**Lemma 2.1 (Debrunner's lemma [4]).** *Let  $S$  be a  $d$ -dimensional simplex. The symmetries of  $S$  are in one-to-one correspondence with the symmetries of its Coxeter diagram  $c(S)$  in the following sense: each symmetry  $\varphi$  of  $S$  induces a symmetry  $\Phi$  of  $c(S)$  so that  $\varphi(v_i) = v_j \Leftrightarrow \Phi(F_i) = F_j$ , and vice versa.*

### 3 A simple proof of Theorem 1.1

We proceed as in the original proof, but instead of using the theory of scissor congruence, Jahnel's theorem about values of rational angles and Fiedler's theorem, we only use Debrunner's lemma (Lemma 2.1).

Assume for contradiction that  $S$  is a  $k$ -reptile tetrahedron where  $k$  is not a third power of a positive integer. A dihedral angle  $\alpha$  is called *indivisible* if it cannot be written as a linear combination of other dihedral angles in  $S$  with nonnegative integer coefficients. Call the edges of  $S$  (and of  $c(S)$ ) with dihedral angle  $\alpha$  the  $\alpha$ -edges.

The following lemmas are proved in [13].

**Lemma 3.1 ([13, Lemma 3.1]).** *If  $\alpha$  is an indivisible dihedral angle in  $S$ , then the  $\alpha$ -edges of  $S$  have at least three different lengths.*

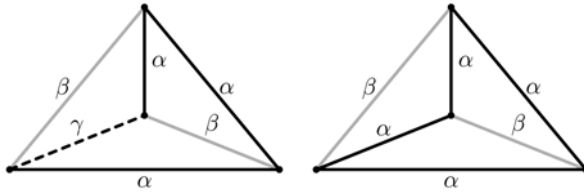
**Lemma 3.2 ([13, Lemma 3.3]).** *One of the following two possibilities occur:*

- (i) *All the dihedral angles of  $S$  are integer multiples of the minimal dihedral angle  $\alpha$ , which has the form  $\frac{\pi}{n}$  for an integer  $n \geq 3$ . If two  $\alpha$ -edges meet at a vertex  $v$  of  $S$ , then the third edge incident to  $v$  has dihedral angle  $\pi - \alpha$ .*
- (ii) *There are exactly two distinct dihedral angles  $\beta_1$  and  $\beta_2$ , each of them occurring three times in  $S$ .*

First we exclude case (ii) of Lemma 3.2. The two triples of edges with dihedral angles  $\beta_1$  and  $\beta_2$  form either a triangle and a claw, or two paths of length three. In both cases, for each  $i \in \{1, 2\}$ , the Coxeter diagram of  $S$  has at least one nontrivial symmetry which swaps two distinct edges with label  $\beta_i$ . By Debrunner's lemma, the corresponding symmetry of  $S$  swaps two distinct edges with dihedral angle  $\beta_i$ , which thus have the

same length. But then the edges with dihedral angle  $\beta_i$  have at most 2 different lengths and this contradicts Lemma 3.1, since the smaller of the two angles  $\beta_1, \beta_2$  is indivisible.

Now we exclude case (i). Since there are at least three  $\alpha$ -edges in  $S$ , there is a vertex  $v$  of  $S$  where two  $\alpha$ -edges meet. Let  $\beta = \pi - \alpha$ . We distinguish several cases depending on the subgraph  $H_\alpha$  of  $c(S)$  formed by the  $\alpha$ -edges.



**Figure 3.1.** The  $\alpha$ -edges form a path (left) or a four-cycle (right) in  $c(S)$ .

- $H_\alpha$  contains three edges incident to a common vertex (which correspond to a triangle in  $S$ ). Then all the other edges must have the angle  $\beta$  and we get the configuration with three-fold symmetry, which we excluded in case (ii).
- $H_\alpha$  contains a triangle (the corresponding edges in  $S$  meet at a single vertex). Then  $\beta = \alpha$ , and thus  $\alpha = \frac{\pi}{2}$ , which contradicts Lemma 3.2 (i).
- $H_\alpha$  is a path of length three (this corresponds to a path in  $S$ , too). Then two edges have the angle  $\beta > \alpha$  and the remaining edge has some angle  $\gamma \neq \alpha$ . See Figure 3.1 (left). The resulting Coxeter diagram has a nontrivial involution swapping two  $\alpha$ -edges. By Debrunner's lemma, this contradicts Lemma 3.1.
- It remains to deal with the case where  $H_\alpha$  is a four-cycle (which corresponds to a four-cycle in  $S$ ). In this case the remaining two edges have dihedral angle  $\beta$ , so the Coxeter diagram has a dihedral symmetry group  $D_4$  acting transitively on the  $\alpha$ -edges. By Debrunner's lemma, all the  $\alpha$ -edges have the same length. This again contradicts Lemma 3.1.

We obtained a contradiction in each of the cases, hence the proof of Theorem 1.1 is finished.

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# Homogeneous selections from hyperplanes

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**Abstract.** Given  $d + 1$  hyperplanes  $h_1, \dots, h_{d+1}$  in general position in  $\mathbb{R}^d$ , let  $\Delta(h_1, \dots, h_{d+1})$  denote the unique bounded simplex enclosed by them. There exists a constant  $c(d) > 0$  such that for any finite families  $H_1, \dots, H_{d+1}$  of hyperplanes in  $\mathbb{R}^d$ , there are subfamilies  $H_i^* \subset H_i$  with  $|H_i^*| \geq c(d)|H_i|$  and a point  $p \in \mathbb{R}^d$  with the property that  $p \in \Delta(h_1, \dots, h_{d+1})$  for all  $h_i \in H_i^*$ .

## 1 The main result

Throughout this paper, let  $H_1, \dots, H_{d+1}$  be finite families of hyperplanes in  $\mathbb{R}^d$  in general position. That is, we assume that (1) no element of  $\cup_{i=1}^{d+1} H_i$  passes through the origin, (2) any  $d$  elements have precisely one point in common, and (3) no  $d + 1$  of them have a nonempty intersection. A transversal to these families is an ordered  $(d + 1)$ -tuple  $h = (h_1, \dots, h_{d+1}) \in \prod_{i=1}^{d+1} H_i$ , where  $h_i \in H_i$  for every  $i$ .

Given hyperplanes  $h_1, \dots, h_{d+1} \subset \mathbb{R}^d$  in general position in  $\mathbb{R}^d$ , there is a unique simplex denoted by  $\Delta = \Delta(h_1, \dots, h_{d+1})$  whose boundary is contained in  $\cup_{i=1}^{d+1} h_i$ . For simpler writing we let  $[n]$  stand for the set  $\{1, 2, \dots, n\}$ . Our main result is the following.

**Theorem 1.1.** *For any  $d \geq 1$ , there is a constant  $c(d) > 0$  with the following property. Given finite families  $H_1, \dots, H_{d+1}$  of hyperplanes in  $\mathbb{R}^d$  in general position, there are subfamilies  $H_i^* \subset H_i$  with  $|H_i^*| \geq c(d)|H_i|$  for  $i = 1, \dots, d+1$  and a point  $p \in \mathbb{R}^d$  such that  $p$  is contained in  $\Delta(h)$  for every transversal  $h \in \prod_{i=1}^{d+1} H_i$ .*

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It follows from the general position assumption that the simplices  $\Delta(h)$  in Theorem 1.1 also have an interior point in common.

It will be convenient to use the language of hypergraphs. Let  $\mathcal{H} = \mathcal{H}(H_1, \dots, H_{d+1})$  be the complete  $(d+1)$ -partite hypergraph with vertex classes  $H_1, \dots, H_{d+1}$ . We refer to  $\mathcal{H}$  as the *hyperplane hypergraph*, or *h-hypergraph* associated with the hyperplane families  $H_1, \dots, H_{d+1}$ . The hyperedges of  $\mathcal{H}$  are the transversals of the families  $H_1, \dots, H_{d+1}$ . Our main result can now be reformulated as follows.

**Theorem 1.2.** *For every positive integer  $d$ , there is a constant  $c(d) > 0$  with the following property. Every complete  $(d+1)$ -partite h-hypergraph  $\mathcal{H}(H_1, \dots, H_{d+1})$  contains a complete  $(d+1)$ -partite h-subhypergraph  $\mathcal{H}^*(H_1^*, \dots, H_{d+1}^*)$  such that  $|H_i^*| \geq c(d)|H_i|$  for all  $i \in [d+1]$  and  $\bigcap_{h \in \mathcal{H}^*} \Delta(h) \neq \emptyset$ .*

In some sense, our theorem extends the following recent and beautiful result of Karasev [5].

**Theorem 1.3.** [5] *Assume  $r$  is a prime power and  $t \geq 2r - 1$ . Let  $\mathcal{H}$  be a complete  $(d+1)$ -partite h-hypergraph with partition classes of size  $t$ . Then there are vertex-disjoint hyperedges (transversals)  $h^1, \dots, h^r$  of  $\mathcal{H}$  such that  $\bigcap_{j=1}^r \Delta(h^j) \neq \emptyset$ .*

Two hyperedges (transversals)  $h$  and  $h'$  of  $\mathcal{H}$  are vertex-disjoint if  $h_i$  and  $h'_i$  are distinct for each  $i$ .

Our Theorem 1.1 implies a weaker version of Karasev's theorem. Namely, the same conclusion holds with arbitrary  $r$  and  $t \geq r/c(d)$ . Since  $c(d)$  will turn out to be doubly exponential in  $d$ , our result is quantitatively much weaker than the bound  $t \geq 4r$  that follows from Karasev's theorem for any  $r$ .

Theorem 1.3 is a kind of dual to Tverberg's famous theorem [8]. In the same sense, our result is dual to the homogeneous point selection theorem of Pach [7] (see also [6]), which guarantees the existence of an absolute constant  $c_d > 0$  with the following property. Let  $X_1, \dots, X_{d+1}$  be finite sets of points in general position in  $\mathbb{R}^d$  with  $|X_i| = n$  for every  $i$ . Then there exist subsets  $X_i^* \subset X_i$  of size at least  $c_d n$  for every  $i \in [d+1]$  and a point  $p \in \mathbb{R}^d$  such that  $p \in \text{conv}\{x_1, \dots, x_{d+1}\}$  for all transversals  $(x_1, \dots, x_{d+1}) \in \prod_{i=1}^{d+1} X_i^*$ . Here the assumption that the sets  $X_i$  are of the same size can be removed (see e.g. [4]).

To establish Theorem 1.2, we need some preparation. Let  $h$  be an edge of the h-hypergraph  $\mathcal{H} = \mathcal{H}(H_1, \dots, H_{d+1})$ . The simplex  $\Delta(h)$  is the

convex hull of the points

$$v_i = \bigcap_{j \neq i} h_j, \quad i \in [d + 1].$$

Similarly, for  $h' \in \mathcal{H}$  the vertices of  $\Delta(h')$  are  $v'_1, \dots, v'_{d+1}$ . Here  $v_i$  (and  $v'_i$ ) is the vertex of  $\Delta(h)$  (and  $\Delta(h')$ ) opposite to the facet contained in  $h_i$  (and  $h'_i$ , respectively). The edges  $h$  and  $h'$  are said to be of the same type if, for each  $i \in [d + 1]$ , the vertices  $v_i$  and  $v'_i$  are not separated by either of the hyperplanes  $h_i$  and  $h'_i$ . We say that the  $h$ -hypergraph  $\mathcal{H}$  is homogeneous if every pair of its edges is of the same type.

The heart of the proof of Theorem 1.2 is the following “same type lemma” for hyperplanes (cf. [2])

**Lemma 1.4.** *For any  $d \geq 1$ , there exists a constant  $b(d) > 0$  with the following property. Every complete  $(d + 1)$ -partite  $h$ -hypergraph  $\mathcal{H}(H_1, \dots, H_{d+1})$  contains a complete  $(d + 1)$ -partite subhypergraph  $\mathcal{H}^*(H_1^*, \dots, H_{d+1}^*)$  with  $|H_i^*| \geq b(d)|H_i|$  for all  $i \in [d + 1]$  which is homogeneous.*

In this extended abstract we show how our main result, Theorem 1.1 follows from the same type lemma for hyperplanes. The proof of the latter appears in the full version of the paper. Actually, there are two separate proofs there. The first one, which provides a better estimate for the value of the constant  $b(d)$ , uses duality and is based on the original same type lemma (for points), [2] (see also [6]). The second proof is shorter, and it utilizes a far reaching generalization of the same type lemma to semialgebraic relations of several variables, found by Fox, Gromov, Lafforgue, Naor, and Pach [4], see also Bukh and Hubard [3] for a quantitative form. The same result for binary semialgebraic relations was first established by Alon, Pach, Pinchasi, Radoičić, and Sharir [1].

## 2 Proof of Theorem 1.2

Here we deduce Theorem 1.2 from Lemma 1.4.

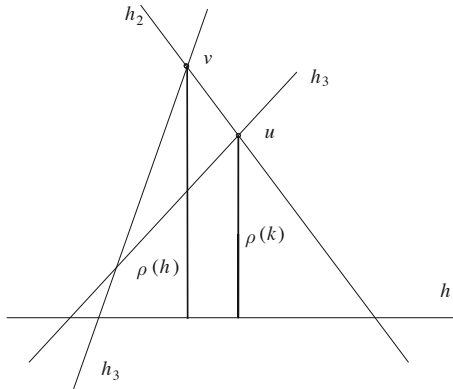
Let  $\mathcal{H}^*$  denote the complete  $(d + 1)$ -partite subhypergraph of  $\mathcal{H}$  whose existence is guaranteed by the lemma. For a fixed  $h \in \mathcal{H}^*$  let  $h_i^+$  denote the half-space bounded by  $h_i$  that contains vertex  $v_i$  of  $\Delta(h)$ , for  $i \in [d + 1]$ . The lemma implies that, for every hyperedge  $k = (k_1, \dots, k_{d+1}) \in \mathcal{H}^*$  and for every  $i$ , the half-space  $h_i^+$  contains the vertex  $u_i$  of  $\Delta(k)$  opposite to hyperplane  $k_i$ . To prove the theorem, it suffices to establish the following claim.

$$\bigcap_{h \in \mathcal{H}^*} \Delta(h) \neq \emptyset.$$

For  $h = (h_1, \dots, h_{d+1}) \in \mathcal{H}^*$ , let  $\rho(h)$  denote the distance between  $h_1$  and  $v_1 = \cap_2^{d+1} h_j$ , and let  $h' \in \mathcal{H}^*$  be the edge for which  $\rho(h)$  is minimal. By the general position assumption, we have  $\rho(h') > 0$ . Set  $v' = \cap_2^{d+1} h'_j$ . We show that  $v' \in \Delta(h)$  for every  $h \in \mathcal{H}^*$ , which implies the claim. To see this, we have to verify that  $v' \in h_i^+$  for every  $h \in \mathcal{H}^*$  and for every  $i$ .

This is trivial for  $i = 1$ . Suppose that  $i \geq 2$ . By symmetry, we may assume that  $i = d + 1$ . We have to show that  $v' \in h_{d+1}^+$  for every  $h_{d+1} \in H_{d+1}^*$ .

Assume to the contrary that  $v' \notin h_{d+1}^+$  for some  $h_{d+1} \in H_{d+1}^*$ . Setting  $k = (h'_1, \dots, h'_d, h_{d+1})$ , we clearly have  $k \in \mathcal{H}^*$ . The simplices  $\Delta(k)$  and  $\Delta(h')$  share the vertex  $v_{d+1} = \cap_1^d h'_i$ . As  $v_{d+1} \in h_{d+1}^+$ , by the construction,  $v' \notin h_{d+1}^+$  implies that  $h_{d+1}$  intersects the segment  $[v_{d+1}, v']$  in a point  $u$  in its relative interior, see Figure 1. On the other hand, we know that  $u = \cap_2^{d+1} k_i$  is the vertex of  $\Delta(k)$  opposite to  $h_1 = k_1$ . Thus,  $u$  is closer to  $h_1 = k_1$  than  $v'$  is. Therefore, we obtain that  $\rho(k) < \rho(h')$ , contradicting the definition of  $h'$ .  $\square$



**Figure 2.1.** Illustration for Theorem 1.2,  $d = 2$ .

It follows from the above proof that Theorems 1.2 and 1.1 hold with  $c(d) = b(d)$ .

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# Conic theta functions and their relations to theta functions

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**Abstract.** It is natural to ask when the spherical volume defined by the intersection of a sphere at the apex of an integer polyhedral cone is rational. We use number theoretic methods to study a new class of polyhedral functions called conic theta functions, which are closely related to classical theta functions. We show that if  $K$  is a Weyl chamber for any finite reflection group, then its conic theta function lies in a graded ring of classical theta functions and in this sense is ‘almost’ modular. It is then natural to ask whether or not the conic theta functions are themselves modular, and we prove that (generally) they are not. In other words, we uncover some connections between the class of integer polyhedral cones that have a rational solid angle, and the class of conic theta functions that are almost modular.

The present investigations arose from an interest in studying the volume of a spherical polytope, also known as a solid angle, and extending the so-called Gram relations by use of conic theta functions [9]. We study the relationship between volumes of spherical polytopes, and ‘almost’ modular conic theta functions associated to them, by considering some connections between the following two a priori different problems:

**Problem 1.** Which lattice polyhedral cones  $K$  give rise to spherical polytopes with a rational volume?

**Problem 2.** Analyzing a certain conic theta function  $\Phi_K$  attached to a polyhedral cone  $K$ , how ‘close’ is  $\Phi_K$  to being modular?

We first recall some of the basic definitions from the combinatorial geometry of cones and the theory of modular forms. Suppose we are

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given a  $d$ -dimensional (simple) **polyhedral cone**, defined by

$$K := \left\{ \sum_{j=1}^d \lambda_j w_j \mid \text{all } \lambda_j \geq 0 \right\},$$

where the edges of the cone are some fixed set of  $d$  linearly independent vectors  $w_j \in \mathbb{R}^d$ . Such a cone  $K$  is called a **pointed cone**, and in the present work every cone has the origin as its vertex. One important special case of a polyhedral cone is the **positive orthant**, defined by  $K_0 := \{(x_1, \dots, x_d) \in \mathbb{R}^d \mid \text{each } x_j \geq 0\} := \mathbb{R}_{\geq 0}^d$ .

For each pointed cone  $K$ , we define its **conic theta function** by:

$$\Phi_{K, \mathcal{L}}(\tau) := \sum_{m \in \mathcal{L} \cap K} e^{\pi i \tau \|m\|^2}, \tag{1}$$

where  $\tau$  lies in the complex upper half plane  $\mathbb{H} := \{\tau := x + iy \mid x \in \mathbb{R}, y \in \mathbb{R}^+\} \subset \mathbb{C}$ , and where  $\mathcal{L}$  is a rank  $d$  lattice in  $\mathbb{R}^d$ . If  $\mathcal{L}$  is clear from the context, we will only write  $\Phi_K$ . The conic theta function  $\Phi_{K, \mathcal{L}}(\tau)$  given in (1) is reminiscent of the modular theta function

$$\theta(\tau) := \sum_{n \in \mathbb{Z}} e^{\pi i \tau n^2}, \tag{2}$$

a statement that we will make more precise in what follows. The function  $\theta(\tau)$  is a classical example of a **modular form**. Loosely speaking, a holomorphic modular form of integer weight  $k$  on a suitable subgroup  $\Gamma \subseteq \text{SL}_2(\mathbb{Z})$  is any holomorphic function  $f : \mathbb{H} \rightarrow \mathbb{C}$  satisfying  $f(\gamma\tau) = (c\tau + d)^k f(\tau)$  for all  $\gamma := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ , as well as a suitable growth condition in the cusps of  $\Gamma$ . It is well known that the **modular group**  $\text{SL}_2(\mathbb{Z})$  acts on  $\mathbb{H}$  by fractional linear transformation  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau := \frac{a\tau + b}{c\tau + d}$ , and so modular forms can be thought of as complex analytic functions that obey a certain symmetry with respect to this action.

To see where the conic theta function naturally comes from, we let  $S^{d-1}$  be the unit sphere centered at the origin. We define the **solid angle**  $\omega_K$  at the vertex of  $K$  by:

$$\omega_K := \frac{\text{vol}(K \cap S^{d-1})}{\text{vol}(S^{d-1})}. \tag{3}$$

In other words,  $\omega_K$  is the normalized volume of a  $(d - 1)$ -dimensional spherical polytope. With this normalization, we note that  $0 \leq \omega_K \leq 1$ , and that in two dimensions we have  $\omega_K = \theta/2\pi$ , where  $\theta$  is the



usual 2-dimensional angle, measured in radians, at the vertex of the 2-dimensional cone  $K$ . It is an elementary fact that

$$\omega_K = \int_K e^{-\pi\|x\|^2} dx, \tag{4}$$

We note that when  $K$  is replaced by all of Euclidean space, the integral (4) becomes  $\int_{\mathbb{R}^d} e^{-\pi\|x\|^2} dx = 1$ , confirming that we do indeed have the proper normalization  $0 \leq \omega_K \leq 1$ . For more information about rational pointed cones, and connections between discrete volumes of polytopes and local spherical angle contributions, the reader may consult [11] and [3]. The papers [7, 8], provide further background for solid angles and their relations.

Thus, the foregoing discussion shows that a strong motivation for defining the conic theta function  $\Phi_K(\tau)$  is that it is essentially a discrete, Riemann sum approximation to the integral definition of the volume  $\omega_K$  of a spherical polytope, as defined by (4). We will make precise sense of this intuition in the following section, which will be used later to consider carefully the putative expansion of  $\Phi_K(\tau)$  at the cusp  $\tau = 0$ .

Conic theta functions are also related to the representation numbers of quadratic forms, a link which we explicate here. From the above definitions, and using the lattice  $\mathcal{L} := A(\mathbb{Z}^d)$ , it is immediate that

$$\Phi_{K,\mathcal{L}}(\tau) = \sum_{m \in \mathcal{L} \cap K} e^{\pi i \tau \|m\|^2} = \sum_{m \in \mathbb{Z}^d \cap K} e^{\pi i \tau (m^t (A^t A) m)} = \sum_{k=0}^{\infty} a(k) q^{k/2}, \tag{5}$$

where  $q = e^{2\pi i \tau}$ , and  $a(k) := \#\{m \in \mathbb{Z}^d \mid m^t (A^t A) m = k, \text{ and } k \in K\}$ . This combinatorial interpretation of the Fourier coefficients tells us that the  $k$ 'th Fourier coefficient is the number of ways to represent the integer  $k$  by the quadratic form  $m^t (A^t A) m$ , while  $m$  is simultaneously constrained to satisfy the finite system of linear inequalities defined by the polyhedral cone  $K$ .

Another very recent analysis of cones from a different perspective takes place in [5]. The authors of [5] define certain zeta functions attached to polyhedral cones and analyze conical zeta values as a geometric generalization of the celebrated multiple zeta values.

Here, we endeavor to show that the conic theta function  $\Phi_K(\tau)$  is never a modular form. However, when  $K$  is a Weyl chamber of a finite reflection group, we show that  $\Phi_K$  nevertheless belongs to a certain graded ring of theta functions (see Theorem 5 below), and it is in this sense that  $\Phi_K$  is ‘almost’ modular. We next give the simplest family of examples of conic theta functions, arising from the positive orthant in each dimension.

**Example 3.** For the conic theta function of the positive orthant  $K_0 := \mathbb{R}_{\geq 0}^d$  (and  $\mathcal{L} = \mathbb{Z}^d$ ), we claim that

$$\Phi_{K_0}(\tau) = \frac{1}{2^d} (\theta(\tau) + 1)^d, \tag{6}$$

where  $\theta(\tau) := \sum_{n \in \mathbb{Z}} e^{\pi i \tau n^2}$ , the classical weight  $1/2$  modular form. In particular,

$$\Phi_{K_0}(\tau) = \frac{1}{2^d} \sum_{k=0}^d \binom{d}{k} \theta^k(\tau), \tag{7}$$

a linear combination of modular forms of distinct weights, with nonzero coefficients, and hence  $\Phi_{K_0}$  is not a modular form.

To see (7), we begin with the case in which the dimension of  $K_0$  equals 1, so that here  $K_0 := \mathbb{R}_{\geq 0}$ . We note that  $1 + \theta(\tau) = 1 + \sum_{n \in \mathbb{Z}} e^{\pi i \tau n^2} = 2 + 2 \sum_{n \geq 1} e^{\pi i \tau n^2}$ . Therefore,  $1 + \theta(\tau) = 2\Phi_{\mathbb{R}_{\geq 0}}(\tau)$ . Finally, the relation  $\Phi_{K_0}(\tau) = \Phi_{\mathbb{R}_{\geq 0}}^d(\tau)$  gives us the desired expansion (6) above.  $\square$

We notice that the positive orthant possesses a lot of symmetry, so it is natural to ask if other cones with less symmetry might not be modular, and for which cones  $K$  might we get a phenomenon similar in spirit to the example above, in the sense that  $\Phi_K$  could be written as a linear combination of “classical” theta functions attached to lower-dimensional lattices.

We also note that, almost by definition, every cone  $K$  which is a Weyl chamber necessarily has a *rational* solid angle  $\omega_K = \frac{1}{|\overline{W}|}$ , because the Weyl group  $W$  tiles  $\mathbb{R}^d$  with isometric copies of the cone  $K$ . For each even integral lattice  $\mathcal{L}$ , we define its usual theta function by:

$$\Theta_{\mathcal{L}}(\tau) := \sum_{n \in \mathcal{L}} e^{\pi i \tau \|n\|^2},$$

where  $\tau$  lies in the upper half plane  $H$ . Next, we quote the standard fact that when  $\mathcal{L}$  is an even integral lattice, the theta function  $\Theta_{\mathcal{L}}(\tau)$  turns out to be a modular form, of weight  $\frac{d}{2}$  and level  $N$ , where  $N$  is the smallest positive integer  $M$  such that  $M(A^t A)^{-1}$  is also even integral.

**Example 4.** Consider the 2-dimensional root system defined by

$$S := \{(1, 1), (-1, 1), (1, -1), (-1, -1), (2, 0), (-2, 0), (0, 2), (0, -2)\},$$

so that we have the root lattice  $\mathcal{L}_{root} := \{m(1, 1) + n(2, 0) \mid m, n \in \mathbb{Z}\}$ . One fundamental domain for this group action on  $\mathbb{R}^2$  is the polyhedral

cone  $K$  whose edge vectors are the roots  $(1, 1)$  and  $(2, 0)$ , and whose (solid) angle is  $\omega = \frac{1}{8}$ . Here, the conic theta function is

$$\Phi_{K, \mathcal{L}_{root}}(\tau) = \sum_{m \geq 0, n \geq 0} e^{\pi i \tau \|m(2,0) + n(1,1)\|^2} \tag{8}$$

$$= \sum_{m \geq 0, n \geq 0} e^{\pi i \tau (4m^2 + 4mn + 2n^2)}. \tag{9}$$

We arrive at the following representation of  $\Phi_{K, \mathcal{L}_{root}}(\tau)$  as a nontrivial rational linear combination of classical theta functions:

$$\begin{aligned} \Phi_{K, \mathcal{L}_{root}}(\tau) &= \frac{3}{8} + \frac{1}{4} \sum_{k \in \mathbb{Z}} e^{\pi i \tau (2k^2)} + \frac{1}{4} \sum_{k \in \mathbb{Z}} e^{\pi i \tau (4k^2)} \\ &+ \frac{1}{8} \sum_{(m,n) \in \mathbb{Z}^2} e^{\pi i \tau (4m^2 + 4mn + n^2)}. \end{aligned} \tag{10}$$

Therefore we see that for this example,  $\Phi_{K, \mathcal{L}_{root}}(\tau)$  is a nontrivial rational linear combination of theta functions of different weights. In particular,  $\Phi_{K, \mathcal{L}_{root}}(\tau)$  is not modular. □

We define  $R$  to be the ring of all finite, rational linear combinations of theta functions  $\Theta_{\mathcal{L}}$ , for any  $d$ -dimensional even integral lattice  $\mathcal{L} \subset \mathbb{R}^d$ , varying over all dimensions  $d$ . The ring  $R$  has a natural grading, namely it is graded by the weight  $k = \frac{d}{2}$  of the relevant theta functions  $\Theta_{\mathcal{L}}$ , for each rank  $d$  lattice  $\mathcal{L} \subset \mathbb{R}^d$ . Equivalently, we may also grade  $R$  by the dimension  $d$  of the lattices  $\mathcal{L} \subset \mathbb{R}^d$ , as  $d$  varies over the positive integers. When a conic theta function lies in  $R$ , it is ‘almost modular’. The two main results here are the following.

**Theorem 5.** *If the polyhedral cone  $K$  is the Weyl chamber of a finite reflection group  $W$ , then the conic theta function  $\Phi_{K, 2\mathcal{L}_{root}}(\tau)$  is in the graded ring  $R$ .*

On the other hand, we also have the following result.

**Theorem 6.** *Suppose that the polyhedral cone  $K \subset \mathbb{R}^d$  has the solid angle  $\omega_K$  at its vertex, located at the origin, and that  $\mathcal{L} := A(\mathbb{Z}^d)$  is an even integral lattice of full rank. If  $\frac{\omega_K}{|\det A|}$  is irrational, then  $\Phi_{K, \mathcal{L}}(\tau)$  is not a modular form of weight  $k$  on any congruence subgroup, and for any  $k \in \frac{1}{2}\mathbb{Z}$ ,  $k \geq \frac{1}{2}$ .*

The tool that we use for the proof of Theorem 6 is the “ $q$ -expansion principle”, due to Deligne and Rapoport [2, Théorème 3.9, p.304], which tells us that if an integer weight modular form  $f$  has rational Fourier coefficients at the cusp  $i\infty$ , then the Fourier expansion of  $f$  at all other cusps must also have rational coefficients.

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# The Carathéodory number of the $P_3$ convexity of chordal graphs

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**Abstract.** If  $S$  is a set of vertices of a graph  $G$ , then the convex hull of  $S$  in the  $P_3$ -convexity of  $G$  is the smallest set  $H_G(S)$  of vertices of  $G$  that contains  $S$  such that no vertex in  $V(G) \setminus H_G(S)$  has at least two neighbors in  $S$ . The Carathéodory number of the  $P_3$  convexity of  $G$  is the smallest integer  $c$  such that for every set  $S$  of vertices of  $G$  and every vertex  $u$  in  $H_G(S)$ , there is a set  $F \subseteq S$  with  $|F| \leq c$  and  $u \in H_G(F)$ . We describe a polynomial time algorithm to determine the Carathéodory number of the  $P_3$  convexity of a chordal graph.

## 1 Introduction

Graph convexities are a well studied topic. For a finite, simple, and undirected graph  $G$  with vertex set  $V(G)$ , a *graph convexity* on  $V(G)$  is a collection  $\mathcal{C}$  of subsets of  $V(G)$  such that

- $\emptyset, V(G) \in \mathcal{C}$  and
- $\mathcal{C}$  is closed under intersections.

The sets in  $\mathcal{C}$  are called *convex sets* and the *convex hull* in  $\mathcal{C}$  of a set  $S$  of vertices of  $G$  is the smallest set  $H_{\mathcal{C}}(S)$  in  $\mathcal{C}$  containing  $S$ .

Several well known graph convexities  $\mathcal{C}$  are defined using some set  $\mathcal{P}$  of paths of the underlying graph  $G$ . In this case, a subset  $S$  of vertices of  $G$  is convex, that is, belongs to  $\mathcal{C}$ , if for every path  $P$  in  $\mathcal{P}$  whose endvertices belong to  $S$  also every vertex of  $P$  belongs to  $S$ . When  $\mathcal{P}$  is the set of all shortest paths in  $G$ , this leads to the *geodetic convexity*

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[2, 8, 12, 14]. The *monophonic convexity* is defined by considering as  $\mathcal{P}$  the set of all induced paths of  $G$  [9, 10, 13]. The set of all paths of  $G$  leads to the *all path convexity* [7]. Similarly, if  $\mathcal{P}$  is the set of all triangle paths in  $G$ , then  $\mathcal{C}$  is the *triangle path convexity* [6]. Here we consider the  $P_3$  convexity of  $G$ , which is defined when  $\mathcal{P}$  is the set of all paths of length two. The  $P_3$  convexity was first considered for directed graphs [11, 15–17]. For undirected graphs, the  $P_3$  convexity was studied in [1, 4, 5].

A famous result about convex sets in  $\mathbb{R}^d$  is *Carathéodory's theorem* [3]. It states that every point  $u$  in the convex hull of a set  $S \subseteq \mathbb{R}^d$  lies in the convex hull of a subset  $F$  of  $S$  of order at most  $d + 1$ . Let  $G$  be a graph and let  $\mathcal{C}$  be a graph convexity on  $V(G)$ . The *Carathéodory number* of  $\mathcal{C}$  is the smallest integer  $c$  such that for every set  $S$  of vertices of  $G$  and every vertex  $u$  in  $H_{\mathcal{C}}(S)$ , there is a set  $F \subseteq S$  with  $|F| \leq c$  and  $u \in H_{\mathcal{C}}(F)$ . A set  $S$  of vertices of  $G$  is a *Carathéodory set* of  $\mathcal{C}$  if the set  $\partial H_{\mathcal{C}}(S)$  defined as  $H_{\mathcal{C}}(S) \setminus \bigcup_{u \in S} H_{\mathcal{C}}(S \setminus \{u\})$  is not empty. This notion allows an alternative definition of the *Carathéodory number* of  $\mathcal{C}$  as the largest cardinality of a Carathéodory set of  $\mathcal{C}$ .

The Carathéodory number was determined for several graph convexities. The Carathéodory number of the monophonic convexity of a graph  $G$  is 1 if  $G$  is complete and 2 otherwise [10]. The Carathéodory number of the triangle path convexity of  $G$  is 2 whenever  $G$  has at least one edge [6]. It is known that the maximum Carathéodory number of the  $P_3$  convexity of a multipartite tournament is 3 [16]. Some general results concerning the Carathéodory number of the  $P_3$  convexity are shown in [1]. On the one hand, [1] contains efficient algorithms to determine the Carathéodory number of the  $P_3$  convexity of trees and, more generally, block graphs. On the other hand, it is NP-hard to determine the Carathéodory number of the  $P_3$  convexity of bipartite graphs [1].

In the present extended abstract we exclusively study the Carathéodory number of  $P_3$  convexities of graphs. Since a graph  $G$  uniquely determines its  $P_3$  convexity  $\mathcal{C}$ , we speak of a Carathéodory set of  $G$  and the Carathéodory number  $c(G)$  of  $G$ . Furthermore, we write  $H_G(S)$  and  $\partial H_G(S)$  instead of  $H_{\mathcal{C}}(S)$  and  $\partial H_{\mathcal{C}}(S)$ , respectively.

Our result is a polynomial time algorithm for the computation of the Carathéodory number of a chordal graph, which extends results from [1].

## 2 Results

The following result from [5] plays a central role in our approach.

**Theorem 2.1 (Centeno et al. [5]).** *If  $u$  and  $v$  are two vertices at distance at most 2 in a 2-connected chordal graph  $G$ , then  $H_G(\{u, v\}) = V(G)$ .*

Let  $G$  be a connected chordal graph. Let  $r$  be a vertex of  $G$ . Let the graph  $G'$  arise from  $G$  by adding edges between all pairs of vertices of  $G$  that lie in the same block of  $G$ . Let  $T$  be the breadth first search tree of  $G'$  rooted in  $r$ .

For a vertex  $u$  of  $G$ , let  $V(u)$  denote the set of vertices of  $G$  that are either  $u$  or a descendant of  $u$  in  $T$ . By the definition of  $T$ , we have that  $V(r) = V(G)$  and if  $u$  is a vertex of  $G$  that is distinct from  $r$ , then  $V(u)$  is the union of  $\{u\}$  and all vertex sets of components of  $G$  that do not contain  $r$ . Furthermore, the set of children of  $u$  in  $T$  is the set of all vertices of  $G$  that belong to  $V(u)$  and lie in a common block of  $G$  with  $u$ . Note that a vertex of  $G$  that is distinct from  $r$  is a leaf of  $T$  if and only if it is no cut vertex of  $G$ .

In order to determine the Carathéodory number of  $G$ , we consider the following three values for every vertex  $u$  of  $G$ :

- $c_{(G,r)}(u)$  is the maximum cardinality of a set  $S$  with
  - $S \subseteq V(u)$  and
  - $u \in \partial H_{G[V(u)]}(S)$ .
  
- $c'_{(G,r)}(u)$  is the maximum cardinality of a set  $S$  with
  - $S \subseteq V(u)$ ,
  - $u \notin H_{G[V(u)]}(S)$ ,
  - $|H_{G[V(u)]}(S) \cap N_G(u)| = 1$ , and
  - $H_{G[V(u)]}(S) \cap N_G(u) \subseteq \partial H_{G[V(u)]}(S)$ .
  
- $c''_{(G,r)}(u)$  is the maximum cardinality of a set  $S$  with
  - $S \subseteq V(u)$ ,
  - $u \in H_{G[V(u)]}(S)$ , and
  - $H_{G[V(u)]}(S) \cap N_G[u] \subseteq \partial H_{G[V(u)]}(S)$ .

Let  $S_{(G,r)}(u)$ ,  $S'_{(G,r)}(u)$ , and  $S''_{(G,r)}(u)$  denote sets of maximum cardinality satisfying the conditions in the above definitions of  $c_{(G,r)}(u)$ ,  $c'_{(G,r)}(u)$ , and  $c''_{(G,r)}(u)$ , respectively, that is, for instance,  $c_{(G,r)}(u) = |S_{(G,r)}(u)|$ . Note that if  $u$  is a leaf of  $T$ , then no set as in the definition of  $c'_{(G,r)}(u)$  exists. In this case, let  $c'_{(G,r)}(u) = -\infty$  and let  $S'_{(G,r)}(u)$  be undefined.

The following lemma describes recursions for  $c_{(G,r)}(u)$ ,  $c'_{(G,r)}(u)$ , and  $c''_{(G,r)}(u)$ , which allow their efficient recursive computation.

**Lemma 2.2.** *Let  $G$ ,  $r$ , and  $T$  be as above. Let  $u$  be a vertex of  $G$ .*

- (a)  $c_{(G,r)}(u)$  is the maximum of the following values:
- (i) 1.
  - (ii)  $c_{(G,r)}(v) + c_{(G,r)}(w)$ , where  $v$  and  $w$  are children of  $u$  in  $T$  and  $\text{dist}_G(v, w) = 2$ .
  - (iii)  $c_{(G,r)}(v) + c'_{(G,r)}(w)$ , where  $v$  and  $w$  are children of  $u$  in  $T$  and  $\text{dist}_G(v, w) = 1$ .
  - (iv)  $c''_{(G,r)}(v) + c''_{(G,r)}(w)$ , where  $v$  and  $w$  are children of  $u$  in  $T$  and  $\text{dist}_G(v, w) = 1$ .
- (b)  $c'_{(G,r)}(u)$  is the maximum of the following values:
- (i)  $-\infty$ .
  - (ii)  $c_{(G,r)}(u')$ , where  $u'$  is a child of  $u$  in  $T$  that is a neighbor of  $u$  in  $G$ .
- (c)  $c''_{(G,r)}(u)$  is the maximum of the following values:
- (i) 1.
  - (ii)  $c_{(G,r)}(v) + c_{(G,r)}(w)$ , where  $v$  and  $w$  are children of  $u$  in  $T$  such that  $v$  and  $w$  are no neighbors of  $u$  in  $G$  and  $\text{dist}_G(v, w) = 2$ .
  - (iii)  $c_{(G,r)}(v) + c'_{(G,r)}(w)$ , where  $v$  and  $w$  are children of  $u$  in  $T$  such that  $v$  is no neighbor of  $u$  in  $G$  and  $\text{dist}_G(v, w) = 1$ .
  - (iv)  $c''_{(G,r)}(v) + c''_{(G,r)}(w)$ , where  $v$  and  $w$  are children of  $u$  in  $T$  such that  $v$  and  $w$  are no neighbors of  $u$  in  $G$  and  $\text{dist}_G(v, w) = 1$ .

**Theorem 2.3.** *The Carathéodory number of a chordal graph can be determined in polynomial time.*

*Proof.* Let  $G$  be a chordal graph. Since the Carathéodory number of  $G$  is the maximum of the Carathéodory numbers of the components of  $G$ , we may assume that  $G$  is connected. Since  $c(G) = \max\{c_{(G,r)}(r) : r \in V(G)\}$ , it suffices to argue that, for every vertex  $r$  of  $G$ , the value  $c_{(G,r)}(r)$  can be determined in polynomial time. In fact, using the recursions given in Lemma 2.2, it is possible to determine  $c_{(G,r)}(r)$  in polynomial time calculating the values  $c_{(G,r)}(u)$ ,  $c'_{(G,r)}(u)$ , and  $c''_{(G,r)}(u)$  for all vertices  $u$  of  $G$  in a bottom up fashion along the corresponding breadth first search tree  $T$ . This completes the proof.  $\square$

While the running time of the algorithm described in the proof of Theorem 2.3 is obviously polynomial, it is possible to reduce it below some immediate estimates, because many values are essentially calculated several times, that is, there are many triples  $(r, s, u)$  of vertices of  $G$  with  $c_{(G,r)}(u) = c_{(G,s)}(u)$ ,  $c'_{(G,r)}(u) = c'_{(G,s)}(u)$ , and  $c''_{(G,r)}(u) = c''_{(G,s)}(u)$ .



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# Locally-maximal embeddings of graphs in orientable surfaces

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## 1 Introduction

The set of all orientable cellular embeddings of a graph has the intrinsic structure of adjacency between embeddings based on elementary operations on rotation schemes. Several types of elementary operations were considered in the past, usually in proofs of interpolation theorems: moving a single arc within its local rotation, moving both ends of an edge in the respective local rotations, and interchanging two arcs in the local rotation at a given vertex, see [2, 6, 7, 10]. We call these operations *rotation moves*. Each type of a rotation move gives rise to the structure of a *stratified graph* on the set of all embeddings of a given graph. Stratified graphs were studied by Gross, Rieper, and Tucker [5, 6, 8], although they were implicit already in the works of Duke [2] and Stahl [10]. Very little is known about stratified graphs in general, although their structure is crucial for understanding the entire system of all embeddings of a given graph. In the present paper we focus on embeddings that correspond to local maxima in stratified graphs. We call a cellular embedding of a graph into an orientable surface *locally maximal* if its genus cannot be raised by moving a single arc within its local rotation. Somewhat surprisingly, the concept of a locally-maximal embedding does not depend on which type of a move is taken as a basis for the stratified graph, indicating its important position in the hierarchy of graph embeddings between the minimum genus and the maximum genus.

The main results of this paper are (1) a characterisation of locally-maximal embeddings, (2) analysis of their relationship to the minimum and the maximum genus of a graph, and (3) a simple greedy 2-approxi-

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mation algorithm for maximum genus based on the notion of a locally-maximal embedding. Full proofs and additional results can be found in the forthcoming papers.

## 2 Fundamentals

The fundamental property of rotation moves is that any rotation move changes the number of faces of an embedding by  $-2$ ,  $0$ , or  $+2$ , changing the genus by  $-1$ ,  $0$ , or  $1$ . This property is well known and was used in several proofs of the interpolation theorem for the genus range of a graph.

The following result asserts that whenever a vertex  $v$  is incident with at least three faces, there is a rotation move at  $v$  that merges three faces into one, thus raising the genus. In this regard, all three types of rotation moves display a similar behaviour.

**Theorem 2.1.** *If a vertex  $v$  is incident with at least three faces of an embedding  $\Pi$ , then there exists a move of an arc at  $v$  that merges three faces at  $v$  into one while leaving all other faces of  $\Pi$  intact. There also exists an interchange of two arcs at  $v$  that merges three faces at  $v$  into one and leaves all other faces intact.*

A particularly useful corollary of our proof of Theorem 2.1 is that moving any arc lying on the boundary of two distinct faces into a corner belonging to a third face merges these three faces into one.

The next result is the cornerstone of our theory of locally-maximal embeddings as it shows that the concept of a locally-maximal embedding is independent on which type of a move is chosen as its basis. At the same time, it provides a characterisation of locally-maximal embeddings in terms of the multiplicity of vertex-face incidences.

**Theorem 2.2.** *For any orientable embedding  $\Pi$  of a connected graph  $G$  the following statements are equivalent.*

- (i) *The embedding  $\Pi$  is locally maximal.*
- (ii) *The genus of  $\Pi$  cannot be raised by interchanging two arcs in a local rotation.*
- (iii) *The genus of  $\Pi$  cannot be raised by moving any edge in the rotation of  $\Pi$ .*
- (iv) *Every vertex of  $G$  is incident with at most two faces of  $\Pi$ .*

A natural question about locally-maximal embeddings concerns their distribution within the embedding range. We therefore define the *locally-maximal genus* of a graph  $G$ ,  $\gamma_L(G)$ , as the minimum among the genera of all locally-maximal embeddings of  $G$ . For a graph  $G$  we fur-

then denote by  $\mu(G)$  the largest number of pairwise disjoint circuits contained in  $G$ ; this number is sometimes known as the *cycle packing number* of  $G$ . We also define the *reduced Betti number*  $\beta'(G)$  by setting  $\beta'(G) = \beta(G) - \mu(G)$ ; note that the reduced Betti number is nonnegative since  $\mu(G) \leq \beta(G)$ . The importance of these two invariants is explained by the following theorem.

**Theorem 2.3.** *The maximum number of faces in a locally-maximal embedding of a graph  $G$  does not exceed  $\mu(G) + 1$  or  $\mu(G)$ , depending on whether  $\beta'(G)$  is even or odd, respectively.*

The previous result enables us to prove that the relationship between the Betti number, the reduced Betti number, and the genus parameters of a graph is governed by the following inequalities.

**Theorem 2.4.** *The following inequalities hold for every connected graph  $G$ :*

- (i)  $\gamma(G) \leq \beta'(G)/2 \leq \gamma_L(G) \leq \gamma_M(G) \leq \beta(G)/2$
- (ii)  $\beta'(G)/2 \leq \gamma_L(G) \leq \gamma_M(G) \leq \beta'(G)$
- (iii)  $\gamma_M(G)/2 \leq \gamma_L(G) \leq \gamma_M(G)$

Graphs  $G$  for which  $\gamma_M(G) = \lfloor \beta(G)/2 \rfloor$  are known as *upper-embeddable* graphs. With this analogy in mind we define a graph  $G$  to be *lower-embeddable* if  $\gamma_L(G) = \lceil \beta'(G)/2 \rceil$ . As with upper-embeddable graphs, many important classes of graphs are lower-embeddable.

**Theorem 2.5.** *All graphs in the following classes are lower embeddable: complete graphs  $K_n$  for all  $n \geq 1$ , complete bipartite graphs  $K_{m,n}$  for all  $m, n \geq 1$ , complete equipartite graphs  $K_{n,\dots,n}$  for all  $n \geq 1$ , and hypercubes  $Q_n$  for all  $n \geq 1$ .*

A connected graph is called a *cycle-tree* if any two of its cycles are vertex disjoint. It turns out that planar locally-maximal embeddings admit a simple characterisation: A connected graph  $G$  has a planar locally-maximal embedding if and only if  $G$  is a cycle-tree.

### 3 Constructions

The proof of the next theorem is based on the well-known edge-addition technique of raising the genus by adding a pair of adjacent edges; for technical details of the method see for example [9], [3], or [1].

We need the following definitions to state and prove our result. A cycle-tree graph is called a *k-cycle-tree* if it contains exactly  $k$  cycles. A component of a graph is called *even* if it has even number of edges.

A face of an embedding is *spanning* if it is incident with each vertex of the graph. The idea of a spanning face was used in [1] to construct embeddings with genus  $\gamma_M(G) - 1$ ;

**Theorem 3.1.** *Let  $G$  be a connected graph. If  $G$  has a spanning  $k$ -cycle-tree  $S$  such that  $G - E(S)$  has only even components, then  $G$  has a locally-maximal embedding with  $k + 1$  faces. In particular,  $\gamma_L(G) \leq (\beta(G) - k)/2$ .*

*Sketch of a proof.* Since each component of  $G - E(S)$  is even,  $G - E(S)$  has a partition  $\mathcal{P}$  into pairs of adjacent edges (see for example [9, Lemma 4]). Let us arrange the pairs from  $\mathcal{P}$  into a linear order, and let  $\{e_i, f_i\}$  be the  $i$ -th pair. Consider the graphs

$$\begin{aligned} G_0 &= S, \\ G_i &= G_{i-1} \cup \{e_i, f_i\} \quad \text{for } i \geq 1. \end{aligned}$$

Assuming that the number of pairs in  $\mathcal{P}$  is  $n$ , we get  $G_n = G$ . The proof is finished by using the edge-addition technique and employing induction to prove that for each  $i \in \{0, 1, \dots, n\}$  the graph  $G_i$  has a locally-maximal embedding with  $i + 1$  faces, at least one of them being spanning.  $\square$

Although Theorem 3.1 cannot be used to construct all locally-maximal embeddings, it is often useful in determining the locally-maximal genus provided that one can obtain good lower bounds. In particular, Theorem 3.1 can be used to prove lower-embeddability in Theorem 2.5, where lower bounds follow from Theorem 2.3. The idea is as follows. Every lower-embeddable graph  $G$  satisfies  $\gamma_L(G) = \lceil \beta'(G)/2 \rceil$ . Therefore, it suffices to construct a spanning cycle-tree  $S$  of  $G$  with  $\mu(G)$  or  $\mu(G) - 1$  cycles such that  $G - E(S)$  consists of even components. In many cases, the graph  $G - E(S)$  is connected. This method is in its nature similar to the one used in the proofs of upper-embeddability, which are often carried out by finding a connected cotree of the given graph.

## 4 Algorithms

Let us start with Greedy-Max-Genus Algorithm described in Figure 4.1. The algorithm repeatedly increases the genus by employing suitable rotation moves. By part (iv) of Theorem 2.2, the genus of an embedding can be raised by a rotation move at a vertex  $v$  if and only if  $v$  is incident with at least three faces. Note that testing whether a vertex is incident with at least three faces, as well as finding and performing a rotation move increasing the genus if such a move exists, can be easily

done in polynomial time. It follows that the running time of Greedy-Max-Genus Algorithm is polynomial. Moreover, by Theorem 2.2, the embedding output by Greedy-Max-Genus Algorithm is locally maximal. Since the genus of any locally-maximal embedding is at least  $\gamma_L(G)$ , and  $\gamma_L(G) \geq \gamma_M(G)/2$  by Theorem 2.4, we obtain the following theorem.

Greedy-Max-Genus Algorithm	
Input:	A connected graph $G$
Output:	An embedding of $G$ and its genus
1:	randomly choose an embedding $\Pi$ of $G$
2:	<b>while</b> there is a rotation move increasing the genus of $\Pi$
3:	apply one of such rotation moves to $\Pi$
4:	output $\Pi$ and the genus of $\Pi$

**Figure 4.1.** 2-approximation algorithm for maximum genus.

**Theorem 4.1.** *The Greedy-Max-Genus Algorithm from Figure 4.1 is a polynomial-time 2-approximation algorithm for maximum genus. Furthermore, the embeddings output by Greedy-Max-Genus Algorithm are precisely the locally-maximal embeddings.*

Part (iii) of Theorem 2.4 enables us to efficiently approximate also the locally-maximal genus using a polynomial-time algorithm computing the maximum genus of an arbitrary graph by Glukhov [4], or Furst et al. [3].

**Theorem 4.2.** *There is a polynomial-time 2-approximation algorithm for locally-maximal genus.* □

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# A characterization of triangulations of closed surfaces

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**Abstract.** In this paper we prove that a finite triangulation of a connected closed surface is completely determined by its intersection matrix. The *intersection matrix* of a finite triangulation,  $K$ , is defined as  $M_K = (\dim(s_i \cap s_j))_{0 \leq i, 0 \leq j}^{n-1}$ , where  $K_2 = \{s_0, \dots, s_{n-1}\}$  is a labelling of the triangles of  $K$ .

## 1 Introduction

Within the theory of convex polytopes, the study of the combinatorial equivalence of  $k$ -skeleta of pairs polytopes which are not equivalent themselves has been of interest, this phenomena is referred to in the literature as ambiguity [3].

It is well known that for  $k \geq \lfloor \frac{d}{2} \rfloor$  the  $k$ -skeleton of a convex polytope is not dimensionally ambiguous, this is, it defines the entire structure of its underlying  $d$ -polytope. However for  $k < \lfloor \frac{d}{2} \rfloor$  the question is much more intricate.

One of the most interesting results in this direction is the solution to Perle's conjecture by P. Blind and R. Mani [1] and, separately, by G. Kalai [2] which states that the 1-skeleta of convex simple  $d$ -polytopes define their entire combinatorial structure. Or, on its dual version, that the dual graph (facet adjacency graph) of a convex simplicial  $d$ -polytope determines its entire combinatorial structure.

## 2 Motivation & contribution

Allured by Perles' conjecture, we decided to explore the extent to which an adequate combination of combinatorial and topological assumptions

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would prove as powerful for characterising certain simplicial complexes. The purpose of this work is to present our first result, product of this exploration.

For topological assumption we will, in this instance, ask for the simplicial complex of study to be a connected closed surface. As for combinatorial assumption, one might be tempted to choose only to have the information provided by its dual graph. However, the dual graph of a triangulation of a closed surface does not provide enough information to characterise it, as there are some dual graphs to triangulations which have been shown in [4] to have combinatorically different polyhedral embeddings.

Therefore, we will need to strengthen the combinatorial hypothesis. In order to do so we will introduce the concept of an intersection preserving mapping of simplices of a simplicial complex.

**Definition 2.1.** A bijective mapping  $f : K_d \rightarrow K'_d$  between the sets of  $d$ -simplices of two simplicial complexes,  $K$  and  $K'$ , is an intersection preserving mapping if for every pair of simplices  $s, t \in K_d$   $\dim(s \cap t) = \dim(f(s) \cap f(t))$ .

Throughout this paper we will use the notation  $K_l$  to refer to the set of  $l$ -dimensional simplices of the complex  $K$ . Additionally, we will define two particular triangulations of the projective plane, which are of interest for this work.

**Definition 2.2.** We define a 10-triangle triangulation of the projective plane,  $T\mathbb{P}_{10}$ , as the triangulation whose triangles have the vertex sets  $(s_i)_0 = \{a_{i \bmod 5}, a_{i+1 \bmod 5}, x\}$ , and  $(r_i)_0 = \{a_{i \bmod 5}, a_{i+1 \bmod 5}, a_{i-2 \bmod 5}\}$  for  $0 \leq i \leq 4$ .

**Definition 2.3.** We define a 12-triangle triangulation of the projective plane,  $T\mathbb{P}_{12}$ , as the triangulation whose triangles have the vertex sets  $(s_i)_0 = \{a_{i \bmod 6}, a_{i+1 \bmod 6}, x\}$ , for  $0 \leq i \leq 5$  and  $(r_i)_0 = \{a_{i \bmod 6}, a_{i+1 \bmod 6}, a_{i+4 \bmod 6}\}$  for  $0 \leq i \leq 4$  even, and  $(r_i)_0 = \{a_{i \bmod 6}, a_{i+1 \bmod 6}, a_{i+3 \bmod 6}\}$  for  $0 \leq i \leq 4$  odd.

We now use the aforementioned definitions to state the main result:

**Theorem 2.4.** *Let  $\|K\|$  and  $\|K'\|$  be geometric realizations of finite triangulations which are homeomorphic to connected closed surfaces, and let  $f : K_2 \rightarrow K'_2$  be an intersection preserving mapping, then one of the following three statements holds:*

- (1)  $f$  can be extended into a bijective simplicial mapping between  $K$  and  $K'$

- (2)  $f$  cannot be extended into a simplicial mapping between  $K$  and  $K'$ , but both  $\|K\|$  and  $\|K'\|$  are  $T\mathbb{P}_{10}$
- (3)  $f$  cannot be extended into a simplicial mapping between  $K$  and  $K'$ , but both  $\|K\|$  and  $\|K'\|$  are  $T\mathbb{P}_{12}$ .

Consider the *intersection matrix*,  $M_K = (\dim(s_i \cap s_j))_{0 \leq i, 0 \leq j}^{n-1}$ , of a finite triangulation,  $K$ , where  $K_2 = \{s_0, \dots, s_{n-1}\}$  is a labelling of the triangles of  $K$  then, in the spirit of Perles' conjecture, we can state the previous theorem as;

**Corollary 2.5.** *A finite triangulation of a connected closed surface is completely determined by its intersection matrix.*

### 3 Preliminaries

One of the peculiarities of triangulations of a closed surface is that the neighbourhood of every vertex is a disk. Furthermore, the triangles incident to any vertex of such surface form the simplest of triangulations of a disk, namely an  $n$ -gon whose vertices are all linked by an edge to a central vertex in the centre of the  $n$ -gon. We start off by analysing the intersection patterns of such a structure.

**Definition 3.1.** An  $n$ -shell is the abstract triangulation  ${}_n\Pi_2^2$  such that  ${}_n\Pi_2^2 = \{s_0, s_1, \dots, s_{n-1}\}$ ,  $\dim(s_i \cap s_{i+1 \bmod n}) = 1$ ,  $\dim(s_i \cap s_j) = 0$  for  $|i - j| \geq 2$  with  $i, j \in \{0, \dots, n - 1\}$ .

We will now focus on studying what other structures can have an intersection pattern equal to that of a triangulated disk.

**Lemma 3.2.** *The vertex sets of the triangles in an  $n$ -shell, with  $n \geq 3$ ,  ${}_n\Pi_2^2 = \{s_0, \dots, s_{n-1}\}$  can only take one of the following three types*

- (1)  $(s_i)_0 = \{a_{i \bmod n}, a_{i+1 \bmod n}, x\}$  for all  $0 \leq i \leq n$ , for any  $n$ ;
- (2)  $(s_0)_0 = \{a_0, a_2, a_1\}$ ,  $(s_1)_0 = \{a_1, a_3, a_2\}$ ,  $(s_2)_0 = \{a_2, a_4, a_3\}$ ,  $(s_3)_0 = \{a_3, a_0, a_4\}$ , and  $(s_4)_0 = \{a_4, a_1, a_0\}$ , when  $n = 5$ ; or
- (3)  $(s_0)_0 = \{a_0, a_1, a_2\}$ ,  $(s_1)_0 = \{a_1, a_2, a_4\}$ ,  $(s_2)_0 = \{a_2, a_3, a_4\}$ ,  $(s_3)_0 = \{a_3, a_0, a_4\}$ ,  $(s_4)_0 = \{a_0, a_5, a_4\}$ , and  $(s_5)_0 = \{a_5, a_2, a_0\}$ , when  $n = 6$ .

The proof of the lemma above consists of several parts and follows largely by a detailed analysis of the combinatorial structure of  $n$ -cycles of triangles.

It is easy to see that geometric realisations of the three types of triangulations associated to puzzles of  $n$ -shells are an  $n$ -triangulation of a disk, a 5-triangulation of a Möbius band and a 6-triangulation of a Möbius band, respectively.

### 4 Proof of the theorem

*Proof.* For each vertex  $x \in K_0$  let  $n_x \Pi$  be the  $n_x$ -cycle around  $x$ , by hypothesis  $\|n_x \Pi\|$  is necessarily a disk.

- (1) If, for all  $x \in K_0$ ,  $\|f(n_x \Pi)\|$  is also a disk, then the mapping  $h : K_0 \rightarrow K'_0$  such that  $h(x) = \bigcap_{i=1}^{n_x} f(s_i)$  is a bijective simplicial mapping.

Assume then that there is a vertex  $x \in K_0$  such that  $\|f(n_x \Pi)\|$  is not a disk.

- (2) Suppose  $\|f(n_x \Pi)\|$  is the 5-triangulation of the Möbius band described in Lemma 3.2.

Let  $n_x \Pi_2 = \{s_0, s_1, s_2, s_3, s_4\}$ , where  $(s_i)_0 = \{a_{i \bmod 5}, a_{i+1 \bmod 5}, x\}$  and  $f(n_x \Pi)_2 = \{s'_0, s'_1, s'_2, s'_3, s'_4\}$  where  $(s'_0)_0 = \{a'_0, a'_2, a'_1\}$ ,  $(s'_1)_0 = \{a'_1, a'_3, a'_2\}$ ,  $(s'_2)_0 = \{a'_2, a'_4, a'_3\}$ ,  $(s'_3)_0 = \{a'_3, a'_0, a'_4\}$ , and  $(s'_4)_0 = \{a'_4, a'_1, a'_0\}$ .

Given that  $K'$  is also a closed surface, then each of the simplices  $s'_0, s'_1, s'_2, s'_3, s'_4$  has got a triangle adjacent to its remaining free edge. Let  $r'_i$  be the simplices such that  $\dim(r'_i \cap s'_i) = 1$ , then  $(r'_0)_0 = \{a'_0, a'_2, x'_0\}$ ,  $(r'_1)_0 = \{a'_1, a'_3, x'_1\}$ ,  $(r'_2)_0 = \{a'_2, a'_4, x'_2\}$ ,  $(r'_3)_0 = \{a'_3, a'_0, x'_3\}$ , and  $(r'_4)_0 = \{a'_4, a'_1, x'_4\}$ . This is  $(r'_i)_0 = \{a'_{i \bmod 5}, a'_{i+2 \bmod 5}, x'_i\}$ . It follows that,  $\dim(r'_i \cap s'_j) \geq 0$  for all  $i \neq j$ .

Note that the interior of each of the edges  $\{a'_{i \bmod 5}, a'_{i+1 \bmod 5}\}$  is in the interior of the Möbius band, thus this edges cannot be repeated in any further simplex in the complex.

This implies that  $x_i \notin \{a'_0, a'_1, a'_2, a'_3, a'_4\}$ , because, if this was the case, at least one of the edges  $\{a'_{i \bmod 5}, a'_{i+1 \bmod 5}\}$  would belong to  $(r'_i)_1$ . Then,  $\dim(r'_i \cap s'_j) = 0$

Let  $r_i = f^{-1}(r'_i)$ , then  $\dim(r_i \cap s_i) = 1$  and  $\dim(r_i \cap s_j) = 0$  for all  $i \neq j$ . As  $(s_i)_0 = \{a_{i \bmod 5}, a_{i+1 \bmod 5}, x\}$  then  $(r_i)_0 = \{a_{i \bmod 5}, a_{i+1 \bmod 5}, x_{i \bmod 5}\}$ .

Here  $\dim(r_i \cap s_{i+1}) \geq 0$  and  $\dim(r_i \cap s_{i-1}) \geq 0$  trivially, hence  $a_{i-1 \bmod 5}, a_{i+2 \bmod 5} \notin (r_i)_0$ . However, for  $\dim(r_i \cap s_{i+2}) = 0$  and  $\dim(r_i \cap s_{i-2}) = 0$  to be accomplished, necessarily  $x_i = a_{i+3 \bmod 5} = a_{i-2 \bmod 5}$ .

That is,  $(r_i)_0 = \{a_{i \bmod 5}, a_{i+1 \bmod 5}, a_{i-2 \bmod 5}\}$ , hence the simplicial complex associated to  $\bigcup_{i=0}^4 r_i$  is a 5-triangulation of a Möbius band, where  $\dim(r_i \cap r_{i+2}) = \dim(r_i \cap r_{i-2}) = 1$ , and as necessarily  $K_2 = \bigcup_{i=0}^4 r_i \cup \bigcup_{i=0}^4 s_i$  and the geometric simplicial complexes associated to  $\bigcup_{i=0}^4 r_i$  and  $\bigcup_{i=0}^4 s_i$  are a Möbius band and a disk, respectively, then  $\|K\|$  is equal to  $T\mathbb{P}_{10}$ .

The above also implies that  $\dim(r'_i \cap r'_{i+2 \bmod 5}) = \dim(r'_i \cap r'_{i-2 \bmod 5}) = 1$  then  $v' = v'_i$  for all  $i = 1, \dots, 4$ , so that  $K'_2 = \bigcup_{i=0}^4 r'_i \cup \bigcup_{i=0}^4 s'_i$ , hence  $\|K'\|$  is also equal to  $T\mathbb{P}_{10}$ .

(3) Suppose  $\|f_{(n_x \Pi)}\|$  is the 6-triangulation of the Möbius band described in Lemma 3.2.

Let  ${}_{n_x}\Pi_2 = \{s_0, s_1, s_2, s_3, s_4, s_5\}$ , where  $(s_i)_0 = \{a_{i \bmod 6}, a_{i+1 \bmod 6}, x\}$  and  $f_{(n_x \Pi)}_2 = \{s'_0, s'_1, s'_2, s'_3, s'_4, s'_5\}$  where  $(s'_0)_0 = \{a'_0, a'_1, a'_2\}$ ,  $(s'_1)_0 = \{a'_1, a'_2, a'_4\}$ ,  $(s'_2)_0 = \{a'_2, a'_3, a'_4\}$ ,  $(s'_3)_0 = \{a'_3, a'_0, a'_4\}$ ,  $(s'_4)_0 = \{a'_0, a'_5, a'_4\}$ , and  $(s'_5)_0 = \{a'_5, a'_2, a'_0\}$ .

As  $\|K'\|$  is a closed surface, then each of the simplices  $\{s'_0, s'_1, s'_2, s'_3, s'_4, s'_5\}$  has got a triangle adjacent to its remaining free edge. Let  $r'_i$  be the simplices such that  $\dim(r'_i \cap s'_i) = 1$ , then  $(r'_0)_0 = \{a'_0, a'_1, x'_0\}$ ,  $(r'_1)_0 = \{a'_1, a'_4, x'_1\}$ ,  $(r'_2)_0 = \{a'_2, a'_3, x'_2\}$ ,  $(r'_3)_0 = \{a'_3, a'_0, x'_3\}$ ,  $(r'_4)_0 = \{a'_4, a'_5, x'_4\}$ , and  $(r'_5)_0 = \{a'_2, a'_5, x'_5\}$ .

Here it follows that,  $\dim(r'_i \cap s'_j) \geq 0$  for all  $i \neq j$ , except for the pairs  $i = 0$  and  $j = 2$ ,  $i = 1$  and  $j = 5$ ,  $i = 2$  and  $j = 4$ ,  $i = 3$  and  $j = 1$ ,  $i = 4$  and  $j = 0$  and  $i = 5$  and  $j = 3$ ; for these exceptions the intersection might be empty.

The above implies that, if  $r_i = f^{-1}(r'_i)$ , then  $\dim(r_i \cap s_i) = 1$  and  $\dim(r_i \cap s_j) \geq 0$  for all  $i \neq j$ , except for the pairs  $i = 0$  and  $j = 2$ ,  $i = 1$  and  $j = 5$ ,  $i = 2$  and  $j = 4$ ,  $i = 3$  and  $j = 1$ ,  $i = 4$  and  $j = 0$ , and  $i = 5$  and  $j = 3$ ; for these exceptions the intersection might be empty.

As  $(s_i)_0 = \{a_{i \bmod 6}, a_{i+1 \bmod 6}, x\}$  then the vertex sets of the  $r_i$ 's are  $(r_i)_0 = \{a_{i \bmod 6}, a_{i+1 \bmod 6}, x_{i \bmod 6}\}$ .

Note that  $x'_0 \notin \{a'_0, a'_1, a'_2, a'_4, a'_5\}$  as the edges  $\{a'_0, a_2\}$ ,  $\{a'_0, a_4\}$ ,  $\{a'_0, a_5\}$ ,  $\{a'_0, a_4\}$  are edges whose interior is in the interior of the Möbius band. Thus we might have  $v'_0 = a'_3$ , however if that was the case  $\dim(r'_0 \cap r'_3) = 1$ , and  $\dim(r_0 \cap r_3) = 1$ , but this is not possible. Then necessarily  $x'_0 \notin \{a'_0, a'_1, a'_2, a'_3, a'_4, a'_5\}$ .

Using an argument analogous to the one in the previous case, we deduce that for each  $i$ ,  $x'_i \notin \{a'_0, a'_1, a'_2, a'_3, a'_4, a'_5\}$ ; so that  $\dim(r'_i \cap s'_j) = 0$  for all  $i \neq j$ , except for the pairs  $i = 0$  and  $j = 2$ ,  $i = 1$  and  $j = 5$ ,  $i = 2$  and  $j = 4$ ,  $i = 3$  and  $j = 1$ ,  $i = 4$  and  $j = 0$ , and  $i = 5$  and  $j = 3$ , for which the intersection is empty.

The above implies  $\dim(r_i \cap s_i) = 1$  and  $\dim(r_i \cap s_j) = 0$  for all  $i \neq j$ , except for the pairs  $i = 0$  and  $j = 2$ ,  $i = 1$  and  $j = 5$ ,  $i = 2$  and  $j = 4$ ,  $i = 3$  and  $j = 1$ ,  $i = 4$  and  $j = 0$ , and  $i = 5$  and  $j = 3$ , for which the intersection is empty. Hence, in order to accomplish the intersection dimensions indicated by the puzzle necessarily,  $x_0 = x_1 = a_4$ ,  $x_2 =$

$x_3 = a_0, x_4 = x_5 = a_2$ , thus;  $r_0 = \{a_0, a_1, a_4\}, r_1 = \{a_1, a_2, a_4\}, r_2 = \{a_2, a_3, a_0\}, r_3 = \{a_3, a_4, a_0\}, r_4 = \{a_4, a_5, a_2\}$ , and  $r_5 = \{a_0, a_5, a_2\}$ .

Therefore, the simplicial complex associated to  $\bigcup_{i=0}^5 r_i$  is a 6-triangulation of a Möbius band and  $K_2 = \bigcup_{i=0}^5 s_i \cup \bigcup_{i=0}^5 r_i$ , so that  $\|K\|$  is equal to  $T\mathbb{P}_{12}$ .

The implication for  $K'$  is that  $\dim(r'_0 \cap r'_1) = 1, \dim(r'_1 \cap r'_4) = 1, \dim(r'_4 \cap r'_5) = 1, \dim(r'_5 \cap r'_2) = 1, \dim(r'_2 \cap r'_3) = 1, \dim(r'_3 \cap r'_0) = 1$ , which in turn implies  $v' = v'_i$  for all  $i \in \{0, \dots, 5\}$  and, further,  $K'_2 = \bigcup_{i=0}^5 s'_i \cup \bigcup_{i=0}^5 r'_i$ , so that  $K'$  is also equal to  $T\mathbb{P}_{12}$ .  $\square$

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# Quasi-perfect linear codes from singular plane cubics

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**Abstract.** We present some recently obtained constructions of quasi-perfect linear codes with small density arising from plane cubic curves defined over finite fields.

## 1 Introduction

Galois Geometry, that is the theory of combinatorial objects embedded in projective spaces over finite fields, is well known to be rich of nice algebraic, combinatorial and group theoretic aspects that have also found wide and relevant applications in Coding Theory and Cryptography; see *e.g.* the monography [5]. In this context an important role is played by plane arcs and their generalizations - especially complete caps, saturating sets and arcs in higher dimensions - since their code theoretic counterparts are distinguished types of error-correcting and covering linear codes. In this extended abstract we present some recent results on small complete caps and quasi-perfect linear codes, obtained in few joint works with N. Anbar, D. Bartoli and I. Platoni, mostly unpublished.

Let  $\mathbb{F}_q$  be the finite field with  $q$  elements and let  $\mathbf{C}$  be an  $[n, k, d]_q$ -code, *i.e.*, a  $q$ -ary linear code of length  $n$ , dimension  $k$  and minimum distance  $d$ . The covering radius of  $\mathbf{C}$  is the minimum integer  $R(\mathbf{C})$  such that for any vector  $\mathbf{v} \in \mathbb{F}_q^n$  there exists  $\mathbf{x} \in \mathbf{C}$  with  $d(\mathbf{v}, \mathbf{x}) \leq R(\mathbf{C})$ . An  $[n, k, d]_q$ -code with covering radius  $R$  is denoted by  $[n, k, d]_q R$ . Let  $t$  be the integer part of  $(d - 1)/2$ . Clearly,  $R(\mathbf{C}) \geq t$  holds and when equality is attained the code  $\mathbf{C}$  is said to be perfect. As there are only finitely many classes of linear perfect codes, of particular interest are those codes  $\mathbf{C}$  with  $R(\mathbf{C}) = t + 1$ , called quasi-perfect codes. One of the parameters characterizing the covering quality of an  $[n, k, d]_q R$ -code  $\mathbf{C}$  is its covering density  $\mu(\mathbf{C})$ , introduced in [3] as the average number of codewords at distance less than or equal to  $R$  from a vector in  $\mathbb{F}_q^n$ . The covering

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density  $\mu(\mathbf{C})$  is always greater than or equal to 1, and equality holds precisely when  $\mathbf{C}$  is perfect. Among codes with the same codimension  $s$  and covering radius  $R$ , the shortest ones have the best covering density. This explains why the problem of determining the minimal length  $n$  for which there exists an  $[n, n - s, d]_q R$ -code with given  $s, q, d$  and  $R$ , has been broadly investigated. Throughout, such minimal length will be denoted as  $l(s, R, q)_d$ .

Here we will restrict our attention to codes with covering radius  $R = 2$  and  $d = 4$ , *i.e.* quasi-perfect linear codes that are both 1-error correcting and 3-error detecting. Interestingly, such codes have a nice geometrical counterpart: the columns of a parity check matrix of an  $[n, n - s, 4]_q 2$ -code can be considered as points of a complete cap of size  $n$  in the finite projective space  $PG(s - 1, q)$ . In particular,  $l(s, 2, q)_4$  coincides with the minimum size of a complete cap in  $PG(s - 1, q)$ . This makes it possible to use methods from both Galois Geometries and Algebraic Geometry in order to investigate covering-radius-2 codes with small density. Here, we are going to discuss some recently obtained upper bounds on the minimum size of a complete cap which are valid for arbitrarily large values of  $q$ . The key tool is the construction of complete caps in higher dimensional spaces from singular plane cubic curves defined over  $\mathbb{F}_q$ .

## 2 Complete caps from bicovering arcs

An  $n$ -cap in an (affine or projective) Galois space over  $\mathbb{F}_q$  is a set of  $n$  points no three of which are collinear. An  $n$ -cap is said to be complete if it is not contained in an  $(n + 1)$ -cap. A plane  $n$ -cap is also called an  $n$ -arc. Let  $t(AG(N, q))$  be the size of the smallest complete cap in the Galois affine space  $AG(N, q)$  of dimension  $N$  over  $\mathbb{F}_q$ . Since the affine space  $AG(N, q)$  is embedded in the projective space  $PG(N, q)$ , a complete cap in  $AG(N, q)$  can be viewed as a cap in  $PG(N, q)$ , whose completeness can be achieved by adding some extra-points at the hyperplane at infinity. Therefore, the following relation holds.

**Proposition 2.1.** *Let  $M(N, q)$  denote the maximal size of a complete cap in  $PG(N - 1, q)$ . Then*

$$l(N + 1, 2, q)_4 \leq t(AG(N, q)) + M(N, q).$$

*In particular,  $l(5, 2, q)_4 \leq t(AG(4, q)) + q^2 + 1$ .*

The trivial lower bound for  $t(AG(N, q))$  is  $\sqrt{2q}^{\frac{N-1}{2}}$ . General constructions of complete caps whose size is close to this lower bound are only known for  $q$  even and  $N$  odd. When  $N$  is even, complete caps of size



of the same order of magnitude as  $cq^{N/2}$ , with  $c$  a constant independent of  $q$ , are known to exist for both the odd and the even order case.

Small complete caps in dimensions  $N \equiv 0 \pmod{4}$  can be obtained from plane arcs via the product method for caps: let  $q' = q^{\frac{N-2}{2}}$  and fix a basis of  $\mathbb{F}_{q'}$  as a linear space over  $\mathbb{F}_q$ ; identify points in  $AG(N, q)$  with vectors of  $\mathbb{F}_{q'} \times \mathbb{F}_{q'} \times \mathbb{F}_q \times \mathbb{F}_q$ ; for an arc  $A$  in  $AG(2, q)$ , let

$$K_A = \{(\alpha, \alpha^2, u, v) \in AG(N, q) \mid \alpha \in \mathbb{F}_{q'}, (u, v) \in A\};$$

then the set  $K_A$  is a cap in  $AG(N, q)$ . For  $q$  odd, the completeness of the cap  $K_A$  depends on the bicovering properties of  $A$  in  $AG(2, q)$ ; see Theorem 2.3 below. According to Segre [6], given three pairwise distinct points  $P, P_1, P_2$  on a line  $\ell$  in  $AG(2, q)$ ,  $P$  is external or internal to the segment  $P_1P_2$  depending on whether  $(x-x_1)(x-x_2)$  is a non-zero square or a non-square in  $\mathbb{F}_q$ , where  $x, x_1$  and  $x_2$  are the coordinates of  $P, P_1$  and  $P_2$  with respect to any affine frame of  $\ell$ .

**Definition 2.2.** Let  $A$  be a complete arc in  $AG(2, q)$ . A point  $P \in AG(2, q) \setminus A$  is said to be bicovered with respect to  $A$  if there exist  $P_1, P_2, P_3, P_4 \in A$  such that  $P$  is both external to the segment  $P_1P_2$  and internal to the segment  $P_3P_4$ . If every  $P \in AG(2, q) \setminus A$  is bicovered by  $A$ , then  $A$  is said to be a bicovering arc.

**Theorem 2.3 ([4]).** *Let  $A$  be a bicovering  $n$ -arc in  $AG(2, q)$ ; then  $K_A$  is a complete cap in  $AG(N, q)$ .*

### 3 Small complete caps from cubic curves

From now on we assume that the characteristic of  $\mathbb{F}_q$  is  $p > 3$ . Let  $\mathcal{X}$  be an irreducible plane cubic curve defined over  $\mathbb{F}_q$ , and consider the set  $G$  of the non-singular  $\mathbb{F}_q$ -rational points of  $\mathcal{X}$ . As it is well known, for any point  $O$  of  $G$  it is possible to give a group structure to  $G$ , by defining a binary operation  $\boxplus$  in such a way that  $(G, \boxplus)$  is an abelian group with neutral element  $O \in G$ . The point  $O$  is usually chosen as an inflection point. One of the main properties of this operation is that three distinct points in  $G$  are collinear if and only if their sum is the neutral element in  $G$ . Then it is easy to see that for a subgroup  $H$  of  $G$  of index  $m$ , with  $(3, m) = 1$ , and a point  $P$  in  $G \setminus H$ , the coset  $K = H \boxplus P$  is an arc.

In order to investigate the covering properties of such an arc, we recall a general method, due to Segre and Lombardo-Radice, that uses Hasse-Weil's Theorem to prove the completeness of arcs contained in conic or cubic curves. This method is based on the following idea for proving that the secants of  $K$  cover a generic point  $P$  off the curve  $\mathcal{X}$ : (1) write  $K$  in an algebraically parametrized form; in the case of cubic curves  $\mathcal{X}$ , this

can be easily done when  $\mathcal{X}$  is singular; (2) construct an algebraic curve  $\mathcal{C}_P$ , defined over  $\mathbb{F}_q$ , describing the collinearity of two points of  $K$  and  $P$ ; (3) show that  $\mathcal{C}_P$  is absolutely irreducible or has at least an absolutely irreducible component defined over  $\mathbb{F}_q$ ; (4) apply the Hasse-Weil bound to guarantee the existence of a suitable  $\mathbb{F}_q$ -rational point of  $\mathcal{C}_P$  (or of its irreducible component): this is sufficient to deduce the collinearity between  $P$  and two points in  $K$ . Finally, in order to obtain the completeness, it might be necessary to extend the arc  $K$  with some points on  $\mathcal{X}$ .

By Theorem 2.3, bicovering arcs in affine planes are a powerful tool to construct small complete caps in  $AG(N, q)$  with  $q$  odd and  $N \equiv 0 \pmod{4}$ . However, to establish whether a complete arc is bicovering can be a difficult task. So far, three different types of irreducible plane cubic curves have been investigated in order to prove the bicovering properties of the associated arcs: non-singular, cuspidal and nodal. The non-singular (or elliptic) case was investigated in [1].

**Theorem 3.1 ([1]).** *Let  $q$  be odd, and let  $m$  be a prime divisor of  $q - 1$ , with  $7 < m < \frac{1}{8}\sqrt[4]{q}$ . Assume that the cyclic group of order  $m$  admits a maximal-3-independent subset of size  $s$ . Then for any positive integer  $N \equiv 0 \pmod{4}$ ,*

$$t(AG(N, q)) \leq s \cdot q^{\frac{N-2}{2}} \cdot \left( \left\lfloor \frac{q - 2\sqrt{q} + 1}{m} \right\rfloor + 31 \right). \tag{3.1}$$

It has been noticed in [7] that in the cyclic group of order  $m$  there exists a maximal 3-independent subset of size  $s \leq (m + 1) / 3$ . For specific values of  $m$ , the upper bound on  $t(AG(N, q))$  can be improved, as there exist maximal-3-independent subsets of the cyclic group of order  $m$  of size significantly less than  $m/3$  (see [1, Table 1]).

The case of a cubic with a cuspidal rational singularity and a rational inflection point is the object of the preprint [2].

**Theorem 3.2 ([2]).** *Let  $q = p^h$  with  $p > 3$  a prime,  $h > 8$ . Let  $N \equiv 0 \pmod{4}$ ,  $N \geq 4$ . Let  $t_h$  be the integer in  $\{1, \dots, 4\}$  such that  $t_h \equiv h \pmod{4}$ . Assume that  $p^{t_h} > 144$ . Then*

$$t(AG(N, q)) \leq 2pq^{\frac{N}{2} - \frac{1}{8}}.$$

We now present some new results on cubics with both a rational node and a rational inflection point, which for infinite  $q$ 's improve both Theorems 3.1 and 3.2.

**Theorem 3.3.** *Let  $m$  be an odd divisor of  $q - 1$  such that  $(3, m) = 1$  and*

$$q + 1 - (12m^2 - 8m + 2)\sqrt{q} \geq 8m^2 + 8m + 1. \tag{3.2}$$

Assume that the cyclic group of order  $m$  admits a maximal 3-independent subset of size  $s$ . Then there exists a bicovering arc in  $AG(2, q)$  of size  $\frac{s(q-1)}{m}$  contained in the cubic curve with equation  $XY = (X - 1)^3$ .

Theorem 3.3 can be used together with Theorem 2.3 in order to construct small complete caps in affine spaces. Note that (3.2) holds whenever  $m \leq \frac{4\sqrt{q}}{3.5}$ .

**Corollary 3.4.** *Let  $m$  be an odd divisor of  $q - 1$  such that  $(3, m) = 1$  and  $m \leq \frac{4\sqrt{q}}{3.5}$ . Assume that the cyclic group of order  $m$  admits a maximal 3-independent subset of size  $s$ . Then for  $N \equiv 0 \pmod{4}$ ,  $N \geq 4$ ,*

$$t(AG(N, q)) \leq \frac{s(q-1)}{m} q^{\frac{N-2}{2}}.$$

In the case where a group  $\mathcal{G}$  is the direct product of two groups  $\mathcal{G}_1 \times \mathcal{G}_2$  of order at least 4, neither of which elementary 3-abelian, there exists a maximal 3-independent subset of  $\mathcal{G}$  of size less than or equal to  $(\#\mathcal{G}_1) + (\#\mathcal{G}_2)$ . Then the following holds.

**Theorem 3.5.** *Let  $m$  be an odd divisor of  $q - 1$  such that  $(3, m) = 1$  and  $m \leq \frac{4\sqrt{q}}{3.5}$ . Assume that  $m = m_1 m_2$  with  $(m_1, m_2) = 1$  and  $m_1, m_2 \geq 4$ . Then for  $N \equiv 0 \pmod{4}$ ,  $N \geq 4$*

$$t(AG(N, q)) \leq \frac{(m_1 + m_2)(q - 1)}{m_1 m_2} q^{\frac{N-2}{2}}$$

**Corollary 3.6.** *Let  $q = \bar{q}^8$  for an odd prime power  $\bar{q}$ . Let  $m = (\bar{q}^2 - 1)/(2^h 3^k)$ , where  $2^h \geq 4$  is the highest power of 2 which divides  $\bar{q}^2 - 1$ , and similarly  $3^k \geq 3$  is the highest power of 3 which divides  $\bar{q}^2 - 1$ . Then for  $N \equiv 0 \pmod{4}$ ,  $N \geq 4$*

$$t(AG(N, q)) \leq (2^{h_2} + 2^{h_1} 3^k) q^{\frac{N}{2} - \frac{1}{8}}.$$

with  $h_1 + h_2 = h$ .

Results on complete arcs contained in cubics with an isolated double point have not appeared in the literature so far. This case is currently under investigation by the authors of [2].

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# Boxicity and cubicity of product graphs

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and Deepak Rajendraprasad<sup>4</sup>

**Abstract.** The *boxicity* (*cubicity*) of a graph  $G$  is the minimum natural number  $k$  such that  $G$  can be represented as an intersection graph of axis-parallel rectangular boxes (axis-parallel unit cubes) in  $\mathbb{R}^k$ . In this article, we give estimates on the boxicity and the cubicity of *Cartesian*, *strong* and *direct products* of graphs in terms of invariants of the component graphs. In particular, we study the growth, as a function of  $d$ , of the boxicity and the cubicity of the  $d$ -th power of a graph with respect to the three products. Among others, we show a surprising result that the boxicity and the cubicity of the  $d$ -th Cartesian power of any given finite graph is in  $O(\log d / \log \log d)$  and  $\Theta(d / \log d)$ , respectively. On the other hand, we show that there cannot exist any sublinear bound on the growth of the boxicity of powers of a general graph with respect to strong and direct products.

## 1 Introduction

Throughout this discussion, a  $k$ -*box* is the Cartesian product of  $k$  closed intervals on the real line  $\mathbb{R}$ , and a  $k$ -*cube* is the Cartesian product of  $k$  closed unit length intervals on  $\mathbb{R}$ . Hence both are subsets of  $\mathbb{R}^k$  with edges parallel to one of the coordinate axes. All the graphs considered here are finite, undirected and simple.

**Definition 1.1 (Boxicity, Cubicity).** A  $k$ -*box representation* ( $k$ -*cube representation*) of a graph  $G$  is a function  $f$  that maps each vertex of  $G$  to a  $k$ -box ( $k$ -cube) such that for any two distinct vertices  $u$  and  $v$  of  $G$ , the pair  $uv$  is an edge in  $G$  if and only if the boxes  $f(u)$  and  $f(v)$  have a non-empty intersection. The *boxicity* (*cubicity*) of a graph  $G$ , denoted by  $\text{boxicity}(G)$  ( $\text{cubicity}(G)$ ), is the smallest natural number  $k$  such that  $G$  has a  $k$ -box ( $k$ -cube) representation.

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It follows from the above definition that complete graphs have boxicity and cubicity 0 and interval graphs (unit interval graphs) are precisely the graphs with boxicity (cubicity) at most 1. The concepts of boxicity and cubicity were introduced by F.S. Roberts in 1969 [9]. He showed that every graph on  $n$  vertices has an  $\lfloor n/2 \rfloor$ -box and a  $\lfloor 2n/3 \rfloor$ -cube representation.

Given two graphs  $G_1$  and  $G_2$  with respective box representations  $f_1$  and  $f_2$ , let  $G$  denote the graph on the vertex set  $V(G_1) \times V(G_2)$  whose box representation is a function  $f$  defined by  $f((v_1, v_2)) = f_1(v_1) \times f_2(v_2)$ . It is not difficult to see that  $G$  is the usual strong product of  $G_1$  and  $G_2$  (cf. Definition 1.2). Hence it follows that the boxicity (cubicity) of  $G$  is at most the sum of the boxicities (cubicities) of  $G_1$  and  $G_2$ . The interesting question here is: *can it be smaller?* We show that *it can be smaller* in general. But in the case when  $G_1$  and  $G_2$  have at least one universal vertex each, we show that the boxicity (cubicity) of  $G$  is equal to the sum of the boxicities (cubicities) of  $G_1$  and  $G_2$  (Theorem 2.1).

**Definition 1.2 (Graph products).** The *strong product*, the *Cartesian product* and the *direct product* of two graphs  $G_1$  and  $G_2$ , denoted respectively by  $G_1 \boxtimes G_2$ ,  $G_1 \square G_2$  and  $G_1 \times G_2$ , are graphs on the vertex set  $V(G_1) \times V(G_2)$  with the following edge sets:

$$\begin{aligned} E(G_1 \boxtimes G_2) &= \{(u_1, u_2)(v_1, v_2) : (u_1 = v_1 \text{ or } u_1v_1 \in E(G_1)) \text{ and} \\ &\quad (u_2 = v_2 \text{ or } u_2v_2 \in E(G_2))\}, \\ E(G_1 \square G_2) &= \{(u_1, u_2)(v_1, v_2) : (u_1 = v_1, u_2v_2 \in E(G_2)) \text{ or} \\ &\quad (u_1v_1 \in E(G_1), u_2 = v_2)\}, \\ E(G_1 \times G_2) &= \{(u_1, u_2)(v_1, v_2) : u_1v_1 \in E(G_1) \text{ and } u_2v_2 \in E(G_2)\}. \end{aligned}$$

The  $d$ -th *strong power*, *Cartesian power* and *direct power* of a graph  $G$  with respect to each of these products, that is, the respective product of  $d$  copies of  $G$ , are denoted by  $G^{\boxtimes d}$ ,  $G^{\square d}$  and  $G^{\times d}$ , respectively. Please refer to [7] to know more about graph products.

Unlike the case in strong product, the boxicity (cubicity) of the Cartesian and direct products can have a boxicity (cubicity) larger than the sum of the individual boxicities (cubicities). For example, while the complete graph on  $n$  vertices  $K_n$  has boxicity 0, we show that the Cartesian product of two copies of  $K_n$  has boxicity at least  $\log n$  and the direct product of two copies of  $K_n$  has boxicity at least  $n-2$ . In this note, we give estimates on boxicity and cubicity of Cartesian and direct products in terms of the boxicities (cubicities) and chromatic number of the component graphs. This answers a question raised by Douglas B. West in 2009 [10].

We also study the growth, as a function of  $d$ , of the boxicity and the cubicity of the  $d$ -th power of a graph with respect to these three products. Among others, we show a surprising result that the boxicity and the cubicity of the  $d$ -th Cartesian power of any given finite graph is in  $O(\log d / \log \log d)$  and  $\Theta(d / \log d)$ , respectively (Corollary 2.7). To get this result, we had to obtain non-trivial estimates on boxicity and cubicity of hypercubes and Hamming graphs and a bound on boxicity and cubicity of the Cartesian product which does not involve the sum of the boxicities or cubicities of the component graphs.

The results are summarised in the next section after a brief note on notations. The proofs and figures are included in the full version of the paper [3].

## 1.1 Notational note

The vertex set and edge set of a graph  $G$  are denoted, respectively, by  $V(G)$  and  $E(G)$ . A pair of distinct vertices  $u$  and  $v$  is denoted at times by  $uv$  instead of  $\{u, v\}$  in order to avoid clutter. A vertex in a graph is *universal* if it is adjacent to every other vertex in the graph. If  $S$  is a subset of vertices of a graph  $G$ , the subgraph of  $G$  induced on the vertex set  $S$  is denoted by  $G[S]$ . If  $A$  and  $B$  are sets, then  $A \Delta B$  denotes their symmetric difference and  $A \times B$  denotes their Cartesian product. The set  $\{1, \dots, n\}$  is denoted by  $[n]$ . All logarithms mentioned are to the base 2.

## 2 Our Results

### 2.1 Strong products

**Theorem 2.1.** *Let  $G_i, i \in [d]$ , be graphs with  $\text{boxicity}(G_i) = b_i$  and  $\text{cubicity}(G_i) = c_i$ . Then*

$$\begin{aligned} \max_{i=1}^d b_i &\leq \text{boxicity}(\boxtimes_{i=1}^d G_i) \leq \sum_{i=1}^d b_i, \text{ and} \\ \max_{i=1}^d c_i &\leq \text{cubicity}(\boxtimes_{i=1}^d G_i) \leq \sum_{i=1}^d c_i. \end{aligned}$$

*Furthermore, if each  $G_i, i \in [d]$  has a universal vertex, then the second inequality in both the above chains is tight.*

If we consider the strong product of a 4-cycle  $C_4$  with a path on 3 vertices  $P_3$ , we get an example where the upper bound in Theorem 2.1 is not tight.

**Corollary 2.2.** *For any given graph  $G$ ,  $\text{boxicity}(G^{\boxtimes d})$  and  $\text{cubicity}(G^{\boxtimes d})$  are in  $O(d)$  and there exist graphs for which they are in  $\Omega(d)$ .*

## 2.2 Cartesian products

We show two different upper bounds on the boxicity and cubicity of Cartesian products. The first and the easier result bounds from above the boxicity (cubicity) of a Cartesian product in terms of the boxicity (cubicity) of the corresponding strong product and the boxicity (cubicity) of a Hamming graph whose size is determined by the chromatic number of the component graphs. The second bound is in terms of the maximum cubicity among the component graphs and the boxicity (cubicity) of a Hamming graph whose size is determined by the sizes of the component graphs. The second bound is much more useful to study the growth of boxicity and cubicity of higher Cartesian powers since the first term remains a constant.

**Theorem 2.3.** *For graphs  $G_1, \dots, G_d$ ,*

$$\begin{aligned} \text{boxicity}(\square_{i=1}^d G_i) &\leq \text{boxicity}(\boxtimes_{i=1}^d G_i) + \text{boxicity}(\square_{i=1}^d K_{\chi_i}) \text{ and} \\ \text{cubicity}(\square_{i=1}^d G_i) &\leq \text{cubicity}(\boxtimes_{i=1}^d G_i) + \text{cubicity}(\square_{i=1}^d K_{\chi_i}) \end{aligned}$$

where  $\chi_i$  denotes the chromatic number of  $G_i$ ,  $i \in [d]$ .

When  $G_i = K_q$  for every  $i \in [d]$ ,  $G = \boxtimes_{i=1}^d G_i$  is a complete graph on  $q^d$  vertices and hence has boxicity and cubicity 0. In this case it is easy to see that both the bounds in Theorem 2.3 are tight.

**Theorem 2.4.** *For graphs  $G_1, \dots, G_d$ , with  $|V(G_i)| = q_i$  and  $\text{cubicity}(G_i) = c_i$ , for each  $i \in [d]$ ,*

$$\begin{aligned} \text{boxicity}(\square_{i=1}^d G_i) &\leq \max_{i \in [d]} c_i + \text{boxicity}(\square_{i=1}^d K_{q_i}), \text{ and} \\ \text{cubicity}(\square_{i=1}^d G_i) &\leq \max_{i \in [d]} c_i + \text{cubicity}(\square_{i=1}^d K_{q_i}). \end{aligned}$$

In wake of the two results above, it becomes important to have a good upper bound on the boxicity and the cubicity of Hamming graphs. The Hamming graph  $K_q^d$  is the Cartesian product of  $d$  copies of a complete graph on  $q$  vertices. We call the  $K_2^d$  the  $d$ -dimensional hypercube.

The cubicity of hypercubes is known to be in  $\Theta\left(\frac{d}{\log d}\right)$ . The lower bound is due to Chandran, Mannino and Oriolo [4] and the upper bound is due to Chandran and Sivadasan [6]. But we do not have such tight estimates on the boxicity of hypercubes. The only explicitly known upper bound is one of  $O(d/\log d)$  which follows from the bound on cubicity since boxicity is bounded above by cubicity for all graphs. The only non-trivial lower bound is one of  $\frac{1}{2}(\lceil \log \log d \rceil + 1)$  due to Chandran, Mathew and Sivadasan [5].

We make use of a non-trivial upper bound shown by Kostochka on the dimension of the partially ordered set (poset) formed by two neighbouring levels of a Boolean lattice [8] and a connection between boxicity and



poset dimension established by Adiga, Bhowmick and Chandran in [1] to obtain the following result.

**Theorem 2.5.** *Let  $b_d$  be the largest dimension possible of a poset formed by two adjacent levels of a Boolean lattice over a universe of  $d$  elements. Then*

$$\frac{1}{2}b_d \leq \text{boxicity}(K_2^d) \leq 3b_d.$$

Furthermore,  $\text{boxicity}(K_2^d) \leq 12 \log d / \log \log d$ .

Below, we extend the results on hypercubes to Hamming graphs.

**Theorem 2.6.** *Let  $K_q^d$  be the  $d$ -dimensional Hamming graph on the alphabet  $[q]$  and let  $K_2^d$  be the  $d$ -dimensional hypercube. Then for  $d \geq 2$ ,*

$$\begin{aligned} \log q &\leq \text{boxicity}(K_q^d) \leq \lceil 10 \log q \rceil \text{boxicity}(K_2^d), \text{ and} \\ \log q &\leq \text{cubicity}(K_q^d) \leq \lceil 10 \log q \rceil \text{cubicity}(K_2^d). \end{aligned}$$

Theorem 2.4, along with the bounds on boxicity and cubicity of Hamming graphs, gives the following corollary which is the main result in this article. The lower bound on the order of growth is due to the presence of  $K_2^d$  as an induced subgraph in the  $d$ -th Cartesian power of any non-trivial graph.

**Corollary 2.7.** *For any given graph  $G$  with at least one edge,*

$$\begin{aligned} \text{boxicity}(G^{\square d}) &\in O(\log d / \log \log d) \cap \Omega(\log \log d), \text{ and} \\ \text{cubicity}(G^{\square d}) &\in \Theta(d / \log d). \end{aligned}$$

### 2.3 Direct products

**Theorem 2.8.** *For graphs  $G_1, \dots, G_d$ ,*

$$\begin{aligned} \text{boxicity}(\times_{i=1}^d G_i) &\leq \text{boxicity}(\boxtimes_{i=1}^d G_i) + \text{boxicity}(\times_{i=1}^d K_{\chi_i}) \text{ and} \\ \text{cubicity}(\times_{i=1}^d G_i) &\leq \text{cubicity}(\boxtimes_{i=1}^d G_i) + \text{cubicity}(\times_{i=1}^d K_{\chi_i}) \end{aligned}$$

where  $\chi_i$  denotes the chromatic number of  $G_i$ ,  $i \in [d]$ .

In the wake of Theorem 2.8, it is useful to estimate the boxicity and the cubicity of the direct product of complete graphs. Before stating our result on the same, we would like to discuss a few special cases. If  $G = \times_{i=1}^d K_2$  then  $G$  is a perfect matching on  $2^d$  vertices and hence has boxicity and cubicity equal to 1. If  $G = K_q \times K_2$ , then it is isomorphic to a graph obtained by removing a perfect matching from the complete bipartite graph with  $q$  vertices on each part. This is known as the *crown graph* and its boxicity is known to be  $\lceil q/2 \rceil$  [2].

**Theorem 2.9.** *Let  $q_i \geq 2$  for each  $i \in [d]$ . Then,*

$$\frac{1}{2} \sum_{i=1}^d (q_i - 2) \leq \text{boxicity}(\times_{i=1}^d K_{q_i}) \leq \sum_{i=1}^d q_i, \text{ and}$$

$$\frac{1}{2} \sum_{i=1}^d (q_i - 2) \leq \text{cubicity}(\times_{i=1}^d K_{q_i}) \leq \sum_{i=1}^d q_i \log(n/q_i),$$

where  $n = \prod_{i=1}^d q_i$  is the number of vertices in  $\times_{i=1}^d K_{q_i}$ .

**Corollary 2.10.** *For graphs  $G_1, \dots, G_d$ ,*

$$\text{boxicity}(\times_{i=1}^d G_i) \leq \sum_{i=1}^d (\text{boxicity}(G_i) + \chi(G_i)).$$

**Corollary 2.11.** *For any given graph  $G$ ,  $\text{boxicity}(G^{\times d})$  is in  $O(d)$  and there exist graphs for which it is in  $\Omega(d)$ .*

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## Planarity



# Planar graphs with $\Delta \geq 8$ are $(\Delta + 1)$ -edge-choosable

Marthe Bonamy<sup>1</sup>

**Abstract.** We consider the problem of *list edge coloring* for planar graphs. Edge coloring is the problem of coloring the edges while ensuring that two edges that are incident receive different colors. A graph is  $k$ -edge-choosable if for any assignment of  $k$  colors to every edge, there is an edge coloring such that the color of every edge belongs to its color assignment. Vizing conjectured in 1965 that every graph is  $(\Delta + 1)$ -edge-choosable. In 1990, Borodin solved the conjecture for planar graphs with maximum degree  $\Delta \geq 9$ , and asked whether the bound could be lowered to 8. We prove here that planar graphs with  $\Delta \geq 8$  are  $(\Delta + 1)$ -edge-choosable.

## 1 Introduction

We consider simple graphs. A  $k$ -edge-coloring of a graph  $G$  is a coloring of the edges of  $G$  with  $k$  colors such that two edges that are incident receive distinct colors. We denote by  $\chi'(G)$  the smallest  $k$  such that  $G$  admits a  $k$ -edge-coloring. Let  $\Delta(G)$  be the maximum degree of  $G$ . Since incident edges have to receive distinct colors in an edge coloring, every graph  $G$  satisfies  $\chi'(G) \geq \Delta(G)$ . A trivial upper-bound on  $\chi'(G)$  is  $2\Delta(G) - 1$ , which can be greatly improved, as follows.

**Theorem 1.1 (Vizing [12]).** *Every graph  $G$  satisfies  $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$ .*

Vizing [13] proved that  $\chi'(G) = \Delta(G)$  for every planar graph  $G$  with  $\Delta(G) \geq 8$ . He gave examples of planar graphs with  $\Delta(G) = 4, 5$  that are not  $\Delta(G)$ -edge-colorable, and conjectured that no such graph exists for  $\Delta(G) = 6, 7$ . This remains open for  $\Delta(G) = 6$ , but the case  $\Delta(G) = 7$  was solved by Sanders and Zhao [10], as follows.

**Theorem 1.2 (Sanders and Zhao [10]).** *Every planar graph  $G$  with  $\Delta(G) \geq 7$  satisfies  $\chi'(G) = \Delta(G)$ .*

An extension of the problem of edge coloring is the *list edge coloring* problem, defined as follows. For any  $L : E \rightarrow \mathcal{P}(\mathbb{N})$  list assignment of colors to the edges of a graph  $G = (V, E)$ , the graph  $G$  is  $L$ -edge-colorable if there exists an edge coloring of  $G$  such that the color of

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every edge  $e \in E$  belongs to  $L(e)$ . A graph  $G = (V, E)$  is said to be *list  $k$ -edge-colorable* (or  *$k$ -edge-choosable*) if  $G$  is  $L$ -edge-colorable for any list assignment  $L$  such that  $|L(e)| \geq k$  for any edge  $e \in E$ . We denote by  $\chi'_\ell(G)$  the smallest  $k$  such that  $G$  is  $k$ -edge-choosable.

One can note that edge coloring is a special case of list edge coloring, where all the lists are equal. Thus  $\chi'(G) \leq \chi'_\ell(G)$ . This inequality is in fact conjectured to be an equality (see [8] for more information).

**Conjecture 1.3 (List Coloring Conjecture).** Every graph  $G$  satisfies  $\chi'(G) = \chi'_\ell(G)$ .

The conjecture is still widely open. Some partial results were however obtained in the special case of planar graphs: for example, the conjecture is true for planar graphs of maximum degree at least 12, as follows.

**Theorem 1.4 (Borodin *et al.* [6]).** Every planar graph  $G$  with  $\Delta(G) \geq 12$  satisfies  $\chi'_\ell(G) = \Delta(G)$ .

There is still a large gap with the lower bound of 7 that should hold by Theorem 1.2 if Conjecture 1.3 were true.

Using Vizing's theorem, the List Coloring Conjecture can be weakened into Conjecture 1.5.

**Conjecture 1.5 (Vizing [14]).** Every graph  $G$  satisfies  $\chi'_\ell(G) \leq \Delta(G) + 1$ .

Conjecture 1.5 has been actively studied in the case of planar graphs with some restrictions on cycles (see for example [11, 15, 16]), and was settled by Borodin [4] for planar graphs of maximum degree at least 9 (a simpler proof was later found by Cohen and Havet [7]).

**Theorem 1.6 (Borodin [4]).** Every planar graph  $G$  with  $\Delta(G) \geq 9$  satisfies  $\chi'_\ell(G) \leq \Delta(G) + 1$ .

Here we prove the following theorem.

**Theorem 1.7.** Every planar graph  $G$  with  $\Delta(G) \leq 8$  satisfies  $\chi'_\ell(G) \leq 9$ .

This improves Theorem 1.6 and settles Conjecture 1.5 for planar graphs of maximum degree 8.

**Corollary 1.8.** Every planar graph  $G$  with  $\Delta(G) \geq 8$  satisfies  $\chi'_\ell(G) \leq \Delta(G) + 1$ .

This answers Problem 5.9 in a survey by Borodin [5]. For small values of  $\Delta$ , Theorem 1.7 implies that every planar graph  $G$  with  $5 \leq \Delta(G) \leq 7$  is also 9-edge-choosable. To our knowledge, this was not known. It is however known that planar graphs with  $\Delta(G) \leq 4$  are  $(\Delta(G) + 1)$ -edge-choosable [9, 14].

## 2 Method

The discharging method was introduced in the beginning of the 20<sup>th</sup> century. It has been used to prove the celebrated Four Color Theorem ([1] and [2]).

We prove Theorem 1.7 using a discharging method, as follows. A graph is *minimal* for a property if it satisfies this property but none of its proper subgraphs does. The first step is to consider a minimal counter-example  $G$  (i.e. a graph  $G$  such that  $\Delta(G) \leq 8$  and  $\chi'_\ell(G) > 9$ , whose every proper subgraph is 9-edge-choosable), and prove it cannot contain some configurations. We assume by contradiction that  $G$  contains one of the configurations. We consider a particular subgraph  $H$  of  $G$ . For any list assignment  $L$  on the edges of  $G$ , with  $|L(e)| \geq 9$  for every edge  $e$ , we  $L$ -edge-color  $H$  by minimality. We show how to extend the  $L$ -edge-coloring of  $H$  to  $G$ , a contradiction.

The second step is to prove that a connected planar graph on at least two vertices with  $\Delta \leq 8$  that does not contain any of these configurations does not satisfy Euler's Formula. To that purpose, we consider a planar embedding of the graph. We assign to each vertex its degree minus six as a weight, and to each face two times its degree minus six. We apply discharging rules to redistribute weights along the graph with conservation of the total weight. As some configurations are forbidden, we can prove that after application of the discharging rules, every vertex and every face has a non-negative final weight. This implies that  $\sum_v (d(v) - 6) + \sum_f (2d(f) - 6) = 2 \times |E(G)| - 6 \times |V(G)| + 4 \times |E(G)| - 6 \times |F(G)| \geq 0$ , a contradiction with Euler's Formula that  $|E| - |V| - |F| = -2$ . Hence a minimal counter-example cannot exist.

The complete proof was omitted due to space limitations. The proof requires eleven forbidden configurations and eleven discharging rules, which we do not present here because they rely on additional definitions. The full proof can be found in [3].

### 3 Conclusion

The key idea in the proof lies in some recoloring arguments using directed graphs. It allowed us to deal with configurations that would not yield under usual techniques, and thus to improve Theorem 1.6. Though this simple argument does not seem to be enough to prove Conjecture 1.5 for  $\Delta = 7$ , it might be interesting to try to improve similarly Theorem 1.4.

Note that the proof could easily be adapted to prove that planar graphs with  $\Delta \geq 8$  are  $(\Delta + 1)$ -edge-choosable. This would however be of little interest considering the simple proof for  $\Delta \geq 9$  presented in [7].

Conjecture 1.5 remains open for  $\Delta = 5, 6$  and  $7$ . It might be interesting to weaken the conjecture and ask whether all planar graphs are  $(\Delta + 2)$ -edge-choosable. This is true for planar graphs with  $\Delta \geq 7$  by Theorems 1.6 and 1.7. What about planar graphs with  $\Delta = 6$ ?

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# Planar emulators conjecture is nearly true for cubic graphs

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**Abstract.** We prove that a cubic nonprojective graph cannot have a finite planar emulator, unless one of two very special cases happen (in which the answer is open). This shows that Fellows' planar emulator conjecture, disproved for general graphs by Rieck and Yamashita in 2008, is nearly true on cubic graphs, and might very well be true there definitely.

## 1 Introduction

A graph  $G$  has a finite *planar emulator*  $H$  if  $H$  is a planar graph and there is a graph homomorphism  $\varphi : V(H) \rightarrow V(G)$  where  $\varphi$  is locally surjective, *i.e.* for every vertex  $v \in V(H)$ , the neighbours of  $v$  in  $H$  are mapped surjectively onto the neighbours of  $\varphi(v)$  in  $G$ . We also say that such a  $G$  is planar-emulable. If we insist on  $\varphi$  being locally bijective, we get a *planar cover*.

The concept of planar emulators was proposed in 1985 by M. Fellows [5], and it tightly relates (although of independent origin) to the better known *planar cover conjecture* of Negami [10]. Fellows also raised the main question: What is the class of graphs with finite planar emulators? Soon later he conjectured that the class of planar-emulable graphs coincides with the class of graphs with finite planar covers (conjectured to be the class of projective graphs by Negami [10]—still open nowadays). This was later restated as follows:

**Conjecture 1.1 (M. Fellows, falsified in 2008).** A connected graph has a finite planar emulator if and only if it embeds in the projective plane.

For two decades the research focus was exclusively on Negami's conjecture and no substantial new results on planar emulators had

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been presented until 2008, when emulators for two nonprojective graphs were given by Rieck and Yamashita [12], effectively disproving Conjecture 1.1.

**Planar emulable nonprojective graphs.** Following Rieck and Yamashita, Chimani et al [2] constructed finite planar emulators of all the minor minimal obstructions for the projective plane except those which have been shown non-planar-emulable already by Fellows ( $K_{3,5}$  and “two disjoint  $k$ -graphs” cases, Def. 2.1), and  $K_{4,4} - e$ . The graph  $K_{4,4} - e$  is thus the only forbidden minor for the projective plane where the existence of a finite planar emulator remains open. Even though we do not have a definite replacement for falsified Conjecture 1.1 yet, the results obtained so far [2, 4] suggest that, vaguely speaking, up to some trivial operations (“planar expansions”), there is only a finite family of nonprojective planar-emulable graphs. A result like that would nicely correspond with the current state-of-art [9] of Negami’s conjecture.

While characterization of planar-emulable graphs has proven itself to be difficult in general, significant progress can be made in a special case. Negami’s conjecture has been confirmed in the case of cubic graphs in [11], and the same readily follows from [9]. Here we prove:

**Definition 1.2.** A *planar expansion* of a graph  $G$  is a graph which results from  $G$  by repeatedly adding a planar graph sharing one vertex with  $G$ , or by replacing an edge or a cubic vertex with a connected planar graph with its attachments (two or three, resp.) on the outer face.

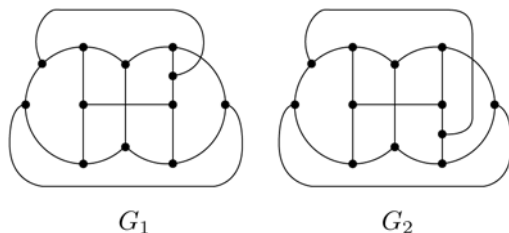
**Theorem 1.3.** *If a cubic nonprojective graph  $H$  has a finite planar emulator, then  $H$  is a planar expansion of one of two minimal cubic nonprojective graphs shown in Figure 1.1.*

A computerized search for possible counterexamples to Conjecture 1.1, carried out so far [4], shows that a nonprojective planar-emulable graph  $G$  cannot be cubic, unless  $G$  contains a minor isomorphic to  $\mathcal{E}_2$ ,  $K_{4,5} - 4K_2$ , or a member of the so called “ $K_7 - C_4$  family”. Our new approach, Theorem 1.3, dismisses the former two possibilities completely and strongly restricts the latter one.

## 2 Cubic planar-emulable graphs

The purpose of this section is to prove Theorem 1.3. In order to do so, we need to introduce some basic related concepts.

**Definition 2.1.** Graph  $G$  is said to *contain two disjoint  $k$ -graphs* if there exist two vertex-disjoint subgraphs  $J_1, J_2 \subseteq G$  such that, for  $i = 1, 2$ , the graph  $J_i$  is isomorphic to a subdivision of  $K_4$  or  $K_{2,3}$ , the subgraph



**Figure 1.1.** Two (out of six in total) cubic irreducible obstructions for the projective plane [6]. Although these graphs result by splitting nonprojective graphs for which we have finite planar emulators [2] (namely  $K_7 - C_4$  and its “relatives”), it is still open whether they are planar-emulable.

$G - V(J_i)$  is connected and adjacent to  $J_i$ , and contracting in  $G$  all the vertices of  $V(G) \setminus V(J_i)$  into one results in a nonplanar graph.

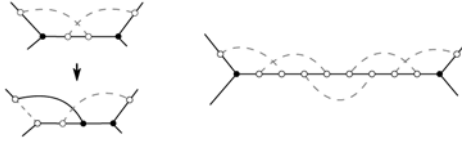
**Proposition 2.2 (Fellows, unpublished).**

- a) *The class of planar-emulable graphs is closed under taking minors.*
- b) *If  $G$  is projective and connected, then  $G$  has a finite planar emulator in form of its finite planar cover.*
- c) *If  $G$  contains two disjoint  $k$ -graphs or a  $K_{3,5}$  minor, then  $G$  is not planar-emulable.*
- d)  *$G$  is planar-emulable if, and only if, so is any planar expansion of  $G$ .*

**Proof of Theorem 1.3.** Glover and Huneke [6] characterized the cubic graphs with projective embedding by giving a set  $\mathcal{I}$  of six cubic graphs such that; if  $H$  is a cubic graph that does not embed in the projective plane, then  $H$  contains a graph  $G \in \mathcal{I}$  as a topological minor.

Let us point out that four out of the six graphs in  $\mathcal{I}$  contain two disjoint  $k$ -graphs, and so only the remaining two—  $G_1 \in \mathcal{I}$  and  $G_2 \in \mathcal{I}$  of Figure 1.1, can potentially be planar-emulable. Hence the cubic graph  $H$  in Theorem 1.3 contains one of  $G_1, G_2$  as a topological minor. In other words, there is a subgraph  $G' \subseteq H$  being a subdivision of a cubic  $G \in \{G_1, G_2\}$ .

We call a *bridge* of  $G'$  in  $H$  any connected component  $B$  of  $H - V(G')$  together with all the incident edges. In a degenerate case,  $B$  might consist just of one edge from  $E(H) \setminus E(G')$  with both ends in  $G'$ . We would like, for simplicity, to speak about positions of bridges with respect to the underlying cubic graph  $G$ : Such a bridge  $B$  connects to vertices  $u$  of  $G'$  which subdivide edges  $f$  of  $G$ —this is due to the cubic degree bound, and we (with neglectable abuse of terminology) say that  $B$  *attaches* to this edge  $f$  in  $G$  itself.



**Figure 2.1.** Illustration for Lemma 2.3. The trivial bridge on the left takes over the role of a branch vertex of  $G$  in the subdivision, resulting in existence of a nontrivial bridge. The other case shows when the transitive closure of declared attachment becomes important.

A bridge  $B$  is *nontrivial* if  $B$  attaches to some two nonadjacent edges of  $G$ , and  $B$  is *trivial* otherwise. For a trivial bridge  $B$ ; either  $B$  attaches to only one edge in  $G$ , and we say *exclusively*, or all the edges to which  $B$  attaches in  $G$  have a vertex  $w$  in common and we say that  $B$  *attaches to this  $w$* .

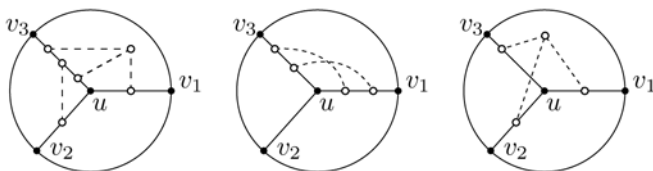
We divide the rest of the proof into two main cases; that either some bridge of  $G'$  in  $H$  is nontrivial or all such bridges are trivial. In the “all-trivial” case one more technical condition has to be observed: Suppose  $B_1, B_2$  are bridges such that  $B_1$  attaches to  $w$  and  $B_2$  attaches to an edge  $f$  incident to  $w$  in  $G$  (perhaps  $B_2$  exclusively to  $f$ ). On the path  $P_f$  which replaces (subdivides)  $f$  in  $G'$ , suppose that  $B_2$  connects to some vertex which is closer to  $w$  on  $P_f$  than some other vertex to which  $B_1$  connects to. Then we *declare that  $B_2$  attaches to  $w$* , too. This is well defined because of the following (Figure 2.1):

**Claim 2.3.** Let  $G' \subseteq H$  be a subdivision of  $G$  where  $G, H$  are cubic graphs. Suppose that all bridges of  $G'$  in  $H$  are trivial, and that a bridge  $B_0$  attaches (is declared to) both to  $w_1$  and  $w_2$ , where  $w_1 w_2 \in E(G)$ . Then there is  $G'' \subseteq H$  which is a subdivision of  $G$ , too, and a nontrivial bridge of  $G''$  in  $H$  exists.

In the described situation, we call  $B_0$  a *conflicting* bridge. We then continue with the following claim obtained by routine examination of (collections of) trivial bridges in view of Definition 2.1 (Figure 2.2).

**Claim 2.4.** Let  $G' \subseteq H$  be a subdivision of  $G$  where  $G, H$  are cubic nonprojective graphs and  $G$  does not contain two disjoint  $k$ -graphs. Suppose that all bridges of  $G'$  in  $H$  are trivial, and no one is conflicting (cf. Lemma 2.3). Then  $H$  does not contain two disjoint  $k$ -graphs if, and only if,  $H$  is a planar expansion of  $G$ .

After that, we use an exhaustive computerized enumeration of nontrivial bridges to conclude the following. We would like to point out that



**Figure 2.2.** Illustration of three collections of trivial bridges that attach to a cubic vertex  $u$ . The first collection gives a planar expansion, while the other two are the “minimal” non-planar-expansion cases.

due to necessity of the  $K_{3,5}$  case, there is likely no simple handwritten argument summarizing the cases similarly as done in Lemma 2.4.

**Claim 2.5.** Let  $G' \subseteq H$  be a subdivision of  $G$  where  $G, H$  are cubic nonprojective graphs. If there exists a nontrivial bridge of  $G'$  in  $H$ , then  $H$  contains two disjoint  $k$ -graphs or a  $K_{3,5}$  minor.

### 3 Conclusions

We identified two graphs (Figure 1.1), for which existence of finite planar emulator now becomes extremely interesting. We would like to point out that similarity of these two graphs suggest that if one has a finite planar emulator, so does the other one. If we however elaborate on this idea and attempt to “unify” the graphs in the form of a common supergraph, we have to use a nontrivial bridge. Perhaps, this provides a clue that these two graphs should not be planar-emulable. Thus, providing an answer for any of these two graphs would bring a better insight to the problem of planar emulations not only for the cubic case, but also in general.

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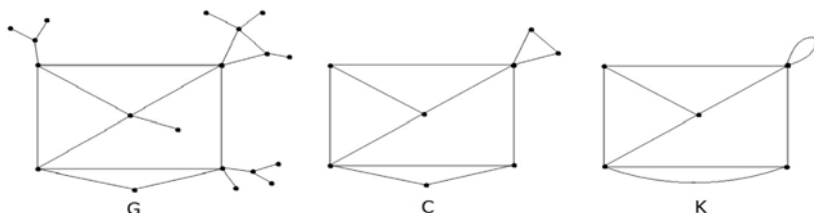
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# Random planar graphs with minimum degree two and three

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## 1 Main results

The main goal of this paper is to enumerate planar graphs subject to a condition on the minimum degree  $\delta$ . Asking for  $\delta \geq 1$  is not very interesting, since a random planar graph only contains a constant number of isolated vertices. The condition  $\delta \geq 2$  is directly related to the concept of the core of a graph. Given a connected graph  $\mathcal{G}$ , its *core* (also called 2-core in the literature) is the maximum subgraph  $\mathcal{C}$  with minimum degree at least two. The core  $\mathcal{C}$  is obtained from  $\mathcal{G}$  by repeatedly removing vertices of degree one. Conversely,  $\mathcal{G}$  is obtained by attaching rooted trees at the vertices of  $\mathcal{C}$ . The *kernel* of  $\mathcal{G}$  is obtained by replacing each maximal path of vertices of degree two in  $\mathcal{C}$  by a single edge. The kernel has minimum degree at least three, and  $\mathcal{C}$  can be recovered from  $\mathcal{K}$  by replacing edges by paths. As shown in the figure below, the kernel may have loops and multiple edges, which must be taken into account since our goal is to analyze simple graphs. Another difficulty is that when replacing loops and multiple edge by paths the same graph can be produced several times. To this end we weight multigraphs appropriately according to the number of loops and edges of each multiplicity. We remark that the concepts of core and kernel of a graph are instrumental in the theory of random graphs [3,5].



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Notice that  $\mathcal{G}$  is planar if and only if  $\mathcal{C}$  is planar, if and only if  $\mathcal{K}$  is planar. It is convenient to introduce the following definitions: a *2-graph* is a connected graph with minimum degree at least two, and a *3-graph* is a connected graph with minimum degree at least three. In order to enumerate planar 2- and 3-graphs, we use generating functions. From now on all graphs are labelled and generating functions are of the exponential type. Let  $c_n$ ,  $h_n$  and  $k_n$  be, respectively, the number of planar connected graphs, 2-graphs and 3-graphs with  $n$  vertices, and let

$$C(x) = \sum c_n \frac{x^n}{n!}, \quad H(x) = \sum h_n \frac{x^n}{n!}, \quad K(x) = \sum k_n \frac{x^n}{n!}$$

be the associated generating functions. Also, let  $t_n = n^{n-1}$  be the number of (labelled) rooted trees with  $n$  vertices and let  $T(x) = \sum t_n x^n / n!$ . The decomposition of a connected graph into its core and the attached trees implies the following equation

$$C(x) = H(T(x)) + U(x), \tag{1.1}$$

where  $U(x) = T(x) - T(x)^2/2$  is the generating function of unrooted trees. Since  $C(x)$  is known completely [2], we have access to  $H(x)$ . Since  $T(x) = xe^{T(x)}$ , we can invert the above relation and obtain

$$H(x) = C(xe^{-x}) - x + \frac{x^2}{2}.$$

The equation defining  $K(x)$  is more involved and requires the bivariate generating function

$$C(x, y) = \sum c_{n,k} y^k \frac{x^n}{n!},$$

where  $c_{n,k}$  is the number of connected planar graphs with  $n$  vertices and  $k$  edges. We can express  $K(x)$  in terms of  $C(x, y)$  as

$$K(x) = C(A(x), B(x)) + E(x), \tag{1.2}$$

where  $A(x)$ ,  $B(x)$ ,  $E(x)$  are certain elementary functions.

The knowledge of  $C(x)$  as the solution of a system of functional-differential equations [2] leads to the following expression:

$$c_n \sim \kappa n^{-7/2} \gamma^n n!,$$

where  $\kappa \approx 0.4261 \cdot 10^{-5}$  and  $\gamma \approx 27.2269$  are computable constants. Analyzing the enriched generating function  $C(x, y)$  it is possible to obtain results on the number of edges and other basic parameters in random

planar graphs. Our main goal is to extend these results to planar 2-graphs and 3-graphs.

Using Equations (1.1) and (1.2) we obtain precise asymptotic estimates for the number of planar 2- and 3-graphs.

**Theorem 1.1.** *The numbers of planar 2- and 3-graphs are asymptotically*

$$\begin{aligned} h_n &\sim \kappa_2 n^{-7/2} \gamma_2^n n!, & \gamma_2 &\approx 26.2076, \\ k_n &\sim \kappa_3 n^{-7/2} \gamma_3^n n!, & \gamma_3 &\approx 21.3102. \end{aligned}$$

As is natural to expect,  $h_n$  and  $k_n$  are exponentially smaller than  $c_n$ . Also, the number of 2-connected planar graphs is known to be asymptotically  $c \cdot n^{-7/2} (26.1849)^n n!$  (see [2]). This is consistent with the estimate for  $h_n$ , since a 2-connected has minimum degree at least two.

By enriching Equations (1.1) and (1.2) taking into account the number of edges, we prove the following.

**Theorem 1.2.** *The number of edges in random planar 2-graphs and 3-graphs are both asymptotically normal with linear expectation and variance. The respective expected values are asymptotically  $\mu_2 n$  and  $\mu_3 n$ , where  $\mu_2 \approx 2.2614$  and  $\mu_3 \approx 2.3227$ .*

The number of edges in connected planar graphs was shown to be [2] asymptotically normal with expectation  $\sim \mu n$ , where  $\mu \approx 2.2133$ . This conforms to our intuition that increasing the minimum degree also increases the expected number of edges.

It is also of interest to analyze the size  $X_n$  of the core in a random connected planar graph, and the size  $Y_n$  of the kernel in a random planar 2-graph.

**Theorem 1.3.** *The variables  $X_n$  and  $Y_n$  are asymptotically normal with linear expectation and variance and*

$$\begin{aligned} \mathbf{E}X_n &\sim \lambda_2 n, & \lambda_2 &\approx 0.9626, \\ \mathbf{E}Y_n &\sim \lambda_3 n, & \lambda_3 &\approx 0.8259. \end{aligned}$$

We remark that the value of  $\lambda_2$  has been recently found by McDiarmid [4] using alternative methods. Also, we remark that the expected size of the largest block (2-connected component) in random connected planar graphs is asymptotically  $0.9598n$ . Again this is consistent since the largest block is contained in the core.

The picture is completed by analyzing the size of the trees attached to the core.



**Theorem 1.4.** *The number of trees with  $k$  vertices attached to the core of a random planar connected graph is asymptotically normal with linear expectation and variance. The expected value is asymptotically*

$$C \frac{k^{k-1}}{k!} \rho^k n,$$

where  $C > 0$  is a constant  $\rho \approx 0.03673$  corresponding to the radius of convergence of  $C(x)$ .

For  $k$  large, the previous quantity grows like

$$\frac{C}{\sqrt{2\pi}} \cdot k^{-3/2} (\rho e)^k n.$$

This quantity is negligible if  $k \gg \log(n)/(\log(1/\rho e))$ . Using the first and second moment method, it can be shown that the size of the largest tree attached to the core is in fact asymptotically

$$\frac{\log(n)}{\log(1/\rho e)}.$$

Our last result concerns the distribution of the vertex degrees in random planar 2-graphs and 3-graphs. It was proved in [1] that the probability that a random vertex has degree  $k \geq 1$  in a random planar tends to a constant  $d(k)$ . The probability generating function

$$p(w) = \sum_{k \geq 1} d(k) w^k$$

is explicit although very complicated [1]. He prove a similar result for planar 2- and 3-graphs.

**Theorem 1.5.** *For each fixed  $k \geq 2$  the probability that a random vertex has degree  $k$  in a random planar 2-graph tends to a constant  $d_H(k)$ , and for each fixed  $k \geq 3$  the probability that a random vertex has degree  $k$  in a random planar 3-graph tends to a constant  $d_K(k)$ . Moreover  $\sum_{k \geq 2} d_H(k) = \sum_{k \geq 3} d_K(k) = 1$ , and the probability generating functions*

$$p_H(w) = \sum_{k \geq 2} d_H(k) w^k, \quad p_K(w) = \sum_{k \geq 3} d_K(k) w^k$$

are computable in terms the probability generating function  $p(w)$  of connected planar graphs (and other constants related to the enumeration of planar graphs).

We show that almost all planar 2-graphs have a vertex of degree two, and almost all planar 3-graphs have a vertex of degree three. Hence asymptotically all our results hold for planar graphs with minimum degree exactly two and three, respectively. In addition, all the results for connected planar graphs extend easily to arbitrary planar graphs.

## 2 Results for maps

We also find analogous results for planar maps. They are simpler to derive and serve as a preparation for the results on planar graphs, while at the same time they are new and interesting by themselves. A planar map is a connected planar multigraph (loops and multiple edges are allowed) embedded in the plane up to homeomorphism. A map is rooted if one of the edges is distinguished and given a direction. In this way a rooted map has a root edge and a root vertex (the tail of the root edge). A rooted map has no automorphisms, in the sense that every vertex, edge and face is distinguishable.

The enumeration of rooted planar maps was started by Tutte in his seminal paper [6]. If  $m_n$  is the number of maps with  $n$  edges, then

$$m_n = \frac{2 \cdot 3^n}{(n+2)(n+1)} \binom{2n}{n}, \quad n \geq 0.$$

The generating function  $M(z) = \sum_{n \geq 0} m_n z^n$  is equal to

$$M(z) = \frac{18z - 1 + (1 - 12z)^{3/2}}{54z^2}. \quad (2.1)$$

Either from the explicit formula or from the expression for  $M(z)$  and singularity analysis, it follows that

$$m_n \sim \frac{2}{\sqrt{\pi}} n^{-5/2} 12^n.$$

If  $m_{n,k}$  is the number of maps with  $n$  edges and degree of the root face equal to  $k$ , then  $M(z, u) = \sum m_{n,k} u^k z^n$  satisfies the equation

$$M(z, u) = 1 + zu^2 M(z, u)^2 + uz \frac{uM(z, u) - M(z, 1)}{u - 1}. \quad (2.2)$$

By duality,  $M(z, u)$  is also the generating function of maps in which  $u$  marks the degree of the root vertex.

The core and the kernel of a map are defined as for connected graphs. The situation is simpler for kernels since loops and multiple edges are allowed. We define a 2-map as a map with minimum degree at least two, and a 3-map as a map with minimum degree at least three.

**Theorem 2.1.** *Let  $h_n$  and  $k_n$  be, respectively, the number of 2-maps and 3-maps with  $n$  edges. The associated generating functions  $H(z)$  and  $K(z)$  are given by*

$$H(x) = \frac{1-x}{1+x} M\left(\frac{x}{(1+x)^2}\right) + x - 1,$$

$$K(x) = \frac{H\left(\frac{x}{1+x}\right) - x}{1+x}.$$

The following estimates hold:

$$h_n \sim \kappa_2 n^{-5/2} (5 + 2\sqrt{6})^n, \quad k_n \sim \kappa_3 n^{-5/2} (4 + 2\sqrt{6})^n,$$

where

$$\kappa_2 = \frac{2}{\sqrt{\pi}} \left(\frac{2}{3}\right)^{5/4}, \quad \kappa_3 = \frac{2}{\sqrt{\pi}} \left(4 - 4\sqrt{\frac{2}{3}}\right)^{5/2}.$$

The same estimates hold for maps with minimum degree exactly two and three, respectively.

We also prove a limit law for the size of the core and the kernel in random maps.

**Theorem 2.2.** *The size  $X_n$  of the core of a random map with  $n$  edges, and the size  $Y_n$  of the kernel of a random 2-map with  $n$  edges are asymptotically Gaussian with*

$$\mathbf{E}X_n \sim \frac{\sqrt{6}}{3}n, \quad \mathbf{Var}(X_n) \sim \frac{n}{6},$$

$$\mathbf{E}Y_n \sim (2\sqrt{6} - 4)n, \quad \mathbf{Var}(Y_n) \sim (18\sqrt{6} - 44)n.$$

Furthermore, the size  $Z_n$  of the kernel of a random map with  $n$  edges is also Gaussian with

$$\mathbf{E}Z_n \sim \left(4 - \frac{4\sqrt{6}}{3}\right)n, \quad \mathbf{Var}(Z_n) \sim \left(\frac{128}{3} - \frac{52}{3}\sqrt{6}\right)n.$$

We also analyze the size of the trees attached to the core of a random map.

**Theorem 2.3.** *Let  $X_{n,k}$  count trees with  $k$  edges attached to the core of a random map with  $n$  edges. Then  $X_{n,k}$  is asymptotically normal and*

$$\mathbf{E}X_{n,k} \sim \alpha_k n$$

where

$$\alpha_k = \left(4 + \frac{5}{3}\sqrt{6}\right) \frac{1}{k+1} \binom{2k}{k} \left(\frac{1}{12}\right)^k.$$

By the same argument as in the previous section, the size of the largest tree attached to the core is asymptotically  $\log n / \log 3$ .

Our last result in this section deals with the distribution of the degree of the root vertex in 2-maps and 3-maps. The limiting probability  $p_M(K)$  that a random map has a root vertex (or face) of degree  $k$  exists for all  $k \geq 1$ . The probability generating function of the distribution is known to be

$$p_M(u) = \sum p_M(k)u^k = \frac{u\sqrt{3}}{\sqrt{(2+u)(6-5u)^3}}.$$

The tail of the distribution is asymptotically

$$p_M(k) \sim \frac{1}{2\sqrt{10\pi}} k^{1/2} \left(\frac{5}{6}\right)^k.$$

We obtain analogous results for 2-maps and 3-maps.

**Theorem 2.4.** *Let  $p_M(u)$  be as before, and let  $p_H(u)$  and  $p_K(u)$  be the probability generating functions for the distribution of the root degree in 2-maps and 3-maps, respectively. Then we have*

$$p_H(u) = \frac{p_M\left(\frac{w(1+\sigma)}{1+w\sigma}\right) \frac{1+\sigma}{1+w\sigma} - w\sigma}{1-\sigma},$$

$$p_K(u) = \frac{p_H(u) - \sigma u^2}{1-\sigma}.$$

Furthermore, the limiting probabilities that the degree of the root vertex is equal to  $k$  exists, both for 2-maps and 3-maps, and are asymptotically

$$p_H(k) \sim \frac{1}{8} \sqrt{\frac{3(1-\sigma)}{\pi}} k^{1/2} \left(\sqrt{\frac{2}{3}}\right)^k,$$

$$p_K(k) \sim \frac{1}{8} \sqrt{\frac{3}{(1-\sigma)\pi}} k^{1/2} \left(\sqrt{\frac{2}{3}}\right)^k.$$

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# Degenerated induced subgraphs of planar graphs

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**Abstract.** A graph  $G$  is *k-degenerated* if it can be deleted by subsequent removals of vertices of degree  $k$  or less. We survey known results on the size of maximal  $k$ -degenerated induced subgraph in a planar graph. In addition, we sketch the proof that every planar graph of order  $n$  has a 4-degenerated induced subgraph of order at least  $8/9 \cdot n$ . We also show that in every planar graph with at least 7 vertices, deleting a suitable vertex allows us to subsequently remove at least 6 more vertices of degree four or less.

## 1 Introduction

A graph  $G$  is *k-degenerated* if every subgraph of  $G$  has a vertex of degree  $k$  or less. Equivalently, a graph is *k-degenerated* if we can delete the whole graph by subsequently removing vertices of degree at most  $k$ . The reverse of this sequence of removed vertices can be used to colour (or even list-colour)  $G$  with  $k + 1$  colours in a greedy fashion. Graph degeneracy is therefore a natural bound on both chromatic number and list chromatic number. In fact, for some problems graph degeneracy provides the best known bounds on the choice number [3].

Every planar graph has a vertex of degree at most 5. Since a subgraph of a planar graph is planar, it also has a vertex of degree at most 5. Therefore, every planar graph is 5-degenerated. If  $k < 5$ , we can still choose at least some of the vertices of  $G$ , and if these vertices induce a

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$k$ -degenerated graph, then they can be greedily coloured by  $k + 1$  colours. Thus an interesting question is how large  $k$ -degenerated subgraph can be guaranteed in a planar graph  $G$ .

Without the restriction to planar graphs Alon, Kahn, and Seymour [2] determined exactly how large  $k$ -degenerated induced subgraph one can guarantee depending only on the degree sequence of  $G$ . For planar graphs the only settled case is  $k = 0$ , due to four colour theorem. Except for  $k = 4$  all other values of  $k$  were examined. Most attention is devoted to the case  $k = 1$ . The Albertson-Berman conjecture [1] asserts that every planar graph has an induced forest with at least half of the vertices. The best known bound, guaranteeing a forest of size at least  $2/5 \cdot |V(G)|$ , is implied by the fact that planar graphs are acyclic 5-colourable [4].

We do not know of any results on maximum 4-degenerated induced subgraphs of planar graphs. A likely reason is that such a bound is not interesting for list-colouring applications: Thomassen [5] proved that every planar graph is 5-choosable. The rest of this paper focuses on degeneracy 4.

We define two operations for vertex removal: deletion and collection. To *delete* a vertex  $v$ , we remove  $v$  and its incident edges from the graph. To *collect* a vertex  $v$  is the same as to delete  $v$ , but to be able to collect  $v$  we require  $v$  to be of degree at most 4. The collected vertices induce a 4-degenerated subgraph. Vertices that are deleted or collected are collectively called *removed*.

Our main results are the following two theorems.

**Theorem 1.1.** *In every planar graph  $G$  we can delete at most  $1/9$  of its vertices in such a way that we can collect all the remaining ones.*

**Theorem 1.2.** *In every planar graph with at least 7 vertices we can delete a vertex in such a way that we can subsequently collect at least 6 vertices.*

These results are probably not the best possible. The worst example known to us is the icosahedron from which we need to delete one vertex out of twelve to be able to collect the remaining eleven. We believe that this is the worst case possible.

**Conjecture 1.3.** In each planar graph  $G$  we can delete at most  $1/12$  of its vertices in such a way that we can collect all the remaining ones.

**Conjecture 1.4.** In each planar graph with at least 12 vertices we can delete a vertex in such a way that we can subsequently collect at least 11 vertices.

## 2 Sketch of the proof

To prove Theorem 1.1 we incorporate the degrees of the vertices of  $G$  to the statement; let

$$\Phi(G) = \sum_{v \in V(G)} (\deg(v) - 5). \tag{2.1}$$

We prove the following theorem by induction on the number of vertices of  $G$ . The function  $\text{tc}(G)$  stands for the number of tree components of  $G$ .

**Theorem 2.1.** *If  $G$  is a planar graph, then there is a set  $S \subset V(G)$  with at most  $\Gamma(G) = |V(G)|/12 + 1/36 \cdot \Phi(G) + 1/18 \cdot \text{tc}(G)$  vertices such that if we delete  $S$  we can subsequently collect all the vertices of  $G$ .*

Since  $\Phi(G) + 2 \text{tc}(G) \leq |V(G)|$ , Theorem 2.1 implies Theorem 1.1. Theorem 1.2 can be proved alongside Theorem 2.1.

We prove Theorem 2.1 by a discharging procedure. Let  $G$  be a minimal counterexample to Theorem 2.1 (with respect to the number of vertices); it can be easily shown that  $G$  is connected with minimal degree 5. We embed  $G$  into the plane (for now, we let the embedding be arbitrary).

Type ( $t$ )	Degree (deg)	Min. number of non-tr. faces ( $n_{\sqcup}$ )	Max. number of $V_5$ neigh. ( $n_5$ )	Maximal charge (mc)
10a	10+	0	3	1
10b	10+	0	$\infty$	1/2
9a	9	1	3	1
9b	9	0	2	1
9c	9	0	3 if consecutive	9/10, 1, 9/10
9d	9	0	9	1/2
8a	8	0	1	1
8b	8	1	2	1
8c	8	2	3 if consecutive	9/10, 1, 9/10
8d	8	0	2	9/10
8e	8	0	8	1/2
7a	7	0	1	4/5
7b	7	1	2	13/20
7c	7	0	2	2/5
7d	7	0	7	1/3
6a	6	1	1	2/5
6b	6	0	6	0

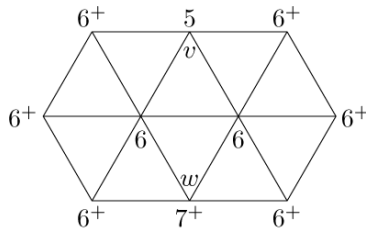
**Table 1.** Maximal charges that can be send to a vertex.

Each vertex of degree at least 6 is assigned a certain *type* according to Table 1. If  $w$  is of degree  $d$ , is contained on at least  $n_{\sqcup}$  non-triangular



faces, and has at most  $n_5$  neighbours from  $V_5$ , then  $w$  can be of type  $t$ . If  $w$  can have more than one type, then the type of  $w$  is the type that occurs first in the table. Let  $vw$  be an edge such that  $v$  is of degree 5 and  $w$  is of degree at least 6. For every such edge we define the maximal charge  $mc(v, w)$  that  $v$  can send to  $w$ . This maximal charge is given in the last column of Table 1.

We start by assigning initial charges to the vertices and faces of  $G$ . Each vertex  $v$  of degree  $d$  receives charge  $6 - d$  and each face of length  $\ell$  receives charge  $2(3 - \ell)$ . According to Euler's theorem, the total initial charge is equal to 12. In the discharging procedure, we redistribute the charges between vertices and faces in a certain way such that no charge is created or lost. The discharging procedure consists of the following three steps.



**Figure 2.1.** Distance discharging ( $d^+$  denotes a vertex of degree at least  $d$ ).

**Step 1: Discharging to faces.** For each vertex  $v$  and for each non-triangular face that contains  $v$ , send  $1/2$  from  $v$  to  $f$ , except for two cases: if  $v$  is of degree 6, then send  $2/5$ ; and if  $v$  is not of degree 6 but both its neighbours on  $f$  are of degree 6, then send  $3/5$ .

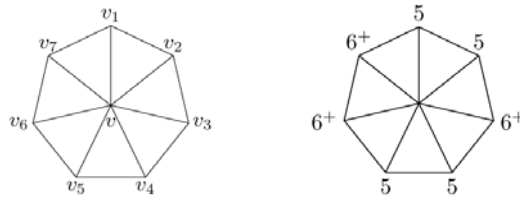
**Step 2: Distance discharging.** In every subgraph of  $G$  isomorphic to the configuration in Figure 2.1 send  $1/5$  from vertex  $v$  to vertex  $w$  (vertices are denoted as in Figure 2.1; the depicted vertices are pairwise distinct; the numbers indicate degrees).

**Step 3: Final discharging of the vertices of degree five.** For each vertex  $v$  of degree 5 carry out the following procedure. Order the neighbours  $w$  of  $v$  which have degree at least 6 according to the value of  $mc(v, w)$  starting with the largest value; let  $w_1, w_2, \dots$  be the resulting ordering. If the value of  $mc(v, w)$  is the same for two neighbours of  $v$ , then we order them arbitrarily. For  $i = 1, 2, \dots$ , send  $\max\{mc(v, w_i), ch_a(v)\}$  from  $v$  to  $w_i$ , where  $ch_a(v)$  denotes the current charge of  $v$ .

Since no face can have positive final charge, there exists a vertex with positive charge. A very technical examination allows us to show that

if we start with a certain embedding of  $G$ , then there is a vertex with positive final charge not contained in any  $C_3$ -cut or  $C_4$ -cut of  $G$ .

We examine  $v$  together with its neighbourhood; this gives rise to a number of cases, each of them leading to a contradiction. We demonstrate this analysis on Configuration 7-3 shown in Figure 2.2 (the numbers in the right part of the figure indicate the degrees of the vertices shown in the left part). Since  $G$  contains no  $C_3$ -cut, the triangles depicted in Figure 2.2 are faces and contain no vertices inside. The vertex  $v$  is of type  $7d$ .



**Figure 2.2.** Configuration 7-3.

Configuration 7-3: Suppose first that the vertex  $v_7$  has degree at most 7. Delete  $v_6$ . Then we can collect  $v_5, v_4, v, v_1, v_2,$  and  $v_7$ . We get a new graph  $G'$  smaller than  $G$ , so there is a set  $S'$  in  $G'$  with at most  $\Gamma(G')$  vertices such that if we delete  $S'$ , we can collect the rest of  $G'$ . If  $\Gamma(G) - \Gamma(G') \geq 1$ , we can extend the set  $S'$  by  $v_6$  and obtain a contradiction with the fact that  $G$  is a smallest counterexample to Theorem 2.1.

We want show that  $\Gamma(G) - \Gamma(G') \geq 1$ . The hardest part is typically to compute  $\Phi(G) - \Phi(G')$ . Among the vertices removed from  $G$ , four vertices have degree at least 5, two vertices have degree at least 6, and one vertex has degree at least 7. After removing these vertices from the sum (2.1) the value of  $\Phi$  decreases by at least 4. Moreover, all the neighbours of the removed vertices have smaller degree in  $G'$  than in  $G$ . From the fact that  $v$  is in no  $C_3$ -cuts we know that there is no extra edge between the neighbours of  $v$ ; all such edges are shown in Figure 2.2. This decreases  $\Phi$  further by at least 17. Therefore  $\Phi(G) - \Phi(G') \geq 21$ . It can be shown that no new tree components are created, so together  $\Gamma(G) - \Gamma(G') \geq 42/36 \geq 1$ .

We are left with the case where  $v_7$  has degree at least 8. If  $v_7$  has another neighbour  $w$  of degree 5 besides  $v_1$ , then we can delete  $v_7$  and collect  $v_1, v_2, v, v_4, v_5,$  and  $w$  (again we need to check that  $\Gamma$  decreases by at least 1). Otherwise,  $v_7$  has only one neighbour of degree 5, the vertex  $v_1$ . According to Table 1,  $mc(v_1, v_7) = 1$ , so  $v_1$  discharges nothing into  $v$  in Step 3 and the final charge of  $v$  is at most 0. This contradicts the fact that  $v$  has positive charge after the discharging.

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# Strong chromatic index of planar graphs with large girth

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**Abstract.** Let  $\Delta \geq 4$  be an integer. We prove that every planar graph with maximum degree  $\Delta$  and girth at least  $10\Delta + 46$  is strong  $(2\Delta - 1)$ -edge-colorable, that is best possible (in terms of number of colors) as soon as  $G$  contains two adjacent vertices of degree  $\Delta$ . This improves [2] when  $\Delta \geq 6$ .

## 1 Introduction

A *strong  $k$ -edge-coloring* of a graph  $G$  is a mapping from  $E(G)$  to  $\{1, 2, \dots, k\}$  such that every two adjacent edges or two edges adjacent to a same edge receive two distinct colors. The *strong chromatic index* of  $G$ , denoted by  $\chi'_s(G)$ , is the smallest integer  $k$  such that  $G$  admits a strong  $k$ -edge-coloring.

Strong edge-colorability was introduced by Fouquet and Jolivet [7, 8] and was used to solve the frequency assignment problem in some radio networks.

An obvious upper bound on  $\chi'_s(G)$  (given by a greedy coloring) is  $2\Delta(\Delta - 1) + 1$  where  $\Delta$  is the maximum degree of  $G$ . The following conjecture was posed by Erdős and Nešetřil [4, 5] and revised by Faudree, Schelp, Gyárfás and Tuza [6]:

**Conjecture 1.1 (See [4–6]).** If  $G$  is a graph with maximum degree  $\Delta$ , then

$$\chi'_s(G) \leq \begin{cases} \frac{5}{4}\Delta^2 & \text{if } \Delta \text{ is even} \\ \frac{1}{4}(5\Delta^2 - 2\Delta + 1) & \text{otherwise.} \end{cases}$$

Moreover, they gave examples of graphs whose strong chromatic indices reach the upper bounds.

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The strong chromatic index was studied for different families of graphs, as cycles, trees,  $d$ -dimensional cubes, chordal graphs, Kneser graphs, see [10]. For complexity issues, see [9, 10].

Faudree, Schelp, Gyárfás exhibited, for every integer  $\Delta \geq 2$ , a planar graph with maximum degree  $\Delta$  and strong chromatic index  $4\Delta - 4$ . They established the following upper bound:

**Theorem 1.2 ([6]).** *Planar graphs with maximum degree  $\Delta$  are strong  $(4\Delta + 4)$ -edge-colorable.*

The purpose of this paper is to prove that if the girth is large enough, then the upper bound can be strengthened to  $2\Delta - 1$ , which is best possible as soon as  $G$  contains two adjacent vertices of degree  $\Delta$ . A first attempt was done Borodin and Ivanova [2] who proved: *every planar graph with maximum degree  $\Delta$  is strong  $(2\Delta - 1)$ -edge-colorable if its girth is at least  $40 \lfloor \frac{\Delta}{2} \rfloor + 1$* . Here we improved the girth condition for every  $\Delta \geq 6$ :

**Theorem 1.3.** *Let  $\mathcal{F}_\Delta$  be the family of planar graphs with maximum degree at most  $\Delta$ . Every graph of  $\mathcal{F}_\Delta$  with girth at least  $10\Delta + 46$  admits a strong  $(2\Delta - 1)$ -edge-coloring when  $\Delta \geq 4$ .*

Next section is devoted to the outline of the proof of Theorem 1.3. For the full version of this paper, see [3].

## 2 On planar graphs with large girth

A *walk* in a graph is a sequence of edges where two consecutive edges are adjacent. Throughout the paper, by *path* we mean a walk where every two consecutive edges are distinct. So a vertex or an edge can appear more than once in a path. By *cycle* we mean a closed path (the first and last edges of the sequence are adjacent).

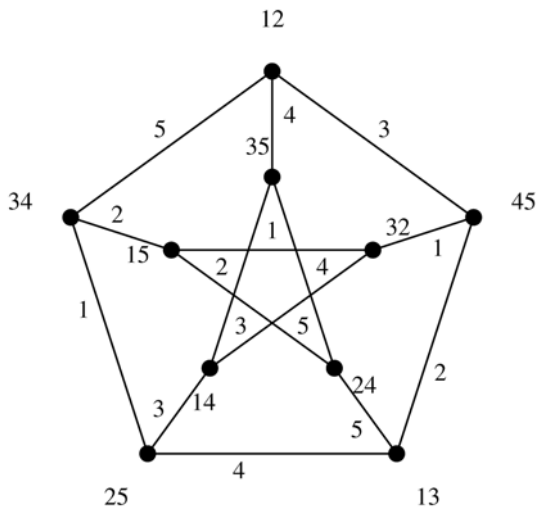
The proof of Theorem 1.3 is based on some properties of *odd graphs*.

Let  $n$  be an integer; the odd graph  $O_n$  may be defined as follows:

- the vertices are the  $(n - 1)$ -subsets of  $\{1, 2, \dots, 2n - 1\}$ .
- two vertices are adjacent if and only if the corresponding subsets are disjoint.

The odd graph  $O_n$  is  $n$ -regular and distance transitive. Moreover its odd-girth is  $2n - 1$  and its even-girth is 6 [1]. We will use the notation  $S(x)$  to denote the subset assigned to the vertex  $x$  in  $O_n$ . Also we can label every edge  $xy$  by the label  $\{1, \dots, 2n - 1\} \setminus (S(x) \cup S(y))$ . Remark that the obtained edge-labeling is a strong edge-coloring. As example,  $O_3$

(the Petersen graph) is depicted in Fig 2.1. To prove Theorem 1.3, we establish that there is a path of length exactly  $2(n - 1)$  between every pair of vertices (not necessarily distinct) in the odd graph  $O_n$  ( $n \geq 4$ ).



**Figure 2.1.** The odd graph  $O_3$  and its edge labeling.

In the following, we consider the case  $\Delta \geq 4$ .

Let  $G \in \mathcal{F}_\Delta$  be a counterexample to Theorem 1.3 with the minimum order. Clearly  $G$  is connected and has minimum degree at least 1.

(1)  $G$  does not contain a vertex  $v$  adjacent to  $d(v) - 1$  vertices of degree 1.

By the way of contradiction, suppose  $G$  contains such a vertex  $v$ . Let  $u$  be a vertex of degree 1 adjacent to  $v$ . By minimality of  $G$ ,  $G' = G - u$  admits a strong  $(2\Delta - 1)$ -edge-coloring. By a simple counting argument, it is easy to see that we can extend the coloring to  $uv$ , a contradiction.

Consider now  $H = G - \{v : v \in G, d_G(v) = 1\}$ .

(2) The minimum degree of  $H$  is at least 2 (by (1)). Graph  $H$  is planar and has the same girth as  $G$ .

The following observation is well-known ([11], Lemma 5):

(3) Every planar graph with minimum degree at least 2 and girth at least  $5d + 1$  contains a path consisting of  $d$  consecutive vertices of degree 2.

Let  $d = 2\Delta + 9$ . It follows from the assumption on the girth, (2) and (3) that  $H$  contains a path  $v_0v_1v_2 \dots v_{d+1}$  in which every vertex  $v_i$  for  $1 \leq i \leq d$  has degree 2. In  $G$ , the path  $v_1 \dots v_d$  is an induced path and

every  $v_i$  ( $1 \leq i \leq d$ ) may be adjacent to some vertices of degree 1, by definition of  $H$  and (1).

Now, consider  $G'$  obtained from  $G$  by

- removing all the pendant vertices adjacent to  $v_1 \dots v_d$
- removing the vertices  $v_2$  to  $v_{d-1}$ .

By minimality of  $G$ ,  $G'$  admits a strong  $(2\Delta - 1)$ -edge coloring  $\phi$ . Our aim is to extend  $\phi$  to  $G$  and get a contradiction.

Let  $c_\phi(u)$  be the set of colors of the edges incident to  $u$ . We can assume that  $|c_\phi(v_0)| = |c_\phi(v_{d+1})| = \Delta$  (by adding vertices of degree 1 adjacent to  $v_0$  and  $v_{d+1}$  in  $G'$  as  $2\Delta < d$  and so  $|V(G')| < |V(G)|$ ). Let  $x = \phi(v_0v_1)$  and  $y = \phi(v_dv_{d+1})$ . For a set  $S$  of colors, define  $\overline{S} = \{1, \dots, 2\Delta - 1\} \setminus S$ .

Extending  $\phi$  to  $G$  is equivalent to find a special path  $P$  in the odd graph  $O_\Delta$ . This path  $P$  must have the following properties:

- P1. its length is  $d + 1$ . Let  $P = u_0u_1 \dots u_{d+1}$ ;
- P2.  $u_0$  is the vertex of  $O_\Delta$  such that  $S(u_0) = c_\phi(v_0)$ ;
- P3.  $u_{d+1}$  is the vertex of  $O_\Delta$  such that  $S(u_{d+1}) = c_\phi(v_{d+1})$ ;
- P4. the edge  $u_0u_1$  is labeled with  $x$ ;
- P5. the edge  $u_du_{d+1}$  is labeled with  $y$ .

Informally speaking, this path may be seen as a mapping of  $v_0 \dots v_{d+1}$  into  $O_\Delta$ . If such a path exists, then one can extend  $\phi$  to  $G$  by coloring the edges incident to  $v_i$  with colors of  $\overline{S}(u_i)$ ; the edge  $v_iv_{i+1}$  is colored with the label of the edge  $u_iu_{i+1}$  in  $O_\Delta$ . In the following, if  $P$  is a path, then we will denote by  $\|P\|$  the length (number of edges) of the path  $P$ .

As  $O_n$  is distance transitive and its even-girth is 6 [1], we have:

- (4) Let  $xyz$  be a simple path of length 2 of  $O_n$  with  $n \geq 3$ . Then  $xyz$  is contained in a cycle of length 6.
- (5) Let  $x$  be a vertex of  $O_n$  with  $n \geq 3$ . Then  $x$  is contained in a cycle of length  $2k$  for any integer  $k \geq 3$ .

The core of the proof is the following lemma (whose proof is omitted in this extended abstract):

**Lemma 2.1.** *Let  $u$  and  $v$  be two (not necessarily distinct) vertices of  $O_n$  with  $n \geq 4$ . There exists a simple path linking  $u$  and  $v$  of length exactly  $2(n - 1)$ .*

We are now able to exhibit the path  $P$  linking  $u_0$  and  $u_{d+1}$ . Let  $P_s = u_0s_1 \dots s_{2(\Delta-1)-1}u_{d+1}$  be a path linking  $u_0$  and  $u_{d+1}$  of length  $2(\Delta - 1)$

in  $O_\Delta$ . Let  $u_1$  be the neighbor of  $u_0$  so that the edge  $u_0u_1$  is labeled with  $x$ . Let  $u_d$  be the neighbor of  $u_{d+1}$  so that the edge  $u_du_{d+1}$  is labeled with  $y$ . As  $\Delta \geq 3$ , let  $t$  be a neighbor of  $u_0$  distinct from  $u_1$  and  $s_1$ , and let  $w$  be a neighbor of  $u_{d+1}$  distinct from  $u_d$  and  $s_{2(\Delta-1)-1}$ . Finally, by (4), let  $C_1$  be a 6-cycle containing  $tu_0u_1$  and let  $C_2$  be a 6-cycle containing  $wu_{d+1}u_d$ .

1. We first start from  $u_0$  making a loop around  $C_1$  going through first  $u_1$ . Hence P2 and P4 are satisfied.
2. We then leave  $u_0$  to  $u_{d+1}$  going through  $P_s$ .
3. Finally we make a loop around  $C_2$  going through first  $w$ . Hence P3 and P5 are satisfied.

Finally observe that  $\|P\| = 6 + \|P_s\| + 6 = 2(\Delta - 1) + 12 = 2\Delta + 10 = d + 1$  as required by P1.

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# On homomorphisms of planar signed graphs to signed projective cubes

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**Abstract.** We conjecture that every planar signed bipartite graph of unbalanced girth  $2g$  admits a homomorphism to the signed projective cube of dimension  $2g - 1$ . Our main result is to show that for a given  $g$ , this conjecture is equivalent to the corresponding case ( $k = 2g$ ) of a conjecture of Seymour claiming that every planar  $k$ -regular multigraph with no odd edge-cut of less than  $k$  edges is  $k$ -edge-colorable.

## 1 Preliminaries

It is a classic result of Tait from 1890 that the Four-Color Theorem (Conjecture at that time) is equivalent to the statement that every cubic bridgeless planar graph is 3-edge-colorable. It is easily observed that if a  $k$ -regular multigraph is  $k$ -edge-colorable, then the number of edges with exactly one end in  $X$ , assuming  $|X|$  is odd, is at least  $k$ . Seymour conjectured in 1975 that for planar graphs the converse is also true:

**Conjecture 1.1 (Seymour [6]).** Every  $k$ -regular planar multigraph with no odd edge-cut of less than  $k$  edges is  $k$ -edge-colorable.

In this work we focus on a direct extension of the Four-Color Theorem using homomorphisms of signed graphs and its relation with Seymour's conjecture. The theory of homomorphisms of signed graphs includes in particular the theory of graph homomorphisms. Here we introduce the

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basic notations but we refer to [4], to [5] and to references mentioned there for more details.

Given a graph  $G$ , a *signature* on  $G$  is a mapping that assigns to each edge of  $G$  either a positive or a negative sign. A signature is normally denoted by the set  $\Sigma$  of negative edges. Given a signature  $\Sigma$  on a graph  $G$ , *resigning* at a vertex  $v$  is to change the sign of each edge incident with  $v$ . Two signatures  $\Sigma_1$  and  $\Sigma_2$  on  $G$  are *equivalent* if one can be obtained from the other by a sequence of resignings or, equivalently, by changing the signs of the edges of an edge-cut. A *signed graph*, denoted  $(G, \Sigma)$ , is a graph together with a class of equivalent signatures where  $\Sigma$  could be any member of the class.

An *unbalanced cycle* in a signed graph  $(G, \Sigma)$  is a cycle having an odd number of negative edges. Note that this is independent of the choice of a representative signature. Furthermore, the notion of unbalanced cycle is, in some sense, an extension of the classic notion of odd cycle, as a cycle of  $(G, E(G))$  is unbalanced if and only if it is an odd cycle. The *unbalanced-girth* of  $(G, \Sigma)$  is then the shortest length of an unbalanced cycle of  $(G, \Sigma)$ .

**Theorem 1.2 (Zaslavsky [7]).** *Two signatures  $\Sigma_1$  and  $\Sigma_2$  on a graph  $G$  are equivalent if and only if they induce the same set of unbalanced cycles.*

A cycle that is not unbalanced, i.e., a cycle that has an even number of negative edges (possibly none) is called *balanced*.

An important subclass of signed graphs, called *consistent signed graphs*, is the class of signed graphs whose balanced cycles are all of even length and whose unbalanced cycles are all of the same parity. This class itself consists of two parts. When all the unbalanced cycles are odd, then the set of unbalanced cycles of  $(G, \Sigma)$  is exactly the set of odd cycles of  $G$ , thus in this case, by Theorem 1.2,  $E(G)$  is a signature. Such a signed graph will then be called an *odd signed graph*. When all balanced and unbalanced cycles are even, the graph  $G$  must be bipartite, and  $\Sigma$  can be any subset of edges. Such a signed graph will be called an *even signed graph*.

Given two graphs  $G$  and  $H$ , a *homomorphism* of  $G$  to  $H$  is a mapping  $\phi : V(G) \rightarrow V(H)$  such that if  $xy \in E(G)$  then  $\phi(x)\phi(y) \in E(H)$ . Given two signed graphs  $(G_1, \Sigma_1)$  and  $(G_2, \Sigma_2)$  we say that  $(G_1, \Sigma_1)$  admits a *signed homomorphism*, or a homomorphism for short, to  $(G_2, \Sigma_2)$  if there are signatures  $\Sigma'_1$  and  $\Sigma'_2$  equivalent to  $\Sigma_1$  and  $\Sigma_2$ , respectively, and a homomorphism  $\varphi$  of  $G_1$  to  $G_2$  such that  $\varphi$  also preserves the signs of edges given by  $\Sigma'_1$  and  $\Sigma'_2$ . The binary relation  $(G_1, \Sigma_1) \rightarrow (G_2, \Sigma_2)$ ,

which denotes the existence of a homomorphism of  $(G_1, \Sigma_1)$  to  $(G_2, \Sigma_2)$ , is an associative one.

Using the definition of *signed projective cube* from the later paragraph, the following conjecture is the main concern of this work:

**Conjecture 1.3.** Every consistent planar signed graph of unbalanced girth  $k$  admits a homomorphism to the signed projective cube of dimension  $k - 1$ .

In an unpublished manuscript [1], B. Guenin, after introducing the notion of signed graph homomorphisms, proposed that the larger class of consistent signed graphs with an additional property (having no  $(K_5, E(K_5))$ -signed-minor) satisfies the same conclusion. He has shown relations between his conjecture and several other conjectures.

The case  $k = 3$  of Conjecture 1.3 is the Four-Color Theorem. For odd values of  $k$  it turns out that all the edges of both sides are negative, and thus the problem is reduced to a graph homomorphism problem. Therefore, by [3], for each such  $k$  this conjecture is equivalent to Conjecture 1.1 with the corresponding value of  $k$ . In this work we prove the analog for even values of  $k$  and thus we examine only the following case of the conjecture.

**Conjecture 1.3 (even case).** Every planar even signed graph of unbalanced girth at least  $2g$  admits a homomorphism to  $S\mathcal{PC}_{2g-1}$ .

We now introduce signed projective cubes and some of their properties.

The *projective cube* of dimension  $d$ , denoted  $\mathcal{PC}_d$ , is the Cayley graph  $(\mathbb{Z}_2^d, \{e_1, e_2, \dots, e_d\} \cup \{J\})$  where  $e_i$  is the vector of  $\mathbb{Z}_2^d$  with the  $i$ -th coordinate being 1 and other coordinates being 0 and  $J = (1, 1, 1, \dots, 1)$ . Let  $\mathcal{J}$  be the set of edges corresponding to  $J$ . We define the *signed projective cube* of dimension  $d$ , denoted  $S\mathcal{PC}_d$ , to be the signed graph  $(\mathcal{PC}_d, \mathcal{J})$ .

**Theorem 1.4 ([5]).** *All balanced cycles of  $S\mathcal{PC}_d$  are of even length, all unbalanced cycles of  $S\mathcal{PC}_d$  are of the same parity, and the unbalanced girth of  $S\mathcal{PC}_d$  is  $d + 1$ . Furthermore, for each unbalanced cycle  $UC$  of  $S\mathcal{PC}_d$  and for each  $x \in \{e_1, e_2, \dots, e_d\} \cup \{J\}$ , there is an odd number of edges of  $UC$  which correspond to  $x$ .*

Using Theorem 1.4 and Theorem 1.2 we get that the signed projective cube  $S\mathcal{PC}_{2d}$  (resp.  $S\mathcal{PC}_{2d-1}$ ) is an odd (resp. even) signed graph and thus if a signed graph  $(G, \Sigma)$  admits a homomorphism to  $S\mathcal{PC}_{2d}$  (resp.  $S\mathcal{PC}_{2d-1}$ ), we conclude that  $(G, \Sigma)$  must also be odd (resp. even). Thus, in general, consistent signed graphs are the only graphs that can map to signed projective cubes. The following theorem shows that the problem

of finding a mapping of a consistent signed graph to a signed projective cube is equivalent to a packing problem.

**Theorem 1.5 ([1,5]).** *An even (resp. odd) signed graph admits a homomorphism to  $SPC_{2d-1}$  (resp.  $SPC_{2d}$ ) if and only if it admits at least  $2d - 1$  (resp.  $2d$ ) edge disjoint signatures.*

## 2 An extension of the Four-Color Theorem

A key lemma in the study of homomorphism properties of a planar graph is the folding lemma of Klostermeyer and Zhang [2]. By considering unbalanced cycles instead of odd cycles in the lemma, using the ideas of Section 4 in [2] we will get the same result for the class of planar even signed graphs.

**Lemma 2.1 (Folding lemma).** [5] *Let  $G$  be a planar even signed graph of unbalanced girth at least  $g$ . If  $C = v_0 \cdots v_{r-1} v_0$  is a facial cycle of  $G$  with  $r \neq g$ , then there is an integer  $i \in \{0, \dots, r - 1\}$  such that the graph  $G'$  obtained from  $G$  by identifying  $v_{i-1}$  and  $v_{i+1}$  (subscripts are taken modulo  $r$ ) is still of unbalanced girth  $g$ .*

By repeated application of this lemma we conclude that:

**Corollary 2.2.** *Given a planar even signed graph  $(G, \Sigma)$  of unbalanced girth at least  $g$ , there is a homomorphic image  $(H, \Sigma_1)$  of  $(G, \Sigma)$  such that: (i)  $H$  is planar, (ii)  $(H, \Sigma_1)$  is an even signed graph, (iii)  $(H, \Sigma_1)$  is of unbalanced girth  $g$  and (iv) every facial cycle of  $(H, \Sigma_1)$  is an unbalanced cycle of length  $g$ .*

We are now ready to prove the main theorem of the paper.

**Theorem 2.3.** *The following two statements are equivalent:*

- (i) *Every planar  $2g$ -regular multigraph with no odd edge-cut of less than  $2g$  edges is  $2g$ -edge-colorable.*
- (ii) *Every planar even signed graph of unbalanced girth at least  $2g$  admits a homomorphism to  $SPC_{2g-1}$ .*

*Proof.* First assume that every planar even signed graph of unbalanced girth at least  $2g$  admits a homomorphism to  $SPC_{2g-1}$  and let  $G$  be a planar  $2g$ -regular multigraph with no odd edge-cut of less than  $2g$  edges. Using Tutte's matching theorem we can easily verify that  $G$  admits a perfect matching. Let  $M$  be a perfect matching of  $G$ . Let  $G^D$  be the dual of  $G$  with respect to some embedding of  $G$  on the plane. Since  $G$  is  $2g$ -regular,  $G^D$  is clearly bipartite. Let  $M^D$  be the edges in  $G^D$  corresponding to the edges of  $M$ . It is now easy to check that  $(G^D, M^D)$  is

a planar even signed graph of unbalanced girth  $2g$ . Therefore, by our main assumption,  $(G^D, M^D)$  admits a homomorphism to  $\mathcal{SPC}_{2g-1}$ . This mapping induces a  $2g$ -edge-coloring on  $G^D$  (not necessarily a proper edge-coloring) using colors  $e_1, \dots, e_{2g-1}, J$ . By Theorem 1.4 every unbalanced cycle has received exactly  $2g$  different colors. In particular each face of  $G^D$ , which is an unbalanced cycle of length  $2g$ , has received all  $2g$  colors. Thus reassigning these colors to their corresponding edges in  $G$  will result in a proper  $2g$ -edge-coloring of  $G$ .

Now we assume that every planar  $2g$ -regular multigraph with no odd edge-cut of less than  $2g$  edges is (properly)  $2g$ -edge-colorable. Let  $(G, \Sigma)$  be a plane even signed graph of unbalanced girth  $2g$ . We would like to prove that this signed graph admits a homomorphism to  $\mathcal{SPC}_{2g-1}$ . By Corollary 2.2 we may assume that each face of  $(G, \Sigma)$  is an unbalanced cycle of length exactly  $2g$ . Let  $G^D$  be the dual of  $G$  with respect to its embedding on the plane. Obviously  $G^D$  is a  $2g$ -regular multigraph, furthermore it is easy to check that  $G^D$  has no odd edge-cut of strictly less than  $2g$  edges (this is the dual of having unbalanced girth at least  $2g$ ). Thus, by our main assumption,  $G^D$  is  $2g$ -edge colorable. Let  $M_i$  be one of the color classes, which, therefore, is a perfect matching. Let  $\Sigma_i$  be the edges of  $G$  corresponding to the edges of  $G^D$  in  $M_i$ . We claim that  $\Sigma_i$  is equivalent to  $\Sigma$ . This is the case because in both  $(G, \Sigma_i)$  and  $(G, \Sigma)$  each face is an unbalanced cycle, and any other cycle is unbalanced if and only if it bounds an odd number of faces. That means that the sets of unbalanced cycles in both signatures are the same and the claim follows by Theorem 1.2. To complete the proof note that we have partitioned edges of  $G$  into  $2g$  sets  $\Sigma_i$  each being a signature of  $(G, \Sigma)$ . Thus, by Theorem 1.5,  $(G, \Sigma)$  admits a homomorphism to  $\mathcal{SPC}_{2g-1}$ .  $\square$

Note that if  $G$  is a simple bipartite graph, then the unbalanced girth of  $(G, \Sigma)$  is at least 4. Furthermore, note that  $\mathcal{SPC}_3$  is isomorphic to  $(K_{4,4}, M)$  where  $M$  is a perfect matching of  $K_{4,4}$ . Therefore:

**Corollary 2.4.** *Every planar even signed graph admits a homomorphism to  $(K_{4,4}, M)$ .*

Using Theorem 6.2 of [4] it follows that this corollary is stronger than the Four-Color Theorem. This fact, in the edge-coloring formulation, was already proved by P. Seymour [6].

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# Classification of $k$ -nets embedded in a plane

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**Abstract.** We present some recent, partly unpublished, results on  $k$ -nets embedded in a projective plane  $PG(2, \mathbb{K})$  defined over a field  $\mathbb{K}$  of any characteristic  $p \geq 0$ , obtained in collaboration with G.P. Nagy and N. Pace.

## 1 Introduction

A general problem in finite geometry is to determine geometric structures, such as graphs, designs and incidence geometries, which can be embedded in a projective plane. Here we deal with  $k$ -nets embedded in a projective plane  $PG(2, \mathbb{K})$  defined over a field  $\mathbb{K}$  of any characteristic  $p \geq 0$ . They are line configurations in  $PG(2, \mathbb{K})$  consisting of  $k$  pairwise disjoint line-sets, called components, such that any two lines from distinct families are concurrent with exactly one line from each component. The size of each component of a  $k$ -net is the same, the order of the  $k$ -net. The concept of a  $k$ -net arose in classical Differential geometry, and there is a long history about finite  $k$ -nets in Combinatorics, especially for  $k = 3$ , related to affine planes, latin squares, loops and strictly transitive permutation sets. In recent years a strong motivation for investigation of  $k$ -nets embedded in  $PG(2, \mathbb{K})$  came from Algebraic geometry and Resonance theory see [2, 8, 9, 13, 14].

## 2 $k$ -nets embedded in $PG(2, \mathbb{K})$

The Stipins-Yuzvinsky theorem states that no embedded  $k$ -net for  $k \geq 5$  exists when  $p = 0$ ; see [11, 14]. Our present investigation of  $k$ -nets embedded in  $PG(2, \mathbb{K})$  includes groundfields  $\mathbb{K}$  of positive characteristic

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$p$ , and as a matter of fact, many examples. This phenomena is not unexpected since  $PG(2, \mathbb{K})$  with  $\mathbb{K}$  of characteristic  $p > 0$  contains an affine subplane  $AG(2, \mathbb{F}_p)$  of order  $p$  from which  $k$ -nets for  $3 \leq k \leq p + 1$  arise taking  $k$  parallel line classes as components. Similarly, if  $PG(2, \mathbb{K})$  also contains an affine subplane  $AG(2, \mathbb{F}_{p^h})$ , in particular if  $\mathbb{K} = \mathbb{F}_q$  with  $q = p^r$  and  $h|r$ , then  $k$ -nets of order  $p^h$  for  $3 \leq k \leq p^h + 1$  exist in  $PG(2, \mathbb{K})$ . Actually, more families of  $k$ -nets embedded in  $PG(2, \mathbb{F}_q)$  when  $q = p^r$  with  $r \geq 3$  arise from Lunardon's work; see [5, 6]. On the other hand, no 5-net of order  $n$  with  $p > n$  is known to exist. This suggests that for sufficiently large  $p$  compared with  $n$ , the Stipins-Yuzvinsky theorem remains valid in  $PG(2, \mathbb{K})$ . Our main result [4] in this direction proves it:

**Theorem 2.1.** *If  $p > 3^{\varphi(n^2-n)}$  where  $\varphi$  is the classical Euler function then no  $k$ -net with  $k \geq 5$  is embedded in  $PG(2, \mathbb{K})$ .*

Our approach also works in zero characteristic and provides a new proof for the Stipins-Yuzvinsky theorem. A key idea in our proof is to consider the cross-ratio of four concurrent lines from different components of a 4-net. We prove that the cross-ratio remains constant when the four lines vary without changing component. In other words, every 4-net in  $PG(2, \mathbb{K})$  has constant cross-ratio. In zero characteristic, and in characteristic  $p$  with  $p > 3^{\varphi(n^2-n)}$ , the constant cross-ratio is restricted to two values only, namely to the roots of the polynomial  $X^2 - X + 1$ . From this, the non-existence of  $k$ -nets for  $k \geq 5$  easily follows both in zero characteristic and in characteristic  $p$  with  $p > 3^{\varphi(n^2-n)}$ . It should be noted that without a suitable hypothesis on  $n$  with respect to  $p$ , the constant cross-ratio of a 4-net may assume many different values, even for finite fields.

In the complex plane, there is known only one 4-net up to projectivity; see [11–14]. This 4-net, called the classical 4-net, has order 3 and it exists since  $PG(2, \mathbb{C})$  contains an affine subplane  $AG(2, \mathbb{F}_3)$  of order 3, unique up to projectivity, and the four parallel line classes of  $AG(2, \mathbb{F}_3)$  are the components of a 4-net in  $PG(2, \mathbb{K})$ . It has been conjectured that the classical 4-net is the only 4-net embedded in  $PG(2, \mathbb{C})$ .

### 3 3-nets embedded in $PG(2, \mathbb{K})$

There are known plenty of 3-nets embedded in  $PG(2, \mathbb{K})$ . One infinite family arises from plane cubic curves. More precisely, let  $\mathcal{C}$  be a plane (possible reducible) cubic curve equipped with its abelian group  $(G, +)$  defined on the set of the nonsingular points of  $\mathcal{C}$ , take three distinct cosets  $H + a$ ,  $H + b$ ,  $H + c$  of a subgroup  $H$  of  $G$  of order  $n$ , such

that  $a + b + c = 0$ . Then, in the dual plane, the lines corresponding to the points in these three cosets are the component of a 3-net. Such embedded 3-nets in  $PG(2, \mathbb{K})$  are called *algebraic*. The isotopy class of quasigroups coordinatizing an algebraic 3-net contains a group isomorphic to  $H$ . Another infinite family arise from tetrahedrons of  $PG(3, \mathbb{K})$  by projection, and called of *tetrahedron type*. In the dual plane, the components  $\Lambda_1, \Lambda_2, \Lambda_3$  of a tetrahedron type embedded 3-net lie on the six sides (diagonals) of a non-degenerate quadrangle such a way that  $\Lambda_i = \Delta_i \cup \Gamma_i$  with  $\Delta_i$  and  $\Gamma_i$  lying on opposite sides, for  $i = 1, 2, 3$ . More precisely,  $(\Lambda_1, \Lambda_2, \Lambda_3)$  can be lifted to the fundamental tetrahedron of  $PG(3, \mathbb{K})$  so that the projection  $\pi$  from the point  $P_0 = (1, 1, 1, 1)$  on the plane  $X_4 = 0$  returns  $(\Lambda_1, \Lambda_2, \Lambda_3)$ . For this purpose, it is enough to define the sets lying on the edges of the fundamental tetrahedron:

$$\begin{aligned} \Gamma'_1 &= \{(\xi, 0, 1, 0) \mid \xi \in L_1\}, & \Gamma'_2 &= \{(0, \eta, 1, 0) \mid \eta \in L_2\}, \\ \Gamma'_3 &= \{(1, -\zeta, 0, 0) \mid \zeta \in L_3\}, & \Delta'_1 &= \{(0, \alpha - 1, 0, -1) \mid \alpha \in M_1\}, \\ \Delta'_2 &= \{(\beta - 1, 0, 0, -1) \mid \beta \in M_2\}, & \Delta'_3 &= \{(0, 0, \gamma - 1, -1) \mid \gamma \in M_3\}, \end{aligned}$$

and observe that  $\pi(\Gamma'_i) = \Gamma_i$  and  $\pi(\Delta'_i) = \Delta_i$  for  $i = 1, 2, 3$ . Moreover, a triple  $(P_1, P_2, P_3)$  of points with  $P_i \in \Gamma_i \cup \Delta_i$  consists of collinear points if and only if their projection does. Hence,  $(\Gamma'_1 \cup \Gamma'_2, \Gamma'_3 \cup \Delta'_1, \Delta'_2 \cup \Delta'_3)$  can be viewed as a “spatial” dual 3-net realizing the same group  $H$ . Clearly,  $(\Gamma'_1 \cup \Gamma'_2, \Gamma'_3 \cup \Delta'_1, \Delta'_2 \cup \Delta'_3)$  is contained in the sides of the fundamental tetrahedron. We claim that these sides minus the vertices form an infinite spatial dual 3-net realizing the dihedral group  $2.\mathbb{K}^*$ . To prove this, parametrize the points as follows.

$$\begin{aligned} \Sigma_1 &= \{x_1 = (x, 0, 1, 0), (\varepsilon x)_1 = (0, 1, 0, x) \mid x \in \mathbb{K}^*\}, \\ \Sigma_2 &= \{y_2 = (1, y, 0, 0), (\varepsilon y)_2 = (0, 0, 1, y) \mid y \in \mathbb{K}^*\}, \\ \Sigma_3 &= \{z_3 = (0, -z, 1, 0), (\varepsilon z)_3 = (1, 0, 0, -z) \mid z \in \mathbb{K}^*\}. \end{aligned} \tag{3.1}$$

Then,

$$\begin{aligned} x_1, y_2, z_3 \text{ are collinear} &\Leftrightarrow z = xy, \\ (\varepsilon x)_1, y_2, (\varepsilon z)_3 \text{ are collinear} &\Leftrightarrow z = xy \Leftrightarrow \varepsilon z = (\varepsilon x)y, \\ x_1, (\varepsilon y)_2, (\varepsilon z)_3 \text{ are collinear} &\Leftrightarrow z = x^{-1}y \Leftrightarrow \varepsilon z = x(\varepsilon y), \\ (\varepsilon x)_1, (\varepsilon y)_2, z_3 \text{ are collinear} &\Leftrightarrow z = x^{-1}y \Leftrightarrow z = (\varepsilon x)(\varepsilon y). \end{aligned}$$

Thus,  $(\Gamma'_1 \cup \Gamma'_2, \Gamma'_3 \cup \Delta'_1, \Delta'_2 \cup \Delta'_3)$  is a dual 3-subnet of  $(\Sigma_1, \Sigma_2, \Sigma_3)$  and  $H$  is a subgroup of the dihedral group  $2.\mathbb{K}^*$ . As  $H$  is not cyclic but it has a cyclic subgroup of index 2, we conclude that  $H$  is itself dihedral.

The isotopy class of quasigroups coordinatizing a tetrahedron type 3-net is a dihedral group. We also know a sporadic example; namely the

Urzúa 3-net of order 8 coordinatized by the quaternion group of order 8; see [12].

In [3] we are dealt with 3-nets embedded in  $PG(2, \mathbb{K})$  which are coordinatized by groups. Our main result is a complete classification for  $p = 0$ :

**Theorem 3.1.** *In the projective plane  $PG(2, \mathbb{K})$  defined over an algebraically closed field  $\mathbb{K}$  of characteristic  $p \geq 0$ , let  $\Lambda$  be an embedded 3-net of order  $n \geq 4$  coordinatized by a group  $G$ . If either  $p = 0$  or  $p > n$  then one of the following holds.*

- (I)  $G$  is either cyclic or the direct product of two cyclic groups, and  $\Lambda$  is algebraic.
- (II)  $G$  is dihedral and  $\Lambda$  is of tetrahedron type.
- (III)  $G$  is the quaternion group of order 8, and  $\Lambda$  is the Urzúa 3-net of order 8..
- (IV)  $G$  has order 12 and is isomorphic to  $\text{Alt}_4$ .
- (V)  $G$  has order 24 and is isomorphic to  $\text{Sym}_4$ .
- (VI)  $G$  has order 60 and is isomorphic to  $\text{Alt}_5$ .

A computer aided exhaustive search shows that if  $p = 0$  then (IV) (and hence (V), (VI)) does not occur, see [7]. Theorem 3.1 shows that every realizable finite group can act in  $PG(2, \mathbb{K})$  as a projectivity group. This confirms Yuzvinsky's conjecture for  $p = 0$ .

A combinatorial characterization of algebraic 3-nets contained in a reducible plane cubic is given in [1]:

**Theorem 3.2.** *Let  $p > n$  or  $p = 0$ . If a component of a dual 3-net is contained in a pencil, then the other two components consist of lines tangents to a unique conic, and the converse also holds.*

The proofs of Theorems 3.1 and 3.2 use several results on collineation groups of  $PG(2, \mathbb{K})$  together with the classification of subgroups of  $PGL(2, \mathbb{K})$  due to Dickson. Furthermore, the proof of Theorem 3.2 also uses the "Rédei polynomial approach" of Szőnyi for the study of blocking-sets in  $PG(2, q)$ ; see [10].

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# An improved lower bound on the maximum number of non-crossing spanning trees

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**Abstract.** We address the problem of counting geometric graphs on point sets. Using analytic combinatorics we show that the so-called double chain point configuration of  $N$  points has  $\Omega^*(12.31^N)$  non-crossing spanning trees and  $\Omega^*(13.40^N)$  non-crossing forests. This improves the previous lower bounds on the maximum number of non-crossing spanning trees and of non-crossing forests among all sets of  $N$  points in general position given by Dumitrescu, Schulz, Sheffer and Tóth in 2011. A new upper bound of  $O^*(22.12^N)$  for the number of non-crossing spanning trees of the double chain is also obtained.

## 1 Introduction

A geometric graph on a point set  $S$  (throughout,  $S$  has no three collinear points) is a graph with vertex set  $S$  and whose edges are straight-line segments with endpoints in  $S$ . A geometric graph is called *non-crossing* (nc- for short) if no two edges intersect except at common endpoints. Counting nc-geometric graphs is a prominent problem in combinatorial geometry. In [7] it is proved that no set of  $N$  points has more than  $O^*(141.07^N)$  nc-spanning trees.<sup>3</sup> The maximum number of nc-spanning trees (among all sets of  $N$  points) is very likely much smaller. The point set with most nc-spanning trees known so far is the so-called *double chain*. The double chain of  $N = 2n$  points consists of two sets of  $n$  points each, one forming a convex chain (the lower chain) and one forming a concave chain (the upper chain). Furthermore, each straight-line defined by two points from the upper chain leaves all the points from the lower chain on the same side, and reversely; see Figure 2.1. Counting nc-geometric graphs

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<sup>3</sup> We use the  $O^*$ -,  $\Theta^*$ -, and  $\Omega^*$ -notation to describe the asymptotic growth of the number of geometric graphs as a function of the number  $N$  of points, neglecting polynomial factors.

on the double chain was initiated by García et al. [6] who proved that it has  $\Theta^*(8^N)$  triangulations,  $\Omega^*(9.35^N)$  nc-spanning trees and  $\Omega^*(4.64^N)$  nc-polygonizations, where the latter bound also is the current best lower bound on the maximum number of nc-polygonizations. The lower bound for the number of nc-spanning trees of the double chain was subsequently improved to  $\Omega^*(10.42^N)$  [2] and to  $\Omega^*(12.0026^N)$  [3]. We further improve this bound to  $\Omega^*(12.31^N)$  and also show a new lower bound of  $\Omega^*(13.40^N)$  for the maximum number of nc-forests among all sets of  $N$  points. The methods we use are analytic combinatorics and singularity analysis, with the same spirit and techniques as in [4]. We finally provide a new upper bound of  $O^*(22.12^N)$  for the number of nc-spanning trees of the double chain.

## 2 The lower bound

**Theorem 2.1.** *The double chain on  $N$  points has  $\Omega^*(12.31^N)$  non-crossing spanning trees.*

In this section we describe a family of trees that gives the desired bound. Our construction depends on some parameters that need to be maximized later.

For any spanning tree of the double chain, the vertices on the upper and lower chains induce two forests  $F_U$  and  $F_L$  on a set of  $n$  points in convex position; there are also some edges with one endpoint on each chain (the interior edges). The first restriction is that we consider trees where only one vertex in each component of  $F_U$  is incident to interior edges; this vertex will be called the *mark* of the component and we call  $F_U$  a *marked forest*.

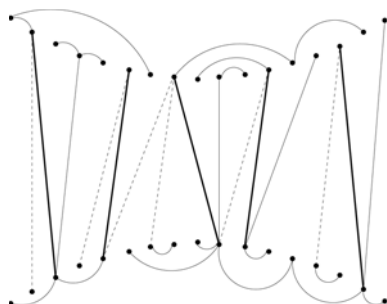
Of the several interior edges that are incident to a mark  $v$ , let  $e_v$  be the rightmost one. The set of edges  $M_1 = \{e_v : v \text{ is a mark}\}$  induces a forest; this forest is uniquely determined by the leftmost edge in each component (assuming the set of marks is known). The set of these edges is denoted  $M_2$ .

Now we pose two further restrictions on our trees. First, no edge  $e_v$  is incident with an isolated vertex of  $F_L$ . Second, for each component  $C$  of  $F_L$  that is not incident to an edge  $e_v$ , there is a unique edge joining this component to a mark  $m_C$  in  $F_U$ , and this edge has as endpoint in  $F_L$  the leftmost vertex  $v_C$  in  $C$ . Moreover,  $m_C$  is as to the right as possible.

See Figure 2.1 for an example of such a tree.

We claim that to uniquely determine such a tree, it is enough to give:

- s1) A marked nc-forest  $F_U$  with at least  $k$  components on a set of  $n$  points in convex position,



**Figure 2.1.** A spanning tree of the double chain. The bold edges are the edges of  $M_2$  and the thin continuous interior edges are the other edges of  $M_1$ . The edges in  $F_U$  and  $F_L$  are drawn as arcs.

- s2) a subset of  $k$  of the marks in  $F_U$ ,
- s3) a nc-forest  $F_L$  on a set of  $n$  points in convex position, and
- s4) a subset  $M_L$  of  $k$  vertices in  $F_L$  such that none of these vertices induces a component of  $F_L$ .

Indeed, it suffices to do the following:

- t1) Match the  $k$  marks from s2) with the  $k$  vertices in  $F_L$  (corresponding to the edges of  $M_2$ );
- t2) join the other marks in  $F_U$  to the leftmost visible vertex in  $M_L$  (corresponding to the edges of  $M_1 \setminus M_2$ );
- t3) for each component of  $F_L$  that has no vertex in  $M_L$ , take its leftmost vertex and join it to the rightmost visible mark;
- t4) if the result is not connected, let  $C_1, \dots, C_r$  be its connected components, from left to right as encountered in the upper chain; for  $i \geq 2$ , join the leftmost mark in  $C_i$  to the rightmost visible vertex of  $M_L$  in  $C_{i-1}$ .

It can be shown using the methods of Section 3 that the number of nc-forests on  $n$  points in convex position where each component has a mark is  $\Theta^*(9.5816 \dots^n)$ , and that for  $n$  sufficiently large, at least 40% of them have  $0.2237n$  or more components each. However, we choose  $F_U$  in two steps, as it actually gives us more choice. First we select  $\ell$  vertices out of the  $n$  in the upper chain, and mark all of them, and then choose a marked forest  $F'_U$  on  $n - \ell$  vertices, such that all components in this forest have at least two vertices.

To determine the forest  $F_L$  in s3) and s4) above, we choose  $m$  of the  $n$  vertices of the lower chain that will be isolated, and then choose a forest without isolated vertices on the remaining  $n - m$  points.

The following estimates will be proved in Section 3.

**Proposition 2.2.**

- (i) *The number of marked non-crossing forests on  $n$  points in convex position such that no component is an isolated vertex is  $cn^{-3/2}\omega_U^n(1+O(1/n))$ , where  $\omega_U = 8.5816\dots$  and  $c$  is a constant. Moreover, for  $n$  large enough at least 40% of these forests have  $0.1332n$  or more components.*
- (ii) *The number of non-crossing forests on  $n$  points in convex position such that no component is an isolated vertex is  $dn^{-3/2}\omega_L^n(1+O(1/n))$ , where  $\omega_L = 7.2246\dots$  and  $d$  is a constant.*

Setting  $k = \beta n$ ,  $\ell = \gamma n$  and  $m = \alpha n$ , and ignoring subexponential terms, we get the following lower bound on the number of nc-spanning trees of the double chain:

$$\binom{n}{\gamma n} 8.5816^{n-\gamma n} \binom{0.1332(n-\gamma n)+\gamma n}{\beta n} \binom{n}{\alpha n} 7.2246^{n-\alpha n} \binom{n-\alpha n}{\beta n}.$$

Note that when choosing the  $k = \beta n$  marks from the upper forest  $F_U$  we are actually not using all the available marks, but only the ones that come from isolated vertices and the first  $0.1332(n - \gamma n)$  marks of the marked forest  $F'_U$ .

Using the binary entropy function  $H(x) = -x \log_2(x) - (1-x) \log_2(1-x)$ , we can estimate a binomial coefficient as  $\binom{\epsilon n}{\delta n} \approx 2^{\epsilon H(\frac{\delta}{\epsilon})n}$ .

The values  $\alpha = 0.09$ ,  $\beta = 0.256$ ,  $\gamma = 0.258$  give  $\Omega^*(2^{7.244139N/2}) = \Omega^*(12.3126^N)$  nc-spanning trees on the double chain on  $N$  vertices, thus proving Theorem 2.1.

Using a similar construction we can also prove a lower bound on the number of nc-forests of the double chain. The details are omitted due to lack of space and will appear elsewhere.

**Theorem 2.3.** *The double chain on  $N$  points has  $\Omega^*(13.40^N)$  non-crossing forests.*

The bound of  $\Theta^*(9.5816\dots^n)$  marked forests on  $n$  points in convex position implies a bound on the number of nc-spanning trees of the *single chain*. This point set has triangular convex hull and all but one point  $p$  of the set form a convex chain; also see [1]. To count the number of nc-spanning trees of this set, connect each mark of a tree in a forest on the convex chain to  $p$ .

**Corollary 2.4.** *The single chain on  $N$  points has  $\Theta^*(9.5816\dots^N)$  non-crossing spanning trees.*



### 3 Non-crossing forests of points in convex position

In this section we use generating functions and the techniques of analytic combinatorics to prove Proposition 2.2.

Consider a set of  $n$  points in convex position, labelled counterclockwise  $p_1, \dots, p_n$ ; the vertex  $p_1$  is called the root vertex. A systematic study of several classes of non-crossing graphs with generating functions was undertaken by Flajolet and Noy in [4]; the results in this section are an extension of theirs using the same techniques (see also the book [5] by Flajolet and Sedgewick).

Let  $\mathcal{T}$  be a set of nc-trees and let  $\mathcal{F}_{\mathcal{T}}$  be the set of those nc-forests such that its connected components belong to  $\mathcal{T}$  (by taking as the root of each component the vertex with smallest label and relabelling the other vertices suitably). Let  $T(z)$  and  $F_{\mathcal{T}}(z)$  be the corresponding generating functions, that is,

$$T(z) = \sum_{n \geq 1} t_n z^n, \quad F_{\mathcal{T}}(z) = \sum_{n \geq 0} f_n z^n,$$

where  $t_n$  and  $f_n$  denote the number of  $n$ -vertex graphs in  $\mathcal{T}$  and  $\mathcal{F}_{\mathcal{T}}$ , respectively. For technical convenience, we set  $f_0 = 1$  (but  $t_0 = 0$ ).

We have the following key relation

$$F_{\mathcal{T}}(z) = 1 + T(zF_{\mathcal{T}}(z)). \quad (3.1)$$

The combinatorial explanation is as follows (see Figure 3.1). Given a forest in  $\mathcal{F}_{\mathcal{T}}$ , let  $t_1$  be the connected component that contains the root vertex; this component is of course a tree of  $\mathcal{T}$ . Now, the vertices (if any) that lie strictly between any two consecutive vertices of  $t_1$  induce a nc-forest, which belongs to  $\mathcal{F}_{\mathcal{T}}$ . The substitution  $zF_{\mathcal{T}}(z)$  in  $T(z)$  reflects this (where the term  $z$  corresponds to a vertex of  $t_1$ ). Thus, if an expression or an equation for  $T(z)$  is known, we immediately get from equation (3.1) an equation for  $F_{\mathcal{T}}(z)$ .

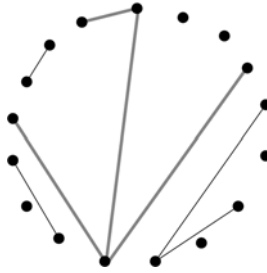
It is well-known that the generating function  $T(z)$  for nc-trees satisfies

$$T(z)^3 - zT(z) + z^2 = 0. \quad (3.2)$$

Let  $\mathcal{T}_1$  be the class of nc-trees with more than one vertex; clearly  $T_1(z) = T(z) - z$ . From equations (3.2) and (3.1) it follows that  $Y = F_{\mathcal{T}_1}(z)$  satisfies

$$(1+z)^3 Y^3 - (3z^2 + 7z + 3)Y^2 + (4z + 3)Y - 1 = 0. \quad (3.3)$$

Now let  $\mathcal{T}_1^*$  be the class of marked nc-trees; then  $T_1^*(z) = zT'(z) - z$ . By differentiating equation (3.2) and eliminating  $T(z)$  we get an equation for



**Figure 3.1.** A nc-forest decomposes as a nc-tree (in grey) with a (possibly empty) nc-forest between any two of its consecutive vertices.

$T'(z)$ , namely,

$$(27z^2 - 4z)T'(z)^3 + (1 - 6z)T'(z) - 1 + 8z = 0,$$

from which an equation for  $T_1^*(z)$  follows immediately. Again from (3.1) we obtain an equation for  $Y = F_{T_1^*}(z)$ :

$$27z(1 + z)^3Y^4 - (83z^3 + 180z^2 + 93z + 4)Y^3 + (99z^2 + 106z + 12)Y^2 - (12 + 40z)Y + 4 = 0. \tag{3.4}$$

Once an algebraic equation for  $F_T(z)$  is known, it is routine to obtain an asymptotic estimate of the coefficients of  $F_T(z)$ . The method we apply is the one described in [4, Section 4] or, more generally, in Lemma VII.3 and Corollary VI.1 of [5]. The main idea is that the singularity of  $F_T(z)$  with smallest modulus determines the asymptotic behaviour of the coefficients of  $F_T(z)$ . More concretely, if  $\rho$  is this singularity, and under certain conditions that in our case are immediate to check, then  $[z^n]F_T(z) = \gamma n^{-3/2} \rho^{-n} (1 + \mathcal{O}(1/n))$ , for some constant  $\gamma$ . To find  $\rho$  we use the fact that it must be one of the roots of the discriminant of the equation satisfied by  $F_T(z)$ , as explained in [5, Section VII.7.1]. Carrying out the calculations for equations (3.3) and (3.4), we obtain the values given in item (ii) and in the first part of item (i) in Proposition 2.2.

As for the number of components, we consider the bivariate generating function

$$F_T(z, w) = \sum_{n \geq 0} f_{n,k} z^n w^k,$$

where  $f_{n,k}$  stands for the number of forests in  $\mathcal{F}_T$  with  $n$  vertices and  $k$  components. It is easy to see that equation (3.1) becomes

$$F_T(z, w) = 1 + wT(zF_T(z, w)). \tag{3.5}$$

Let  $X_{n,k} = [z^n w^k]F_T(z, w)/[z^n]F_T(z)$  and let  $\mu_n$  be the mean of  $X_{n,k}$ . Then  $\mu_n \sim \kappa n$ , where  $\kappa$  is obtained as follows. Let  $\rho(w)$  be the dominant singularity of  $F_T(z, w)$  (it thus satisfies  $\rho(1) = \rho$ ). Then  $\kappa = -\rho'(1)/\rho$ . Moreover,  $X_{n,k}$  converges in law to a Gaussian law. (See [4, Section 5] and [5, Section IX.7.3] for details.) This implies that for each positive  $\varepsilon$ ,  $1/2 - \varepsilon$  of the forests in  $\mathcal{F}_T$  with  $n$  vertices have at least  $\mu_n n$  components, for sufficiently large  $n$ .

To finish the proof of item (i) of Proposition 2.2, we use equation (3.5) to find an equation satisfied by  $F_1^*(z, w)$  and then carry out the necessary calculations.

### 4 The upper bound

**Theorem 4.1.** *The double chain on  $N$  points has  $O^*(22.12^N)$  non-crossing spanning trees.*

We only give a sketch of the proof. Recall that for any nc-spanning tree of the double chain, the vertices on the upper and lower chains induce two forests  $F_U$  and  $F_L$  on a set of  $n$  points in convex position and a forest  $F_I$  formed by interior edges, *i.e.*, edges with one endpoint on each chain. The product of the numbers of forests  $F_U$ ,  $F_L$  and  $F_I$  gives an upper bound on the number of nc-spanning trees of the double chain. We partition the set of nc-spanning trees into  $2n - 1$  classes, according to the number of edges of  $F_I$ . For asymptotic counting, it is sufficient to only consider the one class of nc-spanning trees, with  $k$  edges in forest  $F_I$ , that contains most spanning trees. For a fixed value of  $k$ , we can show that the number of nc-spanning trees of the double chain is maximized when both  $F_U$  and  $F_L$  are forests with  $\frac{k}{2}$  components. We also show that the number of forests  $F_I$  with  $k$  edges is

$$\approx \sum_{\ell=1}^{\min\{k,n\}} \binom{n}{\ell}^2 \binom{2n-3-\ell}{k-\ell} = \sum_{\ell=1}^{\min\{k,n\}} g(\ell).$$

Then the value  $\ell = \ell(k)$  that maximizes this sum is determined. Using the formula for  $F(n, c)$ , the number of nc-forests with  $c$  components on a set of  $n$  points in convex position [4], we determine the value  $k$  that maximizes  $g(\ell(k))(F(n, k/2))^2$ , which leads to the claimed bound  $O^*(22.12^N)$ .

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# On the structure of graphs with large minimum bisection

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**Abstract.** Bounded degree trees and bounded degree planar graphs on  $n$  vertices are known to admit bisections of width  $\mathcal{O}(\log n)$  and  $\mathcal{O}(\sqrt{n})$ , respectively. We investigate the structure of graphs that meet this bound. In particular, we show that such a tree must have diameter  $\mathcal{O}(n/\log n)$  and such a planar graph must have tree width  $\Omega(\sqrt{n})$ . To show the result for trees, we derive an inequality that relates the width of a minimum bisection with the diameter of a tree.

## 1 Introduction and results

A *bisection* of a graph  $G$  is a partition of its vertex set into two sets  $L$  and  $R$  of sizes differing by at most one. The *width* of a bisection is defined to be the number of edges with one vertex in  $L$  and one vertex in  $R$ , and the minimum width of a bisection in  $G$  is denoted by  $\text{MinBis}(G)$ . Determining a bisection of minimum width is a famous optimization problem that is (unlike the Minimum Cut Problem) known to be NP-hard [4].

In this paper, we will concentrate on minimum bisections of trees and planar graphs. Jansen et al. showed that dynamic programming gives an algorithm with running time  $\mathcal{O}(2^t n^3)$  for an arbitrary graph on  $n$  vertices when a tree decomposition of width  $t$  is provided as input [5]. Thus, the problem becomes polynomially tractable for graphs of constant tree

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width. On the other hand, it is open whether finding a minimum bisection for a planar graph is in P or is NP-hard. Currently, the best known approximation algorithm achieves an approximation ratio of  $\mathcal{O}(\log n)$  for arbitrary graphs on  $n$  vertices [6], and nothing better has been established for planar graphs. Further, Berman and Karpinski showed that the Minimum Bisection Problem restricted to 3-regular graphs is as hard to approximate as its general version [3]. Moreover, Arora et al. presented a PTAS for finding a minimum bisection in graphs with a linear minimum degree [1]. Thus, we focus on graphs that have small (in fact, constant) maximum degree and still a large minimum bisection width. If  $T$  is a tree on  $n$  vertices and maximum degree  $d$ , one can show that owing to the existence of a *separating vertex* (i.e. a vertex whose removal leaves no connected component of size greater than  $n/2$ ), we always have  $\text{MinBis}(T) \leq d \cdot \log_2 n$ . Similarly, using the Planar Separator Theorem [2], this idea can be generalized to give an  $\mathcal{O}(\sqrt{n})$  upper bound on  $\text{MinBis}(G)$  for planar graphs  $G$  with bounded maximum degree. Both bounds are tight up to a constant factor as one can show that the perfect ternary tree on  $n$  vertices has minimum bisection width  $\Omega(\log n)$  and the square grid on  $n$  vertices has minimum bisection width  $\Omega(\sqrt{n})$ .

Our aim is to investigate the structure of bounded degree graphs that have a large minimum bisection, in other words, trees  $T$  with  $\text{MinBis}(T) = \Omega(\log n)$  and planar graphs  $G$  with  $\text{MinBis}(G) = \Omega(\sqrt{n})$ . For example, we will show that in a bounded degree tree with minimum bisection of order  $\Omega(\log n)$ , the length of any path must be bounded by  $\mathcal{O}(n/\log n)$ . More generally, we establish the following inequality:

**Theorem 1.1.** *Let  $T$  be a tree on  $n$  vertices and denote by  $\Delta(T)$  its maximum degree and by  $\text{diam}(T)$  its diameter. Then*

$$\text{MinBis}(T) \leq \frac{8\Delta(T)n}{\text{diam}(T)}.$$

In the case of planar graphs, we can no longer use the diameter to control the minimum bisection. For example, a graph consisting of a square grid on  $\frac{3}{4}n$  vertices connected to a path on  $\frac{n}{4}$  vertices has linear diameter but does not allow a bisection of constant width. However, we can show that a planar graph with a minimum bisection width close to the upper bound  $\mathcal{O}(\sqrt{n})$  must indeed be far away from a tree-like structure.

**Theorem 1.2.** *For every  $d \in \mathbb{N}$  and every  $c > 0$ , there is a  $\gamma = \gamma(c) > 0$  such that for all planar graphs  $G$  on  $n$  vertices with  $\Delta(G) \leq d$ , we have*

$$\text{MinBis}(G) \geq c\sqrt{n} \quad \Rightarrow \quad \text{tw}(G) \geq \gamma\sqrt{n}.$$

Using Theorem 6.2 of Robertson, Seymour, and Thomas in [7], this immediately implies that there is a constant  $\gamma' = \gamma'(c) > 0$  such that every such planar graph contains a square grid on  $\gamma'n$  vertices as minor.

## 2 Trees

Although the statement in Theorem 1.1 looks like an elementary inequality, its proof is somewhat lengthy and we need to introduce a few more definitions to be able to sketch it. First, we define the *relative diameter* of a graph to be

$$\text{diam}^*(G) := \frac{1}{|V(G)|} \sum_{G': \text{component of } G} (\text{diam}(G') + 1).$$

Observe that, in a tree, there is only one component to consider and thus the relative diameter of a tree denotes the fraction of the vertices in a longest path of the tree, but we will need this parameter also for graphs that may not be connected.

Moreover, we need to take a more general approach and consider partitions where the size of the classes can be specified by an input parameter  $m$ . Furthermore, we denote by  $e_G(V_1, \dots, V_k)$  the number of edges in  $G$  that have their vertices in two different sets  $V_i \neq V_j$  and define  $[n] = \{1, 2, 3, \dots, n\}$  for  $n \in \mathbb{N}$ . The following theorem is the driving engine for the proof of Theorem 1.1.

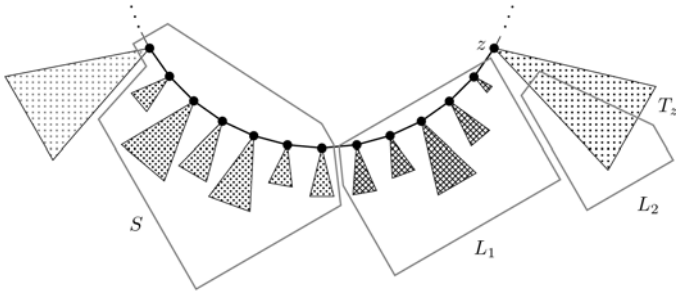
**Theorem 2.1.** *For all trees  $T$  on  $n \geq 3$  vertices and for all  $m \in [n]$ , the vertex set of  $T$  can be partitioned into three classes  $L \cup R \cup S$  such that one of the following two options occurs:*

- (i)  $S = \emptyset$  and  $|L| = m$  and  $e_T(L, R, S) \leq 2$ .
- (ii)  $S \neq \emptyset$  and  $|L| \leq m \leq |L| + |S|$  and  $e_T(L, R, S) \leq \frac{2}{\text{diam}^*(T)} \Delta(T)$  and  $\text{diam}^*(T[S]) \geq 2 \text{diam}^*(T)$ .

This result states that we can either find a partition into two sets  $L$  and  $R$  with exactly the right cardinality by cutting very few edges, or there is a partition with an additional set  $S$ , such that the set  $L$  is smaller and the set  $L \cup S$  is larger than the required size  $m$ , as well as the additional feature that the relative diameter of  $T[S]$  is at least twice as large as that of  $T$ . Using Theorem 2.1 recursively for the graph  $T[S]$ , the relative diameter can therefore be doubled in each round, until it exceeds  $1/2$ , at which point Option (ii) in Theorem 2.1 is no longer feasible, which will then prove Theorem 1.1.

We conclude this section with a few words about how to prove Theorem 2.1. Consider a longest path  $P$  in  $T$  and denote by  $x_0$  and  $y_0$  its first

and last vertex. For each vertex  $z \in V(P)$ , let  $T_z$  be the component of  $T - E(P)$  that contains  $z$  and call  $z$  the root of  $T_z$ . We label the vertices of  $T$  with  $1, 2, \dots, n$  so that  $x_0$  receives label 1; for each  $z \in V(P)$ , the vertices of  $T_z$  receive consecutive labels and  $z$  receives the largest label among those; for all  $z, z' \in V(P)$  with  $z \neq z'$ , if  $x_0$  is closer to  $z$  than to  $z'$ , then the label of  $z$  is smaller than the label of  $z'$ . Given this labeling, we now define for every  $x \in [n]$  the vertex  $f(x) := x + m$ . If for some vertex  $x \in V(P)$  the vertex  $f(x)$  lies also in  $P$ , then we are done by choosing  $L := \{x + 1, \dots, x + m\}$  and  $R := [n] \setminus L$ , which satisfies all requirements of Option (i) in Theorem 2.1. Otherwise, one can show that there exists a vertex  $z \in V(P)$  such that all vertices of  $P$  that are mapped into  $V(T_z)$  and the vertex sets of all trees that are mapped completely into  $T_z$  by  $f$  form a set  $S$  such that the condition on  $\text{diam}^*(T[S])$  is satisfied. Furthermore, this vertex  $z$  has the property that the set  $L$ , which consists of the vertex sets of all trees whose roots have labels strictly between  $z - m$  and  $z$  and of some additional vertices from  $T_z$ , satisfies the remaining conditions of Option (ii) in Theorem 2.1.



**Figure 2.1.** Construction of  $S$  and  $L := L_1 \cup L_2$ .

### 3 Planar graphs

To sketch the proof for Theorem 1.2, consider its contrapositive: for all  $d \in \mathbb{N}$  and for all  $c > 0$ , we choose  $\gamma > 0$  so small that

$$\gamma \left( 2 \log_2 \frac{3}{\gamma \sqrt{2}} + 7 \right) d \leq c.$$

We claim that then, for every planar graph  $G$  on  $n$  vertices with  $\Delta(G) \leq d$ , the following holds:

$$\text{tw}(G) + 1 \leq \gamma \sqrt{n} \quad \Rightarrow \quad \text{MinBis}(G) \leq c \sqrt{n}. \quad (3.1)$$



To explain how we find a bisection of sufficiently small width, let us assume that, for some  $0 < \delta < 1$ , we can find  $\delta$ -separators of a given size in the graph  $G$  and its subgraphs, *i.e.* a vertex subset whose removal leaves connected components of size at most  $\delta n$ . Having removed such a separator  $S'$  from the graph, we can assign the vertices of all but one component of  $G - S'$  to  $L$  and  $R$ , so that  $|L| \leq \lceil \frac{n}{2} \rceil$  and  $|R| \leq \lfloor \frac{n}{2} \rfloor$ . We then continue recursively with the remaining component, which has size at most  $\delta n$ . At the end, we denote by  $S$  the union of the various separators and distribute the vertices in  $S$  to the sets  $L$  and  $R$  in such a way that  $L$  and  $R$  form a bisection of  $G$ . It is easy to see that  $e_G(L, R) \leq \Delta(G) \cdot |S|$  and it only remains to find a bound on  $|S|$ .

In each round of a first phase, we use a cluster from an optimal tree decomposition of  $G$  that can serve as a  $1/2$ -separator and, by assumption, has size at most  $\gamma\sqrt{n}$ . Denote by  $n_i$  the number of vertices after the  $i$ -th round. This first phase stops when  $\gamma\sqrt{n} > \frac{3}{\sqrt{2}}\sqrt{n_i}$ . Due to  $n_i \leq n/2^{i-1}$ , it is easy to see that the index  $i^*$ , where this happens for the first time, can be bounded by a constant depending only on  $\gamma$ .

After the  $i^*$ -th round, we switch to  $2/3$ -separators guaranteed by the Planar Separator Theorem [2], which will have size at most  $\frac{3}{\sqrt{2}}\sqrt{n_i}$ . During this second phase,  $n_i \leq (2/3)^{i-i^*}n_{i^*}$  holds and thus the number of vertices collected in  $S$  during the second phase can be bounded from above by

$$\sum_{i=i^*}^{\infty} \frac{3}{\sqrt{2}}\sqrt{n_i} \leq \frac{3}{\sqrt{2}}\sqrt{n_{i^*}} \sum_{i=0}^{\infty} \left(\sqrt{\frac{2}{3}}\right)^i \leq \alpha\sqrt{n_{i^*}}, \quad \text{with } \alpha = \frac{9}{\sqrt{2}} + 3\sqrt{3}.$$

Summing up, we have  $|S| \leq i^*\gamma\sqrt{n} + \alpha\sqrt{n_{i^*}}$  in total. Now, computing an upper bound on  $i^*$  and on  $n_{i^*}$  will give the desired bound.

## 4 Concluding remarks and open questions

In Theorem 1.1 and Theorem 1.2 we have established necessary conditions for trees and planar graphs to have large minimum bisection width, namely a small diameter and a large tree width, respectively. In both cases, these conditions are not sufficient, but it would be interesting to find additional conditions that give characterizations of graphs in certain classes with large minimum bisection.

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## Colorings



# Coloring intersection graphs of arcwise connected sets in the plane

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**Abstract.** A family of sets in the plane is simple if the intersection of its any subfamily is arcwise connected. We prove that the intersection graphs of simple families of compact arcwise connected sets in the plane pierced by a common line have chromatic number bounded by a function of their clique number.

## 1 Introduction

A *proper coloring* of a graph is an assignment of colors to the vertices of the graph such that no two adjacent ones are assigned the same color. The minimum number of colors sufficient to color a graph  $G$  properly is called the *chromatic number* of  $G$  and denoted by  $\chi(G)$ . The maximum size of a clique (a set of pairwise adjacent vertices) in a graph  $G$  is called the *clique number* of  $G$  and denoted by  $\omega(G)$ . It is clear that  $\chi(G) \geq \omega(G)$ . A class of graphs is  $\chi$ -*bounded* if there is a function  $f : \mathbb{N} \rightarrow \mathbb{N}$  such that  $\chi(G) \leq f(\omega(G))$  holds for any graph  $G$  in the class.

In this paper, we focus our attention on the relation between the chromatic number and the clique number for classes of graphs arising from geometry. The *intersection graph* of a family of sets  $\mathcal{F}$  is the graph with vertex set  $\mathcal{F}$  and edge set consisting of pairs of intersecting elements of  $\mathcal{F}$ . We consider families  $\mathcal{F}$  of arcwise connected compact sets in the plane. For simplicity, we identify the family  $\mathcal{F}$  with its intersection graph.

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In the one-dimensional case of subsets of  $\mathbb{R}$ , the only arcwise connected compact sets are closed intervals. They define the class of *interval graphs*, for which  $\chi(G) = \omega(G)$ . The study of the chromatic number of intersection graphs of geometric objects in higher dimensions was initiated in the seminal paper of Asplund and Grünbaum [1], where they proved that the families of axis-aligned rectangles in  $\mathbb{R}^2$  are  $\chi$ -bounded. On the other hand, Burling [2] showed that triangle-free intersection graphs of axis-aligned boxes in  $\mathbb{R}^3$  can have arbitrarily large chromatic number.

Gyárfás [3] proved  $\chi$ -boundedness of the class of graphs defined by intersections of chords of a circle. This was generalized by Kostochka and Kratochvíl [4], who showed that the families of convex polygons inscribed in a circle are  $\chi$ -bounded. McGuinness [5] proved that the families of L-shapes (shapes consisting of a horizontal and a vertical segments of arbitrary lengths, forming the letter ‘L’) all of which intersect a fixed vertical line are  $\chi$ -bounded. Later, McGuinness [6] showed that the simple families  $\mathcal{F}$  of compact arcwise connected sets in the plane pierced by a common line with  $\omega(\mathcal{F}) \leq 2$  have bounded chromatic number. A family is *simple* if the intersection of its any subfamily is arcwise connected, and is *pierced* by a line  $\ell$  if the intersection of its any member with  $\ell$  is a nonempty segment. Suk [8] proved  $\chi$ -boundedness of the simple families of  $x$ -monotone curves intersecting a fixed vertical line.

We generalize the results of McGuinness, allowing any bound on the clique number, and of Suk, getting rid of the  $x$ -monotonicity assumption.

**Theorem 1.1.** *The class of simple families of compact arcwise connected sets in the plane pierced by a common line is  $\chi$ -bounded.*

This contrasts with a recent result due to Pawlik et al. [7] that there are triangle-free intersection graphs of straight-line segments with arbitrarily large chromatic number. This explains why the assumption of Theorem 1.1 that the sets are pierced by a common line is necessary.

The ultimate goal of this quest is to understand the border line between the classes of graphs (and classes of geometric objects) that are  $\chi$ -bounded and those that are not. The authors would like to share two open problems in this context.

**Problem 1.2.** Are the families (not necessarily simple) of  $x$ -monotone curves in the plane pierced by a common vertical line  $\chi$ -bounded?

**Problem 1.3.** Are the families of curves in the plane pierced by a common line  $\chi$ -bounded?

## 2 Preliminaries

First we simplify the setting of Theorem 1.1. Let  $\mathcal{F}$  be a simple family of compact arcwise connected sets in the plane pierced by a common line with  $\omega(\mathcal{F}) \leq k$ . We can assume without loss of generality that this piercing line is the horizontal axis  $\mathbb{R} \times \{0\}$ . Call it the *baseline*. The *base* of a set  $X$ , denoted by  $\text{base}(X)$ , is the intersection of  $X$  with the baseline. The intersection graph of the bases of the members of  $\mathcal{F}$  is an interval graph and thus can be properly colored with  $k$  colors. To find a proper coloring of  $\mathcal{F}$ , we can restrict our attention to one color class in the coloring of this interval graph. Thus we assume that no two members of  $\mathcal{F}$  intersect on the baseline and show that  $\mathcal{F}$  can be colored properly with a bounded number of colors.

Let  $\mathcal{F}^+ = \{X \cap (\mathbb{R} \times [0, +\infty)) : X \in \mathcal{F}\}$  and  $\mathcal{F}^- = \{X \cap (\mathbb{R} \times (-\infty, 0]) : X \in \mathcal{F}\}$ . Clearly,  $\mathcal{F}^+$  and  $\mathcal{F}^-$  are simple families of arcwise connected sets. If we find proper colorings  $\phi^+$  and  $\phi^-$  of  $\mathcal{F}^+$  and  $\mathcal{F}^-$ , respectively, with bounded numbers of colors, then the coloring of  $\mathcal{F}$  by pairs of colors  $(\phi^+, \phi^-)$  is proper on  $\mathcal{F}$ . Thus we assume that  $\mathcal{F} = \mathcal{F}^+$  (the other case  $\mathcal{F} = \mathcal{F}^-$  is symmetric). All geometric objects that we consider from now on are contained in  $\mathbb{R} \times [0, +\infty)$ . Thus we consider families of compact arcwise connected subsets of  $\mathbb{R} \times [0, +\infty)$  all of which are pierced by the baseline. We call such families *attached*.

**Theorem 2.1.** *For  $k \geq 1$ , there is  $\xi_k$  such that  $\chi(\mathcal{F}) \leq 2^{\xi_k}$  holds for any attached family  $\mathcal{F}$  with  $\omega(\mathcal{F}) \leq k$ .*

The case  $k = 1$  is trivial, and the case  $k = 2$  is the result of McGuinness [6]. Our proof of Theorem 2.1 depends heavily on the techniques developed by McGuinness [6] and Suk [8].

Let  $X \prec Y$  denote that  $\text{base}(X)$  is entirely to the left of  $\text{base}(Y)$ . The relation  $\prec$  is a total order on any attached family  $\mathcal{F}$ . For attached sets  $X_1$  and  $X_2$  such that  $X_1 \prec X_2$ , define  $\mathcal{F}(X_1, X_2) = \{Y \in \mathcal{F} : X_1 \prec Y \prec X_2\}$ . For an attached family  $\mathcal{X}$ , we define  $\text{ext}(\mathcal{X})$  to be the only unbounded arcwise connected component of  $(\mathbb{R} \times [0, +\infty)) \setminus \bigcup \mathcal{X}$ .

**Lemma 2.2 ([5]).** *Let  $\mathcal{F}$  be an attached family, let  $a, b \geq 0$ , and suppose  $\chi(\mathcal{F}) > 2^{a+b+1}$ . Then there exists a subfamily  $\mathcal{H}$  of  $\mathcal{F}$  such that  $\chi(\mathcal{H}) > 2^a$  and for any intersecting  $H_1, H_2 \in \mathcal{H}$  we have  $\chi(\mathcal{F}(H_1, H_2)) \geq 2^b$ .*

A subfamily  $\mathcal{G}$  of an attached family  $\mathcal{F}$  is *externally supported* in  $\mathcal{F}$  if for any  $X \in \mathcal{G}$  there exists  $Y \in \mathcal{F}$  such that  $Y \cap X \neq \emptyset$  and  $Y \cap \text{ext}(\mathcal{G}) \neq \emptyset$ .

**Lemma 2.3.** *Let  $\mathcal{F}$  be an attached family, let  $a \geq 0$ , and suppose  $\chi(\mathcal{F}) > 2^{a+1}$ . Then there exists a subfamily  $\mathcal{G}$  of  $\mathcal{F}$  that is externally supported in  $\mathcal{F}$  and satisfies  $\chi(\mathcal{G}) > 2^a$ .*

Let  $\mathcal{F}$  be an attached family. A  $k$ -clique in  $\mathcal{F}$  is a family of  $k$  pairwise intersecting members of  $\mathcal{F}$ . For a  $k$ -clique  $\mathcal{K}$ , define  $\text{int}(\mathcal{K})$  to be the only arcwise connected component of  $(\mathbb{R} \times [0, +\infty)) \setminus \bigcup \mathcal{K}$  containing the part of the baseline between the two least members of  $\mathcal{K}$ . A  $k$ -bracket in  $\mathcal{F}$  is a family  $\mathcal{B} \subseteq \mathcal{F}$  consisting of a  $k$ -clique  $\mathcal{K}$ , a set  $P \subseteq \text{int}(\mathcal{K})$  called *hook*, and a set  $S$  called *support* such that  $S \prec \mathcal{K}$  or  $\mathcal{K} \prec S$  and  $S \cap P \neq \emptyset$ . For such a  $k$ -bracket  $\mathcal{B}$ , define  $\text{int}(\mathcal{B})$  to be the only arcwise connected component of  $(\mathbb{R} \times [0, +\infty)) \setminus \bigcup \mathcal{B}$  containing the part of the baseline between  $S$  and  $\mathcal{K}$ . The following lemma exhibits a crucial property of these two constructs.

**Lemma 2.4.** *If  $\mathcal{S}$  is an attached clique or bracket, then any closed curve  $c$  such that  $\mathcal{S} \cup \{c\}$  is simple,  $\text{int}(\mathcal{S}) \cap c \neq \emptyset$ , and  $\text{ext}(\mathcal{S}) \cap c \neq \emptyset$  intersects all members of  $\mathcal{S}$ .*

### 3 Proof sketch of Theorem 2.1

The proof of Theorem 2.1 proceeds by induction on  $k$ . The case  $k = 1$  is trivial. Thus assume for the remainder of this section that  $k \geq 2$  and the statement of the theorem holds for  $k - 1$ . A typical application of the induction hypothesis looks as follows: if  $\mathcal{F}$  is an attached family with  $\omega(\mathcal{F}) \leq k$ ,  $\mathcal{G} \subseteq \mathcal{F}$ , and there is  $X \in \mathcal{F} \setminus \mathcal{G}$  intersecting all members of  $\mathcal{G}$ , then  $\omega(\mathcal{G}) \leq k - 1$  and thus  $\chi(\mathcal{G}) \leq 2^{\xi_{k-1}}$ .

Define  $\beta_k = 5\xi_{k-1} + \xi_2 + k + 7$ ,  $\gamma_k = 2\xi_{k-1} + k + 5$ ,  $\delta_{k,1} = \xi_{k-1} + \beta_k + \gamma_k + 2$ ,  $\delta_{k,i} = \delta_{k,i-1} + \beta_k + \gamma_k + 2$  for  $i \geq 2$ , and finally  $\xi_k = \delta_{k,k+1}$ .

For a  $k$ -clique or  $k$ -bracket  $\mathcal{S}$ , define  $\mathcal{F}(\mathcal{S}) = \{X \in \mathcal{F} : \text{base}(X) \subseteq \text{int}(\mathcal{S})\}$ . The following technical fact (considered in the induction context) is a generalization of an analogous statement in [8], with a similar proof.

**Lemma 3.1.** *Let  $\mathcal{F}$  be an attached family with  $\omega(\mathcal{F}) \leq k$  and  $\mathcal{S}$  be a  $k$ -clique or  $k$ -bracket in  $\mathcal{F}$ . Let  $\mathcal{R} = \{R \in \mathcal{F}(\mathcal{S}) : R \cap \text{ext}(\mathcal{S}) \neq \emptyset\}$  and  $\mathcal{D} = \{D \in \mathcal{F}(\mathcal{S}) : D \cap \bigcup(\mathcal{R} \cup \mathcal{S}) \neq \emptyset\}$ . Then  $\chi(\mathcal{D}) \leq 2^{\beta_k}$ .*

A *fancy  $k$ -clique* in an attached family  $\mathcal{F}$  consists of a  $k$ -clique  $\mathcal{K}$ , a set  $P \subseteq \text{int}(\mathcal{K})$  called *hook*, two intersecting sets  $X_1, X_2 \in \mathcal{F}(-\infty, K_1)$  called *left guards*, and two intersecting sets  $Y_1, Y_2 \in \mathcal{F}(K_k, +\infty)$  called *right guards*, where  $K_1$  and  $K_k$  are the least and the greatest elements of  $\mathcal{K}$ , respectively.

**Claim 3.2.** Any attached family  $\mathcal{F}$  with  $\omega(\mathcal{F}) \leq k$  and  $\chi(\mathcal{F}) > 2^{\gamma k}$  contains a fancy  $k$ -clique.

*Proof.* Find in  $\mathcal{F}$  sets  $X_1 \prec X_2 \prec Y_1 \prec Y_2$  so that  $X_1 \cap X_2 \neq \emptyset$ ,  $Y_1 \cap Y_2 \neq \emptyset$ , and  $\chi(\mathcal{F}(X_2, Y_1)) \geq \chi(\mathcal{F}) - 4 > 2^{2\xi_{k-1}+k+1}$ . Apply Lemma 2.2 to find  $\mathcal{H} \subseteq \mathcal{F}(X_2, Y_1)$  such that  $\chi(\mathcal{H}) > 2^{\xi_{k-1}}$  and for any intersecting  $H_1, H_2 \in \mathcal{H}$  we have  $\chi(\mathcal{F}(H_1, H_2)) \geq 2^{\xi_{k-1}+k}$ . It follows that  $\mathcal{H}$  contains a  $k$ -clique  $\mathcal{K}$  such that  $\chi(\mathcal{F}(\mathcal{K})) \geq 2^{\xi_{k-1}+k} > 2^{\xi_{k-1}k}$ . Since  $|\mathcal{K}| = k$ , the members of  $\mathcal{F}(\mathcal{K})$  that intersect  $\bigcup \mathcal{K}$  can be properly colored with  $2^{\xi_{k-1}k}$  colors. Hence there exists  $P \in \mathcal{F}(\mathcal{K})$  disjoint from  $\bigcup \mathcal{K}$ , so that  $P \subseteq \text{int}(\mathcal{K})$ . The clique  $\mathcal{K}$  with hook  $P$ , left guards  $X_1, X_2$ , and right guards  $Y_1, Y_2$  forms a fancy  $k$ -clique in  $\mathcal{F}$ .  $\square$

A  $(k, i)$ -bracket system in an attached family  $\mathcal{F}$  consists of  $k$ -brackets  $\mathcal{B}_1, \dots, \mathcal{B}_i$  with pairwise intersecting supports, and two intersecting sets  $X_1, X_2 \in \mathcal{F}(\mathcal{B}_1) \cap \dots \cap \mathcal{F}(\mathcal{B}_i)$  called *guards*.

**Claim 3.3.** Any attached family  $\mathcal{F}$  with  $\omega(\mathcal{F}) \leq k$  and  $\chi(\mathcal{F}) > 2^{\delta_{k,i}}$  contains a  $(k, i)$ -bracket system.

*Proof.* The proof goes by induction on  $i$ . We start with  $i = 1$ . First, apply Lemma 2.3 to find  $\mathcal{G} \subseteq \mathcal{F}$  that is externally supported in  $\mathcal{F}$  and satisfies  $\chi(\mathcal{G}) > 2^{\xi_{k-1}+\beta_k+\gamma k+1}$ . Next, apply Lemma 2.2 to find  $\mathcal{H} \subseteq \mathcal{G}$  such that  $\chi(\mathcal{H}) > 2^{\xi_{k-1}}$  and for any intersecting  $H_1, H_2 \in \mathcal{H}$  we have  $\chi(\mathcal{G}(H_1, H_2)) \geq 2^{\beta_k+\gamma k}$ . It follows that  $\mathcal{H}$  contains a  $k$ -clique  $\mathcal{K}$  such that  $\chi(\mathcal{G}(\mathcal{K})) \geq 2^{\beta_k+\gamma k}$ . Let  $\mathcal{R} = \{R \in \mathcal{F}(\mathcal{K}) : R \cap \text{ext}(\mathcal{K}) \neq \emptyset\}$  and  $\mathcal{D} = \{D \in \mathcal{G}(\mathcal{K}) : D \cap \bigcup(\mathcal{K} \cup \mathcal{R}) \neq \emptyset\}$ . Lemma 3.1 yields  $\chi(\mathcal{D}) \leq 2^{\beta_k}$ . Let  $\mathcal{G}' = \mathcal{G}(\mathcal{K}) \setminus \mathcal{D}$ . It follows that  $\chi(\mathcal{G}') \geq 2^{\beta_k+\gamma k} - 2^{\beta_k} > 2^{\gamma k}$ . Claim 3.2 guarantees a fancy  $k$ -clique  $\mathcal{K}'$  with hook  $P$ , left guards  $X_1, X_2$ , and right guards  $Y_1, Y_2$  in  $\mathcal{G}'$ . Since  $\mathcal{G}$  is externally supported in  $\mathcal{F}$ , there exists  $S \in \mathcal{F}$  that intersects  $P$  and  $\text{ext}(\mathcal{G})$ . Since  $P \notin \mathcal{D}$  and  $S \cap P \neq \emptyset$ , we have  $S \notin \mathcal{R}$ . This and  $S \cap \text{ext}(\mathcal{K}) \supseteq S \cap \text{ext}(\mathcal{G}) \neq \emptyset$  imply  $S \notin \mathcal{F}(\mathcal{K})$ . Therefore,  $\mathcal{K}'$  with hook  $P$ , support  $S$ , and guards  $X_1, X_2$  or  $Y_1, Y_2$  forms a  $(k, 1)$ -bracket system in  $\mathcal{F}$ .

Now, suppose  $i \geq 2$ . As above, find  $\mathcal{G} \subseteq \mathcal{F}$  externally supported in  $\mathcal{F}$  and  $\mathcal{H} \subseteq \mathcal{G}$  such that  $\chi(\mathcal{H}) > 2^{\delta_{k,i-1}}$  and for any intersecting  $H_1, H_2 \in \mathcal{H}$  we have  $\chi(\mathcal{G}(H_1, H_2)) \geq 2^{\beta_k+\gamma k}$ . By the induction hypothesis,  $\mathcal{H}$  contains a  $(k, i-1)$ -bracket system with brackets  $\mathcal{B}_1, \dots, \mathcal{B}_{i-1}$  and guards  $X_1, X_2$ . Thus  $\chi(\mathcal{G}(X_1, X_2)) \geq 2^{\beta_k+\gamma k}$ . Again,  $\mathcal{G}$  has a fancy  $k$ -clique  $\mathcal{K}'$  with hook  $P$ , left guards  $X_1, X_2$ , and right guards  $Y_1, Y_2$ , and there exists  $S \in \mathcal{F} \setminus \mathcal{F}(X_1, X_2)$  intersecting  $P$  and  $\text{ext}(\mathcal{G})$ . By Lemma 2.4,  $S$  intersects all the supports of  $\mathcal{B}_1, \dots, \mathcal{B}_{i-1}$  (it cannot intersect the  $k$ -cliques of  $\mathcal{B}_1, \dots, \mathcal{B}_{i-1}$  as  $\omega(\mathcal{F}) \leq k$ ). Thus the  $k$ -bracket  $\mathcal{B}_i$  with  $k$ -clique  $\mathcal{K}'$ ,



hook  $P$  and support  $S$  together with  $\mathcal{B}_1, \dots, \mathcal{B}_{i-1}$  and guards  $X_1, X_2$  or  $Y_1, Y_2$  forms a  $(k, i)$ -bracket system in  $\mathcal{F}$ .  $\square$

To complete the proof of Theorem 2.1, observe that if  $\chi(\mathcal{F}) > 2^{\xi_k} = 2^{\delta_{k,k+1}}$ , then by Claim 3.3  $\mathcal{F}$  contains a  $(k, k+1)$ -bracket system, which contains  $k+1$  pairwise intersecting supports, contradicting  $\omega(\mathcal{F}) \leq k$ .

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# A characterization of edge-reflection positive partition functions of vertex-coloring models

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**Abstract.** Szegedy (B. Szegedy, Edge coloring models and reflection positivity, *Journal of the American Mathematical Society* **20**, 2007, 969–988.) showed that the partition function of any vertex-coloring model is equal to the partition function of a complex edge-coloring model. Using some results in geometric invariant theory, we characterize for which vertex-coloring model the edge-coloring model can be taken to be real valued that is, we characterize which partition functions of vertex-coloring models are edge-reflection positive. This answers a question posed by Szegedy.

## 1 Introduction

Partition functions of vertex- and edge-coloring models are graph invariants introduced by de la Harpe and Jones [4]. In fact, in [4] they are called spin and vertex models respectively. Both models give a rich class of graph invariants. But they do not coincide. For example the number of matchings in a graph is the partition function of a real edge-coloring model but not the partition function of any real vertex-coloring model. This can be deduced from the characterization of partition functions of real vertex-coloring models by Freedman, Lovász and Schrijver [3]. (It is neither the partition function of any complex vertex-coloring model, but we will not prove this here.) Conversely, the number of independent sets is not the partition function of any real edge-coloring model, as follows from Szegedy’s characterization of partition functions of real edge-coloring models [7], but it is the partition function of a (real) vertex-coloring model.

However, Szegedy [7] showed that the partition function of any vertex-coloring model can be obtained as the partition function of a complex edge-coloring model. Moreover, he gave examples when the edge-coloring model can be taken to be real valued. This made him ask the question which partition functions of real vertex-coloring models are par-

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tion functions of real edge-coloring models (*cf.* [7, Question 3.2]). In fact, he phrased his question in terms of edge-reflection positivity. We will not say anything about this here. See [7] or the full version of the present extended abstract [6].

In this note we completely characterize for which vertex-coloring models there exists a real edge-coloring model such that their partition functions coincide, answering Szegedy's question.

The organization of this paper is as follows. In the next section we give definitions of partition functions of edge and vertex-coloring models and state our main result (*cf.* Theorem 2.4). In Section 3 we give a sketch of the proof of Theorem 2.4.

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## 2 Partition functions of edge and vertex-coloring models

We give the definitions of edge and vertex-coloring models and their partition functions. After that we describe Szegedy's result how to obtain a complex edge-coloring model from a vertex-coloring model such that their partition functions are the same. (The existence also follows from the characterization of partition functions of complex edge-coloring models given in [1], but Szegedy gives a direct way to construct the edge-coloring model from the vertex-coloring model.) And finally we will state our main result saying which partition functions of vertex-coloring models are partition function of real edge-coloring models.

Let  $\mathcal{G}$  be the set of all graphs, allowing multiple edges and loops. Let  $\mathbb{C}$  denote the set of complex numbers and let  $\mathbb{R}$  denote the set of real numbers. If  $V$  is a vector space we write  $V^*$  for its dual space, but by  $\mathbb{C}^*$  we mean  $\mathbb{C} \setminus \{0\}$ . For a matrix  $U$  we denote by  $U^*$  its conjugate transpose and by  $U^T$  its transpose.

Let  $\mathbb{F}$  be a field. An  $\mathbb{F}$ -valued *graph invariant* is a map  $p : \mathcal{G} \rightarrow \mathbb{F}$  which takes the same values on isomorphic graphs.

Throughout this paper we set  $\mathbb{N} = \{1, 2, \dots\}$  and for  $n \in \mathbb{N}$ ,  $[n]$  denotes the set  $\{1, \dots, n\}$ . We will now introduce partition functions of vertex and edge-coloring models.

Let  $a \in (\mathbb{C}^*)^n$  and let  $B \in \mathbb{C}^{n \times n}$  be a symmetric matrix. We call the pair  $(a, B)$  an *n-color vertex-coloring model*<sup>2</sup>. If moreover,  $a$  is positive and  $B$  is real, then we call  $(a, B)$  a *real n-color vertex-coloring model*.

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<sup>2</sup> Vertex-coloring models with  $a$  equal to the all ones vector were introduced by de la Harpe and Jones in [4] where they are called spin models.

When talking about a vertex-coloring model, we will sometimes omit the number of colors. The *partition function* of an  $n$ -color vertex-coloring model  $(a, B)$  is the graph invariant  $p_{a,B} : \mathcal{G} \rightarrow \mathbb{C}$  defined by

$$p_{a,B}(H) := \sum_{\phi:V(H)\rightarrow[n]} \prod_{v\in V(H)} a_{\phi(v)} \cdot \prod_{uv\in E(H)} B_{\phi(u),\phi(v)}, \tag{2.1}$$

for  $H \in \mathcal{G}$ .

We can view  $p_{a,B}$  in terms of weighted homomorphisms. Let  $G(a, B)$  be the complete graph on  $n$  vertices (including loops) with vertex weights given by  $a$  and edge weights given by  $B$ . Then  $p_{a,B}(H)$  can be viewed as counting the number of weighted homomorphisms of  $H$  into  $G(a, B)$ . In this context  $p_{a,B}$  is often denoted by  $\text{hom}(\cdot, G(a, B))$ . So in particular the number of proper  $k$ -colorings is the partition function of a vertex-coloring model. The vertex-coloring model can also be seen as a statistical mechanics model where vertices serve as particles, edges as interactions between particles, and colors as states or energy levels.

Let for a field  $\mathbb{F}$ ,

$$R(\mathbb{F}) := \mathbb{F}[x_1, \dots, x_k] \tag{2.2}$$

denote the polynomial ring in  $k$  variables. We will only consider  $\mathbb{F} = \mathbb{R}$  and  $\mathbb{F} = \mathbb{C}$ . Note that there is a one-to-one correspondence between linear functions  $h : R(\mathbb{F}) \rightarrow \mathbb{F}$  and maps  $h : \mathbb{N}^k \rightarrow \mathbb{F}$ ;  $\alpha \in \mathbb{N}^k$  corresponds to the monomial  $x^\alpha := x_1^{\alpha_1} \cdots x_k^{\alpha_k} \in R(\mathbb{F})$  and the monomials form a basis for  $R(\mathbb{F})$ . We call any  $h \in R(\mathbb{C})^*$  a  $k$ -color edge-coloring model<sup>3</sup>. Any  $h \in R(\mathbb{R})^*$  is called a *real*  $k$ -color edge-coloring model. When talking about an edge-coloring model, we will sometimes omit the number of colors. The *partition function* of a  $k$ -color edge-coloring model  $h$  is the graph invariant  $p_h : \mathcal{G} \rightarrow \mathbb{C}$  defined by

$$p_h(G) = \sum_{\phi:E(G)\rightarrow[k]} \prod_{v\in V(G)} h\left(\prod_{e\in\delta(v)} x_{\phi(e)}\right), \tag{2.3}$$

for  $G \in \mathcal{G}$ . Here  $\delta(v)$  is the multiset of edges incident with  $v$ . By convention, a loop is counted twice.

Let us give a few examples.

**Example 2.1 (Counting perfect matchings).** Let  $k = 2$ . Define the the edge-coloring model  $h : \mathbb{R}[x_1, x_2] \rightarrow \mathbb{R}$  by  $h(x_1^{a_1} x_2^{a_2}) = \delta_{a_1,1}$ . Then  $p_h(G)$  is equal to the number of perfect matchings of  $G$ . To see this,

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<sup>3</sup> Edge-coloring models were introduced by de la Harpe and Jones in [4] where they are called vertex models.

note that for an assignment of the colors to the edges of  $G$  there is a contribution in the sum (2.3) if and only if at each vertex there is a unique edge which is colored with 1, that is, if and only if the edges colored with 1 form a perfect matching.

**Example 2.2 (Counting proper  $k$ -edge-colorings).** Let  $k \in \mathbb{N}$ . Define the  $k$ -color edge-coloring model  $h$  by

$$h(x_1^{a_1} \cdots x_k^{a_k}) = \begin{cases} 1 & \text{if } a_i \leq 1 \text{ for all } i \\ 0 & \text{else} \end{cases} .$$

Then  $p_h(G)$  is equal to the number of proper  $k$ -edge-colorings of  $G$ .

For more examples we refer to [4] and [7]. The edge-coloring model can also be considered as a statistical mechanics model, where edges serve as particles, vertices as interactions between particles, and colors as states or energy levels.

We will now describe a result of Szegedy [7] (see also [8]) showing that partition functions of vertex-coloring models are partition functions of edge-coloring models.

Let  $(a, B)$  be an  $n$ -color vertex-coloring model. As  $B$  is symmetric we can write  $B = U^T U$  for some  $k \times n$  (complex) matrix  $U$ , for some  $k$ . Let  $u_1, \dots, u_n \in \mathbb{C}^k$  be the columns of  $U$ . Define the edge-coloring model  $h$  by  $h := \sum_{i=1}^n a_i \text{ev}_{u_i}$ , where for  $u \in \mathbb{C}^k$ ,  $\text{ev}_u \in R(\mathbb{C})^*$  is the linear map defined by  $p \mapsto p(u)$  for  $p \in R(\mathbb{C})$ .

**Lemma 2.3 (Szegedy [7]).** *Let  $(a, B)$  and  $h$  be as above. Then  $p_{a,B} = p_h$ .*

Let  $(a, B)$  be an  $n$ -color vertex-coloring model. We say that  $i, j \in [n]$  are *twins* of  $(a, B)$  if  $i \neq j$  and the  $i$ th row of  $B$  is equal to the  $j$ th row of  $B$ . If  $(a, B)$  has no twins we call the model *twin free*. Suppose now  $i, j \in [n]$  are twins of  $(a, B)$ . If  $a_i + a_j \neq 0$ , let  $B'$  be the matrix obtained from  $B$  by removing row and column  $i$  and let  $a'$  be the vector obtained from  $a$  by setting  $a'_j := a_i + a_j$  and then removing the  $i$ th entry from it. In case  $a_i + a_j = 0$ , we remove the  $i$ th and the  $j$ th row and column from  $B$  to obtain  $B'$  and we remove the  $i$ th and the  $j$ th entry from  $a$  to obtain  $a'$ . Then  $p_{a',B'} = p_{a,B}$ . So for every vertex-coloring model with twins, we can construct a vertex-coloring model with fewer colors which is twin free and which has the same partition function.

We need a few more definitions to state our main result. For a  $k \times n$  matrix  $U$  we denote its columns by  $u_1, \dots, u_n$ . Let, for any  $k$ ,  $(\cdot, \cdot)$  denote the standard bilinear form on  $\mathbb{C}^k$ . We call the matrix  $U$  *nondegenerate* if the span of  $u_1, \dots, u_n$  is nondegenerate with respect to  $(\cdot, \cdot)$ . In other

words, if  $\text{rk}(U^T U) = \text{rk}(U)$ . For  $l \in \mathbb{N}$ , let  $O_l(\mathbb{C})$  be the complex orthogonal group, i.e.  $O_l(\mathbb{C}) := \{g \in \mathbb{C}^{l \times l} \mid (gv, gv) = (v, v) \text{ for all } v \in \mathbb{C}^l\}$ . We think of vectors in  $\mathbb{C}^k$  as vectors in  $\mathbb{C}^l$  for any  $l \geq k$ . We can now state our main result.

**Theorem 2.4.** *Let  $(a, B)$  be a twin-free  $n$ -color vertex-coloring model. Let  $U$  be a nondegenerate  $k \times n$  matrix such that  $U^T U = B$ . Then the following are equivalent:*

- (i)  $p_{a,B} = p_y$  for some real edge-coloring model  $y$ ,
- (ii) there exists  $l \geq k$ ,  $g \in O_l(\mathbb{C})$  such that the set  $\left\{ \begin{pmatrix} gu_i \\ a_i \end{pmatrix} \mid i = 1, \dots, n \right\}$  is closed under complex conjugation,
- (iii) there exists  $l \geq k$ ,  $g \in O_l(\mathbb{C})$  such that  $\sum_{i=1}^n a_i \text{ev}_{gu_i}$  is real.

If moreover,  $UU^* \in \mathbb{R}^{k \times k}$ , then we can take  $g$  equal to the identity in (ii) and (iii).

Note that for  $h := \sum_{i=1}^n a_i \text{ev}_{gu_i}$  in Theorem 2.4, we have, by Lemma 2.3,  $p_h = p_{a,B}$ . Moreover, observe that if the set of columns of  $gU$  are closed under complex conjugation, then  $gU(gU)^*$  is real. So the existence of a nondegenerate matrix  $U$  such that  $U^T U = B$  and  $UU^*$  is real, is a necessary condition for  $p_{a,B}$  to be the partition function of a real edge-coloring model.

In case  $B$  is real, there is an easy way to obtain a  $k \times n$  rank  $k$  matrix  $U$ , where  $k = \text{rk}(B)$ , such that  $UU^* \in \mathbb{R}^{k \times k}$  and  $U^T U = B$ , using the spectral decomposition of  $B$ . So by Theorem 2.4, we get the following characterization of partition functions of real vertex-colorings that are partition functions of real edge-coloring models. We will state it as a corollary.

**Corollary 2.5.** *Let  $(a, B)$  be a twin-free real  $n$ -color vertex-coloring model. Then  $p_{a,B} = p_h$  for some real edge-coloring model  $h$  if and only if for each  $i \in [n]$  there exists  $j \in [n]$  such that*

- (i)  $a_i = a_j$ ,
- (ii) for each eigenvector  $v$  of  $B$  with eigenvalue  $\lambda$  :  $\begin{cases} \lambda > 0 \Rightarrow v_i = v_j, \\ \lambda < 0 \Rightarrow v_i = -v_j. \end{cases}$

### 3 A sketch of the proof of Theorem 2.4

Here we give short sketch of the proof of Theorem 2.4. For the full proof see [6]. First of all, Lemma 2.3 shows that (iii) implies (i). The following easy lemma gives the equivalence between (ii) and (iii) for the same  $g$  in Theorem 2.4.

**Lemma 3.1.** *Let  $u_1, \dots, u_n \in \mathbb{C}^k$  be distinct vectors, let  $a \in (\mathbb{C}^*)^n$  and let  $h := \sum_{i=1}^n a_i \text{ev}_{u_i}$ . Then  $h$  is a real edge-coloring model if and only if the set  $\left\{ \begin{pmatrix} u_i \\ a_i \end{pmatrix} \mid i = 1, \dots, n \right\}$  is closed under complex conjugation.*

*Proof.* Suppose first that the set  $\left\{ \begin{pmatrix} u_i \\ a_i \end{pmatrix} \mid i = 1, \dots, n \right\}$  is closed under complex conjugation. Then for  $p \in R(\mathbb{R})$ ,  $h(p) = \sum_{i=1}^n a_i p(u_i) = \sum_{i=1}^n \overline{a_i p(u_i)} = \overline{h(p)}$ . Hence,  $h(p) \in \mathbb{R}$ . So  $h$  is real valued.

Now the 'only if' part. By possibly adding some vectors to  $\{u_1, \dots, u_n\}$  and extending the vector  $a$  with zero's, we may assume that  $\{u_1, \dots, u_n\}$  is closed under complex conjugation. We must show that  $u_i = \overline{u_j}$  implies  $a_i = \overline{a_j}$ . We may assume that  $u_1 = \overline{u_2}$ . Using Lagrange interpolating polynomials we find  $p \in R(\mathbb{C})$  such that  $p(u_j) = 1$  if  $j = 1, 2$  and 0 else. Let  $p' := 1/2(p + \overline{p})$ . Then  $p' \in R(\mathbb{R})$  and consequently,  $h(p') = \sum_{i=1}^n a_i p(u_i) = a_1 + a_2 \in \mathbb{R}$ . Similarly, there exists  $q \in R(\mathbb{C})$  such that  $q(u_1) = i$ ,  $q(u_2) = -i$  and  $q(u_j) = 0$  if  $j > 2$ . Setting  $q' := 1/2(q + \overline{q})$  and applying  $h$  to it, we find that  $i(a_1 - a_2) \in \mathbb{R}$ . So we conclude that  $a_1 = \overline{a_2}$ . Continuing this way proves the lemma.  $\square$

What remains is the proof of (i) implies (iii), which is the hardest part. Our proof of this is based some on fundamental results in geometric invariant theory. We will now sketch our approach.

Let  $h$  be a real  $l$ -color edge-coloring model such that  $p_h = p_{a,B}$ . Then, using that  $U$  is nondegenerate and that the columns of  $U$  are distinct, we can apply a result from [2] and a result from [5] to find  $g \in O_l(\mathbb{C})$  such that  $h = \sum_{i=1}^n a_i \text{ev}_{gu_i}$ .

Suppose now that  $UU^* \in \mathbb{R}^{k \times k}$ . Define for an  $l \times n$  matrix  $W$  the function  $f_W : O_l(\mathbb{C}) \rightarrow \mathbb{R}$  by  $g \mapsto \text{tr}(gW(gW)^*)$ , where  $\text{tr}$  denotes the trace. The function  $f_W$  was introduced by Kempf and Ness [5] in the context of reductive algebraic groups acting on finite dimensional vector spaces. Denote by  $\text{Stab}(W)$  the subgroup of  $O_l(\mathbb{C})$  that leaves  $W$  invariant. Let  $e \in O_l(\mathbb{C})$  denote the identity matrix.

**Lemma 3.2.** *The function  $f_W$  has the following properties:*

- (i)  $\inf_{g \in O_l(\mathbb{C})} f_W(g) = f_W(e)$  if and only if  $WW^* \in \mathbb{R}^{l \times l}$ ,
- (ii) If  $WW^* \in \mathbb{R}^{l \times l}$ , then  $f_W(e) = f_W(g)$  if and only if  $g \in O_l(\mathbb{R}) \cdot \text{Stab}(W)$ .

To prove that we can take  $g$  equal to the identity, we first find, as above,  $g \in O_l(\mathbb{C})$  such that  $h = \sum_{i=1}^n a_i \text{ev}_{gu_i}$ . By Lemma 3.1, this implies that  $gU(gU)^*$  is real. Hence, by Lemma 3.2 (i)  $f_{gU}$  attains its infimum at the identity. Equivalently,  $f_U$  attains its infimum at  $g$ . So by Lemma 3.2 (ii),  $g \in O_l(\mathbb{R}) \cdot \text{Stab}(U)$ . From this we can conclude that  $h$  is real valued.

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# Adjacent vertex-distinguishing edge coloring of graphs

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**Abstract.** An adjacent vertex-distinguishing edge coloring (AVD-coloring) of a graph is a proper edge coloring such that no two neighbors are adjacent to the same set of colors. Zhang *et al.* [17] conjectured that every connected graph on at least 6 vertices is AVD  $(\Delta + 2)$ -colorable, where  $\Delta$  is the maximum degree.

In this paper, we prove that  $(\Delta + 1)$  colors are enough when  $\Delta$  is sufficiently larger than the maximum average degree, denoted  $\text{mad}$ . We also provide more precise lower bounds for two graph classes: planar graphs, and graphs with  $\text{mad} < 3$ . In the first case,  $\Delta \geq 12$  suffices, which generalizes the result of Edwards *et al.* [7] on planar bipartite graphs. No other results are known in the case of planar graphs. In the second case,  $\Delta \geq 4$  is enough, which is optimal and completes the results of Wang and Wang [14] and of Hocquard and Montassier [9].

## 1 Introduction

In the following, a graph is a connected simple graph on at least three vertices. A (*proper*) *edge  $k$ -coloring* of a graph is a coloring of its edges using at most  $k$  colors, where any two incident edges receive distinct colors. The *chromatic index* of a graph  $G$ , denoted by  $\chi'(G)$ , is the smallest integer  $k$  such that  $G$  admits an edge  $k$ -coloring. Let  $\Delta(G)$  be the maximum degree of  $G$ . Since incident edges receive distinct colors in an edge coloring, every graph  $G$  satisfies  $\chi'(G) \geq \Delta(G)$ . The Vizing's theorem ensures that the reverse inequality is nearly true, more precisely:

**Theorem 1.1 ([12]).** *Every graph  $G$  satisfies  $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$ .*

An *adjacent vertex-distinguishing edge  $k$ -coloring* (AVD  $k$ -coloring for short) is a proper edge  $k$ -coloring such that, for any two adjacent vertices  $u$  and  $v$ , the set of colors assigned to edges incident to  $u$  differs from the set of colors assigned to edges incident to  $v$ . The *AVD-chromatic index* of  $G$ , denoted by  $\chi'_{\text{avd}}(G)$ , is the smallest integer  $k$  such that  $G$  admits an AVD  $k$ -coloring. It should be noted that, while an isolated edge admits no AVD coloring, the AVD-chromatic index is finite for all connected graphs on at least three vertices. AVD colorings are also known as *adjacent strong edge colorings* [17] and *1-strong edge colorings* [1].

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Note that AVD colorings are special cases of *vertex-distinguishing proper edge colorings*. Such colorings are proper edge colorings such that no two (not necessarily adjacent) vertices are adjacent to the same set of colors. The corresponding chromatic index is called the *observability* and was studied for different graph classes [3, 5, 6, 8].

Since an AVD coloring is a proper edge coloring, every graph  $G$  satisfies  $\chi'_{avd}(G) \geq \Delta(G)$ . In addition, every graph  $G$  with two adjacent vertices of degree  $\Delta(G)$  satisfies  $\chi'_{avd}(G) \geq \Delta(G) + 1$ . Zhang *et al.* [17] completely determined the AVD-chromatic index of paths, cycles, trees, complete graphs, and complete bipartite graphs. They noted that a cycle of length five requires five colors, but conjectured that it is the only graph with such a gap between  $\chi'_{avd}(G)$  and  $\Delta(G)$ .

**Conjecture 1.2 ([17]).** Every graph  $G$  on at least 6 vertices satisfies  $\chi'_{avd}(G) \leq \Delta(G) + 2$ .

Balister *et al.* [2] proved Conjecture 1.2 for graphs with  $\Delta(G) = 3$  and for bipartite graphs.

For edge coloring, Theorem 1.1 ensures that the chromatic index of a graph can only have two values:  $\Delta(G)$  or  $\Delta(G) + 1$ , and the classification of graphs depending on this received considerable interest (for instance [11]). For AVD coloring, if Conjecture 1.2 holds then the AVD chromatic index of a graph can only have three values:  $\Delta(G)$ ,  $\Delta(G) + 1$  or  $\Delta(G) + 2$ . When considering a given graph class that allows two vertices of maximum degree to be adjacent, there are only two possible upper bounds:  $\Delta(G) + 1$  or  $\Delta(G) + 2$ . Similarly, the classification of graph classes depending on this received subsequent interest, for instance:

**Theorem 1.3 ([7]).** Every (connected) bipartite planar graph  $G$  with  $\Delta(G) \geq 12$  satisfies  $\chi'_{avd}(G) \leq \Delta(G) + 1$ .

Let  $\text{mad}(G) = \max \left\{ \frac{2|E(H)|}{|V(H)|}, H \subseteq G \right\}$  be the *maximum average degree* of the graph  $G$ , where  $V(H)$  and  $E(H)$  are the sets of vertices and edges of  $H$ , respectively. Wang and Wang [14] made the link between maximum average degree and AVD-chromatic index and proved Conjecture 1.2 for graphs with  $\Delta(G) \geq 3$  and  $\text{mad}(G) < 3$ .

**Theorem 1.4 ([14]).** Every (connected) graph  $G$  with  $\Delta(G) \geq 3$  and  $\text{mad}(G) < 3$  satisfies  $\chi'_{avd}(G) \leq \Delta(G) + 2$ .

They also gave sufficient conditions for graphs of bounded maximum average degree to be AVD  $(\Delta(G) + 1)$ -colorable. Combined with results of Hocquard and Montassier [9], we have:

**Theorem 1.5 ([9, 14]).** Every (connected) graph  $G$  with  $\Delta(G) \geq 3$  and  $\text{mad}(G) < 3 - \frac{2}{\Delta(G)}$  satisfies  $\chi'_{avd}(G) \leq \Delta(G) + 1$ .

Two main questions arise from these partial results: can this threshold of 3 as an upper-bound on  $\text{mad}(G)$  be reached with a sufficiently large lower-bound on  $\Delta(G)$  in the case of Theorem 1.5, and broken in the case of Theorem 1.4? We answer positively to these questions with Theorem 1.6: there is no threshold in the case of Theorem 1.5 (and thus in the case of Theorem 1.4).

**Theorem 1.6.** *Every graph  $G$  with  $\Delta(G) > 3 \times (\text{mad}(G))^2$  satisfies  $\chi'_{\text{avd}}(G) \leq \Delta(G) + 1$ .*

In the case of edge coloring, the best lower bound is due to Woodall [15]: every graph  $G$  with  $\Delta(G) > \frac{3 \times \text{mad}(G)}{2}$  satisfies  $\chi'(G) = \Delta(G)$ . There is a very large gap between this bound and its AVD counterpart, but this is essentially due to the fact that most methods on edge coloring are not transposable to AVD coloring. On the other hand, the gap between the bound for AVD coloring and its list edge counterpart is a mere constant factor [4] (note that list edge coloring is similarly conjectured to be always possible with  $\Delta(G) + 1$  colors [13]).

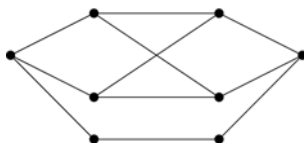
By Theorem 1.6, planar graphs with sufficiently large maximum degree are AVD  $(\Delta(G) + 1)$ -colorable. We provide a more refined lower-bound, and prove here that the bipartite hypothesis in Theorem 1.3 is unnecessary.

**Theorem 1.7.** *Every planar graph  $G$  with  $\Delta(G) \geq 12$  satisfies  $\chi'_{\text{avd}}(G) \leq \Delta(G) + 1$ .*

In the case of graphs with maximum average degree at most 3, we improve Theorem 1.5 by showing that  $\Delta(G) \geq 4$  is enough to reach the threshold of 3, as follows.

**Theorem 1.8.** *Every graph  $G$  with  $\Delta(G) \geq 4$  and  $\text{mad}(G) < 3$  satisfies  $\chi'_{\text{avd}}(G) \leq \Delta(G) + 1$ .*

Note that Theorem 1.8 is best possible since Figure 1.1 provides a sub-cubic graph with  $\text{mad}(G) = \frac{11}{4} < 3$  that is not AVD  $(\Delta + 1)$ -colorable.



**Figure 1.1.** A graph  $G$  with  $\Delta(G) = 3$  and  $\text{mad}(G) = \frac{11}{4} < 3$  such that  $\chi'_{\text{avd}}(G) = 5$ .

## 2 Method

We prove Theorems 1.6, 1.7 and 1.8 using a discharging method. We first choose a partial order on graphs. It depends on the theorem to prove but is basically a customization of the lexicographic order on the number of

vertices of each degree. In a first time, we consider by contradiction a “minimal” counter-example, and prove that it has some strong structural properties. We then prove that a graph with those structural properties cannot satisfy the assumptions of the theorem, which provides a contradiction. Due to the limited number of pages, the complete proofs of Theorems 1.6, 1.7 and 1.8 are omitted. We nevertheless sketch the proof of Theorem 1.8.

The four following lemmas provide the most relevant examples of structural properties that we proved on a minimal counter-example of any of Theorems 1.6, 1.7 and 1.8, where the number of colors is  $k + 1$ , for  $k \geq 4$ . In each case, we assume by contradiction that the graph contains such a configuration, we color by minimality a smaller graph and prove that the coloring can be extended to the whole graph.

**Lemma 2.1.** *No vertex  $v_2$  is adjacent to two vertices  $v_1$  and  $v_3$ , with  $d(v_1)$ ,  $d(v_2)$  and  $d(v_3) \leq \frac{k}{2}$ .*

Lemma 2.1 follows from a recoloring algorithm. The proof is quite involved, when a simple proof exists when  $\frac{k}{2}$  is replaced by  $\frac{k}{4}$ , but this bound is decisive for Theorems 1.6 and 1.7.

**Lemma 2.2.** *No vertex has at least  $\frac{k}{2}$  neighbors of degree 1.*

**Lemma 2.3.** *No vertex of degree 2 is adjacent to two vertices of degree at most  $\frac{k}{2} + 1$ .*

Lemmas 2.2 and 2.3 follow from a simple combinatorial argument, except in the case of a vertex of degree 2 adjacent to a vertex of degree exactly  $\frac{k}{2} + 1$ , where the result is derived from a recoloring argument and from the 2-connectivity of a minimal counter-example.

**Lemma 2.4.** *No vertex is adjacent to two vertices  $u, v$  with  $d(u) = 2$  and  $d(v) \leq 2$ .*

Lemma 2.4 follows from a simple reduction to a smaller graph, where the choice of the partial order is decisive. Those four lemmas are not enough for Theorems 1.6 and 1.7. Lemma 2.1 is decisive for Theorems 1.6 and 1.7, and Lemmas 2.2 to 2.4 suffice for Theorem 1.8.

We consider a graph  $G$  with  $\Delta(G) \leq k$  that satisfies Lemmas 2.2, 2.3 and 2.4, and assign to each vertex its degree as weight. We then design discharging rules to rearrange the weight along the graph so as to derive that  $\text{mad}(G) \geq 3$ . The following observation is instrumental in the proofs of Theorems 1.6 and 1.8.

**Observation 2.5.** For any vertex partition  $(V_1, V_2)$  of a graph  $G$ , if every vertex  $v$  has an initial weight of  $d(v)$ , and the weight can be rearranged along the graph so that every vertex  $v_1$  of  $V_1$  has a weight of at least

$2 \times d(v_1)$  and every vertex  $v_2$  of  $V_2$  has a weight of at least  $m$ , then  $\text{mad}(G) \geq m$ .

Thus, to prove Theorem 1.8, we design a single discharging rule stating that a vertex  $u$  with  $d(u) \geq 3$  that has a neighbor  $v$  of degree 1 or 2 gives a weight of 1 to  $v$ . We consider  $V_1$  to be the set of vertices of degree 1, and  $V_2$  the set of vertices of degree at least 2. By Lemma 2.2, the vertices incident to a vertex of degree 1 are of degree at least 4, and in particular  $V_1$  is an independent set. Let  $u$  be a vertex of  $G$ . If  $u \in V_1$ , since  $V_1$  is an independent set, the vertex  $u$  gives nothing and receives  $d(u)$ , so it has a final weight of  $2 \times d(u)$ . If  $u \in V_2$  with  $d(u) = 2$ , by Lemma 2.3, the vertex  $u$  is adjacent to at least one vertex of degree at least 3, and receives 1 from it. Since it gives nothing, it has a final weight of at least 3. If  $u \in V_2$  with  $d(u) \geq 3$  and  $u$  has a neighbor of degree 2, by Lemmas 2.3 and 2.4, the vertex  $u$  is adjacent to no other vertex of degree at most 2, and  $d(u) \geq 4$ , so  $u$  has a final weight of at least 3. If  $u \in V_2$  and has no neighbor of degree 2 with  $d(u) \geq 3$ , by Lemma 2.2, the vertex  $u$  has at most  $d(u) - 3$  neighbors of degree 1, so  $u$  has a final weight of at least 3. By Observation 2.5,  $\text{mad}(G) \geq 3$ .

For Theorem 1.7, we use a combinatorial argument to prove that a vertex cannot have too many small vertices, with an optimal bound (optimal for a combinatorial argument not involving any recoloring algorithm) depending on the respective degrees and on the number of colors. For Theorem 1.6, we use a method from a beautiful proof by Borodin, Kostochka and Woodall [4] (later simplified in [15]) of a similar result on list edge coloring.

### 3 Conclusion and perspectives

With Theorem 1.6, we made a significant step towards Conjecture 1.2, by proving that there are many graphs that need one less color. Our methods will however not be sufficient for the conjecture itself, as they require sparsity hypotheses. However, we could aim at proving that all planar graphs are AVD  $(\Delta(G) + 2)$ -colorable, by further developing the proof of Theorem 1.7. We conclude with two conjectures.

**Conjecture 3.1.** For any graph  $G$ , if the set of vertices of maximum degree in  $G$  forms an independent set, then  $\chi'_{\text{avd}}(G) \leq \Delta(G) + 1$ .

**Conjecture 3.2.** For any graph  $G$ , if  $G$  admits a subgraph  $H$  such that  $\chi'_{\text{avd}}(H) > \chi'_{\text{avd}}(G)$ , then either  $H$  is not connected or  $\Delta(H) = 2$ .

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# Rainbow path and minimum degree in properly edge colored graphs

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**Abstract.** A rainbow path in a properly edge colored graph is a path in which all the edges are colored with distinct colors. Let  $G$  be a properly edge colored graph with minimum degree  $\delta$  and let  $t$  be the maximum length of a rainbow path in  $G$ . In this paper, we show that  $t \geq \lfloor \frac{3\delta}{5} \rfloor$ . It is easy to see that there exist graphs for which  $t \leq \delta$ ; with respect to some proper edge coloring. For example,  $\delta$ -regular graphs of chromatic index  $\delta$  are graphs for which  $t \leq \delta$ , with respect to their optimum proper edge coloring. We leave open the question of getting a lower bound as close to  $\delta$  as possible.

## 1 Introduction

Given a graph  $G = (V, E)$ , a map  $c : E \rightarrow \mathbb{N}$  ( $\mathbb{N}$  is the set of non-negative integers) is called a *proper edge coloring* of  $G$  if for every two adjacent edges  $e_1$  and  $e_2$  of  $G$ , we have  $c(e_1) \neq c(e_2)$ . If  $G$  is assigned such a coloring  $c$ , then we say that  $G$  is a *properly edge colored graph*. We denote the color of an edge  $e \in E(G)$  by  $color(e)$ .

A path in an edge colored graph with no two edges sharing the same color is called a *rainbow path*. Similarly, a cycle in an edge colored graph is called a *rainbow cycle* if no two edges of the cycle share the same color. Given a coloring of the edges of  $G$ , a *rainbow matching* is a matching whose edges have distinct colors. Rainbow cycles, rainbow paths and rainbow matching in properly edge colored complete graphs are related to partial transversal of latin squares. Latin squares have been a popular topic in combinatorics at least since the times of Euler, who studied them extensively. A survey on rainbow paths, cycles and other rainbow sub-graphs can be found in [6]. Several theorems and conjectures on rainbow cycles can be found in a paper by Akbari, Etesami, Mahini and Mahmoodi in [1].

Maximum length rainbow paths and rainbow cycles are studied extensively in properly edge colored complete graphs. Hahn conjectured that every proper edge coloring of  $K_n$  ( $K_n$  is a complete graph having  $n$  vertices) admits a Hamiltonian rainbow path (a rainbow path visiting every

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vertex of  $K_n$ ). Later, Maamoun and Meyniel [7] disproved this conjecture by constructing counterexamples for the case where  $n$  is a power of two. Still it is widely believed that in every proper edge coloring of  $K_n$ , there is a rainbow path on  $n - 1$  vertices, though this is far from proved. Gyárfás and Mhalla in [5] have shown that the number of vertices in a maximum rainbow path in (a properly edge colored)  $K_n$  is at least  $(2n + 1)/3$ . Very recently in [4], H. Gebauer and F. Mousset have improved this bound and have shown that, in every proper edge coloring of  $K_n$ , there is a rainbow path of length  $(\frac{3}{4} - o(1))n$ .

### 1.1 Our results

In this paper, we consider the problem of finding a maximum length rainbow path in a properly edge colored graph  $G$ . Let  $\delta$  be the minimum degree of  $G$ . The main result of this paper is the following:

**Theorem 1.1.** *If  $t$  is the maximum length of a rainbow path in a properly edge colored graph  $G$ , then  $t \geq \lfloor \frac{3}{5}\delta \rfloor$ .*

We are unable to show that the bound given in this paper is tight. On the other hand, it is easy to see that, we cannot expect to get a lower bound greater than  $\delta$ . For example, consider  $\delta$  regular graphs of chromatic index  $\delta$  which are optimally properly edge colored. Clearly for these cases maximum rainbow path length cannot exceed  $\delta$ . We leave open the question of getting a lower bound of  $\delta - c$ , for some constant  $c$ .

### 1.2 Preliminaries

All graphs considered in this paper are finite, simple and undirected. A graph is a tuple  $(V, E)$ , where  $V$  is the finite set of vertices and  $E$  is the set of edges. For a graph  $G$ , we use  $V(G)$  and  $E(G)$  to denote its vertex set and edge set, respectively. The neighborhood  $N(v)$  of a vertex  $v$  is the set of vertices adjacent to  $v$  but not including  $v$ . The degree of a vertex  $v$  is  $d_v = |N(v)|$ . We write  $|V(G)|$ ,  $\delta$ ,  $\Delta$  for the order, minimum degree and maximum degree of  $G$ , respectively. A path is a non-empty graph  $P = (V, E)$  of the form  $V = \{p_1, p_2, \dots, p_k\}$  and  $E = \{\{p_1, p_2\}, \{p_2, p_3\}, \dots, \{p_{k-1}, p_k\}\}$ , which we usually denote by the sequence  $\{p_1, p_2, \dots, p_k\}$ . The *length* of a path is its number of edges. If  $P = \{p_1, p_2, \dots, p_k\}$  is a path, then the graph  $C = P \cup \{p_k, p_1\}$  is a cycle, and  $|E(C)|$  is the length of  $C$ . We represent this cycle by the cyclic sequence of its vertices, for example  $C = \{p_1, p_2, \dots, p_k, p_1\}$ .

## 2 Proof of the main results

Let  $G$  be a properly edge colored graph with the length of maximum rainbow path equal to  $t$ . In this Section,  $C$  stands for the set of colors



used in the proper edge coloring of the graph  $G$ . The following lemma ensures a rainbow path of length  $\lceil \frac{\delta+1}{2} \rceil$  starting from any vertex in a properly edge colored graph  $G$ . Due to page restriction we omitted the proof.

**Lemma 2.1.** *Given any vertex  $x$  in  $G$ , there exists a rainbow path of length at least  $\lceil \frac{\delta+1}{2} \rceil$  starting from  $x$ .*

The following lemma ensures that if the maximum length of a rainbow path is small enough, then we can convert the maximum rainbow path into a rainbow cycle by some simple modifications.

**Lemma 2.2.** *Let  $G$  be a properly edge colored graph with a rainbow path of maximum length, say  $t$ . If  $t < \lfloor \frac{3}{5}\delta \rfloor$ , then  $G$  contains a rainbow cycle of length  $(t + 1)$ .*

*Proof.* Assume for contradiction that there is no rainbow cycle of length  $t + 1$  in  $G$ . Let  $P = \{u_0(= x), u_1, u_2, \dots, u_t(= y)\}$  be a rainbow path of length  $t$  in  $G$ . Let  $U = \{color(u_i, u_{i+1}), 0 \leq i \leq t - 1\}$  and  $U^c = C \setminus U$ , where  $C$  is the set of colors used to color the edges of  $G$  properly. Clearly  $|U| = t$ . Let  $T_x = \{u_i : 0 \leq i \leq t, (x, u_i) \in E(G) \text{ and } color(x, u_i) \in U^c\}$  and let  $T_y = \{u_i : 0 \leq i \leq t, (y, u_i) \in E(G) \text{ and } color(y, u_i) \in U^c\}$ . First note that  $|\{(x, z) \in E(G) : color(x, z) \in U^c\}| \geq \delta - t$ . Moreover, if  $(x, z) \in E(G)$  with  $color(x, z) \in U^c$ , then  $z \in V(P)$ , i.e.,  $z = u_i$  for some  $1 \leq i \leq t$ , since otherwise we would have rainbow path of length  $t + 1$  in  $G$ . It follows that  $|T_x| \geq \delta - t$ . By a similar argument, we get  $|T_y| \geq \delta - t$ . Note that  $u_0, u_1 \notin T_x$  since  $u_0 = x$  and  $color(x, u_1) \in U$ . Also,  $u_t \notin T_x$ , since if  $(x, u_t)$  is an edge and is colored using a color from  $U^c$ , then we already have a  $t + 1$  length rainbow cycle, contrary to the assumption. So, we can write  $T_x = \{u_i : 2 \leq i \leq t - 1 \text{ and } color(x, u_i) \in U^c\}$ . By similar reasoning, we can write,  $T_y = \{u_i : 1 \leq i \leq t - 2 \text{ and } color(y, u_i) \in U^c\}$ . Define  $M_x = \{u_j : u_{j+1} \in T_x\}$ .

**Observation 1.**  $|M_x| = |T_x| \geq \delta - t$ .

**Claim 1.**  $M_x \cap T_y \neq \emptyset$ .

If possible suppose  $M_x \cap T_y = \emptyset$ . Now,  $|M_x| + |T_y| \leq t - 1$ , as both  $M_x \subset V(P)$  and  $T_y \subset V(P)$  and number of vertices on  $P$  excluding  $x$  and  $y$  is  $t - 1$ . (Note that  $x, y \notin M_x$  and  $x, y \notin T_y$ .) As  $|M_x| \geq \delta - t$  and  $|T_y| \geq \delta - t$  and  $M_x \cap T_y = \emptyset$  by assumption, we have  $\delta - t + \delta - t \leq t - 1$ . That is,  $2\delta \leq 3t - 1$ . So,  $t \geq \frac{2\delta+1}{3}$ . This is a contradiction to the fact that  $t < \lfloor \frac{3}{5}\delta \rfloor$ . Hence Claim 1 is true.

**Claim 2.** If  $u_i \in M_x \cap T_y$ , then  $color(y, u_i) = color(x, u_{i+1})$ .

If possible suppose Claim 2 is false. That is,  $u_i \in M_x \cap T_y$  and  $color(y, u_i) \neq color(x, u_{i+1})$ .

Now consider the cycle:  $CL = \{x, u_1, \dots, u_i, y, u_t, u_{t-1}, \dots, u_{i+1}, x\}$ . Clearly  $CL$  is a rainbow cycle, as  $color(y, u_i) \neq color(x, u_{i+1})$  and  $color(y, u_i) \in U^c$  and  $color(x, u_{i+1}) \in U^c$ . Note that, the length of  $CL$  is  $t + 1$ , as we removed exactly one edge, namely  $(u_i, u_{i+1})$  from  $P$  and added two new edges, namely  $(y, u_i)$  and  $(x, u_{i+1})$  to  $CL$ . So, the length of  $CL$  is  $t - 1 + 2 = t + 1$ , contradiction to the assumption. Hence, we can infer that if  $u_i \in M_x \cap T_y$ , then  $color(y, u_i) = color(x, u_{i+1})$ . Let  $S_y = \{v \in V(P) - M_x : color(y, v) \in U - \{color(y, u_{t-1})\}\}$ .

**Observation 2.**  $|M_x| + |T_y| + |S_y| - |M_x \cap T_y| \leq t - 1$ .

Proof: This is because  $S_y$  is disjoint from  $M_x \cup T_y$  and  $S_y \cup M_x \cup T_y \subseteq V(P) - \{y, u_{t-1}\}$ . (Note that  $y = u_t$  and  $u_{t-1}$  do not appear in  $M_x, T_y$  or  $S_y$ .)

We partition the set  $M_x \cap T_y$  as follows. Let  $u_i \in M_x \cap T_y$ . If  $color(u_i, u_{i+1})$  appears in one of the edges incident on  $y$ , then  $u_i \in A$  otherwise  $u_i \in B$ .

**Observation 3.**  $|T_y| \geq \delta - t + |B|$ .

Proof: To see this first note that there are at least  $\delta$  edges incident on  $y$  and at most  $t - |B|$  of them can get the colors from  $U$ , since  $|B|$  colors in  $U$  do not appear on the edges incident on  $y$ , by the definition of  $B$ . So, at least  $\delta - t + |B|$  of the edges incident on  $y$  have colors from  $U^c$ , and clearly any  $w$ , such that  $(y, w)$  is an edge, colored by a color in  $U^c$  has to be on  $P$ , since otherwise we have a longer rainbow path.

**Claim 3.** If  $u_i \in A$ , then the edge incident on  $y$  with color  $color(u_i, u_{i+1})$  has its other end point on the rainbow path  $P$ . That is, if  $w$  is such that  $(y, w)$  is an edge and  $color(u_i, u_{i+1}) = color(y, w)$ , then  $w \in V(P)$ .

If possible suppose Claim 3 is false.

Let  $(y, w) \in E(G)$  with  $color(y, w) = color(u_i, u_{i+1})$  and  $w \notin V(P)$ . Now, consider the path:  $P' = \{w, y, u_{t-1}, u_{t-2}, \dots, u_{i+1}, x (= u_0), u_1, \dots, u_i\}$ . Clearly  $P'$  is a rainbow path as  $color(u_i, u_{i+1}) = color(y, w)$ , the edge  $(u_i, u_{i+1}) \notin E(P')$  and  $color(u_{i+1}, x) \in U^c$ , since  $u_i \in M_x$ . Note that, the length of  $P'$  is  $t + 1$ . This is a contradiction to the fact that  $t$  is the maximum length rainbow path in  $G$ . Hence Claim 3 is true.

Now, partition  $A$  as follows: if  $u_i \in A$ , then by the above claim the edge incident on  $y$  with the color  $color(u_i, u_{i+1})$  has its other end point say  $w$ , on  $P$ . If  $w \in M_x$ , then let  $u_i \in A_1$ , else  $u_i \in A_2$ .

**Observation 4.**  $|M_x \cap T_y| = |A| + |B| = |A_1| + |A_2| + |B|$ .

**Observation 5.**  $|S_y| \geq |A_2|$ . To see this, recall that  $S_y = \{v \in V(P) - M_x : color(y, v) \in U - \{color(y, u_{t-1})\}\}$ . By definition of  $A_2$ , for

each  $u_i \in A_2$  there exists a unique vertex  $w = w(u_i) \in V(P) - M_x$  such that  $(y, w)$  is an edge and  $color(y, w) = color(u_i, u_{i+1}) \in U$ . Since  $u_i \in A_2 \subset M_x, i < t - 1$  and thus  $color(u_i, u_{i+1}) \neq color(y, u_{t-1})$ . It follows that  $\{w(u_i) : u_i \in A_2\} \subseteq S_y$ , and therefore we have  $|S_y| \geq |A_2|$ .

**Claim 4.**  $|A_1| \leq \frac{|M_x|}{2}$ .

Recall that, for each  $u_i \in A_1$ , there is a unique vertex  $w = w(u_i)$  such that  $(y, w)$  is an edge with  $color(u_i, u_{i+1}) = color(y, w)$ . Moreover,  $w \in M_x$ , by the definition of  $A_1$  and  $A_1 \cup \{w(u_i) : u_i \in A_1\} \subseteq M_x$ . Note that  $w(u_i)$  is uniquely defined for  $u_i$  since it is the end point of the edge incident on  $y$  colored with the color of the edge  $(u_i, u_{i+1})$ . Moreover,  $A_1 \cap \{w(u_i) : u_i \in A_1\} = \emptyset$ , since  $A_1$  contains vertices which are end points of edges from  $y$ , colored by the colors in  $U^c$  whereas each  $w(u_i)$  is the end point of some edge from  $y$  which is colored by a color in  $U$ . It follows that  $2|A_1| \leq |M_x|$ . That is,  $|A_1| \leq \frac{|M_x|}{2}$ , as required.

Now, substituting  $\delta - t + |B|$  for  $|T_y|$  (by Observation 3),  $|A_2|$  for  $|S_y|$  (by Observation 5), and  $|A_1| + |A_2| + |B| = |M_x \cap T_y|$  (by Observation 4) in the inequality of Observation 2, and simplifying we get  $|M_x| + \delta - t - |A_1| \leq t - 1$ . Now using  $|A_1| \leq |M_x|/2$  (Claim 4) and simplifying we get  $\frac{|M_x|}{2} + \delta - t \leq t - 1$ . Recall that  $|M_x| \geq \delta - t$  (Observation 1). Substituting and simplifying we get,  $t \geq \frac{3\delta + 2}{5} \geq \lfloor \frac{3\delta}{5} \rfloor$ , contradicting the initial assumption. Hence the Lemma is true.  $\square$

**Theorem 2.3.** *Let  $G$  be a properly edge colored graph with minimum degree  $\delta$ . If  $t$  is the maximum length of a rainbow path in  $G$ , then  $t \geq \lfloor \frac{3\delta}{5} \rfloor$ .*

*Proof.* If possible suppose  $t < \lfloor \frac{3\delta}{5} \rfloor$ . By Lemma 2.2,  $G$  contains a rainbow cycle of length  $t + 1$ . Let  $CL$  be this cycle. Note that,  $CL$  contains  $(t + 1)$  vertices and  $(t + 1)$  edges. Now,  $t + 1 \leq \lfloor \frac{3\delta}{5} \rfloor$ . Let  $CL = \{u_0, u_1, \dots, u_t, u_0\}$  and  $V(CL^c) = V(G) \setminus V(CL)$ . Let  $U = \{color(e) : e \in E(CL)\}$  and  $U^c = C \setminus U$ , where  $C$  is the set of colors used to color the edges of  $G$  properly. Let  $F_i = \{z \in V(CL^c) : (u_i, z) \in E(G)\}$ .

**Claim 1.**  $|F_i| \geq \lceil \frac{2\delta}{5} \rceil$ . Moreover, for  $z \in F_i, color(u_i, z) \in U$ .

First part follows from the fact that  $degree(u_i) \geq \delta$  and there are at most  $\lfloor \frac{3\delta}{5} \rfloor$  vertices in  $CL$ . If possible suppose  $color(u_i, z) \in U^c$ . Now consider the path  $P' = \{z, u_i, u_{i+1}, u_{i+2}, \dots, u_t, u_0, \dots, u_{i-1}\}$ . Clearly,  $P'$  is a rainbow path as  $color(u_i, z) \in U^c$  and  $\{u_i, u_{i+1}, u_{i+2}, \dots, u_t, u_0, \dots, u_{i-1}\}$  is already a rainbow path being a part of the rainbow cycle  $CL$ . Note that, the length of  $P'$  is  $t + 1$ . This is a contradiction to the assumption that  $t$  is the maximum length rainbow path in  $G$ . Hence Claim 1 is true.

Let  $G' = (V', E')$ , where  $V'(G') = V(G)$  and  $E'(G') = E(G) \setminus \{e \in E(G) : color(e) \in U\}$ . Clearly, in  $G'$  there is no edge between  $V(CL)$  to  $V(CL^c)$ , since by Claim 1, every such edge is colored by a color in  $U$ . Consider the induced subgraph of  $V(CL^c)$  in  $G'$ . Let  $G'' = G'[V(CL^c)]$ . Let  $\delta'$  be the minimum degree of  $G''$ .

**Observation 1.**  $\delta' \geq \lceil \frac{2\delta}{5} \rceil$ .

Proof: Clearly  $\delta' \geq \delta - |U| = \delta - (t + 1) \geq \delta - \lfloor \frac{3\delta}{5} \rfloor \geq \lceil \frac{2\delta}{5} \rceil$ . Consider the following subset  $U_0$  of  $U$ , defined by  $U_0 = U_1 \cup U_2$ , where  $U_1 = \{color(u_i, u_{i+1}) : 0 \leq i \leq \lfloor \frac{\delta}{5} \rfloor - 1\}$  and  $U_2 = \{color(u_i, u_{i+1}) : (t + 1) - \lfloor \frac{\delta}{5} \rfloor \leq i \leq t - 1\} \cup \{color(u_t, u_0)\}$ .

**Claim 2.**  $\{color(u_0, z) : z \in F_0\} \cap U_0 = \emptyset$ .

Suppose not. Let  $z \in F_0$  be such that  $color(u_0, z) \in U_0$ . Without loss of generality assume that  $color(u_0, z) \in U_1$ . Then consider the path  $P^* = (u_{\lfloor \frac{\delta}{5} \rfloor}, \dots, u_t, u_0, z)$ , which is clearly a rainbow path, since the edge of  $CL$  with its color equal to  $color(u_0, z)$  is not there in this path. Also the length of  $P^*$  is  $t + 1 - \lfloor \frac{\delta}{5} \rfloor$ . By Observation 1,  $G''$  has minimum degree at least  $\lceil \frac{2\delta}{5} \rceil$ , and therefore by Lemma 2.1,  $G''$  has a rainbow path of length at least  $\lceil \frac{\delta}{5} \rceil$ , let us call this path  $P''$ . Clearly concatenating the path  $P''$  with  $P^*$  we get a rainbow path since colors used in  $P^*$  belong to  $U$  whereas the colors used in  $P''$  belong to  $U^c$ . Moreover, the length of this rainbow path is at least  $t + 1$ , a contradiction, to the assumption that  $t$  is the length of the maximum rainbow path in  $G$ .

Now we complete the proof as follows: In view of Claim 2, and Claim 1, we know that  $|F_0| \leq |U - U_0|$ . But  $|U - U_0| \leq \lfloor \frac{3\delta}{5} \rfloor - \lceil \frac{2\delta}{5} \rceil \leq \lceil \frac{\delta}{5} \rceil < \lceil \frac{2\delta}{5} \rceil$  (since we can assume  $\delta \geq 5$ : for smaller values of  $\delta$ , the Theorem is trivially true). This is a contradiction to the first part of Claim 1.  $\square$

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# b-coloring graphs with girth at least 8

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**Abstract.** A b-coloring of a graph is a proper coloring of its vertices such that every color class contains a vertex that has neighbors in all other color classes. The b-chromatic number of a graph is the largest integer  $b(G)$  such that the graph has a b-coloring with  $b(G)$  colors. This metric is upper bounded by the largest integer  $m(G)$  for which  $G$  has at least  $m(G)$  vertices with degree at least  $m(G) - 1$ . There are a number of results reporting that graphs with high girth have high b-chromatic number when compared to  $m(G)$ . Here, we prove that every graph with girth at least 8 has b-chromatic number at least  $m(G) - 1$ . This proof is constructive and yields a polynomial time algorithm to find the b-chromatic number of  $G$ . Furthermore, we improve known partial results related to reducing the girth requirement of our proof.

## 1 Introduction

Let  $G$  be a simple graph and consider the traditional definitions of proper coloring and chromatic number. Suppose that we have a proper coloring of  $G$  and there exists a color  $c$  such that every vertex  $v$  with color  $c$  is not adjacent to at least one other color (which may depend on  $v$ ); then we can change the color of these vertices and thus obtain a proper coloring with fewer colors. This heuristic can be applied iteratively, but we cannot expect to reach the chromatic number of  $G$ , since the coloring problem is  $\mathcal{NP}$ -hard. On the basis of this idea, Irving and Manlove introduced the notion of b-coloring [6]. Intuitively, a b-coloring is a proper coloring that cannot be improved by the above heuristic, and the b-chromatic number measures the worst possible such coloring. More formally, consider a proper coloring  $\psi$  of  $G$ . A vertex  $u$  is said to be a *b-vertex* in  $\psi$  if  $u$  has a neighbor in each color class different from its own. A *b-coloring* of  $G$

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is a proper coloring of  $G$  such that each color class contains a  $b$ -vertex. A *basis* of a  $b$ -coloring  $\psi$  is a subset of  $b$ -vertices of  $\psi$  containing one  $b$ -vertex of each color class. The  *$b$ -chromatic number* of  $G$  is the largest integer  $b(G)$  for which  $G$  has a  $b$ -coloring with  $b(G)$  colors. Computing the  $b$ -chromatic number of a graph  $G$  is  $\mathcal{NP}$ -hard [6], even if  $G$  is bipartite [7] or is a chordal graph [5].

Naturally, a proper coloring of  $G$  with  $\chi(G)$  colors is a  $b$ -coloring of  $G$ ; hence,  $\chi(G) \leq b(G)$ . For an upper bound, let  $m(G)$  be the largest integer such that  $G$  has at least  $m(G)$  vertices with degree at least  $m(G) - 1$ . Then,  $b(G) \leq m(G)$  [6]. This upper bound is called the  $m$ -degree of  $G$ .

The difference between  $b(G)$  and  $m(G)$  can be arbitrarily large. For example, the complete bipartite graph  $K_{n,n}$  has  $b(G) = 2$  and  $m(G) = n + 1$ . Let the *girth* of  $G$ , denoted by  $g(G)$ , be the length of a shortest cycle of  $G$ . Even though deciding if  $b(G) = m(G)$  is  $\mathcal{NP}$ -hard [7], there seems to exist a relation between the girth of  $G$  and how close  $b(G)$  is to  $m(G)$ . In fact, the following two conjectures have received some attention.

**Conjecture 1.1 ([1]).** If  $G$  is a  $d$ -regular graph with girth at least 5 and  $G$  is not the Petersen graph, then  $b(G) = d + 1$ .

**Conjecture 1.2 ([5]).** If  $G = (A, B)$  is a  $C_4$ -free bipartite graph such that  $|A| = m$ ,  $d(u) = m - 1$  for all  $u \in A$  and  $d(v) < m - 1$  for all  $v \in B$ , then  $b(G) \geq m - 1$ .

Note that the  $d$ -regular graph in Conjecture 1.1 has girth at least 5 and  $m$ -degree  $d + 1$ , while the bipartite graph in Conjecture 1.2 has girth at least 6 and  $m$ -degree  $m$ .

Many partial results have been given for Conjecture 1.1 ([1, 2, 7, 8, 12]). In particular, Shaebani [12] gives a lower bound of  $\lfloor \frac{d+3}{2} \rfloor$  for the  $b$ -chromatic number of a  $d$ -regular graph with girth at least 5. Concerning Conjecture 1.2, Lin and Chang [9] proved that the conjecture holds if the famous Erdős-Faber-Lovász Conjecture [4] also holds.

Motivated by these conjectures and by Silva's observation that Irving and Manlove's result on trees actually holds for any graph with girth at least 11 [11], we pose the following question, where  $g, m, b$  stands for girth,  $m$ -degree and  $b$ -chromatic number.

**Question  $g, m, b$ .** What is the minimum value  $g^*$  for which  $g(G) \geq g^*$  implies  $b(G) \geq m(G) - 1$ ?

Observe that the complete bipartite graph implies  $g^* \geq 5$ . We mention that the previous best bound on  $g^*$  is obtained by Campos, Farias and Silva [3] where they prove  $g^* \leq 9$ . The main result in this paper is

to prove that  $g^*$  is either 5, 6, 7 or 8. The proof also yields a polynomial time algorithm to find an optimal  $b$ -coloring of a graph  $G$  with  $g(G) \geq 8$ .

**Theorem 1.3.** *If  $G$  is a graph with girth at least 8, then  $b(G) \geq m(G) - 1$ .*

Improving the best known upper bound for  $g^*$  has a few consequences for other studied problems. In fact, showing that  $g^* \leq 6$  proves Conjecture 1.2. Furthermore, showing that  $g^* \leq 5$  improves the lower bound obtained by Shaebani [12] from  $\lfloor \frac{d+3}{2} \rfloor$  to  $d$ .

Let  $C_3 \square C_3$  denote the Cartesian product of two cycles of length 3. In [10], Maffray and Silva make the following conjecture.

**Conjecture 1.4.** Let  $G$  be any graph with no  $K_{2,3}$  as subgraph. If  $G \neq C_3 \square C_3$ , then  $b(G) \geq m(G) - 1$ .

Although this conjecture is false due to a technical error as the graph  $C_3 \square C_3$  together with isolated vertices is a counter-example, we believe Conjecture 1.4 might be true with a few extra constraints. We propose a reformulation of this conjecture as follows.

**Conjecture 1.5.** Let  $G$  be any graph with no  $K_{2,3}$  as subgraph. Then  $b(G) \geq m(G) - 1$  unless  $G$  contains a component isomorphic to  $C_3 \square C_3$  and all other components have maximum degree at most 2.

Observe that Conjecture 1.5 implies  $g^* \leq 5$ . In fact, if  $g(G) \geq 5$ , then  $G$  contains neither a  $K_{2,3}$  nor a  $C_3 \square C_3$  as subgraphs.

Let  $\mathcal{K}_m$  denote the class of graphs obtained from  $m$  copies of  $K_m$  where any two copies of  $K_m$  intersect in at most one vertex. The famous Erdős-Faber-Lovász Conjecture [4] can be stated as follows.

**EFL Conjecture.** *If  $G \in \mathcal{K}_m$ , then  $\chi(G) = m$ .*

Let  $H \in \mathcal{K}_m$ . Note that each copy of  $K_m$  in  $H$  contains at least one vertex not contained in any other copy of  $K_m$ . Let  $A$  be a set of  $m$  vertices obtained by picking one vertex from each copy of  $K_m$  with this property. Let  $G$  be the graph obtained from  $H$  by deleting all edges with no endpoint in  $A$  and let  $B = V(G) \setminus A$ . Note that  $G$  is bipartite and the bipartition  $(A, B)$  satisfies the properties of Conjecture 1.2 with  $m(G) = m$  as long as no vertex in  $B$  has degree at least  $m - 1$ . Furthermore, if  $b(G) = m$  with  $A$  as a basis, then  $\chi(H) = m$  and EFL is true for  $H$ . Therefore, if  $g^* \leq 6$ , characterizing the graphs  $G$  with  $g(G) \geq 6$  and  $b(G) = m(G) - 1$  determines the possible counter-examples for the EFL Conjecture. We find this to be a strong partial result which could lead to a proof of EFL.



## 2 Improving tool lemmas

We say that a vertex  $u \in V(G)$  is *dense* if  $d(u) \geq m(G) - 1$  and we denote the set of dense vertices of  $G$  by  $D(G)$ . Let  $W$  be a subset of  $D(G)$  and  $u$  be any vertex in  $V(G) \setminus W$ . If all vertices in  $W$  are either adjacent to  $u$  or have a common neighbor  $v \in W$  with  $u$  with  $d(v) = m(G) - 1$ , then we say that  $W$  *encircles*  $u$ . A subset  $W$  of  $D(G)$  of size  $m(G)$  is a *good set* if:

- (a)  $W$  does not encircle any vertex, and
- (b) Every vertex  $x \in V(G) \setminus W$  with  $d(x) \geq m(G)$  is adjacent to  $W$ .

**Lemma 2.1 ([6]).** *Let  $G$  be any graph and  $W$  be a subset of  $D(G)$  with  $m(G)$  vertices. If  $W$  encircles some vertex  $v \in V(G) \setminus W$ , then  $W$  is not a basis of a  $b$ -coloring with  $m(G)$  colors.*

**Lemma 2.2 ([11]).** *If  $G$  is a graph with  $g(G) \geq 8$ , then  $G$  does not have a good set if and only if  $|D(G)| = m(G)$  and  $D(G)$  encircles a vertex. Also, a good set can be found in polynomial time, if one exists.*

**Lemma 2.3 ([11]).** *Let  $G$  be a graph with girth at least 8. If  $G$  has no good set, then  $b(G) = m(G) - 1$ .*

First, we note that Lemma 2.1 states that a basis for a  $b$ -coloring with  $m(G)$  colors cannot encircle any vertex. Therefore, a good set is a possible basis for a  $b$ -coloring with  $m(G)$  colors. Indeed, Theorem 1.3 easily follows from Lemmas 2.2, 2.3 and the one below.

**Lemma 2.4.** *If  $G$  has girth at least 8 and  $W$  is a good set of  $G$ , then there exists a  $b$ -coloring  $\psi$  of  $G$  with  $m(G)$  colors having  $W$  as basis.*

Using the same framework for the proof of Theorem 1.3, we cannot improve this result unless we improve the auxiliary results in Lemmas 2.2 and 2.3. This is done next.

**Lemma 2.5.** *Let  $G$  be a graph with  $g(G) \geq 6$ . Then  $G$  does not have a good set if, and only if, one of the following holds (below  $m = m(G)$ ):*

1.  $|D(G)| = m$  and  $D(G)$  encircles a vertex in  $V(G) \setminus D(G)$ ; or
2.  $g(G) = 6$ ,  $|D(G)| = m + 1$ ,  $d(v) = m - 1$  for all  $v \in D(G)$  and  $D(G)$  induces a matching in  $G$ . Furthermore,  $D(G)$  can be partitioned into stable sets  $X_0$  and  $X_1$  such that  $|X_0| = |X_1|$  and there are vertices  $u_0, u_1 \in V(G) \setminus D(G)$  such that  $u_i$  is adjacent to all vertices in  $X_i$  and to no vertices in  $X_{1-i}$ , for  $i \in \{0, 1\}$ .

*Moreover, a good set can be found in polynomial time, if one exists.*

**Lemma 2.6.** *Let  $G$  be a graph with girth at least 7. If  $G$  has no good set, then  $b(G) = m(G) - 1$ .*

**Lemma 2.7.** *If  $G$  is a graph satisfying property (2) of Lemma 2.5, then  $b(G) = m(G) - 1$ .*

### 3 Comments and open questions

We believe Question  $g, m, b$  to be an interesting problem and relate this question to other studied conjectures in the literature in Section 1. Furthermore we show that proving  $g^* \leq 6$  proves Conjecture 1.2 and, at the end of Section 1, we point out that this could be an important step in proving the famous Erdős-Faber-Lovász Conjecture. In Section 1 we also restate a conjecture by Silva and Maffray [10], which, if true, implies that  $g^* = 5$ .

Then, we prove that  $g^* \leq 8$  and improve the partial results used in this proof to try to build a stepping stone for proving an upper bound of 6 or 7 for  $g^*$  in Section 2. We mention that an equivalent of Lemma 2.4 cannot be obtained for graphs of girth at most 6 [11].

Recall that the EFL Conjecture implies Conjecture 1.2 [9]. We also feel it would be interesting to prove a strengthening of this result. Can we prove  $g^* \leq 6$  if we assume that the EFL Conjecture is true?

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# The circular chromatic index of $k$ -regular graphs

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**Abstract.** The circular chromatic index of a graph  $G$  is the infimum of all rational numbers  $p/q$ , such that there exists a circular  $p/q$ -edge-coloring of the graph  $G$ . A natural problem is to determine the set of possible values of the circular chromatic indices of  $k$ -regular graphs. In this work we construct  $k$ -regular graphs with certain circular chromatic indices of the form  $k + a/p$ , in particular graphs for all  $a \in \{1, 2, \dots, \lfloor k/2 \rfloor\}$  and  $p \geq 2a^2 + a + 1$ .

## 1 Introduction

Graphs considered in this paper are finite. The notion graph always refers to a simple graph. If parallel edges are allowed, the object in question is called a multigraph.

For a real number  $r \geq 1$  we define a *circular  $r$ -edge-coloring* of a graph  $G$  as a mapping  $c: E(G) \rightarrow [0, r)$  satisfying  $1 \leq |c(e) - c(f)| \leq r - 1$  for any pair of adjacent edges  $e$  and  $f$  of  $G$ . The *circular chromatic index* of the graph  $G$ , denoted by  $\chi'_c(G)$ , is the infimum of the set of all real numbers  $r$  such that there exists a circular  $r$ -edge-coloring of  $G$ .

Which real numbers are the circular chromatic indices of  $k$ -regular graphs? Since  $\chi'(G) = \lceil \chi'_c(G) \rceil$ , the Vizing's Theorem implies that  $k \leq \chi'_c(G) \leq k + 1$  for any  $k$ -regular graph  $G$ . Circular edge-colorings are especially interesting for  $k$ -regular graphs with the chromatic index  $k + 1$ , so called  $k$ -regular class 2 graphs. While circular edge-colorings of cubic graphs have been extensively studied (e.g. [1–6]), very little is known about circular edge-colorings of  $k$ -regular graphs with  $k \geq 4$ . Re-

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cently, Lukořka and Mazák [5] have shown that for each given rational number  $r$ , such that  $3 < r < 3 + 1/3$ , there exists an infinite family of cubic graphs with the circular chromatic index  $r$ . On the other hand, by [1], the circular chromatic number of a bridgeless cubic graph is at most  $3 + 2/3$ . There are known cubic graphs with the circular chromatic index  $r$  for infinitely many values  $r$ , such that  $3 + 1/3 < r < 3 + 2/3$  [5], but the global situation in this interval is still open.

It has been conjectured [1] that for any  $k \geq 2$  and an  $\varepsilon_k > 0$  there exists no graph with  $k - \varepsilon_k < \chi'_c < k$ . The values of the circular chromatic index we present in this work belong to the interval  $(k, k + 1/4)$ .

## 2 $k$ -Regular graphs with $\chi'_c = k + a/p$

In this section we provide a construction of  $k$ -regular graphs with the circular chromatic index equal to  $k + a/p$  for all integers  $k \geq 4$ ,  $a \in \{1, 2, \dots, \lfloor k/2 \rfloor\}$  and  $p = (2a + 1)m + an$  such that  $m, n \geq 1$ .

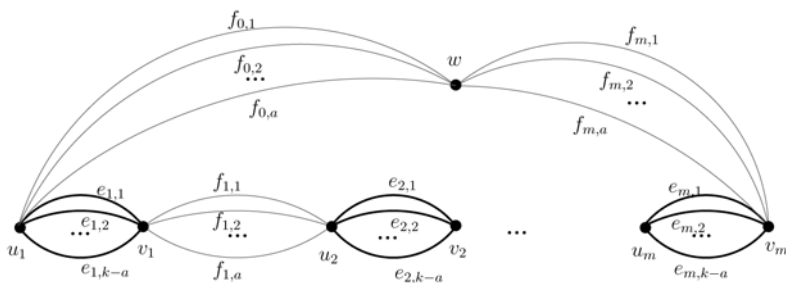
The colors in a circular  $r$ -coloring will be taken modulo  $r$ . Assume that  $a$  and  $b$  are two colors in a circular  $r$ -coloring. The distance of  $a$  and  $b$  is denoted by  $|a - b|_r$  and defined as  $|a - b|_r = \min\{|a - b|, r - |a - b|\}$ .

Let us consider a circular  $r$ -edge-coloring. For any  $a, b \in [0, r)$ , the  $r$ -circular interval  $[a, b]_r$  is defined as follows:

$$[a, b]_r = \begin{cases} [a, b] & (a \leq b), \\ [a, r) \cup [0, b] & (a > b). \end{cases}$$

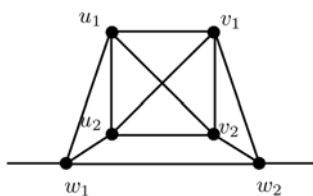
Although our construction aims towards simple graphs, we start with multigraphs. Let  $M_{k,a,m}$  be a multigraph with vertex set  $\{u_1, u_2, \dots, u_m, v_1, v_2, \dots, v_m, w\}$ . The vertex  $w$  is also denoted as  $v_0$  or  $u_{m+1}$ . The edge set of  $M_{k,a,m}$  consists of  $k - a$  parallel  $u_i v_i$ -edges denoted by  $e_{i,1}, e_{i,2}, \dots, e_{i,k-a}$  for each  $i \in \{1, 2, \dots, m\}$  and of  $a$  parallel  $v_i u_{i+1}$ -edges denoted by  $f_{i,1}, f_{i,2}, \dots, f_{i,a}$  for each  $i \in \{0, 1, \dots, m\}$ , see Figure 2.1.

We create the graph  $G_{k,a,m,n}$  from the multigraph  $M_{k,a,m}$  by inserting so called  $\varepsilon_k$ -block between vertices to avoid parallel edges. The  $\varepsilon_k$ -block contains two dangling edges and is  $k$ -regular and  $k$ -edge-colorable. To construct this block we take the union of  $k - 1$  perfect matchings of the complete graph on  $2k - 4$  vertices, which is a  $(k - 1)$ -regular graph that is  $(k - 1)$ -edge-colorable. We add one dangling edge to each vertex and we denote the resulting graph by  $G$ . Let  $w_1$  and  $w_2$  be two vertices disjoint from  $G$  joined by an edge. To create  $\varepsilon_k$ -block, we adjoin half of the dangling edges in  $G$  to the vertex  $w_1$ , the other half to the vertex  $w_2$ , and add a dangling edge, called the *input edge*, to  $w_1$  and a dangling edge, called the *output edge*, to  $w_2$ , see Figure 2.2. Moreover, as we show in

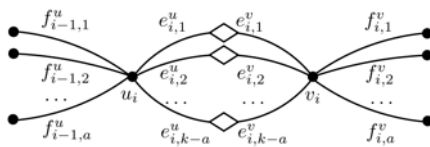


**Figure 2.1.** The multigraph  $M_{k,a,m}$ .

the forthcoming paper, it is possible to divide the dangling edges of  $G$  into those that will be incident with  $w_1$  or with  $w_2$  in such a way that the resulting block is  $k$ -edge-colorable. We will write  $\varepsilon$ -block instead of  $\varepsilon_k$ -block when  $k$  is fixed.



**Figure 2.2.** The  $\varepsilon_4$ -block.



**Figure 2.3.** The subgraph  $H_i$ .

The  $\varepsilon$ -block is inserted into every edge in the multigraph  $M_{k,a,m}$  once, except for the edges  $f_{m,1}, f_{m,2}, \dots, f_{m,a}$  where  $n$  copies of the  $\varepsilon$ -block are inserted into each edge. This way we get the graph  $G_{k,a,m,n}$  that contains no parallel edges at all. The multigraph  $M_{k,a,m}$  contains edges  $e_{i,j}$  for all  $i \in \{1, 2, \dots, m\}$  and  $j \in \{1, 2, \dots, k - a\}$  and  $f_{b,c}$  for all  $b \in \{0, 1, \dots, m\}$  and  $c \in \{1, 2, \dots, a\}$ . In the graph  $G_{k,a,m,n}$ , we denote the  $e$ -edges incident with the vertex  $u_i$  as  $e_{i,j}^u$  and with the vertex  $v_i$  as  $e_{i,j}^v$ . Similarly, the  $f$ -edges incident with  $u_i$  are denoted as  $f_{i-1,j}^u$  and incident with  $v_i$  are  $f_{i,j}^v$ . The edges that are incident with  $\varepsilon$ -blocks inserted into the edge  $f_{m,l}$  are denoted by  $f_{m,l}^1, f_{m,l}^2, \dots, f_{m,l}^{n+1}$  in the order as they occur when proceeding from  $v_m$  to  $w$ .

Let  $H_i$  be the subgraph of  $G_{k,a,m,n}$  induced by the vertices  $u_i, v_i$ , by all  $\varepsilon$ -blocks lying on  $e_{i,j}$  edges for  $j \in \{1, 2, \dots, k - a\}$ , as well as by all the vertices incident with the edges  $u_i$  and  $v_i$  for  $i \in \{1, 2, \dots, m\}$ , see Figure 2.3. Let  $S$  be the subgraph induced by the vertices  $v_m, w$ , and by all  $\varepsilon$ -blocks lying on the  $f_{m,c}$  edges for  $c \in \{1, 2, \dots, a\}$ .

Let  $e_1$  be the input edge and  $e_2$  be the output edge of the  $\varepsilon_k$ -block. Although the edges  $e_1$  and  $e_2$  receive the same color in any  $k$ -edge-coloring of  $\varepsilon_k$ -block, they can slightly differ in a circular  $(k + \varepsilon)$ -edge-coloring  $\varphi$ . We define the *color change in the  $\varepsilon$ -block* as  $|\varphi(e_1) - \varphi(e_2)|_{k+\varepsilon}$ .

**Lemma 2.1.** *Let  $k \geq 4$  and let  $\varepsilon < 1/3$ . The color change in the  $\varepsilon_k$ -block is at most  $\varepsilon$  in any  $(k + \varepsilon)$ -edge-coloring. Moreover, there exists a  $(k + \varepsilon)$ -edge-coloring of the  $\varepsilon_k$ -block with the color change  $\varepsilon$ .*

**Theorem 2.2.** *For each integer  $k \geq 4$ ,  $a \in \{1, 2, \dots, \lfloor k/2 \rfloor\}$ , and  $m, n \geq 1$  there exists a  $k$ -regular graph with circular chromatic index  $k + \frac{a}{(2a+1)m+an}$ .*

*Proof.* Let  $k, a, m, n$  be integers that fulfill conditions of the theorem. Let  $p = (2a + 1)m + an$ ,  $\varepsilon = a/p$ , and  $r = k + \varepsilon$ . First we show that  $\chi'_c(G_{k,a,m,n}) = k + a/p$ . Then we can construct infinitely many  $k$ -regular graphs, which contain  $G_{k,a,m,n}$  as a subgraph, with  $\chi'_c = k + a/p$  [5].

It is possible to construct a circular  $(k + a/p)$ -edge-coloring  $\psi$  of  $G_{k,a,m,n}$  where  $p = (2a + 1)m + an$ , which gives us the upper bound on the circular chromatic index. The  $\varepsilon$ -blocks are colored in accordance with Lemma 2.1, hence their maximal color change is used as shown in our forthcoming paper.

We now derive a lower bound on the circular chromatic index of the graph  $G_{k,a,m,n}$ . Let us consider the subgraph  $H_i$  of the graph  $G_{k,a,m,n}$ . It can be shown that in any circular  $(k + \varepsilon)$ -edge-coloring  $\varphi$  the  $f$ -edges of  $H_i$  can be arranged into pairs in such a way that the colors of the edges in every pair differ by at most  $2\varepsilon$ . Without loss of generality we will assume that the parallel edges with the same second subscript are the ones that differ in color by at most  $2\varepsilon$ . More precisely, we will assume that the edges are denoted in such a way that  $|\varphi(f_{i-1,g}^u) - \varphi(f_{i,g}^v)| \leq 2\varepsilon$  for every  $g \in \{1, 2, \dots, a\}$ .

We say that the edges  $f_{0,t}^x, f_{1,t}^x, \dots, f_{m-1,t}^x, f_{m,t}^1, f_{m,t}^2, \dots, f_{m,t}^{n+1}$ , for  $x \in \{u, v\}$  and  $t \in \{1, 2, \dots, a\}$  belong to the same *coloring line* in the graph  $G_{k,a,m,n}$ .

We say that the edges  $f_{0,c}^u, f_{0,c+1}^u, \dots, f_{0,d}^u$  form a *coloring block* in  $\varphi$ , if  $\varphi(f_{0,c+q}^u) \in [\varphi(f_{0,c}^u) + q - \varepsilon, \varphi(f_{0,c}^u) + q + \varepsilon]$ , for  $c, d \in \{1, 2, \dots, a\}$ , and  $q \in \{1, 2, \dots, d - c\}$  and this set cannot be extended. Roughly speaking, the differences of colors of these edges from  $\varphi(f_{0,1}^u)$  form a non-extendable sequence of numbers that are near to consecutive integers. We assume that all edges  $f_{0,l}^u$  in the coloring  $\varphi$  are divided into  $r$  coloring blocks of sizes  $n_1, n_2, \dots, n_r$  such that  $\sum n_i = a$ . Let  $n_{max} = \max_{i \in \{1, 2, \dots, r\}} \{n_i\}$ . It can be shown that it is sufficient to consider only the case when  $n_{max} = 1$  as the established upper bound is exceeded for greater

values of  $n_{max}$ . The notion of the coloring block is defined analogously for all subgraphs  $H_i$  and the number and sizes of the coloring blocks are the same for all  $i \in \{1, 2, \dots, m\}$ .

We define a *local change* in the subgraph  $H_i$  as  $\sum_{j=1}^a |\varphi(f_{i,j}^v) - \varphi(f_{i-1,j}^u)|_r$ .

We bound the local change by showing that it is at most  $(a + 1)\varepsilon$ .

Let  $\varphi$  be an  $(k + \varepsilon)$ -edge-coloring of the subgraph  $H_i$ . We say that the  $q$ -th coloring line is *increasing* in  $H_i$  if  $\varphi(f_{i,q}^v) \in [\varphi(f_{i-1,q}^u), \varphi(f_{i-1,q}^u) + \varepsilon]$  and *decreasing* otherwise.

If there exist both increasing and decreasing coloring lines in  $H_i$  then it can be shown that the local change is at most  $a\varepsilon$ . More interesting situation happens with the case when all the coloring lines are either increasing or decreasing. We may assume that all coloring lines are increasing and  $\varphi(e_{i,j}^u) \leq \varphi(e_{i,j}^v)$  for every  $j \in \{1, 2, \dots, k - a\}$ .

Let  $\varphi(e_{i,1}^u) = 0$  and  $\varphi(f_{i-1,1}^u) < \varphi(f_{i-1,2}^u) < \dots < \varphi(f_{i-1,a}^u)$ . Then there exist integers  $P_1, P_2, \dots, P_a$  from  $\{1, 2, \dots, k - 1\}$  and real numbers  $p_1, p_2, \dots, p_a$  such that  $\varphi(f_{i-1,j}^u) = P_j + p_j$  for  $j \in \{1, 2, \dots, a\}$  and  $0 \leq p_j \leq \varepsilon$ . Since each coloring block is of size 1, there exist integers  $Q_1, Q_2, \dots, Q_{a-1}$  and real numbers  $q_1, q_2, \dots, q_{a-1}$  such that, for  $j \in \{1, 2, \dots, a - 1\}$  and  $0 \leq q_j \leq \varepsilon$ ,  $P_j + p_j \leq Q_j + q_j \leq P_{j+1} + p_{j+1}$ ,  $0 \leq p_1 \leq q_1 \leq p_2 \leq q_2 \leq \dots \leq p_{a-1} \leq q_{a-1} \leq p_a$  where  $Q_j + q_j$  is the color of  $e_{i,s}^u$  for some  $s$ . (For the sake of convenience we set  $Q_0 = q_0 = 0$  and  $\varphi(e_{i,1}^u) = Q_0 + q_0 = 0$ ). Lemma 2.1 implies that  $Q_j + q'_j$  is the color of  $e_{i,s}^v$  for some  $s$  where  $0 \leq q'_j - q_j \leq \varepsilon$ . Finally, we have  $\varphi(f_{i,j}^v) = P_j + p'_j$  for real numbers  $p'_j$  such that  $Q_{j-1} + q'_{j-1} \leq P_j + p'_j$  for  $j \in \{1, 2, \dots, a\}$  and  $P_j + p'_j \leq Q_j + q'_j$  for  $j \in \{1, 2, \dots, a - 1\}$  and  $0 \leq p'_1 \leq q'_1 \leq p'_2 \leq q'_2 \leq \dots \leq p'_{a-1} \leq q'_{a-1} \leq p'_a \leq q'_0 + \varepsilon$ .

Clearly, the local change is  $\sum_{j=1}^a p'_j - p_j$  and can be bounded as follows.

$$\sum_{j=1}^a (p'_j - p_j) = \sum_{j=1}^{a-1} (p'_j - p_j) + p'_a - p_a \leq \sum_{j=1}^{a-1} (q'_j - q_{j-1}) + p'_a - p_a \leq (a - 2)\varepsilon + (q'_{a-1} - q_0) + (p'_a - q_{a-1}) \leq (a - 1)\varepsilon + (p'_a - q'_0) + (q'_0 - q_0) \leq (a - 1)\varepsilon + \varepsilon + \varepsilon = (a + 1)\varepsilon.$$

There are  $a$  coloring lines in the graph  $G_{k,a,m,n}$  and we need to change the color by at least 1 on each of them. We can obtain the change of at most  $(a + 1)\varepsilon$  in each  $H_i$ . The number of  $\varepsilon$ -blocks outside  $H_i$  subgraphs is  $(m + n)a$ , each with the change of at most  $\varepsilon$ . Summing up we have  $m(a + 1)\varepsilon + an\varepsilon + an\varepsilon \geq a$  and  $\varepsilon \geq \frac{a}{m(2a+1)+an}$ .

Since the lower and the upper bound coincide, the theorem follows.  $\square$

**Corollary 2.3.** *For each integer  $k \geq 3$ ,  $a \in \{1, 2, \dots, \lfloor k/2 \rfloor\}$ , and  $p \geq 2a^2 + a + 1$  there exists a  $k$ -regular graph with circular chromatic index  $k + a/p$ .*



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# Coloring $d$ -Embeddable $k$ -Uniform Hypergraphs

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and Anusch Taraz<sup>4</sup>

**Abstract.** We extend the scenario of the Four Color Theorem in the following way. Let  $\mathcal{H}_{d,k}$  be the set of all  $k$ -uniform hypergraphs that can be linearly embedded into  $\mathbb{R}^d$ . We investigate lower and upper bounds on the maximum (weak and strong) chromatic number of hypergraphs in  $\mathcal{H}_{d,k}$ . For example, we can prove that for  $d \geq 3$  there are hypergraphs in  $\mathcal{H}_{2d-3,d}$  on  $n$  vertices whose weak chromatic number is  $\Omega(\log n / \log \log n)$ , whereas the weak chromatic number for  $n$ -vertex hypergraphs in  $\mathcal{H}_{d,d}$  is bounded by  $\mathcal{O}(n^{(d-2)/(d-1)})$  for  $d \geq 3$ .

## Extended Abstract

The Four Color Theorem [1, 2] has been one of the driving forces in Discrete Mathematics and its theme has inspired many variations. For example, the chromatic number of graphs that are embeddable into a surface of fixed genus has been intensively studied by Heawood [7], Ringel and Youngs [11], and many others.

Here, we consider  $k$ -uniform hypergraphs that are embeddable into  $\mathbb{R}^d$  in such a way that their edges do not intersect (see Definition 1 below). For  $k = d = 2$  the problem specializes to graph planarity. For  $k = 2$  and  $d \geq 3$  it is not a very interesting question because for any  $n \in \mathbb{N}$

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the vertices of the complete graph  $K_n$  can be embedded into  $\mathbb{R}^3$  using arbitrary points on the moment curve  $t \mapsto (t, t^2, t^3)$ .

As a consequence, we focus our attention on hypergraphs, which are in general not embeddable into any specific dimension. Some properties of these hypergraphs (or more generally simplicial complexes) have been investigated (see e.g. [4, 8, 9, 15]), but to our surprise, we have not been able to find any bounds on their (vertex-)chromatic number. However, Grünbaum and Sarkaria (see [6, 12]) have differently generalized the concept of graph colorings to simplicial complexes by coloring faces. They also bound this face-chromatic number subject to embeddability constraints.

We now quickly recall and introduce some useful notation. The pair  $H = (V, E)$  is a  $k$ -uniform hypergraph if the vertex set  $V$  is a finite set and the edge set  $E$  consists of  $k$ -element subsets of  $V$ .

Let  $H$  be a  $k$ -uniform hypergraph. A function  $\kappa : V(H) \rightarrow \{1, \dots, c\}$  is said to be a *strong  $c$ -coloring* if for all  $e \in E(H)$  the property  $|\kappa(e)| = k$  holds. The function  $\kappa$  is said to be a *weak  $c$ -coloring* if  $|\kappa(e)| > 1$  for all  $e \in E(H)$ . The *strong and weak chromatic number* of  $H$  is denoted by  $\chi^s(H)$  and  $\chi^w(H)$ , respectively. For graphs, weak and strong colorings are equivalent.

We next define what we mean when we say that a hypergraph is embeddable into  $\mathbb{R}^d$ . Here, *aff* denotes the affine hull of a set of points and *conv* the convex hull.

**Definition 1 ( $d$ -embeddings).** Let  $H$  be a  $k$ -uniform hypergraph and  $d \in \mathbb{N}$ . A (linear) *embedding* of  $H$  into  $\mathbb{R}^d$  is a function  $\varphi : V(H) \rightarrow \mathbb{R}^d$ , where  $\varphi(A)$  for  $A \subseteq V(H)$  is to be interpreted pointwise, such that  $\dim \text{aff}\varphi(e) = k - 1$  for all  $e \in E(H)$  and  $\text{conv}\varphi(e_1 \cap e_2) = \text{conv}\varphi(e_1) \cap \text{conv}\varphi(e_2)$  for all  $e_1, e_2 \in E(H)$ .

A  $k$ -uniform hypergraph  $H$  is said to be  *$d$ -embeddable* if there exists an embedding of  $H$  into  $\mathbb{R}^d$ . Also, we denote by  $\mathcal{H}_{d,k}$  the set of all  $d$ -embeddable  $k$ -uniform hypergraphs. By Fáry's theorem (see [5]), we have that the  $k = d = 2$  case of this notion of embeddability coincides with the classical concept of planarity.

Our main results are summarized in the Tables 1 and 2, which contain upper or lower bounds for the maximum weak chromatic number of a  $d$ -embeddable  $k$ -uniform hypergraph on  $n$  vertices. All results which only follow non-trivially from prior knowledge are indexed with a number of the theorem in this extended abstract. On the other hand, the trivial entries in the tables are direct consequences of the Menger-Nöbeling Theorem (see [9, page 295] and [10]) which characterizes for which  $d$  all  $k$ -uniform hypergraphs are  $d$ -embeddable. The main results for the

maximum strong chromatic number are Proposition 1 and Theorem 2.

$d \setminus k$	3	4	5	6	7
<b>1</b>	1	1	1	1	1
<b>2</b>	2	1	1	1	1
<b>3</b>	$\Omega\left(\frac{\log n}{\log \log n}\right)_{(5)}$	1	1	1	1
<b>4</b>	$\Omega\left(\frac{\log n}{\log \log n}\right)_{(5)}$	1	1	1	1
<b>5</b>	$\lceil n/2 \rceil$	$\Omega\left(\frac{\log n}{\log \log n}\right)_{(6)}$	1	1	1
<b>6</b>	$\lceil n/2 \rceil$	$\Omega\left(\frac{\log n}{\log \log n}\right)_{(6)}$	1	1	1
<b>7</b>	$\lceil n/2 \rceil$	$\lceil n/3 \rceil$	$\Omega\left(\frac{\log n}{\log \log n}\right)_{(6)}$	1	1
<b>8</b>	$\lceil n/2 \rceil$	$\lceil n/3 \rceil$	$\Omega\left(\frac{\log n}{\log \log n}\right)_{(6)}$	1	1

**Table 1.** Currently known *lower bounds* for the maximum weak chromatic number of a  $d$ -embeddable  $k$ -uniform hypergraph on  $n$  vertices as  $n \rightarrow \infty$ . The number in chevrons indicates the theorem number where we prove this bound.

$d \setminus k$	3	4	5	6	7
<b>1</b>	1	1	1	1	1
<b>2</b>	2	1	1	1	1
<b>3</b>	$\mathcal{O}(n^{\frac{1}{2}})_{(3)}$	$\mathcal{O}(n^{\frac{1}{2}})_{(3)}$	1	1	1
<b>4</b>	$\lceil n/2 \rceil$	$\mathcal{O}(n^{\frac{2}{3}})_{(3)}$	$\mathcal{O}(n^{\frac{2}{3}})_{(3)}$	1	1
<b>5</b>	$\lceil n/2 \rceil$	$\mathcal{O}(n^{\frac{26}{27}})_{(4)}$	$\mathcal{O}(n^{\frac{3}{4}})_{(3)}$	$\mathcal{O}(n^{\frac{3}{4}})_{(3)}$	1
<b>6</b>	$\lceil n/2 \rceil$	$\lceil n/3 \rceil$	$\mathcal{O}(n^{\frac{35}{36}})_{(4)}$	$\mathcal{O}(n^{\frac{4}{5}})_{(3)}$	$\mathcal{O}(n^{\frac{4}{5}})_{(3)}$
<b>7</b>	$\lceil n/2 \rceil$	$\lceil n/3 \rceil$	$\mathcal{O}(n^{\frac{107}{108}})_{(4)}$	$\mathcal{O}(n^{\frac{44}{45}})_{(4)}$	$\mathcal{O}(n^{\frac{5}{6}})_{(3)}$
<b>8</b>	$\lceil n/2 \rceil$	$\lceil n/3 \rceil$	$\lceil n/4 \rceil$	$\mathcal{O}(n^{\frac{134}{135}})_{(4)}$	$\mathcal{O}(n^{\frac{53}{54}})_{(4)}$

**Table 2.** Currently known *upper bounds* for the maximum weak chromatic number of a  $d$ -embeddable  $k$ -uniform hypergraph on  $n$  vertices as  $n \rightarrow \infty$ . The number in chevrons indicates the theorem number where we prove this bound.

For  $d, k, n \in \mathbb{N}$  we define  $\chi_{d,k}^s(n) = \max\{\chi^s(H) : H \in \mathcal{H}_{d,k}, |V(H)| = n\}$  to be the maximum strong chromatic number of a  $d$ -embeddable  $k$ -uniform hypergraph on  $n$  vertices. The maximum weak chromatic number  $\chi_{d,k}^w(n)$  is defined analogously.

**Proposition 1.** For large  $n, d \geq 3$ , and  $d + 1 \geq k$  we have that  $\chi_{d,k}^s(n) \geq \lfloor \sqrt{n - d + 3} \rfloor + d - 3$ .

**Theorem 2.** For large  $n, d \geq 3$ , and  $d \geq k$  we have that  $\chi_{d,k}^s(n) = n$ .

This bound is shown by constructing a sequence of hypergraphs in  $\mathcal{H}_{d,k}$ , which have strong chromatic number equal to the number of their vertices. Thus, except for the cases where  $k = d + 1$ , the maximum strong coloring problem was solved. In particular, we have shown that an unbounded number of colors can be necessary for any strong coloring of a  $d$ -embeddable hypergraph if  $d > 2$ .

**Theorem 3.** *Let  $d \geq 3$ . Then one has*

$$\chi_{d,d}^w(n) \leq \left[ \left( \frac{6ed}{(d-1)!} \right)^{\frac{1}{d-1}} n^{\frac{d-2}{d-1}} \right] = \mathcal{O} \left( \left( \frac{n}{d} \right)^{\frac{d-2}{d-1}} \right).$$

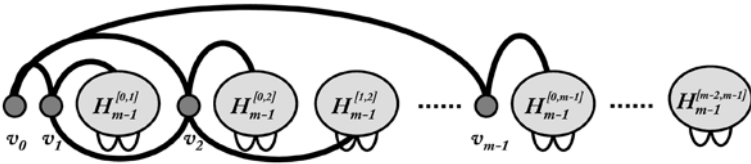
**Theorem 4.** *Let  $d \geq l \geq 3$ . Then one has*

$$\chi_{2d-l,d}^w(n) \leq \left[ (ed)^{\frac{1}{d-1}} n^{1-\frac{3l-1-d}{d-1}} \right] = \mathcal{O} \left( n^{1-\frac{3l-1-d}{d-1}} \right).$$

These two results also holds for piecewise linear embeddings (for a definition e.g. see [8]). To prove them, we first limit the number of edges (relative to the number of vertices) that a hypergraph in  $\mathcal{H}_{d,d}$  and  $\mathcal{H}_{2d-l,d}$  can have. Then, an easy application of the Lovász Local Lemma [3, 14] yields the existence of a weak  $c$ -coloring if  $c$  is as high as requested.

**Theorem 5.** *As  $n \rightarrow \infty$  one has  $\chi_{3,3}^w(n) = \Omega \left( \frac{\log n}{\log \log n} \right)$ .*

*Sketch of proof.* We inductively construct a sequence of 3-uniform, 3-embeddable hypergraphs  $H_m$  which are weakly  $m$ -chromatic. Each new  $H_m$  consists of several copies of  $H_{m-1}$  and a few additional vertices (see Figure 1). The vertices are then arranged on the moment curve and the embeddability is proven using a theorem by Shephard [13].



**Figure 1.** Construction of  $H_m$ .

Note that by monotonicity also  $\chi_{4,3}^w(n) = \Omega \left( \frac{\log n}{\log \log n} \right)$ . Furthermore, it is possible to generalize this result for higher dimensions as follows.

**Theorem 6.** *Let  $d \geq 3$ . Then, as  $n \rightarrow \infty$  one has  $\chi_{2d-3,d}^w(n) = \Omega\left(\frac{\log n}{\log \log n}\right)$ .*

Note that in general there are several other notions of embeddability, the most popular being piecewise linear embeddings and general topological embeddings. A short and comprehensive introduction is given in Section 1 in [8]. Since piecewise linear and topological embeddings are more general than linear embeddings, all lower bounds for chromatic numbers can easily be transferred. Furthermore, we prove all our results on upper bounds for piecewise linear embeddings.

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# Homomorphisms of signed bipartite graphs

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**Abstract.** We study the homomorphism relation between signed graphs where the underlying graph  $G$  is bipartite. We show that this notion captures the notions of chromatic number and graph homomorphisms. In particular we will study Hadwiger's conjecture in this setting. We show that for small values of the chromatic number there are natural strengthenings of this conjecture but such extensions will not work for larger chromatic numbers.

## 1 Homomorphisms

A signature on a graph  $G$  is an assignment of negative or positive sign to the edges. Resigning at a vertex  $v$  is to change the sign of all edges incident to  $v$ . Two signatures are equivalent if one can be obtained from the other by a sequence of resigning. The set of negative edges is normally denoted by  $\Sigma$ . A signed graph, denoted  $(G, \Sigma)$  is a graph  $G$  together with the set of signatures all equivalent to  $\Sigma$ . A signed minor of  $(G, \Sigma)$  is a signed graph obtained from  $(G, \Sigma)$  by a sequence of deleting vertices or edges, contracting positive edges and resigning. Given two signed graphs  $(G, \Sigma)$  and  $(H, \Sigma_1)$  we say there is a homomorphism of  $(G, \Sigma)$  to  $(H, \Sigma_1)$  if there is a signature  $\Sigma'$  equivalent to  $\Sigma$  and a mapping  $\phi : V(G) \rightarrow V(H)$  such that  $\phi$  preserves both adjacency and the sign of an edge with respect to  $\Sigma'$ . If there is a homomorphism of  $(G, \Sigma)$  to  $(H, \Sigma_1)$  we write  $(G, \Sigma) \rightarrow (H, \Sigma_1)$ . This relation is a quasi order on the class of all signed graphs. Thus we may use terms such as *bound* and *maximum*.

A cycle with only one, equivalently odd number of, negative edges is called an *unbalanced cycle* and is denoted by  $UC_k$  if the length of the

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cycle is  $k$ . The following is one of the first theorems in the theory of signed graph homomorphisms.

**Lemma 1.1.** *There is a homomorphism of  $UC_k$  to  $UC_\ell$  if and only if  $k \geq \ell$  and  $k = \ell \pmod{2}$ .*

Let  $G$  be a graph; the signed graph  $S(G) = (G^*, \Sigma)$  is obtained by replacing each edge  $uv$  of  $G$  by an unbalanced 4-cycle on four vertices  $ux_{uv}vy_{uv}$ , where  $x_{uv}$  and  $y_{uv}$  are new and distinct vertices. Let  $(K_{k,k}, M)$  be the signed graph where edges of a perfect matching  $M$  are negative. The following theorem shows how to define  $\chi(G)$  using only the notion of homomorphism between signed bipartite graphs.

**Theorem 1.2.** *For every  $k \geq 3$  and every graph  $G$ ,  $\chi(G) \leq k$  if and only if  $S(G) \rightarrow (K_{k,k}, M)$ .*

*Proof.* The main idea is that if  $S(G) \rightarrow (K_{k,k}, M)$ , then adjacent vertices of  $G$  are mapped to a same side of  $K_{k,k}$  and to distinct vertices.  $\square$

In a similar way we show below that the problem of the existence of a homomorphism of a graph  $G$  into a graph  $H$  is captured by the notion of homomorphism between signed bipartite graphs. For a comprehensive study of graph homomorphisms we refer to [3].

**Theorem 1.3.** *For every two graphs  $G$  and  $H$ ,  $G \rightarrow H$  if and only if  $S(G) \rightarrow S(H)$ .*

*Proof.* The main idea again is to show that a mapping  $S(G) \rightarrow S(H)$  will map  $V(G)$  to  $V(H)$  while preserving adjacency of  $G$  in  $H$ .  $\square$

## 2 Minors

We prove the following minor relation between graphs and their corresponding signed bipartite graphs:

**Theorem 2.1.** *For every integer  $n$  and every graph  $G$ ,  $G$  has a  $K_n$ -minor if and only if  $S(G)$  has a  $(K_n, \Sigma)$ -minor for some  $\Sigma$ .*

*Proof.* First assume  $(K_n, \Sigma)$  is a signed minor of  $S(G)$  for some  $\Sigma$ . We would like to prove that  $K_n$  is a minor of  $G$ . This is clear for  $n = 1, 2$ . So we assume  $n \geq 3$ . Thus, in producing  $(K_n, \Sigma)$  as a signed minor of  $S(G)$  each vertex of degree 2 in  $S(G)$  is either deleted or identified with one of its neighbours as a result of contracting an incident edge. We define a minor of  $G$  as follows: For each edge  $uv$  of  $G$ , if the corresponding unbalanced 4-cycle is deleted in the process of producing  $(K_n, \Sigma)$  as a

signed minor of  $S(G)$ , then delete  $uv$ . If  $u$  and  $v$  are identified through contraction of edges in producing  $(K_n, \Sigma)$  as a signed minor of  $S(G)$ , then contract the edge  $uv$ . Otherwise  $uv$  remains an edge. The resulting minor then must be  $K_n$ .

For the opposite direction, suppose  $K_n$  is a minor of  $G$ . Let  $uv$  be an edge of  $G$ . If the edge  $uv$  is deleted in producing  $K_n$ -minor from  $G$ , then delete all the four edges of corresponding unbalanced 4-cycle. If  $uv$  is contracted, then contract two positive edges of the corresponding unbalanced 4-cycle in  $S(G)$  in such a way that  $u$  and  $v$  are identified after these contractions and delete the other two edges of the unbalanced 4-cycle. Otherwise contract two positive edges of the corresponding unbalanced 4-cycle in such a way that there are two new parallel edges between  $u$  and  $v$ , one positive and one negative. Finally delete all isolated vertices. By allowing multiple edges at the end of this process we get a signed minor of  $S(G)$  which has  $n$  vertices and for each pair  $x$  and  $y$  of vertices two  $xy$  edges, one positive and one negative. For each such pair we delete the negative edge unless  $xy \in \Sigma$  in which case we delete the positive edge. The result is  $(K_n, \Sigma)$  obtained as a signed minor of  $G$ .  $\square$

### 3 Hadwiger's conjecture for signed bipartite graphs

**Conjecture 3.1 (Hadwiger, [2]).** If a graph  $G$  has no  $K_n$ -minor, then it is  $(n - 1)$ -colorable.

By Theorem 1.2, Hadwiger's conjecture can be restated as follows:

**Conjecture 3.2.** Given  $n \geq 4$ , the class  $\mathcal{C} = \{S(G) \mid G \text{ is } K_n\text{-minor-free}\}$  of signed bipartite graphs is bounded by  $(K_{n-1, n-1}, M)$  in the signed graph homomorphism order.

If the conjecture holds, then the next question would be: what is a natural superclass of  $\mathcal{C}$  which is still bounded by  $(K_{n-1, n-1}, M)$ ?

Hadwiger's conjecture is known to be true for  $n \leq 6$ , thus Conjecture 3.2 is also true for  $n \leq 6$ . For  $n = 4$  we have the following generalization.

**Theorem 3.3.** *If  $G$  is a bipartite graph with no  $K_4$ -minor and  $\Sigma$  is any signature on  $G$ , then  $(G, \Sigma) \rightarrow (K_{3,3}, M)$ .*

*Proof.* By adding more edges, if needed, we may assume that  $G$  is edge maximal with respect to being bipartite and having no  $K_4$ -minor. Obviously it is enough to prove the theorem for such edge maximal graphs.

As mentioned before, a classical decomposition theorem for edge-maximal  $K_4$ -minor-free graphs states that every such graph is built from a sequence of triangles starting by one triangle and pasting each new triangle to the graph previously built along an edge. To use the decomposition

theorem we add new edges to  $G$ , of green color, until we reach a maximal  $K_4$ -minor-free graph  $G'$ , which obviously is not bipartite anymore. Let  $G''$  be the edge-colored graph obtained from  $G'$  by coloring original positive edges of  $(G, \Sigma)$  in blue, original negative edges of  $(G, \Sigma)$  in red and keeping the green color for edges not in  $G$ .

We claim that there is no triangle in  $G''$  with exactly two green edges. To see this, suppose that  $v_1v_2$  and  $v_1v_3$  are both green and that  $v_2v_3$  is an edge of  $G$ . Since  $G$  is bipartite  $v_2$  and  $v_3$  are in two different parts and thus  $v_1$  is in a different part with respect to one of them. Without loss of generality assume  $v_1$  and  $v_2$  are in different parts. Consider the graph  $G + \{v_1v_2\}$ . By the choice of  $v_2$  this graph is bipartite and since it is a subgraph of  $G'$ , it has also no  $K_4$ -minor but this contradicts the edge maximality of  $G$ .

We now build a new edge-colored graph  $F$  from  $(K_{3,3}, M)$ . The blue and red edges of  $F$  are defined as before and we add green edges between every pair of vertices non adjacent in  $(K_{3,3}, M)$ . The edge-colored graph  $F$  has three types of triangles: (i) triangles with three green edges, (ii) triangles with one green edge and two blue edges, and (iii) triangles with no two edges of the same color. Furthermore it is not hard to verify that each red edge only belongs to triangles of type (iii), each blue edge belongs to triangles of type (ii) or (iii) and each green edge is contained in triangles of each of the three types.

To prove the theorem we now prove the following stronger statement: there exists a suitable “resigning”  $G^*$  of  $G''$  such that  $G^*$  admits a color-preserving homomorphism to  $F$ . By resigning here we mean exchanging the colors red and blue on edges of an edge cut, this can be regarded as a sequence of vertex resigning.

To prove this stronger statement, let  $T_1, \dots, T_k$  be the sequence of triangles obtained from the decomposition of  $G''$  mentioned above. Note that since  $G$  was bipartite, each such triangle contains a green edge. Consider the triangle  $T_1$ . Either it is one of the three types (i), (ii) or (iii), in which case we simply map it to  $F$ , or it has one green and two red edges. Let  $u$  be the common vertex of these two red edges. After resigning at  $u$  we have a triangle of type (ii) and thus we can map it to  $F$ .

By induction, assume now that the graph  $G''_i$ , obtained by pasting the triangles  $T_1, \dots, T_i, i < k$ , is mapped to  $F$  and assume that  $T_{i+1}$  is pasted to  $G''_i$  along the edge  $e$ . Let  $v$  be the vertex of  $T_{i+1}$  not incident to  $e$ . If  $T_{i+1}$  is a triangle of one the three types, because of the above mentioned property of  $F$ , we can extend the mapping of  $G''_i$  to  $G''_{i+1}$ , where the colors of the two edges of  $T_{i+1}$  incident with  $v$  are preserved. Otherwise  $T_{i+1}$  has exactly two red edges and one green edge. By resigning at  $v$  we get a triangle that has either one or no red edge, thus obtaining a

triangle of type (ii) or (iii). We now extend the homomorphism thanks to the properties of  $F$ . In this process, resigning a vertex would be done at most once, when it is added to the already built part of the graph, so our process is well-defined and the stronger claim is proved.  $\square$

We note that our proof has an algorithmic feature. Given a signed bipartite graph  $(G, \Sigma)$ , where  $G$  is a  $K_4$ -minor-free graph, we can find, in polynomial time, a homomorphism of  $(G, \Sigma)$  to  $(K_{3,3}, M)$ .

Furthermore, we believe that the following stronger statement should also be true:

**Conjecture 3.4.** If  $G$  is bipartite and  $(G, \Sigma)$  has no  $(K_4, E(K_4))$  as a signed minor, then  $(G, \Sigma) \rightarrow (K_{3,3}, M)$ .

For  $n = 5$  it is shown in [4], using the four-color theorem and a result of [1], that the following holds.

**Theorem 3.5.** If  $G$  is a bipartite planar graph and  $\Sigma$  is any signature on  $G$ , then  $(G, \Sigma) \rightarrow (K_{4,4}, M)$ .

For large values of  $n$  ( $n \geq 7$ ) Conjecture 3.2 does not extend so nicely. To show this we use the following signed bipartite graph, *Fano*. That is signed graph on  $K_{7,7}$  where vertices on one side are labeled with points of Fano plane and on the other side with lines of the Fano plane. An edge is negative if it connects a line to a point of a line.

**Theorem 3.6.** There exists no value of  $n$  for which *Fano* admits a homomorphism to  $(K_{n,n}, M)$ .

A proof can be obtained mainly by counting number of distinct copies of  $UC_4$  containing a given edge. For more details we refer to [4]. The following then is an immediate corollary.

**Corollary 3.7.** The class  $\mathcal{C} = \{(G, \Sigma) \mid G \text{ is bipartite and has no } H\text{-minor}\}$  is not bounded by  $(K_{n,n}, M)$  (for no values of  $n$ ) if  $H$  is a graph on at least 15 vertices.

This shows that for  $n \geq 15$  the reformulation of Hadwiger's conjecture given in Conjecture 3.2 cannot be extended to a general minor closed class of signed bipartite graphs. Even though such an extension was possible for small values of  $n$ .

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## Games



# A threshold for the Maker-Breaker clique game

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**Abstract.** We study Maker-Breaker  $k$ -clique game played on the edge set of the random graph  $G(n, p)$ . In this game, two players alternately claim unclaimed edges of  $G(n, p)$ , until all the edges are claimed. Maker wins if he claims all the edges of a  $k$ -clique; Breaker wins otherwise. We determine that the threshold for the graph property that Maker can win is at  $n^{-\frac{2}{k+1}}$ , for all  $k > 3$ , thus proving a conjecture from [5]. More precisely, we conclude that there exist constants  $c, C > 0$  such that when  $p > Cn^{-\frac{2}{k+1}}$  the game is Maker's win a.s., and when  $p < cn^{-\frac{2}{k+1}}$  it is Breaker's win a.s.

For the triangle game, when  $k = 3$ , we give a more precise result, describing the hitting time of Maker's win in the random graph process. We show that, with high probability, Maker can win the triangle game exactly at the time when a copy of  $K_5$  with one edge removed appears in the random graph process. As a consequence, we are able to give an expression for the limiting probability of Maker's win in the triangle game played on the edge set of  $G(n, p)$ .

## 1 Introduction

Let  $X$  be a finite set and let  $\mathcal{F} \subseteq 2^X$  be a family of subsets of  $X$ . In the positional game  $(X, \mathcal{F})$ , two players take turns in claiming one previously unclaimed element of  $X$ . The set  $X$  is called the “board”, and the members of  $\mathcal{F}$  are referred to as the “winning sets”. In a *Maker-Breaker* positional game, the two players are called Maker and Breaker. Maker wins the game if he occupies all elements of some winning set; Breaker wins otherwise. A game  $(X, \mathcal{F})$  is said to be a *Maker's win* if Maker has a strategy that ensures his win against any strategy of Breaker; otherwise it is a *Breaker's win*. Note that  $\mathcal{F}$  alone determines whether the game is Maker's win or Breaker's win.

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A well-studied class of positional games are the *games on graphs*, where the board is the set of edges of a graph. The winning sets are usually representatives of some graph theoretic structure. The first game studied in this area was the connectivity game, a generalization of the well-known Shannon switching game, where Maker's goal is to claim a spanning connected graph by the end of the game. We denote the game by  $(E(K_n), \mathcal{T})$ . Another important game is the Hamilton cycle game  $(E(K_n), \mathcal{H})$ , where  $\mathcal{H} = \mathcal{H}_n$  consists of the edge sets of all Hamilton cycles of  $K_n$ .

In the clique game the winning sets are the edge sets of all  $k$ -cliques, for a fixed integer  $k \geq 3$ . We denote this game with  $(E(K_n), \mathcal{K}_k)$ . Note that the size of the winning sets is fixed and does not depend on  $n$ , which distinguishes it from the connectivity game and the Hamilton cycle game. A simple Ramsey argument coupled with the strategy stealing argument ensures Maker's win if  $n$  is large.

All three games that we introduced are straightforward Maker's wins when  $n$  is large enough. This is however not the end of the story, as there are two general approaches to even out the odds, giving Breaker more power – *biased games* and *random games*. Here, we will stick with the latter.

## 2 Our results

A way to give Breaker more power in a positional game, introduced by the second author and Szabó in [5], is to randomly thin out the board before the game starts, thus eliminating some of the winning sets.

For games on graphs, given a game  $\mathcal{F}$  that is Maker's win when played on  $E(K_n)$ , we want to find the *threshold probability*  $p_{\mathcal{F}}$  so that, if the game is played on  $E(G(n, p))$ , an almost sure Maker's win turns into an almost sure Breaker's win. Such a threshold  $p_{\mathcal{F}}$  exists, as “being Maker's win” is clearly a monotone increasing graph property.

The threshold probability for the connectivity game was determined to be  $\frac{\log n}{n}$  in [5], and shown to be sharp. As for the Hamilton cycle game, the order of magnitude of the threshold was given in [4]. Using a different approach, it was proven in [3] that the threshold is  $\frac{\log n}{n}$  and it is sharp. Finally, as a consequence of a hitting time result, Ben-Shimon *et al.* [2] closed this question by giving a very precise description of the low order terms of the limiting probability.

Moving to the clique game, it was shown in [5] that for every  $k \geq 4$  and every  $\varepsilon > 0$  we have  $n^{-\frac{2}{k+1}-\varepsilon} \leq p_{\mathcal{K}_k} \leq n^{-\frac{2}{k+1}}$ . Moreover, it was proved that there exist a constant  $C > 0$  such that for  $p \geq Cn^{-\frac{2}{k+1}}$  Maker wins the  $k$ -clique game on  $G(n, p)$  a.a.s. The threshold for the triangle



game was determined to be  $p_{\mathcal{K}_3} = n^{-\frac{5}{9}}$ , showing that the behavior of the triangle game is different from the  $k$ -clique game for  $k \geq 4$ .

Our main result is the following theorem. It gives a lower bound on the threshold for the  $k$ -clique game, when  $k \geq 4$ , which matches the upper bound from [5] up to the leading constant.

**Theorem 2.1.** *Let  $k \geq 4$ . There exists a constant  $c > 0$  such that for  $p \leq cn^{-\frac{2}{k+1}}$  Breaker wins the Maker-Breaker  $k$ -clique game played on the edge set of  $G(n, p)$  a.a.s.*

The threshold probability for the  $k$ -clique game for  $k \geq 4$  was conjectured to be  $p_{\mathcal{K}_k} = n^{-\frac{2}{k+1}}$  in [5]. The previous theorem resolves this conjecture in the affirmative. Summing up the results of Theorem 2.1 and Theorem 19 from [5], we now have the following.

**Corollary 2.2.** *Let  $k \geq 4$  and consider the Maker-Breaker  $k$ -clique game on the edge set of  $G(n, p)$ . There exist constants  $c, C > 0$  such that the following hold:*

1. *If  $p \geq Cn^{-\frac{2}{k+1}}$ , then Maker wins a.a.s.;*
2. *If  $p \leq cn^{-\frac{2}{k+1}}$ , then Breaker wins a.a.s.*

A result of this type is sometimes called a “semi-sharp threshold” in the random graphs literature.

**Hitting time of Maker’s win.** Let  $V$  be a set of cardinality  $n$ , and let  $\pi$  be a permutation of the set  $\binom{V}{2}$ . If by  $G_i$  we denote the graph on the vertex set  $V$  whose edges are the first  $i$  edges in the permutation  $\pi$ ,  $G_i = (V, \pi^{-1}([i]))$ , then we say that  $\tilde{G} = \{G_i\}_{i=0}^{\binom{n}{2}}$  is a *graph process*. Given a monotone increasing graph property  $\mathcal{P}$  and a graph process  $\tilde{G}$ , we define the hitting time of  $\mathcal{P}$  with  $\tau(\tilde{G}; \mathcal{P}) = \min\{t : G_t \in \mathcal{P}\}$ . If  $\pi$  is chosen uniformly at random from the set of all permutations of the set  $\binom{V}{2}$ , we say that  $\tilde{G}$  is a *random graph process*. Such processes are closely related to the model of random graph we described above.

Given a positional game, our general goal is to describe the hitting time of the graph property “Maker’s win” in a typical graph process. For a game  $\mathcal{G}$ , by  $\mathcal{M}_{\mathcal{F}}$  we denote the graph property “Maker wins  $\mathcal{G}$ ”. It was shown in [5] that in the connectivity game (with the technical assumption that Breaker is the first to play), for a random graph process  $\tilde{G}$ , we have  $\tau(\tilde{G}; \mathcal{M}_{\mathcal{T}}) = \tau(\tilde{G}; \delta_2)$ , where  $\delta_\ell$  is the graph property “minimum degree at least  $\ell$ ”. Recently, Ben-Shimon *et al.* [2] resolved the same question for the Hamilton cycle game, obtaining  $\tau(\tilde{G}; \mathcal{M}_{\mathcal{H}}) = \tau(\tilde{G}; \delta_4)$ . Note that inequality in one direction for both of these equalities holds trivially.

Moving on to the clique game, we denote the property “the graph contains  $K_5 - e$  as a subgraph” with  $\mathcal{G}_{\mathcal{K}_5^-}$ . We are able to show the following hitting time result for Maker’s win in the triangle game.

**Theorem 2.3.** *For a random graph process  $\tilde{G}$ , the hitting time for Maker’s win in the triangle game is asymptotically almost surely the same as the hitting time for appearance of  $K_5 - e$ , i.e.,  $\tau(\tilde{G}; \mathcal{M}_{\mathcal{K}_3}) = \tau(\tilde{G}; \mathcal{G}_{\mathcal{K}_5^-})$  a.a.s.*

Using this, we are able to give a precise expression for the probability for Maker’s win in the triangle game on  $G(n, p)$ .

**Corollary 2.4.** *Let  $p = p(n)$  be an arbitrary sequence of numbers  $\in [0, 1]$  and let us write  $x = x(n) = p \cdot n^{\frac{2}{k+1}}$ . Then*

$$\lim_{n \rightarrow \infty} \Pr[\text{Maker makes triangle on } G(n, p)] = \begin{cases} 0 & \text{if } x \rightarrow 0, \\ 1 - e^{-\frac{c^5}{3}} & \text{if } x \rightarrow c \in \mathbb{R}, \\ 1 & \text{if } x \rightarrow \infty. \end{cases}$$

### 3 Conclusion and open problems

**Random graph intuition.** In the 70s, Erdős observed the following paradigm which is referred to as the *random graph intuition* in positional game theory. As it turns out for many games on graphs, the inverse of the threshold bias  $b_{\mathcal{G}}$  in the game played on the complete graph is “closely related” to the probability threshold for the appearance of a member of  $\mathcal{G}$  in  $G(n, p)$ . Another parameter that is often “around” is the threshold probability  $p_{\mathcal{G}}$  for Maker’s win when played on  $G(n, p)$ . As we saw, for the two games mentioned in the introduction, the connectivity game and the Hamilton cycle game, all three parameters are equal to  $\frac{\log n}{n}$ .

In the  $k$ -clique game, for  $k \geq 4$ , the threshold bias is  $b_{\mathcal{K}_k} = \Theta(n^{\frac{2}{k+1}})$  and the threshold probability for Maker’s win is the inverse (up to the leading constant),  $p_{\mathcal{K}_k} = n^{-\frac{2}{k+1}}$ , supporting the random graph intuition. But, the threshold probability for appearance of a  $k$ -clique in  $G(n, p)$  is not at the same place, it is  $n^{-\frac{2}{k-1}}$ . And in the triangle game there is even more disagreement, as all three parameters are different – they are, respectively,  $n^{\frac{1}{2}}$ ,  $n^{-\frac{5}{9}}$  and  $n^{-1}$ . Now, more than thirty years after Chvátal and Erdős formulated the paradigm, there is still no general result that would make it more formal. We are curious to the reasons behind the total agreement between the three thresholds in the connectivity game and the Hamilton cycle game, partial disagreement in  $k$ -clique game for  $k \geq 4$ , and the total disagreement in the triangle game.

**Random clique game vs. biased clique game.** Our Corollary 2.2 gives two constants  $c > 0$  and  $C > 0$ , stating that the probability threshold for Maker's win in the  $k$ -clique game on  $G(n, p)$  for  $k \geq 4$  is between  $cn^{-\frac{2}{k+1}}$  and  $Cn^{-\frac{2}{k+1}}$ . In a way, with this result, the game played on the random graph catches up with the biased  $k$ -clique game played on the complete graph, as a result of Bednarska and Łuczak [1] guarantees the existence of constants  $c' > 0$  and  $C' > 0$ , such that the bias threshold for this game is between  $c'n^{\frac{2}{k+1}}$  and  $C'n^{\frac{2}{k+1}}$ , for all  $k \geq 3$ . Both pairs of constants,  $c, C$  and  $c', C'$ , are quite far apart. Also, in both games, the best known strategy for Maker's exploits the same derandomized random strategy approach, proposed in [1].

We know much more for the triangle game on the random graph, as Corollary 2.4 gives the threshold probability quite accurately, and it turns out to be a coarse threshold. The reason for such different behavior (compared to  $k > 3$ ) may lie behind the fact that  $K_3 = C_3$ .

**A more precise result for the  $k$ -clique game when  $k \geq 4$ ?** As we saw, we can say a lot about the threshold probability for the triangle game, the connectivity game and the Hamilton cycle game when the game is played on the random graph. We do not know that much about the  $k$ -clique game, when  $k \geq 4$ , and it would be interesting to see what happens between the bounds given in Corollary 2.2. Also, a graph-theoretic description of the hitting time of Maker's win on the random graph process would be of great importance, as we know very little about Maker's winning strategy at the threshold. What we know is that we cannot hope for a result analogous to Theorem 2.3 – the reason for Maker's win cannot be the appearance of a fixed graph, as we know that Breaker wins on every typical (fixed) subgraph of the random graph on the probability threshold. Hence, Maker's optimal strategy must be of "global nature", taking into account a non-constant part of the random graph to win the game. Having that in mind we propose the following conjecture.

**Conjecture 3.1.** For every  $k \geq 4$  there exists a  $c = c(k)$  such that for any fixed  $\varepsilon > 0$ , if  $p \leq (c - \varepsilon)n^{-\frac{2}{k+1}}$ , then Breaker wins the  $k$ -clique game on  $G(n, p)$  a.a.s, and if  $p \geq (c + \varepsilon)n^{-\frac{2}{k+1}}$ , then Maker wins a.a.s.

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# On the threshold bias in the oriented cycle game

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**Abstract.** In the Oriented cycle game, the two players, called OMaker and OBreaker, alternately direct edges of  $K_n$ . OMaker directs exactly one edge, whereas OBreaker is allowed to claim between one and  $b$  edges. OMaker wins if the final tournament contains a directed cycle, otherwise OBreaker wins. It was shown recently [1] that OMaker has a winning strategy for this game whenever  $b \leq n/2 - 2$ . We show that OBreaker has a strategy whenever  $b > 5n/6$ , and give a non-trivial upper bound when OBreaker is asked to direct exactly  $b$  edges in each of his moves.

## 1 Introduction and Results

We study the oriented cycle game, which is a particular orientation game. Orientation games were studied by Ben-Eliezer, Krivelevich and Sudakov in [1], and we follow their notation. In orientation games, the board consists of the edges of the complete graph  $K_n$ . In the  $(p : q)$  orientation game, the two players called OMaker and OBreaker, orient previously undirected edges alternately. OMaker starts, and in each round, OMaker directs between one and  $p$  edges, and then OBreaker directs between one and  $q$  edges. At the end of the game, the final graph is a tournament on  $n$  vertices. OMaker wins the game if this tournament has some predefined property  $\mathcal{P}$ . Otherwise, OBreaker wins. We study the  $(1 : b)$ -game and refer to it as the *b-biased orientation game*. Increasing  $b$  can only help OBreaker, so the game is *bias monotone*. Therefore, any such game has a threshold  $t(n, \mathcal{P})$  such that OMaker wins the  $b$ -biased game when  $b \leq t(n, \mathcal{P})$  and OBreaker wins the game when  $b > t(n, \mathcal{P})$ .

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In a variant, OBreaker is required to direct exactly  $b$  edges. We refer to this variant as the *strict  $b$ -biased orientation game*. Playing the exact bias in every round may be disadvantageous for OBreaker, so the existence of a threshold as for the monotone rules is not guaranteed in general. We therefore define  $t^+(n, \mathcal{P})$  to be the largest value  $b$  such that OMaker has a strategy to win the strict  $b$ -biased orientation game, and  $t^-(n, \mathcal{P})$  to be the largest integer such that for every  $b \leq t^-(n, \mathcal{P})$ , OMaker has a strategy to win the strict  $b$ -biased orientation game. Trivially,  $t(n, \mathcal{P}) \leq t^-(n, \mathcal{P}) \leq t^+(n, \mathcal{P})$ . The threshold bias  $t(n, \mathcal{P})$  was investigated in [1] for several orientation games. However, the relation between all three parameters in question is still widely open. It is not even clear whether  $t^-(n, \mathcal{P})$  and  $t^+(n, \mathcal{P})$  need to be distinct values.

We focus on the oriented cycle game, in which OMaker wins if the final tournament contains a directed cycle. Let  $\mathcal{P}$  be the property of containing a directed cycle. The strict version of this game was studied by Bollobás and Szabó in [2]. They show that  $t^+(n, \mathcal{P}) \geq \lfloor (2 - \sqrt{3})n \rfloor$ . Moreover, they remark that the proof also works for the monotone rules, which implies that  $t(n, \mathcal{P}) \geq \lfloor (2 - \sqrt{3})n \rfloor$ . For an upper bound, it is rather simple to see that OBreaker wins the  $b$ -biased oriented cycle game for  $b \geq n - 2$ , even when the strict rules apply. Therefore,  $t^+(n, \mathcal{P}) \leq n - 3$ . Bollobás and Szabó conjecture that this upper bound is tight. In [1], Ben-Eliezer, Krivelevich and Sudakov show that for  $b \leq n/2 - 2$ , OMaker has a strategy guaranteeing a cycle in the  $b$ -biased orientation game, i.e.  $t(n, \mathcal{P}) \geq n/2 - 2$ . We give a strategy for OBreaker in the  $b$ -biased oriented cycle game when  $b \geq 5n/6 + 1$ .

**Theorem 1.1.** *For  $b \geq 5n/6 + 1$ , OBreaker has a strategy to prevent OMaker from closing a directed cycle in the  $b$ -biased orientation game. In particular,  $t(n, \mathcal{P}) \leq 5n/6$ .*

Furthermore, we adjust our strategy to the strict rules and show the following.

**Theorem 1.2.** *For  $b \geq n - c\sqrt{n}$ , where  $0 < c < 1$  is a constant, OBreaker has a strategy to prevent OMaker from closing a directed cycle in the strict  $b$ -biased orientation game. In particular,  $t^+(n, \mathcal{P}) \leq n - c\sqrt{n} - 1$ .*

Theorem 1.2 refutes the above conjecture of Bollobás and Szabó.

## 2 Outline of the proofs

For both proofs, we need to provide OBreaker with a strategy to prevent OMaker from closing a directed cycle, no matter how she plays. This is equivalent to constructing the transitive tournament.

There are two essential concepts to our proofs, so called *UDB*'s and  $\alpha$ -structures. Suppose the game is in play, and let  $G$  denote the subgraph of already directed edges (by either player). For a directed edge  $e \in G$ , we write  $e^+$  for its tail and  $e^-$  for its head, i.e.  $e = (e^+, e^-)$ . For two disjoint subsets  $A, B \subseteq V$ , we call the pair  $(A, B)$  a *uniformly directed biclique* (or short *UDB*), if for all  $a \in A, b \in B$  the edge  $(a, b) \in E(G)$  is present in  $G$  already. Our goal is to create a *UDB*  $(A, B)$  such that both parts fulfil  $|A|, |B| \leq b$  and  $A \cup B = V$ . Suppose both sets  $A$  and  $B$  would be independent. OBreaker could then follow the "trivial strategy" inside  $A$  and  $B$  respectively (as OBreaker wins on  $K_{b+2}$ ). However, while building such a *UDB*, OMaker will direct edges inside these sets, and OBreaker needs to control those. To handle this obstacle, we introduce  $\alpha$ -structures. Let  $V' \subseteq V$  and  $E = E(G[V'])$ . Then the set  $E$  is called an  $\alpha$ -structure in  $V'$  of size  $k$  if there exist edges  $e_1, \dots, e_k \in E$  such that

- $(\alpha_1)$  for every directed path  $P = (e_{i_1}, \dots, e_{i_k})$ :  $i_1 > \dots > i_k$ ;
- $(\alpha_2)$  for every  $1 \leq i < j \leq k$ :  $(e_j^+, e_i^-) \in E$ ;
- $(\alpha_3)$  and no other edges are present in  $G[V']$ .

In our strategy, the edges  $e_1, \dots, e_k$  will be the edges directed by OMaker (though not necessarily in that order), and the edges of "type"  $(\alpha_2)$  are the ones directed by OBreaker. It is easy to verify that  $(\alpha_1)$ - $(\alpha_3)$  imply that  $E$  does not contain a directed cycle, nor can OMaker close one in her next move inside  $V'$ . Suppose  $E = E(G[V'])$  is an  $\alpha$ -structure inside some subset  $V' \subseteq V$ , and let  $e = (v, w)$  be the edge OMaker directed in her previous move. If  $v, w \in V'$ , we provide OBreaker with a *procedure*  $\alpha$  to add  $e$  to the  $\alpha$ -structure. This includes providing  $e$  with an appropriate index  $\ell \in [k + 1]$  (and an index shift of the existing edges  $e_1, \dots, e_k$ ) such that

- (i) OBreaker needs to direct at most  $k$  new edges  $\{f_1, \dots, f_\ell\}$  and
- (ii)  $E \cup \{e, f_1, \dots, f_\ell\}$  forms an  $\alpha$ -structure in  $V'$  again.

The details are straight-forward though technical, so we omit them here. We are now ready to describe the *global* strategies of OBreaker.

### Strategy for Theorem 1.1

The strategy is divided into three stages. In Stage I, OBreaker maintains a *UDB*  $(A, B)$  such that after each of his moves

- $E(G[A])$  is an  $\alpha$ -structure of size  $k$  in  $V \setminus B$ ,
- $E(G[B])$  is an  $\alpha$ -structure of size  $\ell$  in  $V \setminus A$ ,
- $|A| - k$  and  $|B| - \ell$  increase by at least 1 in every round,
- $k + \ell \leq |A| - k = |B| - \ell \leq \frac{n}{6}$ .

Note that property  $(\alpha_3)$  implies that  $E(G) \subseteq (A \cup B) \times (A \cup B)$ . Let  $e = (v, w)$  be the edge OMaker directed in her previous move. Since  $(A, B)$  is a  $UDB$ , either  $\{v, w\} \subseteq V \setminus A$  or  $\{v, w\} \subseteq V \setminus B$ . Let  $\{v, w\} \subseteq V \setminus A$ . Then OBreaker adds  $e$  to the  $\alpha$ -structure  $E(G[A])$  by procedure  $\alpha$ . As noted above, this takes him at most  $k$  edges to direct. He then directs all edges  $(v, b), (w, b)$  for  $b \in B$  and thus adds  $v$  and  $w$  to  $A$ . If  $v$  (or  $w$  respectively) was already an element of  $A$ , OBreaker picks an arbitrary new vertex  $v' \in V \setminus (A \cup B)$  (or  $w'$  respectively), and directs all edges  $(v, b)$  for  $b \in B$ . Furthermore, he picks an arbitrary element  $b' \in V \setminus (A \cup B)$  and directs all edges  $(a, b')$  for  $a \in A$ . This way,  $|A| - k$  and  $|B| - \ell$  increase by 1. Furthermore, since  $|A| - k, |B| - \ell$ , and  $k + \ell$  are bounded by  $\frac{n}{6}$  and since  $b \geq \frac{5n}{6} + 1$ , OBreaker can follow the strategy in Stage I. The analysis for  $\{v, w\} \subseteq V \setminus B$  is similar. As soon as  $|A| - k, |B| - \ell \geq n/6$ , OBreaker proceeds to Stage II.

In Stage II, OBreaker stops increasing the values  $|A| - k$  and  $|B| - \ell$ . He now maintains a  $UDB$   $(A, B)$  such that after each of his moves

- $E_A := E(G[V \setminus B])$  is an  $\alpha$ -structure of size  $k$  in  $V \setminus B$ ,
- for all  $e^+ \in e \in E_A$ :  $e^+ \in A$ ,
- $E(G[B])$  is an  $\alpha$ -structure of size  $\ell$  in  $V \setminus A$ ,
- $|A| - k$  and  $|B| - \ell$  do not decrease,
- i.e.  $|A| - k = |B| - \ell \geq \frac{n}{6} = n - b$ .

Again, let  $e = (v, w)$  be the edge OMaker directed in her previous move and assume w.l.o.g. that  $\{v, w\} \subseteq V \setminus B$ . Then OBreaker adds  $e$  to the  $\alpha$ -structure  $E(G[A])$  by procedure  $\alpha$ . Furthermore, he adds  $v$  to  $A$  by directing all edges  $(v, b)$  for  $b \in B$ . If  $v \in A$  already, OBreaker picks a new vertex  $v' \in V \setminus (A \cup B)$  and directs all edges  $(v', b)$  for  $b \in B$ . This way,  $|A| - k$  does not decrease. Stage II ends when  $A \cup B = V$ . Since the strategy asks to direct at most  $|B| + k \leq V - (|A| - k) \leq b$  edges, OBreaker can follow that strategy.

In the final Stage, the situation is as follows: The vertex set  $V$  can be partitioned into two sets  $A$  and  $B$  such that

- all edges  $(a, b) \in E(G)$  are already directed,
- $E(G[A])$  is an  $\alpha$ -structure in  $V \setminus B = A$ ,
- $E(G[B])$  is an  $\alpha$ -structure in  $V \setminus A = B$ ,
- $|A|, |B| \leq b$  (since  $|A|, |B| \geq n - b$  and  $V = A \dot{\cup} B$ ).

OBreaker now follows procedure  $\alpha$  inside  $A$  or  $B$  respectively, depending on the part OMaker plays in. It is evident that OMaker cannot close a cycle: Since she only plays one edge in every round, since  $G[V \setminus (A \cup B)] = \emptyset$  throughout the whole game, and since  $(A, B)$  is a  $UDB$ , any



cycle she could close lies completely inside  $V \setminus A$  or  $V \setminus B$ . But the edge sets  $E_A := E(G[V \setminus B])$  and  $E_B := E(G[V \setminus A])$  form  $\alpha$ -structures inside  $V \setminus B$  and  $V \setminus A$  respectively, so OMaker cannot close a cycle inside these sets.

### Strategy for Theorem 1.2

In our strategy for Theorem 1.1, we heavily use that OBreaker may direct fewer than  $b$  edges in each round. This way, we have complete control over the structures that evolve. For the strict game, OBreaker needs to be a lot more careful where to put the remaining edges, as OMaker could make use of them to create a directed cycle. Here, we split the strategy into two stages. Stage I consists of exactly one move in which OBreaker claims a  $UDB(A, B)$  such that  $|A|, |B| \geq n - b \sim \sqrt{n}$ . Similar to the proof for the monotone rules, he now plays either inside  $V \setminus A$ , or  $V \setminus B$ , depending on the placement of OMaker's directed edge. For the exact details, we refer the reader to our paper [3].

### 3 Concluding remarks

The upper bound of  $5n/6$  in Theorem 1.1 is not tight. As one might expect, playing almost the full bias from the beginning is advantageous for OBreaker. However, the upper bound improves only to roughly  $0.82n$  when optimizing Stage I. Our strategy for OBreaker utterly fails when  $b \leq 2n/3$ . We therefore conjecture that  $t(n, \mathcal{P}) \geq 2n/3 + o(n)$ .

Concerning the strict rules, OBreaker has to be a lot more careful where to put remaining edges, since any additional edge can be used by OMaker to her advantage. We conjecture that there is a constant  $\varepsilon > 0$  such that  $t^+(n, \mathcal{P}) \leq (1 - \varepsilon)n$ .

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# Building spanning trees quickly in Maker-Breaker games

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**Abstract.** For a tree  $T$  on  $n$  vertices, we study the Maker-Breaker game, played on the edge set of the complete graph on  $n$  vertices, which Maker wins as soon as the graph she builds contains a copy of  $T$ . We prove that if  $T$  has bounded maximum degree, then Maker can win this game within  $n + 1$  moves. Moreover, we prove that Maker can build almost every tree on  $n$  vertices in  $n - 1$  moves and provide non-trivial examples of families of trees which Maker can build in  $n - 1$  moves.

## 1 Introduction

Let  $X$  be a finite set and let  $\mathcal{F} \subseteq 2^X$  be a family of subsets. In the Maker-Breaker game  $(X, \mathcal{F})$ , two players, called Maker and Breaker, take turns in claiming a previously unclaimed element of  $X$ , with Breaker going first. The set  $X$  is called the board of the game and the members of  $\mathcal{F}$  are referred to as the winning sets. Maker wins this game as soon as she claims all elements of some winning set. If Maker does not fully

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claim any winning set by the time every board element is claimed by some player, then Breaker wins the game. We say that the game  $(X, \mathcal{F})$  is Maker's win if Maker has a strategy that ensures her win in this game (in some number of moves) against any strategy of Breaker, otherwise the game is Breaker's win. One can also consider a *biased* version in which Maker claims  $p$  board elements per move (instead of just 1) and Breaker claims  $q$  board elements per move. We refer to this version as a  $(p : q)$  game. For a more detailed discussion, we refer the reader to [3].

The following game was studied in [8]. Let  $T$  be a tree on  $n$  vertices. The board of the *tree embedding game*  $(E(K_n), \mathcal{T}_n)$  is the edge set of the complete graph on  $n$  vertices and the minimal (with respect to inclusion) winning sets are the labeled copies of  $T$  in  $K_n$ . Several variants of this game were studied by various researchers (see *e.g.* [2,4]).

It was proved in [8] that for any real numbers  $0 < \alpha < 0.005$  and  $0 < \varepsilon < 0.05$  and a sufficiently large integer  $n$ , Maker has a strategy to win the  $(1 : q)$  game  $(E(K_n), \mathcal{T}_n)$  within  $n + o(n)$  moves, for every  $q \leq n^\alpha$  and every tree  $T$  with  $n$  vertices and maximum degree at most  $n^\varepsilon$ . The bounds on the duration of the game, on Breaker's bias and on the maximum degree of the tree to be emdedded, do not seem to be best possible. Indeed, it was noted in [8] that it would be interesting to improve each of these bounds, even at the expense of the other two. In this paper we focus on the duration of the game, while we restrict our attention to the case of bounded degree trees and to unbiased games (that is, the case  $q = 1$ ). The smallest number of moves Maker needs in order to win some Maker-Breaker game is an important game invariant which has received a lot of attention in recent years (see *e.g.* [5–10]).

It is obvious that Maker cannot build any tree on  $n$  vertices in less than  $n - 1$  moves. This trivial lower bound can be attained for some trees. For example, it was proved in [9] that Maker can build a Hamilton path of  $K_n$  in  $n - 1$  moves. On the other hand it is not hard to see that there are trees on  $n$  vertices which Maker cannot build in less than  $n$  moves, *e.g.* the complete binary tree on  $n$  vertices. In this paper we prove the following general upper bound which is only one move away from the aforementioned lower bound.

**Theorem 1.1.** *Let  $\Delta$  be a positive integer. Then there exists an integer  $n_0 = n_0(\Delta)$  such that for every  $n \geq n_0$  and for every tree  $T = (V, E)$  with  $|V| = n$  and  $\Delta(T) \leq \Delta$ , Maker has a strategy to win the game  $(E(K_n), \mathcal{T}_n)$  within  $n + 1$  moves.*

As mentioned before, it can be shown that there exist trees on  $n$  vertices which Maker cannot claim in less than  $n$  moves. Nevertheless, the following theorem suggests that such examples are quite rare.

**Theorem 1.2.** *Let  $T$  be a tree, chosen uniformly at random from the class of all labeled trees on  $n$  vertices. Then asymptotically almost surely,  $T$  is such that Maker has a strategy to win the game  $(E(K_n), \mathcal{T}_n)$  in  $n - 1$  moves.*

Moreover, we construct a non-trivial family of trees for which Maker wins the tree embedding game within  $n - 1$  moves. We call a path  $P$  inside a tree  $T$  a *bare path* if all its inner vertices have degree 2 in  $T$ . Our result then is the following generalization of Theorem 1.4 from [9].

**Theorem 1.3.** *Let  $\Delta$  be a positive integer. Then there exists an integer  $m_1 = m_1(\Delta)$  and an integer  $n_1 = n_1(\Delta, m_1)$  such that the following holds for every  $n \geq n_1$  and for every tree  $T = (V, E)$  such that  $|V| = n$  and  $\Delta(T) \leq \Delta$ . If  $T$  admits a bare path of length  $m_1$ , such that one of its endpoints is a leaf of  $T$ , then Maker has a strategy to win the game  $(E(K_n), \mathcal{T}_n)$  in  $n - 1$  moves.*

## 2 Outline of the proofs

**A fast winning strategy.** The proof of Theorem 1.1 highly depends on the existence of a long bare path.

Assume first that there is a bare path  $P \subseteq T$  whose length is at least  $C_1 = C_1(\Delta)$ , where  $C_1$  is a large constant depending only on  $\Delta$ . Then we split  $T$  into this bare path  $P$  and a forest of two subtrees  $T_1 = (V_1, E_1)$  and  $T_2 = (V_2, E_2)$ . In a first step, we show that Maker can claim a copy of  $T_1 \cup T_2$  within  $|V_1| + |V_2| - 2$  moves, while ensuring that Breaker does not claim too many edges that might become dangerous for Maker with respect to the still necessary embedding of  $P$ . To do so, we use a nice trick and consider the *method of potential functions*, studied intensively in [3]. Throughout the first step, Maker consistently increases a set  $S \subseteq V_1 \cup V_2$  of embedded vertices, which means that Maker claims a copy of the induced subgraph  $(T_1 \cup T_2)[S]$ , and she also considers a set  $U$  consisting of all *available* vertices (*i.e.* vertices in  $K_n$  that are not part of the embedding) plus two vertices  $x_1, x_2$  corresponding to the endpoints of  $P$ . For every vertex  $u$  she defines its potential  $\phi(u)$  that measures those Breaker edges incident to  $u$  which might become dangerous in the proceeding game. Further, she considers a cumulative potential  $\psi = e_B(U) + \sum_{u \text{ open}} \phi(u)$ , where  $e_B(U)$  denotes the number of Breaker edges inside  $U$  and where *open* means that  $u$  is already part of the embedding, while Maker still needs to claim edges incident to  $u$  (in order to complete the embedding of  $T$ ). Now Maker's strategy essentially is based on the idea to embed  $T_1 \cup T_2$  step by step and to ensure in parallel that  $\psi$  never exceeds a given constant  $C_2 = C_2(\Delta)$ . In a second step, using  $e_B(U) \leq$

$C_2$  after  $T_1 \cup T_2$  is fully embedded, we prove that Maker can claim a path from  $x_1$  to  $x_2$  through all remaining available vertices, wasting at most one move. On the one hand this gives us a strengthening of the fast winning result in the Hamilton cycle game from [10], on the other hand this finishes our proof in the first case.

Assume then that the tree  $T$  does not admit a bare path of length  $C_1$ . We conclude that  $T$  contains  $\Theta(n)$  leaves and, since its maximum degree is bounded,  $T$  also contains a matching  $M$  of size  $\Theta(n)$  where each edge is incident with some leaf of  $T$ . In a first step, we introduce a *danger function*, calling a vertex dangerous if its Breaker degree exceeds some large constant  $C_3 = C_3(\Delta)$ . With some technical argumentation we prove that Maker can claim a copy of some subtree  $T'' \subseteq T$  within  $|V(T'')| - 1$  moves such that  $E(T) \setminus E(T'')$  is a subset of  $M$  of size  $\Theta(n)$ . Moreover, she can do it in such a way that, at the time when  $T''$  is fully embedded, there is neither an open nor an available vertex that is dangerous. In a second step, giving a stronger inductive statement, we prove that Maker can claim a perfect matching between the available and the open vertices, wasting at most two moves. This way, we complete our proof in the second case, but also give a fast winning result for the perfect matching game on nearly complete bipartite graphs which strengthens the results from [9].

**Building trees in optimal time.** Recall that in the case where Maker wants to claim a tree with a long bare path, she only wastes one move when she tries to create a path between two designated vertices. A central ingredient in the proofs of Theorem 1.2 and Theorem 1.3 is Maker's ability to build a Hamilton path with exactly one designated vertex as an endpoint in optimal time. The latter is proven by taking a closer look at the Hamilton path game, initiated in [9]. Here, we generalize the known result and give a Maker strategy consisting of five stages that looks more carefully at the actions of Breaker throughout the Hamilton path game.

Assume now that  $T$  is a tree, chosen uniformly at random from the class of all labeled trees on  $n$  vertices. In order to show that Maker can win in optimal time, we use a nice mixture of results from general graph theory and random graph theory, coupled with our methods introduced in the previous proof. By [1] and [11] we at first observe that asymptotically almost surely the maximum degree is given by  $\Delta(T) = (1 + o(1)) \log(n) / \log \log(n)$ , while  $T$  also contains a large family  $\mathcal{P}$  of edge disjoint bare paths, with each path being of size at least  $C_4 = C_4(\Delta)$  and having a leaf of  $T$  as one of its endpoints. Then, in the first two steps of the game Maker claims a copy of some subtree  $T' \subseteq T$  within  $|V(T')| - 1$  moves such that  $T \setminus T'$  is a union of  $\Theta(n)$  bare paths from  $\mathcal{P}$ .

Moreover, again using a danger function, we guarantee that, when  $T'$  is embedded, the Breaker degree at every open/available vertex is bounded by some sufficiently small function  $f(n)$ . In a third step, using general graph theory, we partition the set of open vertices and available vertices into sets of size  $C_4 + 1$ , each having the property that so far Breaker did not claim any edge inside and each containing exactly one open vertex. Finally, Maker plays on each of these sets separately, always claiming a Hamilton path with one designated vertex in optimal time. This finalizes the embedding of  $T$ .

### 3 Concluding remarks and open problems

**Building trees in the shortest possible time.** As noted in the introduction, there are trees  $T$  on  $n$  vertices with bounded maximum degree which Maker cannot build in  $n - 1$  moves. In this paper we proved that Maker can build such a tree  $T$  in at most  $n + 1$  moves. We do not believe that there are bounded degree trees that require Maker to waste more than one move. This leads us to make the following conjecture.

**Conjecture 3.1.** Let  $\Delta$  be a positive integer. Then there exists an integer  $n_0 = n_0(\Delta)$  such that for every  $n \geq n_0$  and for every tree  $T = (V, E)$  with  $|V| = n$  and  $\Delta(T) \leq \Delta$ , Maker has a strategy to win the game  $(E(K_n), \mathcal{T}_n)$  within  $n$  moves.

**Strong tree embedding games.** In the *strong game*  $(X, \mathcal{F})$ , two players, called Red and Blue, take turns in claiming one previously unclaimed element of  $X$ , with Red going first. The winner of the game is the first player to fully claim some  $F \in \mathcal{F}$ . If neither player is able to fully claim some  $F \in \mathcal{F}$  by the time every element of  $X$  has been claimed by some player, the game ends in a *draw*.

Strong games are notoriously hard to analyze. However, the use of explicit very fast winning strategies for Maker in a weak game for devising an explicit winning strategy for Red in the corresponding strong game was initiated in [6]. This idea was used to devise such strategies for the strong perfect matching and Hamilton cycle games [6] and for the  $k$ -vertex-connectivity game [7]. Since it was proved in [8] that Maker has a strategy to win the weak tree embedding game  $(E(K_n), \mathcal{T}_n)$  within  $n + o(n)$  moves, it was noted in [7] that one could be hopeful about the possibility of devising an explicit winning strategy for Red in the corresponding strong game. The first step towards this goal is to find a much faster strategy for Maker in the weak game  $(E(K_n), \mathcal{T}_n)$ . This was accomplished in the current paper.

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# Dicots, and a taxonomic ranking for misère games

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**Abstract.** We study combinatorial games in misère version. In a general context, little can be said about misère games. For this reason, several universes were earlier considered for their study, which can be ranked according to their inclusion ordering. We study in particular a special universe of games called dicots, which turns out to be the known universe of lowest rank modulo which equivalence in misère version implies equivalence in normal version. We also prove that modulo the dicot universe, we can define a canonical form as the reduced form of a game that can be obtained by getting rid of dominated options and most reversible options. We finally count the number of dicot equivalence classes of dicot games born by day 3.

We study combinatorial games in misère version, and in particular a special universe (*i.e.* family) of games called dicots. We first recall basic definitions, following [1, 3, 4].

A combinatorial game is a finite two-player game with no chance and perfect information. The players, called Left and Right, alternate moves until one player has no available move. Under the normal convention, the last player to move wins the game while under the misère convention, that player loses the game.

A game can be defined recursively by its sets of options  $G = \{G^L | G^R\}$ , where  $G^L$  is the set of games reachable in one move by Left (called Left options), and  $G^R$  the set of games reachable in one move by Right (called Right options). The zero game  $0 = \{\cdot | \cdot\}$ , is the game with no options. The birthday of a game is defined recursively as  $birthday(G) = 1 + \max_{G' \in G^L \cup G^R} birthday(G')$ , with 0 being the only game with birthday 0. We say a game  $G$  is born on day  $n$  if  $birthday(G) = n$ , and that it is born by day  $n$  if  $birthday(G) \leq n$ . The games born on day 1 are

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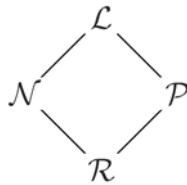
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$\{0|\cdot\} = 1, \{\cdot|0\} = \bar{1}$  and  $\{0|0\} = *$ .

Given two games  $G = \{G^L|G^R\}$  and  $H = \{H^L|H^R\}$ , we recursively define the (disjunctive) sum of  $G$  and  $H$  as  $G + H = \{G^L + H, G + H^L | G^R + H, G + H^R\}$  (where  $G^L + H$  is the set of sums of  $H$  and an element of  $G^L$ ), *i.e.* the game where each player chooses on his turn which one of  $G$  and  $H$  to play on. One of the main objectives of combinatorial game theory is to determine for a game  $G$  the outcome of its sum with any other game.

For both conventions, there are four possible outcomes for a game. Games for which Left player has a winning strategy whatever Right does have outcome  $\mathcal{L}$  (for *left*). Similarly,  $\mathcal{N}$ ,  $\mathcal{P}$  and  $\mathcal{R}$  (for *next, previous* and *right*) denote respectively the outcomes of games for which the first player, the second player, and Right has a winning strategy. We note  $o^+(G)$  the normal outcome of a game  $G$  *i.e.* its outcome under the normal convention and  $o^-(G)$  the misère outcome of  $G$ . Outcomes are partially ordered according to Figure 1, with greater games being more advantageous for Left. Note that there is no general relationship between the normal outcome and the misère outcome of a game.



**Figure 1.** Partial ordering of outcomes.

Given two games  $G$  and  $H$ , we say that  $G$  is greater than or equal to  $H$  in misère play whenever Left prefers the game  $G$  rather than the game  $H$ , that is  $G \geq^- H$  if for every game  $X, o^-(G + X) \geq o^-(H + X)$ . We say that  $G$  and  $H$  are equivalent in misère play, denoted  $G \equiv^- H$ , when for every game  $X, o^-(G + X) = o^-(H + X)$  (*i.e.*  $G \geq^- H$  and  $H \geq^- G$ ). Inequality and equivalence are defined similarly in normal convention, using superscript  $+$  instead of  $-$ .

General equivalence and comparison are very limited in misère play (see [5, 10]), this is why Plambeck and Siegel defined in [8, 9] an equivalence relationship under restricted universes, leading to a breakthrough in the study of misère play games.

**Definition 1 ([8,9]).** Let  $\mathcal{U}$  be a universe of games,  $G$  and  $H$  two games. We say  $G$  is greater than or equal to  $H$  modulo  $\mathcal{U}$  in misère play and write  $G \geq^- H \pmod{\mathcal{U}}$  if  $o^-(G + X) \geq o^-(H + X)$  for every  $X \in \mathcal{U}$ . We

say  $G$  is equivalent to  $H$  modulo  $\mathcal{U}$  in misère play and write  $G \equiv^- H \pmod{\mathcal{U}}$  if  $G \geq^- H \pmod{\mathcal{U}}$  and  $H \geq^- G \pmod{\mathcal{U}}$ .

For instance, Plambeck and Siegel [8, 9] considered the universe of all positions of given games, especially octal games. Other universes have been considered, including the universes of impartial games  $\mathcal{I}$  [3, 4], dicot games  $\mathcal{D}$  [2, 6], dead-ending games  $\mathcal{E}$  [7], and all games  $\mathcal{G}$  [10]. These classes are ordered (ranked) by inclusion as follows:

$$\mathcal{I} \subset \mathcal{D} \subset \mathcal{E} \subset \mathcal{G}.$$

The canonical form of a game is the simplest game of its equivalence class. It is therefore natural to consider canonical forms modulo a given universe. In normal play, impartial games have the same canonical form when considered modulo the universe of impartial games or modulo the universe of all games. In misère play, the corresponding canonical forms are different.

In the following, we focus on the universe of dicots. A game is said to be dicot either if it is  $\{\cdot|\cdot\}$  or if it has both Left and Right options and all these options are dicot. Note that the universe of dicots, denoted  $\mathcal{D}$  is closed under sum of games and taking option.

**Theorem 2.** *Let  $G$  and  $H$  be any games. If  $G \geq_{\mathcal{D}}^- H$ , then  $G \geq^+ H$ .*

The dicot universe is the universe of lowest rank known to have this property.

Modulo the dicot universe, we propose a reduced form of a game that can be obtained by getting rid of dominated options and most reversible options.

**Theorem 3.** *Consider two dicot games  $G$  and  $H$ . If  $G \equiv_{\mathcal{D}}^- H$  and both are in reduced form, then either  $G$  and  $H$  are the games  $0 = \{\cdot|\cdot\}$  and  $\{*\mid*\}$ , or there exists a bijection between the Left (resp. Right) options of  $G$  and of  $H$  such that an option and its image are equivalent modulo  $\mathcal{D}$ .*

As a consequence, we can define the canonical form of a game as its reduced form, except when the game reduces to  $\{*\mid*\}$ , in which case the canonical form is 0.

Thanks to that result, we are able to count the number of dicot equivalence classes (modulo  $\mathcal{D}$ ) of games born by day 3, improving the bound of 5041 proposed by Milley in [6].

**Theorem 4.** *The 1046530 dicot games born by day 3 are distributed among 1214 equivalence classes modulo  $\mathcal{D}$ .*

By comparison, Milley proved in [6] that the number of misère dicot equivalence classes of dicot games born by day 2 is 9. In normal play,

there are 50 non-equivalent dicot games born by day 3 (both modulo the universe of all games or the universe of dicots).

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# Avoider-Enforcer star games

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**Abstract.** We study  $(1 : b)$  Avoider-Enforcer games played on the edge set of the complete graph  $K_n$ , on  $n$  vertices, where Avoider's goal is to avoid claiming a copy of some small fixed graph  $G$ . In particular, we give explicit winning strategies for both players in the  $k$ -star game, where  $G$  is a  $K_{1,k}$ , for constant  $k \geq 2$  under both strict and monotone rules. We also give the winning strategies for both players in another two related monotone games.

## 1 Introduction

Let  $V$  be a finite set and let  $\mathcal{F} \subseteq 2^V$ . Consider a hypergraph  $\mathcal{H} = (V, \mathcal{F})$  with vertex set  $V$  and edge set  $\mathcal{F}$ . The set  $V$  is called the *board* and  $\mathcal{F}$  the family of *losing sets*. Two players, Avoider and Enforcer, take turns in claiming unoccupied vertices of  $V$  until all vertices are claimed and Avoider starts the game. Avoider's goal in the game is to avoid claiming all the elements of any losing set in  $\mathcal{F}$ , while Enforcer's goal is to force him to do so before the end of the game. Avoider-Enforcer games can be played by two different sets of rules [9]. Let  $a$  and  $b$  be positive integers called the *biases* of Avoider, respectively Enforcer. By the first set of rules, in the  $(a : b)$  Avoider-Enforcer game, Avoider claims exactly  $a$

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vertices and Enforcer claims exactly  $b$  vertices per move. This set of rules is called the *strict rules*, and games played by these rules are referred to as *strict games*. By the second set of rules, the *monotone rules*, Avoider and Enforcer claim at least  $a$ , respectively at least  $b$  vertices, per move, and games played by monotone rules are called *monotone games*. In any move of either player, if there are less unclaimed vertices than what a player should claim in his turn, he must claim all the remaining vertices and the game is then over. The game is an *Enforcer's win* if, at any point of the game Avoider has claimed all the vertices of at least one losing set  $F \in \mathcal{F}$ . Otherwise, the game is an *Avoider's win*.

One of the main advantages of the monotone rules is that they are bias monotone. In monotone Avoider-Enforcer games, if the  $(a : b)$  game is an Enforcer's win, then the  $(a + 1 : b)$  and  $(a : b - 1)$  games are also won by Enforcer. Similarly, if the  $(a : b)$  game is an Avoider's win, then the  $(a : b + 1)$  and  $(a - 1 : b)$  games are also won by Avoider.

When  $a = b = 1$ , we call such games *unbiased*. Unbiased Avoider-Enforcer games were studied *e.g.* in [3] and [11]. In an unbiased game, it is also interesting to see how fast Enforcer can force Avoider to lose, or to see how long can Avoider defend himself before losing. These type of problems were considered in [1, 7].

In [11], Lu proved that Beck's [2] generalization of the Erdős-Selfridge criterion [6] gave sufficient conditions for an Avoider's win in the  $(1 : 1)$  Avoider-Enforcer game. In [10], Hefetz, Krivelevich and Szabó gave a general winning criterion for Avoider in  $(a : b)$  Avoider-Enforcer games played by both sets of rules. This criterion takes only Avoider's bias into account. In [4] a new criterion for Avoider's win in both strict and monotone  $(a : b)$  games on  $\mathcal{H}$  is introduced, which depends on both biases  $a$  and  $b$ . Note, however, that these criteria are non-constructive and do not give the strategy of the winning player.

The focus of our research is on  $(1 : b)$  Avoider-Enforcer games, played by both monotone and strict rules on a graph where the board is the set of edges of the complete graph on  $n$  vertices, *i.e.*  $V = E(K_n)$ . This type of game appears frequently in the literature (see, for example [8] and [9]).

We follow the terminology for strict Avoider-Enforcer games introduced by Hefetz, Krivelevich and Szabó in [10]. The *upper threshold bias*  $f_{\mathcal{H}}^+$  is the smallest integer such that for every integer  $b, b > f_{\mathcal{H}}^+$ , the  $(1 : b)$  game on  $\mathcal{H}$  is an Avoider's win. The *lower threshold bias*  $f_{\mathcal{H}}^-$  is the largest integer such that for every integer  $b, b \leq f_{\mathcal{H}}^-$ , the  $(1 : b)$  game on  $\mathcal{H}$  is an Enforcer's win. The inequality  $f_{\mathcal{H}}^- \leq f_{\mathcal{H}}^+$  always holds, but the upper and lower threshold biases can be close to each other in the case of some games, like for example in Connectivity game [10] or be far apart. When  $f_{\mathcal{H}}^- = f_{\mathcal{H}}^+$  we call this number  $f_{\mathcal{H}}$  and refer to it as the

threshold bias of the game  $\mathcal{H}$ . This threshold bias may not exist for some games.

For monotone  $(1 : b)$  Avoider-Enforcer game on  $\mathcal{H}$ , there is a unique monotone threshold bias  $f_{\mathcal{H}}^{mon}$  defined as the largest integer value such that for every integer  $b$ ,  $b \leq f_{\mathcal{H}}^{mon}$ , the game is an Enforcer’s win. The question that arises is whether  $f_{\mathcal{H}}^- \leq f_{\mathcal{H}}^{mon} \leq f_{\mathcal{H}}^+$  hold for all  $(1 : b)$  Avoider-Enforcer games. The results from [9] and [10] show that it does not hold in general.

We are interested in  $(1 : b)$  Avoider-Enforcer games where Avoider wants to avoid claiming a copy of some small fixed graph  $G$ . This problem was studied in [9] for some graphs  $G$ . Let  $\mathcal{K}_G$  denote the hypergraph whose edges are the edge sets of all the copies of  $G$  in  $K_n$ . In [9] the authors analysed games where  $G$  is  $K_3$  and  $P_3$  respectively and gave the thresholds for both the monotone and strict game  $\mathcal{K}_{P_3}$  and for the monotone game  $\mathcal{K}_{K_3}$ . They showed that

$$f_{\mathcal{K}_{P_3}}^{mon} = \binom{n}{2} - \lfloor \frac{n}{2} \rfloor - 1, f_{\mathcal{K}_{P_3}}^+ = \binom{n}{2} - 2, f_{\mathcal{K}_{P_3}}^- = \Theta(n^{\frac{3}{2}}) \text{ and } f_{\mathcal{K}_{K_3}}^{mon} = \Theta(n^{\frac{3}{2}}).$$

Bednarska-Bzdęga in [4] showed that  $f_{\mathcal{K}_{K_3}}^- = \Omega(n^{\frac{1}{2}})$ . Clemens et al. in [5] showed that in a monotone game where losing sets are the edges of all  $P_4$ , the threshold bias is  $f_{\mathcal{K}_{P_4}}^{mon} = \frac{1}{2} \binom{n}{2} - \frac{n}{2} \left( \frac{1}{\sqrt{2}} - o(1) \right)$ . Moreover, in [9], the authors conjectured that in general  $f_{\mathcal{K}_G}^+$  and  $f_{\mathcal{K}_G}^-$  are not of the same order and asked about the strategies in games where  $G$  is some fixed graph on more than 4 vertices. Bednarska-Bzdęga established in [4] general upper and lower bounds on  $f_{\mathcal{K}_G}^+$ ,  $f_{\mathcal{K}_G}^-$  and  $f_{\mathcal{K}_G}^{mon}$  for every fixed graph  $G$ , but these bounds are not tight.

## 2 Results

In the present paper, we study the game  $\mathcal{K}_G$  where  $G$  is a  $k$ -star  $K_{1,k}$ , for some fixed  $k \geq 2$  and denote it by  $\mathcal{S}_k$ . We call the game  $k$ -star game.

In order to state our main result, we have to introduce some functions: let us define  $r = r(n, b)$  by  $1 \leq r \leq b + 1$  and  $\binom{n}{2} \equiv r \pmod{b + 1}$ . Note that  $r$  is the number of edges that Avoider is allowed to choose from in his last move when playing the strict  $(1 : b)$  game. Let

$$e_{n,k}^+ = \max \left\{ b : r(b + 1) < \frac{1}{8} \frac{n^k}{(2b)^{k-2}} \right\}, \text{ and}$$

$$e_{n,k}^- = \max \left\{ b' : \forall b, \frac{1}{4} n^{\frac{k+1}{k}} \leq b \leq b' : r(b + 1) < \frac{1}{8} \frac{n^k}{(2b)^{k-2}} \right\}.$$

The main result in our paper is the following theorem.

**Theorem 2.1.** *In  $(1 : b)$   $k$ -star game  $\mathcal{S}_k$ ,  $k \geq 2$ , we have*

(i)  $f_{\mathcal{K}_{\mathcal{S}_k}}^{mon} = \Theta(n^{\frac{k}{k-1}}),$

- (ii)  $e_{n,k}^+ \leq f_{\mathcal{K}_{S_k}}^+ = O(n^{\frac{k}{k-1}})$  holds for all values of  $n$ , and  $f_{\mathcal{K}_{S_k}}^+ = \Theta(n^{\frac{k}{k-1}})$  holds for infinitely many values of  $n$ ,
- (iii)  $\max\{\frac{1}{2}n^{\frac{k+1}{k}}, e_{n,k}^-\} \leq f_{\mathcal{K}_{S_k}}^- = O(n^{\frac{k+1}{k}} \log n)$  holds for all values of  $n$ , and  $f_{\mathcal{K}_{S_k}}^- = \Theta(n^{\frac{k+1}{k}})$  holds for infinitely many values of  $n$ .

We also consider two more monotone games similar to the  $k$ -star game. Let the *double star*  $\mathcal{S}_{k,k}$  be a graph on  $2k$  vertices  $u, u_1, \dots, u_{k-1}, v, v_1, \dots, v_{k-1}$  such that the edge set of  $\mathcal{S}_{k,k}$  is  $\{(uv)\} \cup \{(uu_i) : 1 \leq i \leq k-1\} \cup \{(vv_i) : 1 \leq i \leq k-1\}$  (see Figure 2.1) and let  $\mathcal{K}_{S_{k,k}}$  be the hypergraph of the game.

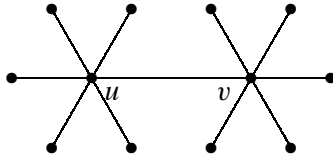


Figure 2.1.  $\mathcal{S}_{6,6}$  on vertices  $(u, v)$ .

Let the *path double star*  $\mathcal{PS}_{k,k}$  be a graph on  $2k+1$  vertices  $w, u, u_1, \dots, u_{k-1}, v, v_1, \dots, v_{k-1}$  such that  $E(\mathcal{PS}_{k,k}) = \{(u, u_i), 1 \leq i \leq k-1\} \cup \{(v, v_i), 1 \leq i \leq k-1\} \cup (v, w) \cup (u, w)$ , as shown in Figure 2.2, and let  $\mathcal{K}_{\mathcal{PS}_{k,k}}$  be the hypergraph of the game.

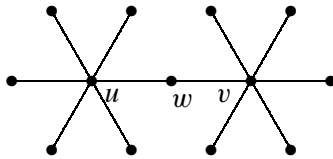


Figure 2.2.  $\mathcal{PS}_{6,6}$  on vertices  $(u, v, w)$ .

**Theorem 2.2.** *Let  $k \geq 2$ . In  $(1 : b)$  double star  $\mathcal{S}_{k,k}$  and path double star  $\mathcal{PS}_{k,k}$  games, we have*

- (i)  $f_{\mathcal{K}_{S_{k,k}}}^{mon} = \Theta(n^{\frac{k}{k-1}})$ ,
- (ii)  $f_{\mathcal{K}_{\mathcal{PS}_{k,k}}}^{mon} = \Theta(n^{\frac{k+1}{k}})$ .

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# Algebra and Polynomials

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# Fooling-sets and rank in nonzero characteristic

Mirjam Friesen<sup>1</sup> and Dirk Oliver Theis<sup>2</sup>

**Abstract.** An  $n \times n$  matrix  $M$  is called a *fooling-set matrix of size  $n$* , if its diagonal entries are nonzero, whereas for every  $k \neq \ell$  we have  $M_{k,\ell}M_{\ell,k} = 0$ . Dietzfelbinger, Hromkovič, and Schnitger (1996) showed that  $n \leq (\text{rk}M)^2$ , regardless of over which field the rank is computed, and asked whether the exponent on  $\text{rk}M$  can be improved.

We settle this question for nonzero characteristic by constructing a family of matrices for which the bound is asymptotically tight. The construction uses linear recurring sequences.

## 1 Introduction

An  $n \times n$  matrix  $M$  over some field  $\mathbb{K}$  is called a *fooling-set matrix of size  $n$*  if

$$M_{kk} \neq 0 \quad \text{for all } k \text{ (its diagonal entries are all nonzero), and (1.1a)}$$

$$M_{k,\ell}M_{\ell,k} = 0 \quad \text{for all } k \neq \ell. \quad (1.1b)$$

Note that the definition depends only on the zero-nonzero pattern of  $M$ . The word “fooling set” originates from Communication Complexity, but the concept is used under different names in other contexts.

In Communication Complexity and Combinatorial Optimization, one is interested in finding a large fooling-set (sub-)matrix contained in a given matrix  $A$  (permutation of rows and columns is allowed), as its size provides a lower bound to other numerical properties of the matrix. Since large fooling-set submatrices are typically difficult to identify (the problem is equivalent to finding a large clique in a graph of a certain type), one would like to upper-bound the size of a fooling-set matrix one may possibly hope for in terms of easily computable properties of  $A$ .

Dietzfelbinger, Hromkovič, and Schnitger ([4, Thm. 1.4], or see [10, Lemma 4.15]; cf. [5, 8]) proved that the rank of a fooling-set matrix of size  $n$  is at least  $\sqrt{n}$ , i.e.,

$$n \leq (\text{rk}_{\mathbb{K}}M)^2. \quad (1.2)$$

This inequality gives such an upper bound on the largest fooling-set submatrix in terms of the easily computable rank of  $A$ .

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However, it is an open question whether the exponent on the rank in the right-hand side of (1.2) can be improved or not. Dietzfelbinger *et al.* [4, Open Problem 2] were particularly interested in 0/1-matrices and  $\mathbb{K} = \mathbb{F}_2$ , which corresponds to the Communication Complexity situation they dealt with.

Klauck and de Wolf [8] have pointed out the importance for Communication Complexity of the question regarding general (*i.e.*, not 0/1) matrices.

Currently, the examples (attributed to M. Hühne in [4]) of 0/1 fooling-set matrices  $M$  with smallest rank are such that  $n \approx (\text{rk}_{\mathbb{F}_2} M)^{\log_4 6}$  ( $\log_4 6 = 1.292\dots$ ); for general matrices, Klauck and de Wolf [8] have given examples with  $n \approx (\text{rk}_{\mathbb{Q}} M)^{\log_3 6}$  ( $\log_3 6 = 1.63\dots$ ).

In our paper, we settle the question for fields  $\mathbb{K}$  of nonzero characteristic. We prove that inequality (1.2) is asymptotically tight if the characteristic of  $\mathbb{K}$  is nonzero. Notably, not only is the exponent on the rank in inequality (1.2) best possible, but so is the constant (one) in front of the rank.

**Organization of this extended abstract.** In the next section we will explain some of the connections of the fooling-set vs. rank problem with Combinatorial Optimization and Graph Theory concepts. In Section 3, we will sketch the proof of our result. In the final section, we point to some questions which remain open.

## 2 Some remarks on the importance of fooling-set matrices

While the fooling-set size vs. rank problem is of interest in its own right as a minimum-rank type problem in Combinatorial Matrix Theory, fooling-set matrices are connected to other areas of Mathematics and Computer Science.

**In Polytope Theory,** given a polytope  $P$ , sizes of fooling-set submatrices of appropriately defined matrices provide lower bounds to the number of facets of any polytope  $Q$  which can be mapped onto  $P$  by a projective mapping ([14], *cf.* [5]). Similarly, in **Combinatorial Optimization**, sizes of fooling-set matrices are lower bounds to the minimum sizes of Linear Programs for combinatorial optimization problems ([14]). For example, it is an open question whether Edmond's matching polytope for a complete graph on  $n$  vertices admits a fooling-set matrix whose size grows quicker in  $n$  than the dimension of the polytope. Such a fooling-set matrix would yield a fairly spectacular improvement on the currently known lower bounds of sizes of Linear Programming formulations for the matching problem. See [5] for bounds based on fooling sets for a number of combinatorial optimization problems, including bipartite matching.

In the Polytope Theory / Combinatorial Optimization applications, we typically have  $\mathbb{K} = \mathbb{Q}$ , and the rank of the large matrix  $A$  is known. However, since the definition of a fooling-set matrix depends only on the zero-nonzero pattern, changing the field from  $\mathbb{Q}$  to  $\mathbb{K}'$  and replacing the nonzero rational entries of  $A$  by nonzero numbers in  $\mathbb{K}'$  may yield a lower rank and hence a better upper bound on the size of a fooling-set matrix.

**In Computational Complexity,** fooling-set matrices provide lower bounds for the communication complexity of Boolean functions (see, e.g., [1, 4, 8, 10, 12]), and for the number of states of an automaton accepting a given language (e.g., [6]).

As an example from Communication Complexity where the “fooling-set method” can be seen to yield a poor lower bound is the inner product function<sup>3</sup>

$$f(x, y) = \sum_{j=1}^n x_j y_j, \quad \text{for } x, y \in \mathbb{Z}_2^n.$$

The rank of the associated  $2^n \times 2^n$ -matrix is  $n$ , hence, by (1.2), there is no fooling-set sub-matrix larger than  $n^2$ .

**In Graph Theory,** a fooling-set matrix (up to permutation of rows and columns) can be understood as the incidence matrix of a bipartite graph containing a perfect cross-free matching. Recall that a matching in a bipartite graph  $H$  is called *cross-free* if no two matching edges induce a  $C_4$ -subgraph of  $H$ .

Cross-free matchings are best known as a lower bound on the size of biclique coverings of graphs (e.g. [3,7]). A *biclique covering* of a graph  $G$  is a collection of complete bipartite subgraphs of  $G$  such that each edge of  $G$  is contained in at least one of these bipartite subgraphs. If a cross-free matching of size  $n$  is contained as a subgraph in  $G$ , then at least  $n$  bicliques are needed to cover all edges of  $G$ . (For some classes of graphs, this is a sharp lower bound on the biclique covering number [3, 13]).

**In Matrix Theory,** the maximum size of a fooling-set sub-matrix is known under a couple of different names, e.g. as independence number [2, Lemma 2.4]), or as the intersection number. For some semirings, this number provides a lower bound for the so-called factorization rank of the matrix over the semiring.

**In each of these areas,** fooling-set matrices are used as lower bounds. Upon embarking on a search for a big fooling-set matrix in a large, complicated matrix  $A$ , one is interested in an *a priori* upper bound on their sizes and thus the potential usefulness of the lower bound method.

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<sup>3</sup> Thanks to one of the referees for pointing us to this example.

### 3 Fooling-Set Matrices from Linear Recurring Sequences

For a prime number  $p$ , we denote by  $\mathbb{F}_p$  the finite field with  $p$  elements. The following is an accurate statement of our result.

**Theorem 3.1.** *For every prime number  $p$ , there is a family of fooling-set matrices  $M^{(t)}$  over  $\mathbb{F}_p$  of size  $n^{(t)}$ ,  $t = 1, 2, 3, \dots$ , such that  $n^{(t)} \rightarrow \infty$ , and*

$$\frac{n^{(t)}}{(\text{rk}_{\mathbb{F}_p} M^{(t)})^2} \longrightarrow 1.$$

The method used in all the *earlier* examples (mentioned in the introduction) of fooling-set matrices with small rank was the following: One conjures up a single, small fooling-set matrix  $M^0$  (of size, say, 6), determines its rank (say, 3), and then uses the tensor-powers of  $M^0$  (which are fooling-set matrices, too). With these numerical values, from  $M^0$ , one obtains  $\log_3 6$  as a lower bound on the exponent on the rank in (1.2).

Our technique is a departure from that approach. As noted above, we use linear recurring sequences. For every  $t$ , we construct an  $n^{(t)}$ -periodic function, which gives us a fooling-set matrix of size  $n^{(t)}$ .

*We now describe that construction.* Let  $p$  be a prime number and  $r \geq 2$  an integer. Define the function  $f: \mathbb{Z} \rightarrow \mathbb{F}_p$  by the recurrence relation

$$f(k+r) = -f(k) - f(k+1) \quad \text{for all } k \in \mathbb{Z} \quad (3.1a)$$

and the initial conditions

$$f(0) = 1, \text{ and } f(1) = \dots = f(r-1) = 0. \quad (3.1b)$$

Fix an integer  $n > r$ . From the sequence, we define an  $n \times n$  matrix as follows. For ease of notation, the matrix indices are taken to be in  $\{0, \dots, n-1\} \times \{0, \dots, n-1\}$ . We let

$$M_{k,\ell} := f(k-\ell). \quad (3.2)$$

It is fairly easy to see that  $\text{rk} M \leq r$ .

**Lemma 3.2.** *The rank of  $M$  is at most  $r$ .*

*Proof.* From (3.1a), for  $k \geq r$ , we deduce the equation  $M_{k,\star} = -M_{k-r,\star} - M_{k-r+1,\star}$ . Hence, each of the rows  $M_{k,\star}$ ,  $k \geq r$ , is a linear combination of the first  $r$  rows of  $M$ .  $\square$

It can be seen that the rank is, in fact, equal to  $r$ : The top-left  $r \times r$  sub-matrix is non-singular because it is upper-triangular with nonzeros along the diagonal.

Next, we reduce the fooling-set property (1.1) to a property of the function  $f$ .

**Lemma 3.3.** *The matrix  $M$  defined in (3.2) is a fooling-set matrix, if and only if,*

$$f(k)f(-k) = 0 \quad \text{for all } k \in \{1, \dots, n - 1\}. \quad (3.3)$$

*Proof.* It is clear from (3.1b) and (3.2) that  $M_{j,j} = f(0) = 1$  for all  $j = 0, \dots, n - 1$ , so it remains to verify (1.1b). Since

$$M_{i,j}M_{j,i} = f(i - j)f(j - i) = f(i - j)f(-(i - j)),$$

if  $f(k)f(-k) = 0$  for all  $k = 1, \dots, n - 1$ , then  $M_{i,j}M_{j,i}$  is zero whenever  $i \neq j$ . This proves (1.1b). □

Given appropriate conditions on  $r$  and  $n$  (depending on  $p$ ), this condition on  $f$  can indeed be verified:

**Lemma 3.4.** *For all integers  $t \geq 1$ , if we let  $r := p^t + 1$  and  $n := r(r - 1) + 1$ , then  $f(k)f(-k) = 0$  for all  $k \in \mathbb{Z} \setminus n\mathbb{Z}$ .*

Combining the above three lemmas, we can complete the proof of Theorem 3.1.

*Proof of Theorem 3.1.* Let  $p$  be a prime number. For every integer  $t \geq 1$ , let  $r := p^t + 1$  and  $n^{(t)} := r(r - 1) + 1$ , and define the matrix  $M^{(t)} := M$  over  $\mathbb{F}_p$  as in (3.2). By Lemma 3.2, the rank of  $M^{(t)}$  is at most  $r$ , and from Lemmas 3.3 and 3.4 we conclude that  $M^{(t)}$  is a fooling-set matrix. Hence, we have

$$1 \geq \frac{n^{(t)}}{\text{rk}_{\mathbb{F}_p}(M^{(t)})^2} \geq \frac{r^2 - r + 1}{r^2} \geq 1 - p^{-t}/4 \xrightarrow{t \rightarrow \infty} 1,$$

where the left-most inequality is from (1.2). □

To prove Lemma 3.4, we need two more lemmas. The first one states that in every section  $\{jr, \dots, (j + 1)r - 1\}$ ,  $j = 0, 1, \dots$ , there is a block of zeros whose length decreases with  $j$ .

**Lemma 3.5.** *For  $j = 0, \dots, r - 2$ , we have*

$$f(jr + i) = 0 \quad \text{for } i = 1, \dots, r - 1 - j. \quad (3.4)$$

*Proof.* Equation (3.4) is true for  $j = 0$  by (3.1b). Suppose (3.4) holds for some  $j < r - 2$ . Then  $f((j + 1)r + i) = 0$  for  $i = 1, \dots, r - 1 - (j + 1)$ , because, by (3.1a),

$$f((j + 1)r + i) = f(jr + i + r) = -f(jr + i) - f(jr + (i + 1)) = -0 - 0$$

holds. □

Every function on  $\mathbb{Z}$  with values in a finite field which is defined by a (reversible) linear recurrence relation is periodic (cf. e.g. [11]). The second lemma establishes that a specific number  $n$  is a period of  $f$  as defined in (3.1).

**Lemma 3.6.** *If  $r = p^t + 1$  for some integer  $t \geq 1$ , then  $n := r(r - 1) + 1$  is a period of the function  $f$ .*

This lemma is the difficult part of the proof of Theorem 3.1. Due to the space limitations, for its proof, we have to refer to the full paper. At this point, suffice it to say that the argument proceeds by identifying binomial coefficients among the values of  $f$ , and then uses the known periodicity of the binomial coefficients modulo  $p$ .

Lemmas 3.5 and 3.6 allow us to prove Lemma 3.4.

*Proof of Lemma 3.4.* We need to show  $f(k)f(-k) = 0$  whenever  $n \nmid k$ . By Lemma 3.6, this is equivalent to showing  $f(k)f(n - k) = 0$  for  $k = 1, \dots, n - 1$ . Given such a  $k$ , let  $j, i$  be such that  $k = jr + i$  and  $0 \leq i \leq r - 1$ .

If  $i \leq r - 1 - j$ , then  $f(k) = 0$  by Lemma 3.5, and we are done. If, on the other hand,  $i > r - 1 - j$ , then

$$n - k = r^2 - r + 1 - jr - i = (r - 1 - (j + 1))r + (r - i + 1),$$

and  $r - i + 1 \leq j + 1$ , so, by Lemma 3.5, we have  $f(n - k) = 0$ .  $\square$

## 4 Conclusion

Dietzfelbinger *et al.*'s original question regarding the tightness of inequality (1.2) for 0/1-matrices remains open in characteristic  $p > 2$ . For these matrices, it may still be possible that the exponent on the rank in the inequality (1.2) can be improved.

For characteristic zero, Klauck and de Wolf [8] have given an example of a fooling-set matrix of size 6 with entries in  $\{0, \pm 1\}$  which has rank 3. Thus, using the method sketched above (following Theorem 3.1), the exponent on the rank in inequality (1.2) with  $\mathbb{K} := \mathbb{Q}$  for general (*i.e.*, not 0/1) matrices is at least  $\log_3 6 = 1.63\dots$ , while the best known bound for 0/1-matrices is  $\log_4 6 = 1.292\dots$

We would like to point out the possibility that, in characteristic zero, the minimum achievable rank on the right hand side of inequality (1.2) may depend not only on the characteristic, but on the field  $\mathbb{K}$  itself. Indeed, there are examples of zero-nonzero patterns for which the minimum rank of a matrix with that zero-nonzero pattern differs between  $\mathbb{K} = \mathbb{Q}$  and  $\mathbb{K} = \mathbb{R}$ , see e.g. [9]. Hence, for characteristic zero, we ask the following weaker version of Dietzfelbinger *et al.*'s question.

**Question 4.1.** Is there a field  $\mathbb{K}$  (of characteristic zero) over which the fooling-set matrix size vs. rank inequality in (1.2) can be improved?

As mentioned in Section 2, another question in characteristic zero comes from polytope theory. Let  $P$  be a polytope. Let  $A$  be a matrix whose rows are indexed by the facets of  $P$  and whose columns are indexed by the vertices of  $P$ , and which satisfies  $A_{F,v} = 0$ , if  $v \in F$ , and  $A_{F,v} \neq 0$ , if  $v \notin F$ . For any fooling-set submatrix of size  $n$  of  $A$ , the following inequality follows from (1.2) (cf. [5]):

$$n \leq (\dim P + 1)^2. \quad (4.1)$$

The following variant of Dietzfelbinger *et al.*'s question is of pertinence in Polytope Theory and Combinatorial Optimization (see Section 2).

**Question 4.2.** Can the fooling-set size vs. dimension inequality (4.1) be improved (for polytopes)?

To our knowledge, the best known lower bound for the best possible exponent on the dimension in inequality (4.1) is 1.

Finally, the complexity of the Fooling-Set-Submatrix problem is still open:

**Conjecture 4.3.** The *Fooling-Set-Submatrix problem*

**Input:** Integers  $n, m$  and  $m \times m$  0/1-matrix  $A$

**Output:** “Yes”, if a fooling-set submatrix of size  $n$  of  $A$  exists,  
“No” otherwise.

is NP-hard.

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# Krasner near-factorizations and 1-overlapped factorizations

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**Abstract.** Near and/or 1-overlapped factorizations on cyclic groups play important roles both in perfect graph theory and ideal clutter theory. Such a factorization is *Krasner* if its construction does not need any modulo operation (i.e. every addition can be thought as the addition of integers). In this paper, we characterize Krasner near-factorizations and 1-overlapped factorizations, which solves a problem posed by S. Szabó and A.D. Sands [9].

## 1 Introduction

In this paper, let  $G$  denote an abelian group. For two subsets  $S_1$  and  $S_2$  of  $G$ , let  $S_1 + S_2$  denote the multiset  $\{a + b \mid a \in S_1, b \in S_2\}$ . If  $S_2$  has only one element  $g$ , we use  $S_1 + g$  instead of  $S_1 + \{g\}$ . A sum of two sets  $S_1 + S_2$  is *direct* if all elements are distinct. A subset  $S \subset G$  is *symmetric* respect to an element  $x \in G$  if  $S + x = -S - x$ , where this  $x$  is called the *center* of  $A$ . Especially, a subset  $S \subset G$  is *symmetric* if it is symmetric respect to 0. A subset  $S \subset G$  is *shift-symmetric* if there exists an element  $g \in G$  such that  $S = -S + g$ . For a set  $S \subset G$  and an element  $g \in G$ , let  $gS$  or  $Sg$  denote the set  $\{gs \mid s \in S\}$ .

A pair  $(A, B)$  of a finite cyclic group  $\mathbb{Z}_n$  with  $\min\{|A|, |B|\} \geq 2$  is a *factorization* if  $A + B$  equals  $\mathbb{Z}_n$ , and a near (resp. 1-overlapped) factorization if  $A + B$  equals  $G \setminus \{g\}$  (resp.  $G \cup \{g\}$ ) for some element  $g \in \mathbb{Z}_n$  which is called the *uncovered element* (resp. the *doubly covered element*) of  $(A, B)$ . For two integers  $a$  and  $b$ , let  $[a, b]$  be the set  $\{i \in \mathbb{Z} \mid a \leq i \leq b\}$ . A pair  $(A, B)$  of subsets of  $\mathbb{Z}$  is a *factorization of*

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an interval  $[a, b]$  if  $A + B = [a, b]$ . Let  $\phi_n$  be the natural bijection from  $\mathbb{Z}_n$  to  $[0, n - 1]$ , and  $\psi_n$  be the inverse map of  $\phi_n$ . Let us use the abbreviations  $\bar{0}$  and  $\bar{1}$  for  $\psi_n(0)$  and  $\psi_n(1)$ , respectively. A factorization (resp. a near factorization and 1-overlapped factorization)  $(A, B)$  is called *Krasner* if  $0 \leq \phi_n(a) + \phi_n(b) \leq n$  for all  $a \in A$  and  $b \in B$ . Let  $J$  be the  $n$  by  $n$  all one matrix, and  $I$  be the identity matrix. A simple graph is a *normalized partitionable* if its clique incidence matrix is a 0-1 solution of  $XY = J - I$ . A square 0-1 matrix is a *thin Lehman matrix* if it is a solution of  $XY = J + I$ . Normalized partitionable graphs and thin Lehman matrices are rare known methods for constructing minimally imperfect graphs and minimally non-ideal clutters. (See [1], [6].)

Let  $\rho(\geq 1)$  and  $m_1, m_2, \dots, m_{2\rho}(\geq 2)$   $r, s(\geq 2)$  be integers such that  $r = \prod_{i=1}^{\rho} m_{2i-1}, s = \prod_{i=1}^{\rho} m_{2i}$  and  $n = \prod_{i=1}^{2\rho} m_i$ . Let  $\mu_j = \prod_{i=1}^{j-1} m_i$  for  $1 \leq j \leq 2\rho$ . Define a subset  $M_i$  of  $\mathbb{N}$  by  $\{0, 1, 2, \dots, m_i - 1\}\mu_i$ ,  $A' := M_1 + M_3 + \dots + M_{2\rho-1}$  and  $B' := M_2 + M_4 + \dots + M_{2\rho}$ . It is clear that  $(\psi_n(A'), \psi_n(B')), (\psi_{n+1}(A'), \psi_{n+1}(B')), (\psi_{n-1}(A'), \psi_{n-1}(B'))$  are a factorization, a near factorization, and 1-overlapped factorization of cyclic groups with corresponding orders. Let  $(A, B)$  be one of a factorization, a near factorization, or a 1-overlapped factorization of  $\mathbb{Z}_n$ . Then the following three operations carry each factorization to another factorization of the same kind.

- *Shifting*: Consider  $(A + a, B + b)$  for some  $a, b \in \mathbb{Z}_n$ .
- *Scaling*: Consider  $(\lambda A, \lambda B)$  for some  $\lambda \in \mathbb{Z}_n^\times$ .
- *Swapping*: Consider  $(-A, B)$ .

We call a factorization (near factorization, 1-overlapped factorization) constructed by the above method (with shifting, scaling and swapping) a *DBNS factorization (DBNS near factorization, DBNS 1-overlapped factorization)* of  $\mathbb{Z}_n$ .

N. G. De Bruijn proved the following theorem.

**Theorem 1.1.** *Let  $A$  and  $B$  be two subsets of  $\mathbb{Z}$ . If  $A + B$  is direct, and equals  $[\min A + \min B, \max A + \max B]$ , and is direct, there exist parameters  $m_1, m_2, \dots, m_r$  such that*

$$A = \sum_{i:\text{odd}} [0, m_i - 1] \prod_{k=1}^{i-1} m_k + \min A, \quad B = \sum_{i:\text{even}} [0, m_i - 1] \prod_{k=1}^{i-1} m_k + \min B,$$

$$\max A + \max B - \min A - \min B + 1 = \prod_{k=1}^r m_k.$$

*Epecially, each Krasner factorization is a DBNS factorization.*

In comparison to the above, S. Szabó and A.D. Sands [9] recently posed the following problem.

**Problem 1.2.** Characterize Krasner near factorization.

On the other hand, in 1984, C. Grinstead raised the following conjecture.

**Conjecture 1.3 (C. Grinstead (1984)).** Every near-factorization of a finite cyclic group is a DBNS near-factorization.

Certainly, the problem of S. Szabó et al. is corresponding to the most basic case (sub-conjecture) of the above.

## 2 Preliminaries

Here, we introduce some theorems which we use in this article.

**Theorem 2.1 (D. De Caen and el. [3], H. Shinohara [8]).** *Let  $(A, B)$  be a near (resp. 1-overlapped) factorization of  $G$ . There exist two elements  $a, b \in G$  such that the uncovered element (resp. the doubly covered element) of  $(A + a, B + b)$  is 0, and that both of  $A + a$  and  $B + b$  are symmetric.*

A subset  $S$  of  $G$  is an  $(a, i, k)$  arithmetic progression if  $S = \{a + il \mid 0 \leq l \leq k - 1\}$ . A set  $S \subset G$  is partially  $(i, k)$  arithmetic if there exists an  $(a, i, k)$  arithmetic progression  $S_1$  and a subset  $S_2 \subset G$  such that  $S = S_1 + S_2$ .

**Theorem 2.2 (K. Kashiwabara et al. [5], T. Sakuma et al. [7]).** *Let  $(A, B)$  be a near (resp. 1-overlapped) factorization of  $\mathbb{Z}_n$ . If  $A$  is partially  $(i, k)$  arithmetic for some  $i, k$ , then  $(A, B)$  is a DBNS near (resp. 1-overlapped) factorization.*

## 3 Results

In this section, we prove the following theorem, which not only solves the problem of S. Szabó et al. but also settles positively the most basic cases of Grinstead's conjecture and its ideal clutter theoretical analog [7], at the same time.

**Theorem 3.1.** *Every Krasner near (resp. 1-overlapped) factorization is a DBNS near (resp. 1-overlapped) factorization.*

**Lemma 3.2.** *Let  $A, B$  be two sets of  $\mathbb{Z}$  such that  $A$  is shift-symmetric,  $\min A = \min B = 0$  and  $\max B < \max A$ . If  $A + B$  is direct and  $[0, \max A] \subseteq A + B$ , then  $A + B$  is an interval.*

*Proof of Theorem 3.1.* Let  $(A, B)$  be a Krasner near factorization of  $\mathbb{Z}_n$  with uncovered element  $u$ . If  $u$  is 0 then  $(\phi_n(A), \phi_n(B))$  is a factorization of either  $[1, n]$  or  $[0, n - 1]$  and hence this  $(A, B)$  is a DBNS near-factorization. Therefore without loss of generality, we can assume that  $u$  does not equal 0. Furthermore, we can assume that  $\bar{0} \in A \cap B$ , for otherwise we can use either the Krasner factorization  $(A - \bar{1}, B)$  (if  $\bar{0} \notin A$ ) or  $(A, B - \bar{1})$  (if  $\bar{0} \notin B$ ) as a substitute for the  $(A, B)$ . From the assumption  $\bar{0} \in A \cap B$ , we have  $\phi_n(A) + \phi_n(B) \subset [0, n - 1]$ . Thus, either  $\max \phi_n(A)$  or  $\max \phi_n(B)$  is strictly less than  $n/2$ . Without loss of generality, we assume  $\max \phi_n(A) < n/2$ .

Let  $a'$  and  $b'$  be two elements of  $\mathbb{Z}_n$  such that  $(A - a', B - b')$  is a near factorization of  $\mathbb{Z}_n$  whose uncovered element is the identity, and that both of  $A - a'$  and  $B - b'$  are symmetric. Then the uncovered element of  $(A, B)$  is  $a' + b'$ . Since  $a'$  and  $b'$  are the centers of symmetry of  $A$  and  $B$  respectively, we have  $2a' \in A$  and  $2b' \in B$ . Combining this and the assumption  $\max \phi_n(A) < n/2$ , we also have  $\phi_n(2a') = \max(\phi_n(A))$ . Without loss of generality, we can assume that  $\phi_n(a' + b') \leq n/2$ , for otherwise we can use the other Krasner near factorization  $(\psi_n(\max \phi_n(A)) - A, \psi_n(\max \phi_n(B)) - B)$  as a substitute for the  $(A, B)$ . (Note that shifting carries  $(A, B)$  to  $(\psi_n(\max \phi_n(A)) - A, \psi_n(\max \phi_n(B)) - B)$  by Theorem 2.1).

Suppose that there exists an element  $b_1$  such that  $\phi_n(b_1) \leq 2\phi_n(a' + b') < \phi_n(b_1) + \phi_n(2a')$ . Then, since  $b'$  is the center of  $B$ , we have  $2b' - b_1 \in B$  and hence  $(2b' - b_1) + 2a' \in B + A$ . Furthermore, since  $(a' + b') - (2b' - b_1) = (b_1 + 2a') - (a' + b')$ , we have that  $\phi_n(2b' - b_1) < n < \phi_n((2b' - b_1)) + \phi_n(2a')$ , which contradicts the assumption that  $(A, B)$  is Krasner. Thus, there exists a subset  $B' \subset B$  such that  $\phi_n(A) + \phi_n(B') = [2\phi_n(a' + b') + 1, n - 1]$ . In other words,  $(A, B)$  is a DBNS factorization. If  $2\phi_n(a' + b') + 1 < n - 1$ , then, from Theorem 1.1,  $\phi_n(A)$  turns to be partially arithmetic in  $\mathbb{Z}$ . If  $2\phi_n(a' + b') + 1 = n - 1$ , then  $A \subset [0, a' + b' - 1]$  holds. Hence, combining Lemma 3.2 and Theorem 1.1, we have that  $A$  is partially arithmetic in  $\mathbb{Z}$  again. Thus, in any case,  $A$  is also partially arithmetic in  $\mathbb{Z}_n$ . Therefore,  $(A, B)$  is a DBNS near-factorization from Lemma 2.2. We can prove similarly to the above for the case of 1-overlapped factorizations.  $\square$

We also have the following complete descriptions for the DBNS parameters of the Krasner near factorizations; let  $(A, B)$  be a Krasner near factorization (resp. 1-overlapped factorization) of  $\mathbb{Z}_n$  such that  $0 \in A$ , and that  $\max \phi_n(A) \leq \max \phi_n(B) - \min \phi_n(B)$ . Then one of the following holds;

- there are  $2\rho + 1$  integers such that  $m_1 (\geq 1)$ , and that  $m_i \geq 2$  for  $i \in [2, 2\rho - 1]$ , and that  $m_i \geq 0$  for  $i = 2\rho, 2\rho + 1$  which satisfies

$$A = \psi_n \left( \sum_{j=1}^{\rho} [0, m_{2j-1} - 1] \prod_{k=1}^{2j-2} m_k \right)$$

$$B = \psi_n \left( \sum_{j=1}^{\rho} [0, m_{2j} - 1] \prod_{k=1}^{2j-1} m_k \right) \cup \psi_n \left( \sum_{j=1}^{\rho} [0, m_{2j} - 1] \prod_{k=1}^{2j-1} m_k + 1 \right),$$

- there is an integer  $m_1$  at least 2, and nonnegative integers  $m_2, m_3$  such that

$$A = \psi_n(\{0, 2m_1 - 1\})$$

$$B = \psi_n(\{[0, 2m_1 - 2] + ([0, m_2]2(2m_1 - 1) \cup ([m_2 + 1, m_3 - 1]2(2m_1 - 1) - 1)) \setminus \{(2m_1 + 1)(2m_1 - 1) - 1\}\}).$$

Although we also specified the parameters for Krasner 1-overlapped factorizations, its details are omitted here due to space limitations.

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# Correlation inequality for formal series

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**Abstract.** We extend the considerations of the paper [1] and prove two correlation inequalities (statement of lemma below and inequality (8)) for totally ordered set.

First we introduce class of correlation inequalities.

Assume that  $f_1, \dots, f_n$  are nonnegative nondecreasing functions  $2^X \rightarrow R$ . The expectation of a random variable  $f : 2^X \rightarrow R$  with respect to  $\mu$  we denote by  $\langle f \rangle_\mu$ . For a subset  $\delta \in [n]$  define

$$E_\delta = \left\langle \prod_{i \in \delta} f_i \right\rangle_\mu.$$

Let

$$\sigma = \{\sigma_1, \dots, \sigma_\ell\}$$

be a partition of  $[n]$  into disjoint subsets. Define

$$E_\sigma = \prod_{i=1}^{\ell} E_{\sigma_i}.$$

Let  $\lambda_1 = |\sigma_i|$ . We have  $\sum_{i=1}^{\ell} \lambda_i = n$ . Let  $\lambda(\sigma) = (\lambda_1, \dots, \lambda_\ell)$  and  $\lambda_1 \geq \dots \geq \lambda_\ell$ . For a partition  $\lambda$  of number  $n$  define

$$E_\lambda = \sum_{\sigma: \lambda(\sigma)=\lambda} E_\sigma.$$

We need the following

**Lemma 1.** *Consider the totally ordered set  $1, \dots, N$  with probability measure  $\mu$  on it and let's functions  $f_i$ ,  $i = 1, \dots, n$  are nonnegative and monotone nondecreasing. Functional*

$$E_n(f_1, \dots, f_n) = \sum_{\lambda \vdash n} c_\lambda E_\lambda \tag{1}$$

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where

$$c_\lambda = (-1)^{\ell+1} \prod_{i=1}^{\ell} (\lambda_i - 1)!$$

is nonnegative.

In [2] was conjectured that statement of lemma (along with (8)) is true when probability measure  $\mu$  on  $2^X$  satisfies FKG conditions

$$\mu(A \cap B)\mu(A \cup B) \geq \mu(A)\mu(B) \tag{2}$$

and functions  $f_i$  are nonnegative and monotone.

Note that under conditions from the lemma in particular case  $n = 2$  lemma gives Chebyshev inequality

$$\langle f_1 f_2 \rangle_\mu \geq \langle f_1 \rangle_\mu \langle f_2 \rangle_\mu .$$

Hence our proof can be considered as extension of Chebyshev inequality to multiple variables. For monotone functions  $f_i(j)$ ,  $i = 1, \dots, n$ ;  $j = 1, \dots, N$  we put

$$f_i(1) = a_{i,1}, \quad f_i(j) = f_i(j - 1) + a_{i,j}, \quad j = 2, \dots, N, \quad a_{i,j} \geq 0. \tag{3}$$

Then substituting in the formula

$$E_n(f_1, \dots, f_n) = \sum_{\lambda \vdash n} c_\lambda \sum_{\sigma: \lambda(\sigma)=\lambda} \prod_{i=1}^{\ell} \langle \prod_{j \in \sigma_i} f_j \rangle_\mu \tag{4}$$

coefficients  $c_\lambda$  one can show that to prove lemma it is sufficient to prove the inequality

$$\sum_{\{m_s\}} F_{m_1, \dots, m_N}(\mu) \prod_{j=0}^{N-1} \prod_{i=\sum_{s=1}^j m_s+1}^{\sum_{s=1}^{j+1} m_s} \left( \sum_{t=1}^{j+1} a_{i,t} \right) \geq 0, \tag{5}$$

where

$$F_{m_1, \dots, m_N}(\mu) = - \prod_{j=1}^N \prod_{i=1}^{m_j} (i - 1 - \mu(j)).$$

One can check that coefficient before the monomial

$$\prod_{j=0}^{N-1} \prod_{i=\sum_{s=1}^j m_s+1}^{\sum_{s=1}^{j+1} m_s} a_{i,j+1}$$



in the lhs of (5) is

$$\begin{aligned}
 & B(m_1, \dots, m_{N-1}) \\
 & \triangleq - \sum_{\{i_j\}} \prod_{j=1}^{N-1} \binom{\sum_{s=1}^j m_s - \sum_{s=1}^{j-1} i_s}{i_j} \prod_{i=1}^{i_j} (i - 1 - \mu(j)) \\
 & \quad \times \prod_{i=1}^{n - \sum_{s=1}^{N-1} i_s} (i - 1 - \mu(N)).
 \end{aligned} \tag{6}$$

Thus to prove (5) and complete the proof of lemma it is sufficient to prove the inequality

$$B(m_1, \dots, m_{N-1}) \geq 0. \tag{7}$$

We prove this inequality by induction on  $m_j$ . This proves lemma.

Next we consider the set of formal series  $P[[t]]$ , whose coefficients are monotone nondecreasing nonnegative functions on  $2^X$ . Then  $p(A) = p_1(A)t + p_2(A)t^2 + \dots \in P[[t]]$ . In [2] was formulated the following

**Conjecture 1.** *For FKG probability measure  $\mu$  the following inequality is true*

$$1 - \prod_{A \in 2^X} (1 - p(A))^{\mu(A)} \geq 0. \tag{8}$$

The inequality (8) is understood as non negativity of coefficients of formal series obtained by series expansion of the product on the left-hand side of this inequality.

We will prove, that inequality (8) follows from inequalities

$$E_n(f_1, \dots, f_n) \geq 0 \tag{9}$$

for all  $n$  and hence it is sufficient to prove last inequalities and then inequality (8) follows under the same conditions on  $\mu$ .

We make some transformations of the expression in the lhs of (8). We have

$$\begin{aligned}
 & 1 - \prod_{A \in 2^X} (1 - p(A))^{\mu(A)} = 1 - \exp \{ \langle \ln(1 - p) \rangle_\mu \} \\
 & = 1 - \exp \left\{ - \sum_{i=1}^{\infty} \frac{1}{i} \langle p^i \rangle_\mu \right\} = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j!} \left( \sum_{i=1}^{\infty} \frac{1}{i} \langle p^i \rangle_\mu \right)^j \\
 & = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j!} \sum_{\{q_s\}: \sum q_s = j} \binom{j}{q_1, \dots, q_j} \sum_{\{i_s\}} \frac{\prod_{s=1}^j \langle p^{i_s} \rangle_\mu^{q_s}}{(i_1)^{q_1} (i_2)^{q_2} \dots (i_j)^{q_j}}.
 \end{aligned}$$

Next remind that the number of partitions of  $n$  with given set  $\{q_i\}$  of occurrence of  $i$  is equal to

$$\frac{n!}{\prod_i (i!)^{q_i} q_i!}.$$

Continuing the last chain of identities and using last formula we obtain

$$\begin{aligned} & 1 - \prod_{A \in 2^X} (1 - p(A))^{\mu(A)} \\ &= \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\lambda \vdash n} \sum_{\sigma: \lambda(\sigma)=\lambda} (-1)^{\sum_i q_i + 1} \frac{n! \prod_i ((i-1)!)^{q_i}}{\prod_i (i!)^{q_i} q_i!} \prod_i \langle p^i \rangle_{\mu}^{q_i} \\ &= \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\lambda \vdash n} (-1)^{\ell(\lambda)+1} \prod_i (\lambda_i - 1)! \sum_{\sigma: \lambda(\sigma)=\lambda} E_{\sigma}(p, \dots, p) \quad (10) \\ &= \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\lambda \vdash n} c_{\lambda} E_{\lambda}(p, \dots, p) \\ &= \sum_{n=1}^{\infty} \frac{1}{n!} E_n(p, \dots, p). \end{aligned}$$

Hence now to prove the conjecture 3 we need to show that

$$E_n(p, \dots, p) \geq 0. \tag{11}$$

But the coefficients of the formal series  $E_n(p, \dots, p)$  are the sums of  $E_n(p_{i_1}, \dots, p_{i_n})$  for multisets  $\{i_1, \dots, i_n\}$ . This completes the proof that inequality (8) follows from inequalities (9) under the same conditions on  $\mu$ .

Thus because we prove lemma, we prove inequality (8) for totally ordered lattice and this is our main result.

**Remark.** To extend lemma to the conditions (2) one can try to find proper expansion for the monotone functions  $f_i$  which extend expansion (3) to the case of poset  $2^X$ .

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# Covariants of spherical $\Theta$ -orbits for types $E_6, E_7, E_8$

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**Abstract.** We calculate the rings of covariants for spherical orbits in the class of representations of reductive algebraic groups associated to various gradings on simple Lie algebras of type  $E_6, E_7$  and  $E_8$ .

## 1 Introduction

Let  $X_n$  be a Dynkin diagram with a distinguished node  $x \in X_n$  and let  $\mathfrak{g}$  be a complex semisimple Lie algebra corresponding to  $X_n$ . Then  $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\beta \in \Delta} \mathfrak{g}_\beta$  where  $\mathfrak{h}$  is the Cartan subalgebra and  $\Delta$  is a root system of type  $X_n$ . Choose a basis for  $\Delta$  and let  $\alpha$  be a simple root corresponding to  $x$ . For a root  $\beta \in \Delta$ , let the rank of  $\beta$  be the multiplicity of  $\alpha$  in decomposition of  $\beta$  in the basis of simple roots. For  $i \neq 0$ , let  $\mathfrak{g}_i$  be the sum of  $\mathfrak{g}_\beta$  with  $\beta$  of rank  $i$  and let  $\mathfrak{g}_0$  be the sum of  $\mathfrak{h}$  and the sum of  $\mathfrak{g}_\beta$  with  $\beta$  of rank 0. In this way,  $x$  defines grading of  $\mathfrak{g}$  with  $\mathfrak{g}_0$  being a Lie subalgebra of  $\mathfrak{g}$ . Let  $G_0$  be the adjoint group corresponding to  $\mathfrak{g}_0$ . The group  $G = G_0 \times \mathbb{C}^*$  acts on  $\mathfrak{g}_1$  with finitely many orbits. Moreover, V. Kac proved in [1] that almost all actions of reductive algebraic groups with finitely many orbits can be obtained in such a way. The orbits of  $G$  in  $\mathfrak{g}_1$  where parameterized by E.B. Vinberg ([5, 6]) using combinatorics of root subsystems of  $\Delta$ . We call them  $\Theta$ -orbits.

Systematic study of algebraic–geometric properties of orbits connected with exceptional roots systems was undertaken by authors in a series of papers [2–4]. In the following, we use previous results to describe the action of  $G$  on the ring of regular functions on spherical  $\Theta$ -orbits.

## 2 Methodology

Let  $\mathcal{O}$  be an orbit of the action of  $G$  on  $\mathfrak{g}_1$ . Its closure  $\overline{\mathcal{O}}$  is an affine algebraic variety and the group  $G$  acts on the ring  $\mathbb{C}[\overline{\mathcal{O}}]$  of regular functions

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in a standard way. Let  $B \subset G$  be a Borel subgroup.  $B$ -invariant regular functions on  $\overline{\mathcal{O}}$  are called covariants of  $\mathcal{O}$ . The orbit is spherical if and only if it contains dense  $B$ -orbit, or equivalently if the action of  $G$  on  $\mathbb{C}[\overline{\mathcal{O}}]$  is multiplicity free.

Let  $\mathcal{M} = \mathcal{M}_{\mathcal{O}}$  be the set of dominant weights  $\lambda$  of  $G$  such that the irreducible representation  $V(\lambda, G)$  of highest weight  $\lambda$  occurs in  $\mathbb{C}[\overline{\mathcal{O}}]$ . The set  $\mathcal{M}$  is in fact a monoid with respect to usual addition of weights.

Assume that the orbit  $\mathcal{O}$  is spherical. For  $\lambda \in \mathcal{M}$ , we write  $\lambda[d]$  to indicate the degree  $d$  of the homogeneous component of  $\mathbb{C}[\overline{\mathcal{O}}]$  that contains the only copy of  $V(\lambda, G)$ . In that way we obtain a grading  $\mathcal{M} = \bigoplus_{d=0}^{\infty} \mathcal{M}_d$  and

$$\mathbb{C}_d[\overline{\mathcal{O}}] = \bigoplus_{\lambda \in \mathcal{M}_d} V(\lambda, G), \quad d = 0, 1, 2, \dots \tag{2.1}$$

The ring of covariants of  $\mathcal{O}$  is isomorphic to the monoid ring  $\mathbb{C}[\mathcal{M}]$ . We use standard notation  $e^\lambda$  for a weight  $\lambda \in \mathcal{M}$  regarded as an element of  $\mathbb{C}[\mathcal{M}]$ .

Let  $X_n$  be a Dynkin diagram of type  $E_6, E_7$  or  $E_8$  and let  $\mathcal{O}$  be a  $\Theta$ -orbit of type  $X_n$ . The key result of [2–4] is an explicit construction of a desingularisation  $\mathcal{S}$  of  $\overline{\mathcal{O}}$  which is a vector bundle over a homogeneous space  $X$  of  $G$ . Using that desingularisation and a geometric technique described in [7] algebraic properties of the variety  $\overline{\mathcal{O}}$  were derived. In particular Hilbert–Poincaré series  $\mathcal{H}_{\mathcal{O}}(t) = \sum_{n=0}^{\infty} p(n)t^n = \sum_{n=0}^{\infty} \dim \mathbb{C}_n[\overline{\mathcal{O}}]t^n$  was calculated. It is of the form

$$\mathcal{H}_{\mathcal{O}}(t) = \frac{N(t)}{(1-t)^c} = N(t) \cdot \sum_{n=0}^{\infty} \binom{n+c-1}{c-1} t^n,$$

where  $N(t)$  is a polynomial given in [2–4] and  $c = \dim \mathcal{O}$ . It follows that  $p(n)$  is a polynomial function of  $n$  of degree  $\leq c - 1$ .

On the other hand  $\mathbb{C}_n[\overline{\mathcal{O}}] = H^0(X, \text{Sym}_n(\mathcal{S}^*))$  and it can be computed by Bott algorithm, at least for small  $n$ . Assume that we know  $s$  covariants of  $\mathcal{O}$  with linearly independent weights  $\lambda_i[d_i], i = 1, 2, \dots, s$ , and we want to show that in fact the ring of covariants is generated by them. For  $\mathbf{k} = (k_1, k_2, \dots, k_s) \in \mathbb{N}^s$ , let

$$W(k_1, k_2, \dots, k_s) = \dim V(k_1\lambda_1 + k_2\lambda_2 + \dots + k_s\lambda_s, G) \tag{2.2}$$

and let

$$q(n) = \sum_{\substack{(k_1, k_2, \dots, k_s) \\ k_1d_1 + k_2d_2 + \dots + k_sd_s = n}} W(k_1, k_2, \dots, k_s), \quad n = 0, 1, 2, \dots \tag{2.3}$$

In order to prove the statement, it is enough to show that

$$N(t) = (1 - t)^c \sum_{n=0}^{\infty} q(n)t^n. \tag{2.4}$$

It follows from Weyl character formula that  $W(\mathbf{k})$  is polynomial in  $\mathbf{k}$  of degree less or equal to

$$\delta(\lambda_1, \lambda_2, \dots, \lambda_s) = | \Delta^+ \setminus \{ \alpha \in \Delta^+ : (\lambda_1, \alpha) = (\lambda_2, \alpha) = \dots = (\lambda_s, \alpha) = 0 \} | .$$

Let  $\mathcal{P} \subset \mathbb{R}^s$  be a polytope with vertices

$$A_i = \left( 0, \dots, 0, \frac{1}{d_i}, 0, \dots, 0 \right), \quad i = 1, 2, \dots, s.$$

Then (2.3) is equivalent to

$$q(n) = \sum_{\mathbf{k} \in n\mathcal{P} \cap \mathbb{N}^s} W(k_1, k_2, \dots, k_s). \tag{2.5}$$

Recall that a function  $g : \mathbb{N} \rightarrow \mathbb{R}$  is quasi-polynomial of period  $d$  if there exist  $d$  polynomials  $G_0(t), G_1(t), \dots, G_{d-1}(t) \in \mathbb{R}[t]$  such that  $g(n) = G_{n \bmod d}(n)$  for every natural  $n$ . It is known that if  $\mathcal{P} \subset \mathbb{R}^s$  is a rational polytope and  $f = f(x_1, x_2, \dots, x_s)$  is a polynomial function on  $\mathbb{R}^s$  then  $\bar{f}(n) = \sum_{x \in n\mathcal{P} \cap \mathbb{N}^s} f(x_1, x_2, \dots, x_s)$  is quasi-polynomial in  $n$  of degree  $\leq \deg(f) + \dim(\mathcal{P})$ . Moreover, if  $d$  is common denominator of all coordinates of vertices of  $\mathcal{P}$  then  $d$  is a period of  $\bar{f}$ .

Let  $d = \text{lcm}(d_1, d_2, \dots, d_s)$ . Then the equation (2.4) is equivalent to

$$N(t) \equiv (1 - t)^c \sum_{n=0}^{\infty} q(n)t^n \pmod{t^L}, \tag{2.6}$$

where  $L = d \cdot (\delta(\lambda_1, \lambda_2, \dots, \lambda_s) + s)$ . The condition (2.6) can be checked by computer calculation (in Maple).

The approach can be modified to cover cases of generators with relation since they are of very special form. It turns out that we can divide a set of generators of  $\mathcal{M}$  into two groups

$$\lambda_1[d_1], \lambda_2[d_2], \dots, \lambda_s[d_s]$$

and

$$\mu_1[e_1], \mu_2[e_2], \dots, \mu_t[e_t]$$

in such a way that

1.  $\lambda_i$ 's are linearly independent; let  $\mathcal{M}^0$  be a monoid spanned by them;
2. the sets  $\mathcal{M}^0$  and  $\mathcal{M}^i = \mu_i + \mathcal{M}^0, i = 1, 2, \dots, t$ , are disjoint and

$$\mathcal{M} = \mathcal{M}^0 \cup \mathcal{M}^1 \cup \dots \cup \mathcal{M}^t$$

(i.e. minimal relations express sums of two not necessarily different  $\mu_i$ 's as a positive combination of  $\lambda_i$ 's possibly plus another  $\mu_j$ ).

For  $i = 1, 2, \dots, t$ , let

$$W_i(k_1, k_2, \dots, k_s) = \dim V(\mu_i + k_1\lambda_1 + k_2\lambda_2 + \dots + k_s\lambda_s, G),$$

and let

$$q(n) = \sum_{\mathbf{k} \in n\mathcal{P} \cap \mathbb{N}^s} W(k_1, k_2, \dots, k_s) + \sum_{i=1}^t \sum_{\mathbf{k} \in (n-e_i)\mathcal{P} \cap \mathbb{N}^s} W_i(k_1, k_2, \dots, k_s).$$

Denote  $e = \max(e_1, e_2, \dots, e_t)$ . For  $n \geq e$ ,  $q(n)$  is quasi-polynomial and in order to prove that  $\lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_t$  generate the ring of covariants, it is enough to check (2.6) with  $L$  substituted with  $L' = e + d \cdot (\delta(\lambda_1, \lambda_2, \dots, \lambda_s) + s)$ .

### 3 Results

There are 15 spherical  $\Theta$ -orbits of type  $E_6$ , 33 of type  $E_7$  and 42 of type  $E_8$ . For each of them we have found explicit generators and relations of the monoid  $\mathcal{M}$ .

As a corollary, we obtained the following result.

**Theorem 3.1.** *Let  $\mathcal{O}$  be a spherical  $\Theta$ -orbit of type  $E_6, E_7$  or  $E_8$ . Then the ring of covariants of  $\mathcal{O}$  is either a polynomial ring or a quotient of a polynomial ring by the ideal generated by  $2 \times 2$  minors of a symmetric matrix of dimension  $2 \times 2$  or  $3 \times 3$ .*

**Example 3.2.** Let  $\alpha = \alpha_2$  in a root system of type  $E_7$  (in Bourbaki enumeration of simple roots). Then  $G = GL(7)$  and  $\mathfrak{g}_1 = \bigwedge^3 \mathbb{C}^7$ . There are 5 spherical orbits of that type denoted by  $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3, \mathcal{O}_4$  and  $\mathcal{O}_6$  in [3] of dimension 13, 20, 21, 25 and 28, respectively. Let

$$\begin{aligned} \lambda_1 &= (1, 1, 1, 0, 0, 0, 0)[1], & \lambda_2 &= (2, 1, 1, 1, 1, 0, 0)[2], \\ \lambda_3 &= (3, 1, 1, 1, 1, 1, 1)[3], & \lambda_4 &= (2, 2, 2, 1, 1, 1, 0)[3], \\ \lambda_5 &= (3, 3, 2, 2, 1, 1, 0)[4], & \lambda_6 &= (3, 3, 3, 3, 1, 1, 1)[5], \\ \lambda_7 &= (5, 5, 3, 3, 3, 1, 1)[7]; \\ \mu_1 &= (3, 2, 2, 2, 1, 1, 1)[4], & \mu_2 &= (4, 3, 2, 2, 2, 1, 1)[5], \\ \mu_3 &= (4, 4, 3, 3, 2, 1, 1)[6]; \end{aligned}$$

The monoids  $\mathcal{M}$  for the first four orbits are freely generated and their generators are given in the following table.

$\mathcal{O}_1$	$\lambda_1;$
$\mathcal{O}_2$	$\lambda_1, \lambda_2;$
$\mathcal{O}_3$	$\lambda_1, \lambda_2, \lambda_3;$
$\mathcal{O}_4$	$\lambda_1, \lambda_2, \lambda_4, \lambda_5;$

The monoid  $\mathcal{M}$  for  $\mathcal{O}_6$  is generated by  $\lambda_1, \dots, \lambda_7$  and  $\mu_1, \dots, \mu_3$  with relations

$$2\mu_1 = \lambda_3 + \lambda_6 \text{ [8]}, \quad \mu_1 + \mu_2 = \lambda_3 + \mu_3 \text{ [9]}, \quad 2\mu_2 = \lambda_3 + \lambda_7 \text{ [10]},$$

$$\mu_1 + \mu_3 = \mu_2 + \lambda_6 \text{ [10]}, \quad \mu_2 + \mu_3 = \mu_1 + \lambda_7 \text{ [11]}, \quad 2\mu_3 = \lambda_6 + \lambda_7 \text{ [12]}.$$

The ring of covariants of  $\mathcal{O}_6$  is isomorphic to a quotient of the polynomial ring  $\mathbb{C}[e^{\lambda_1}, \dots, e^{\lambda_7}, e^{\mu_1}, \dots, e^{\mu_3}]$  by the ideal generated by  $2 \times 2$  minors of the matrix

$$\begin{bmatrix} e^{\lambda_3} & e^{\mu_1} & e^{\mu_2} \\ e^{\mu_1} & e^{\lambda_6} & e^{\mu_3} \\ e^{\mu_2} & e^{\mu_3} & e^{\lambda_7} \end{bmatrix}.$$

The case of  $(E_8, \alpha_2)$  is analogous:  $G = GL(8)$  and  $\mathfrak{g}_1 = \bigwedge^3 \mathbb{C}^8$ . There are also 5 spherical orbits and their rings of covariants are isomorphic to the rings of corresponding orbits of type  $(E_7, \alpha_2)$  (the orbits are degenerate in a sense of [2, Section 4]).

**Example 3.3.** For  $(E_7, \alpha_5)$  we have  $G = SL(3) \times SL(5) \times \mathbb{C}^*$ ,  $\mathfrak{g}_1 = \mathbb{C}^3 \otimes \bigwedge^2 \mathbb{C}^5$ . The monoid  $\mathcal{M}$  for the orbit  $\mathcal{O}_8$  of dimension 19 is generated by

$$\begin{aligned} \lambda_1 &= (1, 0, 0 \mid 1, 1, 0, 0, 0)[1], & \lambda_2 &= (2, 0, 0 \mid 1, 1, 1, 1, 0)[2], \\ \lambda_3 &= (1, 1, 0 \mid 2, 1, 1, 0, 0)[2], & \lambda_4 &= (1, 1, 1 \mid 2, 2, 2, 0, 0)[3], \\ \lambda_5 &= (1, 1, 1 \mid 3, 1, 1, 1, 0)[3], & \lambda_6 &= (2, 2, 0 \mid 3, 3, 1, 1, 0)[4]; \\ \mu_1 &= (2, 1, 0 \mid 2, 2, 1, 1, 0)[3], & \mu_2 &= (2, 1, 1 \mid 3, 2, 2, 1, 0)[4], \\ \mu_3 &= (2, 2, 1 \mid 4, 3, 2, 1, 0)[5] \end{aligned}$$

with relations

$$\begin{aligned} 2\mu_1 &= \lambda_2 + \lambda_6 \text{ [6]}, & \mu_1 + \mu_2 &= \mu_3 + \lambda_2 \text{ [7]}, \\ 2\mu_2 &= \lambda_2 + \lambda_4 + \lambda_5 \text{ [8]}, & \mu_1 + \mu_3 &= \mu_2 + \lambda_6 \text{ [8]}, \\ \mu_2 + \mu_3 &= \mu_1 + \lambda_4 + \lambda_5 \text{ [9]}, & 2\mu_3 &= \lambda_4 + \lambda_5 + \lambda_6 \text{ [10]}. \end{aligned}$$

The ring of covariants is a quotient of the ring of polynomials in  $e^{\lambda_1}, \dots, e^{\lambda_6}$  and  $e^{\mu_1}, \dots, e^{\mu_3}$  by the ideal generated by  $2 \times 2$  minors of the matrix

$$\begin{bmatrix} e^{\lambda_2} & e^{\mu_2} & e^{\mu_1} \\ e^{\mu_2} & e^{\lambda_4} e^{\lambda_5} & e^{\mu_3} \\ e^{\mu_1} & e^{\mu_3} & e^{\lambda_7} \end{bmatrix}.$$

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# Partition regularity of nonlinear polynomials: a nonstandard approach

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**Abstract.** While the linear partition regular polynomials have been characterized by Richard Rado in the 1930's, very few results are known in the nonlinear case. Our aim is to prove that, on  $\mathbb{N}$ , there are at least two interesting classes of partition regular nonlinear polynomials.

Our approach is based on the study of ultrafilters on  $\mathbb{N}$  from the point of view of Nonstandard Analysis. A particularity of this technique is that the proofs of the main results can be carried out by almost elementary algebraic considerations.

Our aim is to expose a few results regarding the partition regularity of nonlinear polynomials, which have been presented in [1]. We recall the following definitions:

**Definition.** A polynomial  $P(x_1, \dots, x_n)$  is

- **partition regular** (on  $\mathbb{N}$ ) if for every natural number  $r$ , for every partition  $\mathbb{N} = \bigcup_{i=1}^r A_i$ , there is an index  $j \leq r$  and nonzero natural numbers  $a_1, \dots, a_n \in A_j$  such that  $P(a_1, \dots, a_n) = 0$ ;
- **injectively partition regular** (on  $\mathbb{N}$ ) if for every natural number  $r$ , for every partition  $\mathbb{N} = \bigcup_{i=1}^r A_i$ , there is an index  $j \leq r$  and mutually distinct nonzero natural numbers  $a_1, \dots, a_n \in A_j$  such that  $P(a_1, \dots, a_n) = 0$ .

The problem of determining which polynomials are partition regular has been studied since Issai Schur's work [2], and the linear case was settled by Richard Rado in [3]:

**Theorem (Rado).** Let  $P(x_1, \dots, x_n) = \sum_{i=1}^n a_i x_i$  be a linear polynomial with nonzero coefficients. The following conditions are equivalent:

1.  $P(x_1, \dots, x_n)$  is partition regular on  $\mathbb{N}$ ;
2. there is a nonempty subset  $J$  of  $\{1, \dots, n\}$  such that  $\sum_{j \in J} a_j = 0$ .

In his work, Rado also characterized the partition regular finite systems of linear equations and, since then, one of the main streams of the research

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in this field has been the study of infinite systems of linear equations (for a general background on many notions related to this subject see, e.g., [4]). One other possible direction is the study of the partition regularity for nonlinear polynomials, which is the problem that we want to face.

As far as we know, perhaps the most interesting known result in the context of partition regularity of nonlinear polynomials is the following (see [5]):

**Theorem (Hindman).** *For every natural numbers  $n, m \geq 1$ , with  $n + m \geq 3$ , the nonlinear polynomial*

$$\sum_{i=1}^n x_i - \prod_{j=1}^m y_j$$

*is injectively partition regular.*

Our main results generalize the previous Theorem.

Our first result states that, if we start with a partition regular linear polynomial  $P(x_1, \dots, x_n)$ , a finite set of variables  $Y = \{y_1, \dots, y_m\}$  and we multiply each variable  $x_i$  for some of the variables in  $Y$  then we obtain a partition regular polynomial. To precisely state this result we need the following definition:

**Definition.** *Let  $m$  be a positive natural number, and  $\{y_1, \dots, y_m\}$  a set of mutually distinct variables. For every finite set  $F \subseteq \{1, \dots, m\}$ , we denote by  $Q_F(y_1, \dots, y_m)$  the monomial*

$$Q_F(y_1, \dots, y_m) = \begin{cases} \prod_{j \in F} y_j, & \text{if } F \neq \emptyset; \\ 1, & \text{if } F = \emptyset. \end{cases}$$

**Theorem (1).** *Let  $n \geq 2$  be a natural number,  $R(x_1, \dots, x_n) = \sum_{i=1}^n a_i x_i$  a partition regular polynomial, and  $m$  a positive natural number. Then, for every  $F_1, \dots, F_n \subseteq \{1, \dots, m\}$  (with the request that, when  $n = 2$ ,  $F_1 \cup F_2 \neq \emptyset$ ), the polynomial*

$$P(x_1, \dots, x_n, y_1, \dots, y_m) = \sum_{i=1}^n a_i x_i Q_{F_i}(y_1, \dots, y_m)$$

*is injectively partition regular.*

E.g., as a consequence of Theorem (1) we have that the polynomial

$$P(x_1, x_2, x_3, x_4, y_1, y_2, y_3) = 2x_1 + 3x_2y_1y_2 - 5x_3y_1 + x_4y_2y_3$$

is injectively partition regular. We observe that the exposed Theorem of Hindman is a particular case of our result. A particularity of the polynomials considered in Theorem (1) is that the degree of each of their variables is one; to state a result that ensures the partition regularity for many polynomials having variables with degree greater than one we need to introduce a few definitions:

**Definition 1) Let**

$$P(x_1, \dots, x_n) : \sum_{i=1}^k a_i M_i(x_1, \dots, x_n)$$

be a polynomial, and let  $M_1(x_1, \dots, x_n), \dots, M_k(x_1, \dots, x_n)$  be the distinct monic monomials of  $P(x_1, \dots, x_n)$ . We say that  $\{v_1, \dots, v_k\} \subseteq V(P)$  is a **set of exclusive variables** for  $P(x_1, \dots, x_n)$  if, for every  $i, j \leq k$ ,  $d_{M_i}(v_j) \geq 1 \Leftrightarrow i = j$ , where  $d_{M_i}(v_j)$  denotes the degree of the variable  $v_j$  in the monomial  $M_i$ .

In this case we say that the variable  $v_i$  is **exclusive** for the monomial  $M_i(x_1, \dots, x_n)$  in  $P(x_1, \dots, x_n)$ .

2) We say that  $P(x_1, \dots, x_n)$  satisfies **Rado's Condition** if there is a nonempty subset  $F$  of the set of coefficients of  $P(x_1, \dots, x_n)$  such that  $\sum_{c \in F} c = 0$ .

**Definition.** Let  $P(x_1, \dots, x_n) = \sum_{i=1}^k a_i M_i(x_1, \dots, x_n)$  be a polynomial, and let  $M_1(x_1, \dots, x_n), \dots, M_k(x_1, \dots, x_n)$  be the monic monomials of  $P(x_1, \dots, x_n)$ . Then

- $NL(P) = \{x \in V(P) \mid d(x) \geq 2\}$  is the set of nonlinear variables of  $P(x_1, \dots, x_n)$ ;
- for every  $i \leq k$ ,  $\mathbf{l}_i = \max\{d(x) - d_i(x) \mid x \in NL(P)\}$ .

**Theorem (2). Let**

$$P(x_1, \dots, x_n) = \sum_{i=1}^k a_i M_i(x_1, \dots, x_n)$$

be a polynomial, and let  $M_1(x_1, \dots, x_n), \dots, M_k(x_1, \dots, x_n)$  be the monic monomials of  $P(x_1, \dots, x_n)$ . Suppose that  $k \geq 3$ , that  $P(x_1, \dots, x_n)$  satisfies Rado's Condition and that, for every index  $i \leq k$ , in the monomial  $M_i(x_1, \dots, x_n)$  there are at least  $m_i = \max\{1, l_i\}$  exclusive variables with degree equal to 1.

Then  $P(x_1, \dots, x_n)$  is injectively partition regular.

*E.g.*, in consequence of Theorem (2) we get that the polynomial

$$P(x, y, z, t_1, t_2, t_3, t_4, t_5, t_6) = t_1 t_2 x^2 + t_3 t_4 y^2 - t_5 t_6 z^2$$

is injectively partition regular.

The technique we use to prove our main results is based on an approach to combinatorics by means of nonstandard analysis (with something in common with, *e.g.*, [6–8]): the idea behind this approach is that, as it is well-known, problems talking about partition regularity of properties on  $\mathbb{N}$  can be reformulated in terms of ultrafilters and it can be showed that some properties of ultrafilters can be translated, and studied, in terms of sets of hyperintegers. This can be obtained by associating, in particular hyperextensions  ${}^*\mathbb{N}$  of  $\mathbb{N}$ , to every ultrafilter  $\mathcal{U}$  its monad  $\mu(\mathcal{U})$ :

$$\mu(\mathcal{U}) = \{\alpha \in {}^*\mathbb{N} \mid \alpha \in {}^*A \text{ for every } A \in \mathcal{U}\},$$

and then proving that some of the properties of  $\mathcal{U}$  can be deduced by properties of  $\mu(\mathcal{U})$ .

From the point of view of the partition regularity of polyomials, this approach works as follows: first of all, we recall that, given a polynomial  $P(x_1, \dots, x_n)$ , we have that  $P(x_1, \dots, x_n)$  is partition regular if and only if there exists an ultrafilter  $\mathcal{U}$  such that for every set  $A$  in  $\mathcal{U}$  there are elements  $a_1, \dots, a_n \in A$  such that  $P(a_1, \dots, a_n) = 0$ . Then we reformulate this property in terms of monads of ultrafilters, giving one of the basic results that we use to prove Theorems (1) and (2) (as usual, we denote by  $\beta\mathbb{N}$  the space of ultrafilters on  $\mathbb{N}$ ):

**Theorem (3).** *Let  $P(x_1, \dots, x_n)$  be a polynomial, and  $\mathcal{U} \in \beta\mathbb{N}$  an ultrafilter. Then the following two conditions are equivalent:*

1. *for every set  $A$  in  $\mathcal{U}$  there are elements  $a_1, \dots, a_n \in A$  such that  $P(a_1, \dots, a_n) = 0$ ;*
2. *there are elements  $\alpha_1, \dots, \alpha_n$  in  $\mu(\mathcal{U})$  such that  $P(\alpha_1, \dots, \alpha_n) = 0$ .*

The interesting fact is that, whenever it is given an ultrafilter  $\mathcal{U}$  and elements  $\alpha_1, \dots, \alpha_n$  in  $\mu(\mathcal{U})$  for which there is a polynomial  $P(x_1, \dots, x_n)$  such that  $P(\alpha_1, \dots, \alpha_n) = 0$ , we automatically know that the polynomial is partition regular.

Usually, the ultrafilters that we use to prove our results are (or are related to) idempotent ultrafilters which, we recall, are defined as follows:

**Definition.** *An ultrafilter  $\mathcal{U} \in \beta\mathbb{N}$  is called idempotent if  $\mathcal{U} \oplus \mathcal{U} = \mathcal{U}$ , where  $\oplus$  denotes the usual sum of ultrafilters, defined as follows:*

$$\forall \mathcal{U}, \mathcal{V} \in \beta\mathbb{N},$$

$$\forall A \in \wp(\mathbb{N}), A \in \mathcal{U} \oplus \mathcal{V} \Leftrightarrow \{n \in \mathbb{N} \mid \{m \in \mathbb{N} \mid n + m \in A\} \in \mathcal{V}\} \in \mathcal{U};$$

similarly, an ultrafilter  $\mathcal{U}$  is called a multiplicative idempotent if  $\mathcal{U} \odot \mathcal{U} = \mathcal{U}$ , where  $\odot$  denotes the usual product of ultrafilters, defined as follows:

$$\forall \mathcal{U}, \mathcal{V} \in \beta\mathbb{N},$$

$$\forall A \in \wp(\mathbb{N}), A \in \mathcal{U} \oplus \mathcal{V} \Leftrightarrow \{n \in \mathbb{N} \mid \{m \in \mathbb{N} \mid n \cdot m \in A\} \in \mathcal{V}\} \in \mathcal{U}.$$

The last general result about ultrafilters that we need to apply the technique to the problem of the partition regularity of nonlinear polynomials regards the existence of particular ultrafilters on  $\beta\mathbb{N}$ :

**Theorem (4).** *There is a multiplicative idempotent ultrafilter  $\mathcal{U}$  such that, for every linear partition regular polynomial  $P(x_1, \dots, x_n)$ , for every set  $A$  in  $\mathcal{U}$  there are elements  $a_1, \dots, a_n \in A$  such that  $P(a_1, \dots, a_n) = 0$ .*

Theorems (1) and (2) can be proved by easy algebraical considerations on the elements in the monads of the ultrafilter given by Theorem (4) in particular hyperextensions of  $\mathbb{N}$ , called  $\omega$ -hyperextensions (or iterated-hyperextensions). These hyperextensions are constructed by considering an universe  $\mathbb{U}$  containing  $\mathbb{N}$  and a star map  $*$  :  $\mathbb{U} \rightarrow \mathbb{U}$  satisfying the transfer property. Since  $*$  :  $\mathbb{U} \rightarrow \mathbb{U}$ , this map can be iterated: the  $\omega$ -hyperextension of  $\mathbb{N}$  (denoted by  $\bullet\mathbb{N}$ ) is

$$\bullet\mathbb{N} = \bigcup_{n \in \mathbb{N}} S_n(\mathbb{N}),$$

where  $S_0(\mathbb{N}) = \mathbb{N}$  and  $S_{n+1}(\mathbb{N}) = *(S_n(\mathbb{N}))$ . As a consequence of the elementary chain condition, it is easy to prove that  $\bullet\mathbb{N}$  is elementarily equivalent to  $\mathbb{N}$ .

The particularity of these hyperextensions is that the possibility of iterating the map  $*$  translates in a few interesting algebraical properties of the monads, in particular of the monads of idempotent ultrafilters. In fact we have the following useful facts regarding the monads in  $\bullet\mathbb{N}$ :

**Theorem (5).** *If  $\mathcal{U} \in \beta\mathbb{N}$  is an ultrafilter and  $\mu(\mathcal{U})$  is the monad of  $\mathcal{U}$  in  $\bullet\mathbb{N}$ , then we have the following three properties:*

1. *if  $\alpha \in \mu(\mathcal{U})$  then  $*\alpha \in \mu(\mathcal{U})$ ;*
2.  *$\mathcal{U}$  is idempotent if and only if, for every  $\alpha, \beta$  in  $\mu(\mathcal{U}) \cap \bullet\mathbb{N}$ ,  $\alpha + *\beta \in \mu(\mathcal{U})$ ;*
3.  *$\mathcal{U}$  is a multiplicative idempotent if and only if, for every  $\alpha, \beta$  in  $\mu(\mathcal{U}) \cap \bullet\mathbb{N}$ ,  $\alpha \cdot *\beta \in \mu(\mathcal{U})$ .*

The nonstandard technique to prove results in combinatorics that we just described has been applied in [9], [10] to obtain a few results in combinatorics.

As an example of application of this nonstandard approach, we conclude by proving that the nonlinear polynomial

$$P(x, y, z, w) = x + y - zw$$

is injectively partition regular: by Theorems (3) and (4) it follows that there is a multiplicative idempotent ultrafilter  $\mathcal{U}$  and elements  $\alpha, \beta, \gamma \in \mu(\mathcal{U})$  with  $\alpha + \beta - \gamma = 0$ . Then

$$P(\alpha \cdot^* \alpha, \beta \cdot^* \alpha, \gamma \cdot^* \gamma) = \alpha \cdot^* \alpha + \beta \cdot^* \alpha - \gamma \cdot^* \alpha = 0$$

and, by Theorem (5),  $\alpha \cdot^* \alpha, \beta \cdot^* \alpha, \gamma$  and  $\cdot^* \alpha$  are in  $\mu(\mathcal{U})$ . So we can conclude by applying Theorem (3).

The proofs of Theorems (1) and (2) can be carried out by similar considerations.

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## Randomness



# Random subgraphs make identification affordable

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**Abstract.** An identifying code of a graph is a dominating set which uniquely determines all the vertices by their neighborhood within the code. Whereas graphs with large minimum degree have small domination number, this is not true for the identifying code number.

We show that every graph  $G$  with  $n$  vertices, maximum degree  $\Delta = \omega(1)$  and minimum degree  $\delta \geq c \log \Delta$ , for some constant  $c > 0$ , contains a large spanning subgraph which admits an identifying code of size  $O(n \log \Delta / \delta)$ . The result is best possible both in terms of code size and in number of edges deleted. The proof is based on the study of random subgraphs of  $G$  using standard concentration tools and the local lemma.

## 1 Introduction

Consider any graph parameter that is not monotone with respect to graph inclusion. Given a graph  $G$ , a natural problem in this context is to study the minimum value of this parameter over all spanning subgraphs of  $G$ . In particular, how many edge deletions are sufficient in order to obtain from  $G$  a graph with optimal value of the parameter? Herein, we study this question with respect to the identifying code number of a graph, a well-studied non-monotone parameter.

An *identifying code* of a graph is a subset of vertices which is a dominating set  $C$  such that each vertex is uniquely determined by its neighborhood within  $C$ . More formally, each vertex  $x$  of  $V(G) \setminus C$  has at least one neighbor in  $C$  ( $x$  is *dominated*) and for each pair  $u, v$  of vertices of  $G$ ,  $N[u] \cap C \neq N[v] \cap C$ ;  $u, v$  are *separated*. The minimum size of an identifying code in a graph  $G$ , denoted by  $\gamma^{\text{ID}}(G)$ , is the *identifying code number* of  $G$ . Identifying codes were introduced in [5], motivated by

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several applications. Generally speaking, if the graph models a facility or a computer network, identifying codes can be used to detect dangers in facilities [9] or failures in networks [5]. In this context, deleting edges to the underlying graph is particularly meaningful since it may represent the sealing of a door in a facility network, and the removal of a wire in a computer network.

Note that some graphs may not admit an identifying code, in particular when they have pairs of twin vertices (*i.e.* which have the same closed neighborhood). However any twin-free graph is easily seen to admit an identifying code (*e.g.* its vertex set). The identifying code number of a graph  $G$  on  $n$  vertices satisfies  $\log_2(n + 1) \leq \gamma^{\text{ID}}(G) \leq n$ .

There are very dense graphs that have a huge identifying code number; sparse graphs, such as trees and planar graphs, also have a linear identifying code number [10]. On the other hand, one can also find sparse and dense graphs with identifying code number  $O(\log n)$  [4, 8].

It shall be observed from the previous facts that the identifying code number is not a monotone function with respect to the addition (or deletion) of edges. This motivates the following question:

*Given any sufficiently dense graph, can we delete a small number of edges to get a spanning subgraph with a small identifying code? If the answer is positive, how many edges are sufficient (and necessary)?*

In other words, we would like to study the minimum size of an identifying code among all spanning subgraphs of a given graph, and to determine the largest spanning subgraph with an asymptotically optimal identifying code.

Despite being dense, the random graph  $G(n, p)$  (for  $0 < p < 1$ ) has a logarithmic size identifying code, as with high probability,

$$\gamma^{\text{ID}}(G(n, p)) = (1 + o(1)) \frac{2 \log n}{\log(1/q)},$$

where  $q = p^2 + (1 - p)^2$  [4]. This suggests that in a dense graph, the lack of structure implies the existence of a small identifying code number. Hence, introducing some randomness to the structure of a dense graph having large identifying code number might decrease this number. Indeed, this intuition is used in this work.

By studying the behavior of a *random subgraph of a graph with large minimum degree* (see for example [1, 6]), we prove the following:

**Theorem 1.1.** *For any graph  $G$  on  $n$  vertices ( $n$  large enough) with maximum degree  $\Delta = \omega(1)$  and minimum degree  $\delta \geq 66 \log \Delta$ , there exists*

a subset of edges  $F \subset E(G)$  of size

$$|F| = O(n \log \Delta) ,$$

such that

$$\gamma^{ID}(G \setminus F) = O\left(\frac{n \log \Delta}{\delta}\right) .$$

In particular, when the minimum degree is linear,  $\delta = \Theta(n)$ , this shows that it is enough to delete  $O(n \log n)$  edges to get a logarithmic size identifying code. The next theorem shows that Theorem 1.1 cannot be improved much.

**Theorem 1.2.** *For any  $d \geq 2$ , there exists a  $d$ -regular graph  $G_n^d$  on  $n$  vertices with the following properties.*

1. *For any  $M \geq 0$ , there exists a constant  $c > 0$  such that for any set of edges  $F \subset E(G_n^d)$  satisfying  $\gamma^{ID}(G_n^d \setminus F) \leq M \frac{n \log d}{d}$ ,  $|F| \geq cn \log d$ .*
2. *For any spanning subgraph  $H$  of  $G_n^d$ ,  $\gamma^{ID}(H) = \Omega\left(\frac{n \log d}{d}\right)$ .*

When  $\delta = \text{Poly}(\Delta)$ , Theorem 1.2 shows that Theorem 1.1 is tight, that is, we cannot hope for having a smaller identifying code by deleting any set of edges. Moreover, if  $\Delta$  is bounded or  $\delta \leq c' \log \Delta$ , for some small constant  $c' > 0$ , there is no way to improve the size of the identifying code of  $G$  by deleting edges.

## 2 Methods and proofs

The complete proofs can be found in [3].

The proof of Theorem 1.1 focuses on the study of the random spanning subgraph  $G(B, f)$  of  $G$ , where  $B \subseteq V$  and  $f : V(G) \rightarrow \mathbb{R}^+ \cup \{0\}$  is a function. Edges non incident to  $B$  are always present in  $G(B, f)$ , while each incident edge  $uv$  appears in  $G(B, f)$  independently with probability  $1 - p_{uv}$ , where  $p_{uv}$  depends on  $f(u)$ ,  $f(v)$  and the degree of  $u$  and  $v$  in  $B$ ,  $d_B(u)$  and  $d_B(v)$  respectively. The next lemma gives an exponential upper-bound on the probability that two vertices of  $G(B, f)$  are not separated by  $B$ .

**Lemma 2.1.** *Given a graph  $G$  and a subset  $B \subseteq V(G)$ , consider the random subgraph  $G(B, f)$ . For every pair  $u, v$  of distinct vertices with  $d_B(u) \geq d_B(v)$ ,*

$$\Pr(N_{G(B,f)}[u] \cap B = N_{G(B,f)}[v] \cap B) < e^{-3f(u)/16} .$$

The proof of Theorem 1.1 is structured in the following steps:

1. Select a set  $C \subseteq V$  at random, where each vertex is selected independently with probability  $p$ . Using the Chernoff inequality, estimate the probability of the event  $A_C$  that  $C$  is small enough for our purposes. From  $C$ , construct the spanning subgraph  $G(C, f)$  of  $G$ , with  $f(u) = \min(66 \log \Delta, d_B(u))$ .
2. Use Lovász Local Lemma and Lemma 2.1 to lower-bound the probability that the following events (whose disjunction we call  $A_{LL}$ ) hold jointly: 1. in  $G(C, f)$ , each pair of vertices that are at distance at most 2 from each other are separated by  $C$ ; 2. for each such pair and each neighbor of this pair in  $G$ , its degree within  $C$  in  $G$  is close to its expected value  $d(v)p$ . Show that with nonzero probability,  $A_C$  and  $A_{LL}$  hold jointly.
3. Find a dominating set  $D$  with  $|D| = O(|C|)$ ; if  $A_{LL}$  holds,  $C \cup D$  is an identifying code.
4. Show that, if  $A_C$  and  $A_{LL}$  hold, the expected number of deleted edges is small.

To prove Theorem 1.2, we first study the complete graph on  $n$  vertices. We combine the following two lemmata to get as a direct corollary Proposition 2.4.

**Lemma 2.2.** *For any  $M \geq 0$ , there exists a constant  $c_0 > 0$  such that any graph  $G$  with  $\gamma^{ID}(G) \leq M \log n$  contains at least  $c_0 n \log n$  many edges.*

**Lemma 2.3.** *Let  $G$  be a graph and  $\bar{G}$  its complement. If  $G$  and  $\bar{G}$  are twin-free,*

$$\frac{1}{2} \leq \frac{\gamma^{ID}(\bar{G})}{\gamma^{ID}(G)} \leq 2.$$

**Proposition 2.4.** *Let  $K_n$  be the complete graph on  $n$  vertices. For any  $M \geq 0$ , there exists a constant  $c > 0$  such that for any set of edges  $F \subset E(K_n)$  satisfying  $\gamma^{ID}(K_n \setminus F) \leq M \log n$ ,  $|F| \geq cn \log n$ .*

Now, consider the graph  $G_n^d$  to be the disjoint union of cliques of order  $d + 1$ . Since each clique is a connected component, an asymptotically optimal identifying code for  $G_n^d$  must be also asymptotically optimal for each component. By Proposition 2.4, we must delete at least  $\Omega(d \log d)$  edges from each clique to get an identifying code of size  $O(\log d)$  in each component. Thus one must delete at least  $\Omega(n \log d)$  edges from  $G_n^d$  to get an identifying code of size  $O\left(\frac{n \log d}{d}\right)$ , thus, proving Theorem 1.2.

### 3 Concluding remarks

1. In [2], the notion of a *watching system* has been introduced as a relaxation of identifying codes: in a watching system, code vertices (“watchers”) are allowed to identify any subset of their closed neighborhood, and several watchers can be placed in one vertex. Hence, for any spanning subgraph  $G'$  of  $G$  and denoting by  $w(G)$  the minimum size of a watching system of  $G$ , we have  $w(G) \leq w(G') \leq \gamma^{\text{ID}}(G')$ . In particular, the watching number is a monotone parameter with respect to graph inclusion. From Theorem 1.1 we have:

**Corollary 3.1.** *Under the hypothesis of Theorem 1.1,*

$$w(G) = O\left(\frac{n \log \Delta}{\delta}\right).$$

2. Using Lemma 2.3, Theorem 1.1 can be adapted to the case where we want to *add* edges rather than deleting them.

3. Given a graph property  $\mathcal{P}$ , the *resilience* of  $G$  with respect to  $\mathcal{P}$  is the minimum number of edges one has to delete to obtain a graph not satisfying  $\mathcal{P}$ . The resilience of monotone properties is well studied, in particular, in the context of random graphs [11]. Our result can be understood in terms of the resilience of the property  $\mathcal{P}$ ,  $G$  does not admit an small identifying code. For any graph  $G$  satisfying the hypothesis of Theorem 1.1 and Theorem 1.2, the resilience with respect to  $\mathcal{P}$  is at most  $O(n \log \Delta)$ , and this upper bound is attained.

4. The proof of Theorem 1.1 just provides an exponentially small lower bound on the probability that we can find the desired object. However, if we assume that  $\Delta = n$ , this probability can be shown to be  $1 - o(1)$ . In such a case our proof provides a randomized algorithm which constructs a good spanning subgraph and a small identifying code meeting the bounds on Theorem 1.1.

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# On two-point configurations in subsets of pseudo-random sets

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**Abstract.** We prove a transference type result for pseudo-random subsets of  $\mathbb{Z}_N$  that is analogous to the well-known Fürstenberg-Sárközy theorem. More precisely, let  $k \geq 2$  be an integer and let  $\beta$  and  $\gamma$  be real numbers satisfying

$$\gamma + (\gamma - \beta)/(2^{k+1} - 3) > 1.$$

Let  $\Gamma \subseteq \mathbb{Z}_N$  be a set with size at least  $N^\gamma$  and linear bias at most  $N^\beta$ . Then, every  $A \subseteq \Gamma$  with relative density  $|A|/|\Gamma| \geq (\log \log N)^{-\frac{1}{2} \log \log \log \log \log N}$  contains a pair of the form  $\{x, x + d^k\}$  for some nonzero integer  $d$ .

For instance, for squares, *i.e.*,  $k = 2$ , and assuming the best possible pseudo-randomness  $\beta = \gamma/2$  our result applies as soon as  $\gamma > 10/11$ .

Our approach uses techniques of Green as seen in [6] relying on a Fourier restriction type result also due to Green.

## 1 Introduction

A classical result in additive combinatorics, proved independently by Sárközy [10] and Fürstenberg [4], states that subsets of the first  $N$  integers with positive density contains a pair which differ by a perfect  $k$ -th power, *i.e.*, a pair  $\{x, x + d^k\}$  for some  $d > 0$ . As a quantitative version it is known due to [1, 9] that this conclusion already holds for sets  $A$  of density

$$|A|/N \geq (\log N)^{-c \log \log \log \log N} \quad \text{for some } c > 0 \quad (1.1)$$

In this note we consider the problem of extending (1.1) to hold for subsets of vanishing relative density of sparse pseudo-random subsets of

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$\mathbb{Z}_N$ . This type of extensions are commonly called transference results in which an extremal problem known for dense objects is transferred or carried over to sparse objects taken from a well-behaved universe like a random or a pseudo-random set. We refer to [2, 3, 6–8, 11] for further information.

Qualitative transference of (1.1) to sparse random sets were first considered in [7]. Later on, Nguyen [8] proved that with high probability every relatively dense subset of random sets  $R \subset [N]$  of size  $\Omega(N^{1-1/k})$  contains the configuration  $\{x, x + d^k\}$ , for some nonzero  $d$ . Up to a multiplicative constant this density attained for the random host is best possible.

It is also interesting to note that Tao and Ziegler proved that the polynomial Szemerédi theorem also holds in the primes. Their proof relies heavily on pseudo-random properties of the primes, thus, can be seen in the scheme mentioned above. For a classical notion of pseudo-randomness defined by small non-trivial Fourier coefficients, however, nothing is known concerning extensions of (1.1) and our Theorem 1.1 shall give the first nontrivial bound for this setting.

**Our main result.** Before stating the result we require some notation. For the purposes of Fourier analysis we endow  $\mathbb{Z}_N$  with the counting measure and, consequently, endow its dual group  $\widehat{\mathbb{Z}}_N$  with the uniform measure. As a result, given a function  $f: \mathbb{Z}_N \rightarrow \mathbb{C}$  the Fourier transform of  $f$  is defined to be the function  $\widehat{f}: \widehat{\mathbb{Z}}_N \rightarrow \mathbb{C}$  given by  $\widehat{f}(\xi) = \sum_{x \in \mathbb{Z}_N} f(x)e(-\xi x)$  where  $e(x) = e^{2\pi i x/N}$ . We write  $\|f\|_u = \sup_{\widehat{0} \neq \xi \in \widehat{\mathbb{Z}}_N} |\widehat{f}(\xi)|$  to denote magnitude of the second largest Fourier coefficient of  $f$ , and call  $\|f\|_u$  the *linear bias* of  $f$ . Given  $f, g: \mathbb{Z}_N \rightarrow \mathbb{C}$ , the convolution of  $f$  and  $g$  is given by  $f * g(x) = \sum_{y \in \mathbb{Z}_N} f(y)g(x - y)$ .

We identify a set with its characteristic function, *i.e.*, if  $A \subseteq \mathbb{Z}_N$  then  $A$  also denotes a 0, 1-function with  $A(x) = 1$  if and only if  $x \in A$ . Finally, let  $Q_k = \{x^k : x \leq N^{1/k}, x \text{ is integer}\}$  denote the set of  $k$ th powers. Our main result reads as follows.

**Theorem 1.1.** *Let  $k \geq 2$  be an integer and let  $\gamma > \beta > 0$  be reals with  $\gamma + (\gamma - \beta)/(2^{k+1} - 3) > 1$ . Then there is an  $n_0$  such that for all  $N > n_0$  the following holds. Let  $\Gamma \subseteq \mathbb{Z}_N$  satisfy*

$$|\Gamma| \geq N^\gamma \text{ and } \|\Gamma\|_u \leq N^\beta,$$

*and let  $\alpha = \alpha(N) \geq (\log \log N)^{-\frac{1}{2} \log \log \log \log \log N}$ . Then, every subset  $A \subseteq \Gamma$  satisfying  $|A| \geq \alpha|\Gamma|$  contains a pair  $\{x, x + d\} \subseteq A$  where  $d \in Q_k$ .*

It is worth to note that the proof of Theorem 1.1 does not merely guarantee one desired configuration but many. Indeed, the number of such configurations found in the subset  $A$  is at least  $\eta|\Gamma|^2|Q_k|N^{-1}$ , where  $\eta = \eta(\alpha) > 0$  if  $\alpha \neq 0$ . Up to  $\eta(\alpha)$  this bound is clearly best possible. Moreover, let us emphasise that in Theorem 1.1 we can handle subsets of  $\Gamma$  whose relative density is vanishing as  $N$  goes to infinity. This is due to the fact that we are transferring the *quantitative* version of the dense case of the Fürstenberg-Sárközy theorem, namely (1.1).

Due to Parseval’s equality the the parameter  $\beta$  in Theorem 1.1 controlling the pseudo-randomness of  $\Gamma$  satisfies  $\beta \geq \gamma/2$ . That is, one may think of  $\beta = \gamma/2$  as though  $\Gamma$  is “as pseudo-random as possible”. In this case, *i.e.*,  $\beta = \gamma/2$ , we have that Theorem 1.1 is applicable as long as  $\gamma > 1 - \frac{1}{2^{k+2}-5}$ .

## 2 Sketch of the proof of Theorem 1.1

Our approach follows that of Green [6]. Given  $\alpha = \alpha(N)$  we introduce functions  $\eta, \delta$ , and  $\varepsilon$  depending on  $N$  so that  $\alpha \gg \eta \gg \delta \gg \varepsilon > 0$ . Given the set  $A \subseteq \Gamma$  and a set of frequencies  $\emptyset \neq S \subset \mathbb{Z}_N$  we know that there is a  $\varrho = \varrho(N) \in [\varepsilon/2, \varepsilon]$  such that the Bohr set  $B = B(S, \varrho)$  is a *regular* (see, *e.g.* [12], chapter 4.4). Define the function  $a : \mathbb{Z}_N \rightarrow \mathbb{R}$  given by  $a(x) = \frac{N}{|\Gamma||B|} (A * B)(x)$  which can be shown to have certain attributes seen in characteristic functions of *dense* sets; that is  $\|a\|_{\ell^1} \geq \alpha N$  and  $\|a\|_\infty \leq 3$ , provided  $|\Gamma| \| \Gamma \|_u^{-1} > 2(\varepsilon/(20|S|^{1/2}))^{-|S|}$ .

Applying(1.1) combined with a Varnavides type argument then yields a lower bound on the of number of desired configurations “in  $a$ ”

$$\Lambda(a, Q_k) = \sum_{x,d \in \mathbb{Z}_N} a(x)a(x+d)Q_k(d) \geq \eta N|Q_k|. \tag{2.1}$$

Due to the convolution property and assuming that  $A \subseteq \Gamma$  contains less than  $\eta N^{1+1/k}$  pairs of the form  $\{x, x^k\}$ , we have

$$\Lambda(a, Q_k) < \frac{N}{|\Gamma|^2} \sum_{\xi \in \mathbb{Z}_N} |\widehat{A}(\xi)|^2 |\widehat{Q}_k(\xi)| \left| \frac{|\widehat{B}(\xi)|^2}{|B|^2} - 1 \right| + \frac{\eta}{2} N^{1+1/k} \tag{2.2}$$

from which we then derive a contradiction to (2.1).

To this end, we split the sum into two sums; one ranging over  $\xi \in S = \text{Spec}_\delta(A) = \{\xi \in \mathbb{Z}_N : |\widehat{A}(\xi)| \geq \delta|\Gamma|\}$  and another ranging over  $\xi \notin S$ . It can be shown that  $\left| \frac{|\widehat{B}(\xi)|^2}{|B|^2} - 1 \right| \leq 2\varepsilon$  for all  $\xi \in S$  and  $|S| \leq (2/\delta)^{(2-2\beta)/(\gamma-\beta)}$  from which we conclude that for the first sum ranging over  $S$  is at most  $2\varrho|S||\widehat{A}(0)|^2|\widehat{Q}_k(0)| \leq \eta|\Gamma|^2|Q_k|/4$ .



Handling the sum over  $\widehat{\mathbb{Z}}_N \setminus S$ , however, is more complicated. In fact, its proof is a central part of the proof of Theorem 1.1 but due to space limitation we omit it here. Nevertheless, we mention that the main tools in the proof are Waring's theorem and a restriction type result due to Green [5] which is similar to that used in [6]. It is a close adaption of the restriction theorem due to Stein-Tomas [13]. We omit the details.

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# On the giant component of random hyperbolic graphs

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## 1 Introduction

The theory of geometric random graphs was initiated by Gilbert [2] already in 1961 in the context of what is called *continuum percolation*. In 1972, Hafner [4] focused on the typical properties of large but finite random geometric graphs. Here  $N$  points are sampled within a certain region of  $\mathbb{R}^d$  following a certain distribution and any two of them are joined when their Euclidean distance is smaller than some threshold which, in general, is a function of  $N$ . In the last two decades, this class of random graphs has been studied extensively – see the monograph of Penrose [6].

However, what structural characteristics emerge when one considers these points distributed on a curved space where distances are measured through some (non-Euclidean) metric? Such a model was introduced by Krioukov et al. [5] and some typical properties of these random graphs were studied with the use of non-rigorous methods.

### 1.1 Random geometric graphs on a hyperbolic space

The most common representations of the hyperbolic space is the upper-half plane representation  $\{z : \Im z > 0\}$  as well as the Poincaré unit disc which is simply the open disc of radius one, that is,  $\{(u, v) \in \mathbb{R}^2 : 1 - u^2 - v^2 > 0\}$ . Both spaces are equipped with the hyperbolic metric; in the former case this is  $\frac{1}{(\zeta y)^2} dy^2$  whereas in the latter this is  $\frac{4}{\zeta^2} \frac{du^2 + dv^2}{(1 - u^2 - v^2)^2}$ , where  $\zeta$  is some positive real number. It can be shown that the (Gaussian) curvature in both cases is equal to  $-\zeta^2$ . We will denote by  $\mathbb{H}_\zeta^2$  the class of these spaces.

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In this paper, following the definitions in [5], we shall be using the native representation of  $\mathbb{H}_\zeta^2$ . Under this representation, the ground space of  $\mathbb{H}_\zeta^2$  is  $\mathbb{R}^2$  and every point  $x \in \mathbb{R}^2$  whose polar coordinates are  $(r, \theta)$  has hyperbolic distance from the origin  $O$  equal to  $r$ . Also, a circle of radius  $r$  around the origin has length equal to  $\frac{2\pi}{\zeta} \sinh(\zeta r)$  and area equal to  $\frac{2\pi}{\zeta^2} (\cosh(\zeta r) - 1)$ .

Let  $N = \nu e^{\zeta R/2}$ , where  $\nu$  is a positive real number that controls the average degree of the random graph. We create a random graph by selecting randomly  $N$  points from the disc of radius  $R$  centred at the origin  $O$ , which we denote by  $\mathcal{D}_R$ . If such a random point  $u$  has polar coordinates  $(r, \theta)$ , then  $\theta$  is uniformly distributed in  $(0, 2\pi]$ , whereas the probability density function of  $r$ , which we denote by  $\rho(r)$ , is determined by a parameter  $\alpha > 0$  and is equal to

$$\rho(r) = \alpha \frac{\sinh \alpha r}{\cosh \alpha R - 1}. \quad (1.1)$$

When  $\alpha = \zeta$ , then this is the uniform distribution. This set of points will be the vertex set of the random graph and we denote it by  $V_N$ . The random graph  $\mathcal{G}(N; \zeta, \alpha)$  is formed when we join two vertices, if they are within (hyperbolic) distance  $R$ .

Krioukov et al. [5] focus on the degree distribution of  $\mathcal{G}(N; \zeta, \alpha)$ , showing that when  $0 < \zeta/\alpha < 2$  this follows a power law with exponent  $2\alpha/\zeta + 1$ . They also discuss clustering on a smooth version of the above model. Their results have been verified rigorously by Gugelmann et al. [3]. When  $1 < \zeta/\alpha < 2$ , the exponent is between 2 and 3, as is the case in a number of networks that emerge in applications such as computer networks, social networks and biological networks (see for example [1]). Krioukov et al. [5] introduce this model as a geometric framework for the study of complex networks. In fact, they view the degree distribution as well as the existence of clustering at a local level as “natural reflections of the underlying hyperbolic geometry”.

## 1.2 Component structure of $\mathcal{G}(N; \zeta, \alpha)$

This paper focuses on the component structure of  $\mathcal{G}(N; \zeta, \alpha)$  and, in particular, on the size of its largest component. We also denote by  $|L_1|$  the size of a largest connected component of  $\mathcal{G}(N; \zeta, \alpha)$ .

In this contribution, we show that when  $\zeta/\alpha$  crosses 1 a “phase transition” occurs. More specifically, if  $\zeta/\alpha < 1$ , then *asymptotically almost surely (a.a.s.)*, that is, with probability  $1 - o(1)$  as  $N \rightarrow \infty$ ,  $|L_1|$  is bounded by a sublinear function, whereas if  $\zeta/\alpha > 1$ , then  $|L_1|$  is linear.

**Theorem 1.1.** *Let  $\zeta, \alpha$  be positive real numbers. The following hold:*

- *If  $\zeta/\alpha > 1$ , then there exists  $c = c(\zeta, \alpha, v) > 0$  such that a.a.s.  $|L_1| > cN$ .*
- *If  $\zeta/\alpha < 1$ , then a.a.s.  $|L_1| < CR^2N^{\zeta/\alpha}$ , where  $C = C(\zeta, \alpha) > 0$ .*

Furthermore, one can show that when  $\zeta/\alpha > 2$ , then  $\mathcal{G}(N; \zeta, \alpha)$  is a.a.s. connected. We now proceed with a brief sketch of the proof of each part of the above theorem.

## 2 The supercritical regime

For any given point  $v \in \mathcal{D}_R$ , we let  $t_v = R - r_v$ , where  $r_v$  denotes the radius of  $v$  – we call this the *type* of point  $v$ . We define a partition of  $\mathcal{D}_R$  into homocentric bands  $\mathcal{B}_i$ , for  $i = 1, \dots, T$ , where  $T = T(N)$  is a suitably defined function of  $N$ . More specifically, the central band  $\mathcal{B}_0$  consists of all points in  $\mathcal{D}_R$  whose type is larger than  $R/2$ , that is, their radius is less than  $R/2$ . Note that any two vertices in  $\mathcal{B}_0$  are connected by an edge as their hyperbolic distance is less than  $R$ . In other words, the subgraph of  $\mathcal{G}(N; \zeta, \alpha)$  induced by the vertices in  $\mathcal{B}_0$  is the complete graph. To define the remaining bands, we define a decreasing sequence of positive real numbers  $t_0 > t_1 > \dots$ , where  $t_0 := R/2$  and for  $i \geq 1$  we have

$$t_i - \frac{2}{\zeta} \ln \left( \frac{8\pi}{v} \ln t_i \right) = \lambda t_{i-1}, \tag{2.1}$$

where  $\lambda := \frac{2}{\zeta} (\alpha - \frac{\zeta}{2}) < 1$  (as  $\zeta/\alpha > 1$ ) and

$$\mathcal{B}_i = \{v \in \mathcal{D}_R : t_i \leq t_v < t_{i-1}\}.$$

We define  $T$  as the largest  $i$  such that  $e^{-\alpha(t_{i-1}-t_i)} < 1/2$  and  $t_i > e$ . Let  $\mathcal{N}_0$  denote the set of vertices that belong to the set  $\mathcal{B}_0$ . In turn, for  $i > 0$  we let  $\mathcal{N}_i$  denote the set of vertices in  $\mathcal{B}_i$  that have at least one neighbour in  $\mathcal{N}_{i-1}$ . Since  $\mathcal{B}_0$  is a clique, the subgraph of  $\mathcal{G}(N; \zeta, \alpha)$  that is induced by  $\cup_{i=0}^T \mathcal{N}_i$  is connected and has size  $\sum_{i=0}^T |\mathcal{N}_i|$ .

We establish bounds on the sizes of the sets  $\mathcal{N}_i$ , for  $i = 1, \dots, T$ . In particular, we show that the number of vertices in  $\mathcal{N}_i$  stochastically dominates the number of vertices that are contained in a subset of  $\mathcal{B}_i$  that has arc  $\Theta_i$ , where the sequence of  $\Theta_i$  satisfies for  $i \geq 1$

$$\Theta_i \geq \Theta_{i-1} \left( 1 - \exp \left( -\frac{v}{4\pi} (e^{-\alpha t_{i-1}} - e^{-\alpha t_{i-2}}) \theta^{(i)} \right) \right), \quad \Theta_0 := 2\pi$$

and  $\theta^{(i)} := e^{\frac{\zeta}{2}(t_{i-1}+t_i)}$  (here  $t_{-1} := R$ ). We denote this number by  $N'_i$ . A concentration argument shows that a.a.s.  $N'_i \geq \frac{1}{2} N \frac{\Theta_i}{2\pi} (e^{-\alpha t_i} - e^{-\alpha t_{i-1}})$ .

As  $e^{-\alpha(t_{i-1}-t_i)} < 1/2$ , it follows that

$$\begin{aligned} (e^{-\alpha t_{i-1}} - e^{-\alpha t_{i-2}}) \theta^{(i)} &= e^{-\alpha t_{i-1}} (1 - e^{-\alpha(t_{i-2}-t_{i-1})}) \theta^{(i)} \\ &> \frac{1}{2} e^{-\alpha t_{i-1} + \frac{\zeta}{2}(t_{i-1}+t_i)} = \frac{4\pi}{\nu} \ln t_i, \end{aligned}$$

whereby

$$\Theta_i \geq \Theta_{i-1} \left(1 - \frac{1}{t_i}\right) \stackrel{(2.1)}{>} \Theta_{i-1} \left(1 - \frac{1}{\lambda^i t_0}\right).$$

It then follows that for some  $c = c(\zeta, \alpha, \nu) > 0$  a.a.s.

$$\sum_{i=0}^T N'_i \geq N (e^{-\alpha t_T} - e^{-\alpha t_0}) \prod_{i=0}^T \left(1 - \frac{1}{\lambda^i t_0}\right) > cN.$$

The stochastic domination implies this part of the theorem.

### 3 The subcritical regime

A first moment argument shows that all vertices have type at most  $\frac{\zeta}{2\alpha}R + \omega(N)$ , where  $\omega(N)$  is a function such that  $\omega(N) \rightarrow \infty$  as  $N \rightarrow \infty$ . Hence, since  $\zeta/\alpha < 1$ , it follows that all vertices have types which are smaller than and bounded away from  $R/2$ . We consider a vertex  $v$  which has this type and we analyse a *breadth exploration process* through which we bound the *total angle of the component* which  $v$  belongs to. We define the *total angle* of the component of  $v$  to be the largest relative angle between any two of its vertices – if the component has only one vertex, then this is equal to zero. We denote this angle by  $\Theta(v)$ . We show that the assumption that the type of  $v$  is  $\frac{\zeta}{2\alpha}R + \omega(N)$  gives a stochastic upper bound on the total angle of the component of any vertex in  $V_N$ . Our bound is as follows.

**Lemma 3.1.** *Let  $v \in V_N$  be a vertex having  $t_v = \frac{\zeta}{2\alpha}R + \omega(N)$ . There exists a constant  $C' = C'(\zeta, \alpha, \nu) > 0$  such that with probability  $1 - o\left(\frac{1}{N^{1-\zeta/\alpha}}\right)$  we have*

$$\Theta(v) \leq C' \frac{R^2 N^{\zeta/\alpha}}{N}.$$

The *breadth exploration process* is a process that is somewhat similar to the breadth-first search algorithm. More specifically, starting from vertex  $v$  with type as in the above lemma, we expose the vertex of the largest type among those vertices that are within distance  $R$  from  $v$  in clockwise direction. Subsequently, we continue this procedure until a vertex of type  $K$  is reached, where  $K$  is a large constant. We repeat this in anticlockwise

direction. This completes the first phase of the process. Thereafter, we bound the contribution that comes from vertices of type smaller than  $K$ . If these have also vertices within distance  $R$  that have not been covered previously, then we start the first phase again. We show that the number of repetitions of this phase is bounded in probability. A first moment argument shows that the probability that there is a vertex  $v$  with  $\Theta(v)$  larger than the bound of the above lemma is  $o(1)$ . The result provides a bound on the total angle of each component. This needs to be complemented by a result which associates the total angle of a component with the number of vertices. We show that a.a.s. there is no component with total angle at most  $C'R^2N^{\zeta/\alpha}/N$  that has more than  $CR^2N^{\zeta/\alpha}$  vertices, where  $C = C(\zeta, \alpha, \nu) > 0$  is another constant. This completes the proof of Theorem 1.1.

#### 4 Conclusions - Further directions

This contribution focuses on the size of the largest component of random geometric graphs on the hyperbolic plane. We show that when the ratio  $\zeta/\alpha$  crosses 1 a giant component emerges. But is this component unique? What is the size of the largest component in each case? Moreover, our results do not cover the critical case  $\zeta/\alpha = 1$ , that is, when the points are uniformly distributed on  $\mathcal{D}_R$ . This is a natural direction that goes along the lines of the theory of random geometric graphs on Euclidean spaces.

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# Discontinuous bootstrap percolation in power-law random graphs

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and Konstantinos Panagiotou<sup>3</sup>

## 1 Introduction

Bootstrap percolation was introduced by Chalupa, Leath and Reich [6] during the 1970's in the context of magnetic disordered systems and has been re-discovered since then by several authors mainly due to its connections with various physical models. A *bootstrap percolation process* with *activation threshold* an integer  $r \geq 2$  on a graph  $G = G(V, E)$  is a deterministic process which evolves in rounds. Every vertex has two states: it is either *infected* or *uninfected*. Initially, there is a subset  $\mathcal{A}_0 \subseteq V$  which consists of infected vertices, whereas every other vertex is uninfected. Subsequently, in each round, if an uninfected vertex has at least  $r$  of its neighbours infected, then it also becomes infected and remains so forever. This is repeated until no more vertices become infected. We denote the final infected set by  $\mathcal{A}_f$ . Our general assumption will be that the initial set of infected vertices  $\mathcal{A}_0$  is chosen randomly among all subsets of vertices of a certain size.

These processes have been studied on a variety of graphs, such as trees, grids, hypercubes, as well as on several distributions of random graphs. A short survey regarding applications of bootstrap percolation processes can be found in [1].

During the last decade, there has been significant experimental evidence on the structural characteristics of networks that arise in applications such as the Internet, the World Wide Web as well as social networks or even biological networks. One of the fundamental features is their

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degree distribution, which in most cases appears to follow a power law with exponent between 2 and 3 (see for example the article of Albert and Barabási [2]). The theme of this contribution is the study of the evolution of bootstrap percolation processes on random graphs which exhibit this characteristic. We show that this boosts the evolution of the process, resulting in large infected sets starting from a small set of infected vertices.

## 2 Models and results

The random graph model that we consider is asymptotically equivalent to a model considered by Chung and Lu [7], and is a special case of the so-called *inhomogeneous random graph*, which was introduced by Söderberg [9] and was studied in detail by Bollobás, Janson and Riordan in [5].

### 2.1 Inhomogeneous random graphs – The Chung-Lu model

In order to define the model we consider for any  $n \in \mathbb{N}$  the vertex set  $[n] := \{1, \dots, n\}$ . Each vertex  $i$  is assigned a positive weight  $w_i(n)$ , and we will write  $\mathbf{w} = \mathbf{w}(n) = (w_1(n), \dots, w_n(n))$ . We assume in the remainder that the weights are deterministic, and we will suppress the dependence on  $n$ , whenever this is obvious from the context. However, note that the weights could be random variables; we will not consider this case here, although it is very likely that under suitable technical assumptions our results generalize to this case as well. For any  $S \subseteq [n]$ , set

$$W_S(\mathbf{w}) := \sum_{i \in S} w_i.$$

In our random graph model, the event of including the edge  $\{i, j\}$  in the resulting graph is independent of the events of including all other edges, and its probability equals

$$p_{ij}(\mathbf{w}) = \min \left\{ \frac{w_i w_j}{W_{[n]}(\mathbf{w})}, 1 \right\}. \quad (2.1)$$

We will refer to this model as the *Chung-Lu* model, and we shall write  $CL(\mathbf{w})$  for a random graph in which each possible edge  $\{i, j\}$  is included independently with probability as in (2.1).

### 2.2 Power-law degree distributions

Following van der Hofstad [10], we write for any  $n \in \mathbb{N}$  and any sequence of weights  $\mathbf{w} = (w_1(n), \dots, w_n(n))$

$$F_n(x) = n^{-1} \sum_{i=1}^n \mathbf{1}[w_i(n) < x], \quad \forall x \in [0, \infty)$$



for the empirical distribution function of the weight of a vertex chosen uniformly at random. We will assume that  $F_n$  satisfies the following two conditions.

**Definition 2.1.** We say that  $(F_n)_{n \geq 1}$  is *regular*, if it has the following two properties.

- **[Weak convergence of weight]** There is a distribution function  $F : [0, \infty) \rightarrow [0, 1]$  such that for all  $x$  at which  $F$  is continuous  $\lim_{n \rightarrow \infty} F_n(x) = F(x)$ ;
- **[Convergence of average weight]** Let  $W_n$  be a random variable with distribution function  $F_n$ , and let  $W_F$  be a random variable with distribution function  $F$ . Then we have  $\lim_{n \rightarrow \infty} \mathbb{E}[W_n] = \mathbb{E}[W_F]$ .

The regularity of  $(F_n)_{n \geq 1}$  guarantees two important properties. Firstly, the weight of a random vertex is approximately distributed as a random variable that follows a certain distribution. Secondly, this variable has finite mean and therefore the resulting graph has bounded average degree. Apart from regularity, our focus will be on weight sequences that give rise to power-law degree distributions.

**Definition 2.2.** We say that a regular sequence  $(F_n)_{n \geq 1}$  is *of power law with exponent  $\beta$* , if there are  $0 < \gamma_1 < \gamma_2, x_0 > 0$  and  $0 < \zeta \leq 1/(\beta - 1)$  such that for all  $x_0 \leq x \leq n^\zeta$

$$\gamma_1 x^{-\beta+1} \leq 1 - F_n(x) \leq \gamma_2 x^{-\beta+1},$$

and  $F_n(x) = 0$  for  $x < x_0$ , but  $F_n(x) = 1$  for  $x > n^\zeta$ .

We consider the random graph  $CL(\mathbf{w})$  where the weight sequence  $\mathbf{w} = \mathbf{w}(n)$  gives rise to a regular sequence of empirical distribution functions that are of power law with exponent  $\beta$ . We assume that a random set of  $a(n)$  vertices is initially infected. We say that an event occurs *asymptotically almost surely (a.a.s.)*, if it occurs with probability  $\rightarrow 1$  as  $n \rightarrow \infty$ , in the product space of the random graph and the choice of the initially infected vertices.

### 2.3 Results

We determine explicitly a critical function which we denote by  $a_c(n)$  such that when we infect randomly  $a(n)$  vertices in  $[n]$ , then the following threshold phenomenon occurs. If  $a(n) \ll a_c(n)$ , then a.a.s. the infection spreads no further than  $\mathcal{A}_0$ , but when  $a(n) \gg a_c(n)$ , then a linear number of vertices become eventually infected. We remark that  $a_c(n) = o(n)$ . We define the function  $\psi_r(x)$  for  $x \geq 0$  to be equal to the probability that a Poisson-distributed random variable with parameter  $x$  is at least  $r$ .

Also, for a random variable  $X$  with finite expected value and distribution function  $F$ , we (informally) say that  $X^*$  follows the  $F$ -size-biased distribution function, if the distribution of  $X^*$  is weighted by the value of  $X$ .

**Theorem 2.3.** *For any  $\beta \in (2, 3)$  and any integer  $r \geq 2$ , we let  $a_c(n) = n^{\frac{r(1-\zeta)+\zeta(\beta-1)-1}{r}}$  for all  $n \in \mathbb{N}$ . Let  $a : \mathbb{N} \rightarrow \mathbb{N}$  be a function such that  $a(n) \rightarrow \infty$ , as  $n \rightarrow \infty$ , but  $a(n) = o(n)$ . Let also  $\zeta \leq \frac{1}{\beta-1}$ . If we initially infect uniformly at random  $a(n)$  vertices in  $[n]$ , then the following holds:*

- if  $a(n) \ll a_c(n)$ , then a.a.s.  $\mathcal{A}_f = \mathcal{A}_0$ ;
- if  $a(n) \gg a_c(n)$  and also  $\frac{r-1}{2r-\beta+1} < \zeta \leq \frac{1}{\beta-1}$ , then

$$\frac{|\mathcal{A}_f|}{n} \xrightarrow{p} \mathbb{E} [\psi_r(U\hat{y})], \text{ as } n \rightarrow \infty,$$

where  $U$  is a random variable with  $F$  as its distribution function and  $\hat{y}$  is the smallest positive solution of

$$y = \mathbb{E} [\psi_r(Wy)],$$

with  $W$  being a random variable whose law follows the  $F$ -size-biased distribution function.

When  $0 < \zeta \leq \frac{r-1}{2r-\beta+1}$  the second part of the above statement holds with  $a_c^+(n) = n^{1-\zeta \frac{r-\beta+2}{r-1}}$  instead of  $a_c(n)$ .

Note that the above theorem implies that when the maximum weight of the sequence is  $n^{1/(\beta-1)}$ , then the threshold function becomes equal to  $n^{\frac{\beta-2}{\beta-1}}$  and does not depend on  $r$ .

This result is in sharp contrast with the behaviour of the bootstrap percolation process in  $G(n, p)$  random graphs, where every edge on a set of  $n$  vertices is included independently with probability  $p$ . Recently, Janson, Łuczak, Turova and Vallier [8] (see Theorem 5.2 there) showed that when  $p = d/n$ , with  $d > 0$  fixed, if  $|\mathcal{A}_0| = o(n)$ , then typically no evolution occurs. In other words, the density of the initially infected vertices must be positive in order for the density of infected vertices to grow. We note that similar behavior to the case of  $G(n, p)$  has been observed in the case of random regular graphs [4], and in random graphs with given vertex degrees constructed through the configuration model, studied by the first author in [3], when the sum of the square of degrees scales linearly with  $n$ , the size of the graph. The later case includes random graphs

with power-law degree sequence with exponent  $\beta > 3$ . Our results imply that the two regimes  $2 < \beta < 3$  and  $\beta > 3$  have completely different behaviors.

The next theorem complements the above theorem, as it gives a law of large numbers for the size  $\mathcal{A}_f$  when a positive fraction of vertices are initially infected.

**Theorem 2.4.** *Let  $2 < \beta < 3$  and  $r \geq 2$ . If  $a(n) = pn$ , where  $p \in (0, 1)$  is fixed, then*

$$\frac{|\mathcal{A}_f|}{n} \xrightarrow{p} (1-p)\mathbb{E}[\psi_r(U\hat{y})] + p, \text{ as } n \rightarrow \infty,$$

where  $U$  is a random variable having  $F$  as its distribution function with  $\hat{y}$  being the smallest positive solution of

$$y = (1-p)\mathbb{E}[\psi_r(Wy)] + p$$

and  $W$  is a random variable whose law follows the  $F$ -size-biased distribution function.

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# On a conjecture of Graham and Häggkvist for random trees

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**Abstract.** A conjecture of Graham and Häggkvist says that every tree with  $m$  edges decomposes the complete bipartite graph  $K_{m,m}$ . By establishing some properties of random trees with the use of singularity analysis of generating functions, we prove that asymptotically almost surely a tree with  $m$  edges decomposes the complete bipartite graph  $K_{2m,2m}$ .

## 1 Introduction

Given two graphs  $H$  and  $G$  we say that  $H$  decomposes  $G$  if  $G$  is the edge-disjoint union of isomorphic copies of  $H$ . The following is a well-known conjecture of Ringel.

**Conjecture 1.1 (Ringel [12]).** Every tree with  $m$  edges decomposes the complete graph  $K_{2m+1}$ .

The conjecture has been verified by a number of particular classes of trees, see the dynamic survey of Gallian [5]. By using the polynomial method, the conjecture was verified by Kézdy [7] for the more general class of so-called *stunted* trees. As mentioned by the author, this class is still small among the set of all trees.

The following bipartite version of the conjecture was formulated by Graham and Häggkvist.

**Conjecture 1.2 (Graham and Häggkvist [6]).** Every tree with  $m$  edges decomposes the complete bipartite graph  $K_{m,m}$ .

Again the conjecture has been verified by a number of cases; see *e.g.* [9]. Approximate versions of the two conjectures have been also proved [6, 8–10]. However, to our knowledge, there are no results stating that

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every tree decomposes  $K_{cm+1}$  or  $K_{cm,cm}$  for some absolute constant  $c$  of reasonable size. The purpose of this paper is to show such a result for almost all trees.

Let  $\mathcal{T}$  denote the class of (unlabelled) trees and let  $\mathcal{T}_m$  be the class of trees with  $m$  edges. By a random tree with  $m$  edges we mean a tree chosen from  $\mathcal{T}_m$  with the uniform distribution. We say that a random tree satisfies a property  $\mathbb{P}$  asymptotically almost surely (a.a.s) if the probability that a random tree with  $m$  edges satisfies  $\mathbb{P}$  tends to one with  $m \rightarrow \infty$ .

**Theorem 1.3.** *Asymptotically almost surely a tree with  $m$  edges decomposes  $K_{2m,2m}$ .*

The proof of Theorem 1.3 combines a structural analysis of random trees with combinatorial techniques for graph decompositions. In the following two sections we discuss the results and tools used in this proof.

## 2 Stable sets of random trees

The first property we use in the proof of Theorem 1.3 concerns the number of leaves in a random tree. Robinson and Schwenk [13] proved that the average number of leaves in an (unlabelled) random tree with  $m$  edges is asymptotically  $cm$  with  $c \approx 0.438$ . Drmota and Gittenberger [2] showed that the distribution of the number of leaves in a random tree with  $m$  edges is asymptotically normal with variance  $c_2m$  for some positive constant  $c_2$ . Thus, asymptotically almost surely a random tree with  $m$  edges has more than  $2m/5$  leaves.

The second property we use deals with the size of a stable set in the base tree of  $T$  (the tree obtained from  $T$  by deleting its leaves.) Unfortunately this is not a parameter whose analysis can be explicitly found in the literature. We prove the following result.

**Theorem 2.1.** *The stable sets  $A, B$  of the base tree of a random tree with  $m$  edges satisfy a.a.s.*

$$||A| - |B|| \leq \epsilon m,$$

for every fixed  $\epsilon > 0$ .

The proof of Theorem 2.1 is based on the use of generating functions. We first consider the case of rooted trees. Let

$$t(x, w_0, w_1) = \sum_{m, k_0, k_1} t_{m, k_0, k_1} x^{m+1} w_0^{k_0} w_1^{k_1}, \quad (2.1)$$

where  $t_{m, k_0, k_1}$  denotes the number of rooted trees with  $m$  edges and  $k_0$  inner vertices (including the root if the tree has at least one edge even if the

root has degree one) with even distance to the root and  $k_1$  inner vertices with odd distance to the root. Then, by using the recursive description of a tree as a collection of trees hanging from a root (iterated twice to get the proper alignment of stable sets), one can obtain an explicit expression for (2.1).

Recall that we are interested in the difference  $|A| - |B|$  which we can do by setting  $w_0 = w$  and  $w_1 = w^{-1}$ . Hence, if  $T(x, w) = \sum_{m, \ell} T_{m, \ell} x^{m+1} w^\ell$  denotes the generating function, where  $T_{m, \ell}$  denotes the number of rooted trees with  $m$  edges and  $|A| - |B| = \ell$  (where  $\ell$  is some – possibly negative – integer and the root is contained in  $A$  even if the root has degree one) then an explicit expression for  $T(x, w) = t(x, w, w^{-1})$  is also obtained.

As usual we denote by  $a_n = [x^n] a(x)$  the  $n$ -th coefficient of a power series  $a(x) = \sum_{n \geq 0} a_n x^n$ . With the help of this notation it follows that

$$\mathbb{E} w^{|A|-|B|} = \frac{[x^{m+1}] T(x, w)}{[x^{m+1}] T(x, 1)}.$$

This magnitude can be determined asymptotically if  $w$  is close to 1 with the help of standard singularity analysis tools.

From a version of Hwang's Quasi-Power-Theorem (see [1]), one can deduce that the variable  $Z = |A| - |B|$  follows a normal distribution which in our case has zero mean and variance linear in  $m$  from which

$$\Pr(|A| - |B| \geq \varepsilon m) \leq C e^{-c\varepsilon^2 m}$$

for some positive constants  $c$  and  $C$  and for sufficiently small  $\varepsilon > 0$ . Of course this is precisely the statement that we want to prove for unlabelled trees. We translate the above analysis to unrooted unlabeled trees via Otter's bijection (see [11] or [1]). Unfortunately we cannot prove something like a central limit theorem for  $|A| - |B|$  in this case, but it is still possible to keep track of the second moment which, by using Chebyshev inequality, provides a proof of Theorem 2.1.

### 3 The embedding

The general approach to show that a tree  $T$  decomposes a complete graph or a complete bipartite graph consists in showing that  $T$  cyclically decomposes the corresponding graphs. We next recall the basic principle behind this approach in slightly different terminology.

A rainbow embedding of a graph  $H$  into an oriented arc-colored graph  $X$  is an injective homomorphism  $f$  of some orientation  $\vec{H}$  of  $H$  in  $X$  such that no two arcs of  $f(\vec{H})$  have the same color.

Let  $X = \text{Cay}(G, S)$  be a Cayley digraph of an abelian group  $G$  with respect to an antisymmetric subset  $S \subset G$  (that is,  $S \cap -S = \emptyset$ ). We consider  $X$  as an arc-colored oriented graph, by giving to each arc  $(x, x+s)$ ,  $x \in G, s \in S$ , the color  $s$ . Suppose that  $H$  admits a rainbow embedding  $f$  in  $X$ . For each  $a \in G$  the translation  $x \rightarrow x+a, x \in G$ , is an automorphism of  $X$  which preserves the colors and has no fixed points. Therefore, each translation sends  $f(\vec{H})$  to an isomorphic copy which is edge disjoint from it. Thus the sets of translations for all  $a \in G$  give rise to  $n := |G|$  edge-disjoint copies of  $\vec{H}$  in  $X$ . By ignoring orientations and colors, we thus have  $n$  edge disjoint copies of  $H$  in the underlying graph of  $X$ .

We will use the above approach with the Cayley graph  $X = \text{Cay}(\mathbb{Z}_m \times \mathbb{Z}_4, \mathbb{Z}_m \times \{1\})$ . We note that the underlying graph of  $X$  is isomorphic to  $K_{2m, 2m}$ . The strategy of the proof is to show first that the base tree  $T_0$  of a random tree with  $m$  edges admits a rainbow embedding  $f$  into  $X$  in such a way that  $f(T_0) \subset \mathbb{Z}_m \times \{1, 2\}$ . This can actually be achieved greedily as shown in the proof of next Lemma.

**Lemma 3.1.** *Let  $m$  be a positive integer. Let  $T$  be a tree with  $n < 3m/5$  edges and stable sets  $A, B$ . If  $||A| - |B|| \leq m/10$  then there is a rainbow embedding  $f$  of  $T$  into  $X = \text{Cay}(\mathbb{Z}_m \times \mathbb{Z}_4, \mathbb{Z}_m \times \{1\})$  such that  $f(V(T)) \subset (\mathbb{Z}_m \times \{1\}) \cup (\mathbb{Z}_m \times \{2\})$ .*

The second step involves a proper embedding of the leaves of  $T$ . For this we use Häggkvist [6, Corolary 2.8] to get:

**Lemma 3.2.** *Let  $T$  be a tree with  $m$  edges. If the base tree  $T_0$  of  $T$  admits a rainbow embedding  $f$  in  $X = \text{Cay}(\mathbb{Z}_m \times \mathbb{Z}_4, \mathbb{Z}_m \times \{1\})$  such that  $f(V(T_0)) \subset (\mathbb{Z}_m \times \{1\}) \cup (\mathbb{Z}_m \times \{2\})$  then  $T$  decomposes  $K_{2m, 2m}$ .*

The proof of Theorem 1.3 follows now directly from Lemma 3.1 and Lemma 3.2 and the results on random trees from Section 2.

*Proof.* As it has been mentioned in Section 2, a random tree  $T$  with  $m$  edges has a.a.s. more than  $2m/5$  leaves. Furthermore, by Theorem 2.1, the cardinalities of the stable sets of the base tree of  $T$  differ less than  $m/10$  in absolute value a.a.s. By Lemma 3.1, the base tree of  $T$  admits a.a.s. a rainbow embedding in  $\text{Cay}(\mathbb{Z}_m \times \mathbb{Z}_4, \mathbb{Z}_m \times \{1\})$  in such a way that the image of the embedding sits in  $\mathbb{Z}_m \times \{1, 2\}$ . In that case, Lemma 3.2 ensures that the tree  $T$  decomposes  $K_{2m, 2m}$ .  $\square$



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# Sharp threshold functions via a coupling method

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**Abstract.** We will present a new method used to establish threshold functions in the random intersection graph model. The method relies on a coupling of a random intersection graph with a random graph similar to an Erdős and Rényi random graph. Formerly a simple version of the technique was used in the case of homogeneous random intersection graphs. Now it is considerably modified and extended in order to be applied in the general case. By means of the method we are able to establish threshold functions for the general random intersection graph model for monotone properties. Moreover the new approach allows to sharpen considerably the best known results concerning threshold functions for homogeneous random intersection graph. We outline the main results obtained in [10].

## 1 Introduction

Various comparison methods such as equivalence or contiguity have shown to be very helpful in the analysis of the structure of random graph models. For example one may consider a random graph with independent edges instead of the model considered in the seminal papers of Erdős and Rényi [3,4]. In some cases, when the models are neither equivalent nor contiguous the comparison by coupling still may simplify the arguments. This is so in the case of finding threshold functions for random intersection graphs.

The first random intersection graph model was introduced by Karoński, Scheinerman, and Singer–Cohen [7]. Several generalisations of the model have been proposed, mainly in order to adapt it to use in some particular purpose. We consider the  $\mathcal{G}(n, m, \bar{p})$  model studied for example in [1, 2, 8]. In a random intersection graph  $\mathcal{G}(n, m, \bar{p})$  there is a set of  $n$  vertices  $\mathcal{V} = \{v_1, \dots, v_n\}$ , an auxiliary set of  $m = m(n)$  features  $\mathcal{W} = \{w_1, \dots, w_{m(n)}\}$ , and a vector  $\bar{p}(n) = (p_1, \dots, p_{m(n)})$  such that  $p_i \in (0, 1)$ , for each  $1 \leq i \leq m$ . Each vertex  $v \in \mathcal{V}$  adds a feature  $w_i \in \mathcal{W}$  to its feature set  $W(v)$  with probability  $p_i$  independently of all other properties and features. Any two vertices  $v, v' \in \mathcal{V}$

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are connected by an edge in  $\mathcal{G}(n, m, \bar{p})$  if  $W(v)$  and  $W(v')$  intersect. If  $\bar{p}(n) = (p, \dots, p)$  for some  $p \in (0; 1)$  then  $\mathcal{G}(n, m, \bar{p})$  is a random intersection graph defined in [7]. We denote it by  $\mathcal{G}(n, m, p)$ .

For  $m = n^\alpha$  with  $\alpha > 6$ ,  $\mathcal{G}(n, m, p)$  is equivalent to a random graph with independent edges (see [5]) but in some cases the models differ (see [7]). However, even in the case when there is a large correlation between appearance of edges in  $\mathcal{G}(n, m, p)$  (as well as in  $\mathcal{G}(n, m, \bar{p})$ ), interesting results may be obtained by coupling with an auxiliary graph with almost independent edges. It is shown in [10] that the coupling may be used to obtain sharp results on threshold functions for  $\mathcal{G}(n, m, \bar{p})$  for monotone properties. This extends results obtained in [9].

Let  $G_2(n, \hat{p}_2)$  be a random graph with the vertex set  $\mathcal{V}$  in which each edge appears independently with probability  $\hat{p}_2$ . Similarly let  $H_3(n, \hat{p}_3)$  be a random hypergraph in which each 3–element subset of the set of vertices  $\mathcal{V}$  is added to the hyperedge set independently with probability  $\hat{p}_3$ . Define  $G_3(n, \hat{p}_3)$  to be a graph with the vertex set  $\mathcal{V}$  and an edge set consisting of those two element subsets of  $\mathcal{V}$  which are subsets of at least one hyperedge of  $H_3(n, \hat{p}_3)$ . For the family  $\mathcal{G}$  of all graphs with the vertex set  $\mathcal{V}$ , we call  $\mathcal{A} \subseteq \mathcal{G}$  a property if it is closed under isomorphism. Moreover  $\mathcal{A}$  is increasing if  $G \in \mathcal{A}$  implies  $G' \in \mathcal{A}$  for all  $G' \in \mathcal{G}$  such that  $E(G) \subseteq E(G')$  and decreasing if  $\mathcal{G} \setminus \mathcal{A}$  is increasing.

Let  $\bar{p} = (p_1, \dots, p_m)$  be such that  $p_i \in (0, 1)$ , for all  $1 \leq i \leq m$ . Moreover for all  $1 \leq i \leq m$ , let  $X_i$  be the number of vertices which contain  $w_i$  in the feature set and  $\mathbb{I}_{i1}$ ,  $\mathbb{I}_{i2}$  and  $\mathbb{I}_{i\text{odd}}$  be indicator random variables of the events  $\{X_i = 1\}$ ,  $\{X_i = 2\}$  and  $\{X_i \text{ is odd}\}$ , respectively. Define

$$\begin{aligned}
 S_1 &= \mathbb{E} \left( \sum_{i=1}^m (X_i - \mathbb{I}_{i1}) \right) = \sum_{i=1}^m np_i (1 - (1 - p_i)^{n-1}); \\
 S_2 &= \mathbb{E} \left( \sum_{i=1}^m (X_i - \mathbb{I}_{i\text{odd}}) \right) = \sum_{i=1}^m np_i \left( 1 - \frac{1 - (1 - 2p_i)^n}{2np_i} \right); \\
 S_3 &= \mathbb{E} \left( \sum_{i=1}^m (\mathbb{I}_{i\text{odd}} - \mathbb{I}_{i1}) \right) \\
 &= \sum_{i=1}^m np_i \left( \frac{1 - (1 - 2p_i)^n}{2np_i} - (1 - p_i)^{n-1} \right); \\
 S_{1,2} &= \mathbb{E} \left( \sum_{i=1}^m 2\mathbb{I}_{i2} \right) = \sum_{i=1}^m n(n - 1)p_i^2(1 - p_i)^{n-2},
 \end{aligned} \tag{1.1}$$

The following theorem is the main tool of the presented coupling technique. It is an extension of the result obtained in [9].

**Theorem 1.1.** *Let  $S_1, S_2$  and  $S_3$  be given by (1.1). For some function  $\omega$  tending to infinity let*

$$\hat{p} = \frac{S_2 - \omega\sqrt{S_2} - 2S_2^2n^{-2}}{2\binom{n}{2}};$$

$$\hat{p}_2 = \begin{cases} \frac{S_1 - 3S_3 - \omega\sqrt{S_1} - 2S_1^2n^{-2}}{2\binom{n}{2}}, & \text{for } S_3 \gg \sqrt{S_1} \text{ and } \omega^2 \ll S_3/\sqrt{S_1}; \\ \frac{S_1 - \omega\sqrt{S_1} - 2S_1^2n^{-2}}{2\binom{n}{2}}, & \text{for } S_3 = O(\sqrt{S_1}); \end{cases}$$

$$\hat{p}_3 = \begin{cases} \frac{S_3 - \omega\sqrt{S_1} - 6S_3^2n^{-3}}{\binom{n}{3}}, & \text{for } S_3 \gg \sqrt{S_1} \text{ and } \omega^2 \ll S_3/\sqrt{S_1}; \\ 0, & \text{for } S_3 = O(\sqrt{S_1}). \end{cases}$$

If  $S_1 \rightarrow \infty$  and  $S_1 = o(n^2)$  then for any increasing property  $\mathcal{A}$ .

$$\liminf_{n \rightarrow \infty} \Pr \{G_2(n, \hat{p}) \in \mathcal{A}\} \leq \limsup_{n \rightarrow \infty} \Pr \{\mathcal{G}(n, m, \bar{p}) \in \mathcal{A}\},$$

$$\liminf_{n \rightarrow \infty} \Pr \{G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3) \in \mathcal{A}\} \leq \limsup_{n \rightarrow \infty} \Pr \{\mathcal{G}(n, m, \bar{p}) \in \mathcal{A}\}.$$

We use Theorem 1.1 to establish threshold functions for  $\mathcal{G}(n, m, \bar{p})$  for  $k$ -connectivity (denoted  $\mathcal{C}_k$ ), a Hamilton cycle containment (denoted  $\mathcal{HC}$ ), and a perfect matching containment (denoted  $\mathcal{PM}$ ).

**Theorem 1.2.** *Let  $\max_{1 \leq i \leq m} p_i = o((\ln n)^{-1})$  and  $S_1$  and  $S_{1,2}$  be given by (1.1).*

(i) *If  $S_1 = n(\ln n + c_n)$ , then*

$$\lim_{n \rightarrow \infty} \Pr \{\mathcal{G}(n, m, \bar{p}) \in \mathcal{C}_1\} = \begin{cases} 0 & \text{for } c_n \rightarrow -\infty; \\ e^{-e^{-c}} & \text{for } c_n \rightarrow c \in (-\infty; \infty); \\ 1 & \text{for } c_n \rightarrow \infty \end{cases}$$

(ii) *Let  $k$  be a positive integer and  $a_n = \frac{S_{1,2}}{S_1}$ . If*

$$S_1 = n(\ln n + (k - 1) \ln \ln n + c_n),$$

*then*

$$\lim_{n \rightarrow \infty} \Pr \{\mathcal{G}(n, m, \bar{p}) \in \mathcal{C}_k\} = \begin{cases} 0 & \text{for } c_n \rightarrow -\infty \text{ and } a_n \rightarrow a \in (0; 1]; \\ 1 & \text{for } c_n \rightarrow \infty. \end{cases}$$

**Theorem 1.3.** Let  $\max_{1 \leq i \leq m} p_i = o((\ln n)^{-1})$  and  $S_1$  be given by (1.1). If  $S_1 = n(\ln n + c_n)$  then

$$\lim_{n \rightarrow \infty} \Pr \{ \mathcal{G}(2n, m, \bar{p}(2n)) \in \mathcal{PM} \} = \begin{cases} 0 & \text{for } c_{2n} \rightarrow -\infty; \\ e^{-e^{-c}} & \text{for } c_{2n} \rightarrow c \in (-\infty; \infty); \\ 1 & \text{for } c_{2n} \rightarrow \infty. \end{cases}$$

**Theorem 1.4.** Let  $\max_{1 \leq i \leq m} p_i = o((\ln n)^{-1})$ ,  $S_1$  and  $S_{1,2}$  be given by (1.1) and  $a_n = \frac{S_{1,2}}{S_1}$ . If  $S_1 = n(\ln n + \ln \ln n + c_n)$ , then

$$\lim_{n \rightarrow \infty} \Pr \{ \mathcal{G}(n, m, \bar{p}) \in \mathcal{HC} \} = \begin{cases} 0 & \text{for } c_n \rightarrow -\infty \text{ and } a_n \rightarrow a \in (0; 1]; \\ 1 & \text{for } c_n \rightarrow \infty. \end{cases}$$

Even simple corollaries of Theorems 1.2–1.4 give sharp threshold functions for  $\mathcal{G}(n, m, p)$ . However the method of the proof is strong enough to enable to improve the best known results concerning  $\mathcal{G}(n, m, p)$  even more.

**Theorem 1.5.** Let  $m \gg \ln^2 n$  and

$$p(1 - (1 - p)^{n-1}) = \frac{\ln n + \ln \left( \max \left\{ 1, \ln \left( \frac{np e^{-np} \ln n}{1 - e^{-np}} \right) \right\} \right) + c_n}{m}.$$

Then

$$\lim_{n \rightarrow \infty} \Pr \{ \mathcal{G}(n, m, p) \in \mathcal{HC} \} = \begin{cases} 0 & \text{for } c_n \rightarrow -\infty; \\ 1 & \text{for } c_n \rightarrow \infty. \end{cases}$$

**Theorem 1.6.** Let  $m \gg \ln^2 n$ ,  $k$  be a positive integer, and

$$a_n = (np)^{k-1} \left( \left( \frac{e^{-np} \ln n}{1 - e^{-np}} \right)^{k-1} + \frac{e^{-np} \ln n}{1 - e^{-np}} \right)$$

If

$$p(1 - (1 - p)^{n-1}) = \frac{\ln n + \ln(\max \{1, a_n\}) + c_n}{m},$$

then

$$\lim_{n \rightarrow \infty} \Pr \{ \mathcal{G}(n, m, p) \in \mathcal{C}_k \} = \begin{cases} 0 & \text{for } c_n \rightarrow -\infty; \\ 1 & \text{for } c_n \rightarrow \infty. \end{cases}$$

In the proof we modify and extend the techniques used in [9]. First of all, to get the general result, we couple  $\mathcal{G}(n, m, \bar{p})$  with a sum of independent random graphs  $G_2(n, \hat{p}_2)$  and  $G_3(n, \hat{p}_3)$ . Edges in  $G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3)$  are not fully independent, therefore we prove some additional facts about it. Moreover we need sharp bounds on the minimum degree threshold function for  $\mathcal{G}(n, m, \bar{p})$ . Due to edge correlation, estimation of moments of the random variable counting vertices with a given degree in  $\mathcal{G}(n, m, \bar{p})$  is complicated. Therefore we suggest a different approach to resolve the problem. We divide  $\mathcal{G}(n, m, \bar{p})$  into subgraphs and use coupling to relate construction of those subgraphs with a coupon collector process. Finally the solution of a coupon collector problem combined with the method of moments provides the answer.

Concluding, we provide a general method to establish bounds on threshold functions for many properties for  $\mathcal{G}(n, m, \bar{p})$ . By means of the method we are able to obtain sharp thresholds for  $k$ -connectivity, perfect matching containment and hamiltonicity for the general model. Last but not least we considerably improve known results concerning  $\mathcal{G}(n, m, p)$ . In general, the coupling technique provides a very elegant method to get bounds on threshold functions for random intersection graphs for a large class of properties. The full proofs are presented in [10].

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# Analytic description of the phase transition of inhomogeneous multigraphs

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A random graph from the  $G(n, p)$  model has  $n$  vertices and each pair of vertices is linked with probability  $p$ . In [6], Erdős and Rényi located the density of edges at which the first connected component with more than one cycle - called a *complex* component - appears. Using analytic tools, Janson, Knuth, Łuczak and Pittel derived in [11] more precise informations on the structure of a random graph or multigraph near the birth of the first complex component. Söderberg introduced in [13] a model of inhomogeneous random graphs, extended by Bollobás, Janson and Riordan [3]. This model generalizes  $G(n, p)$  in the following way: each vertex receives a type among a set of  $q$  types, and the probability that a vertex of type  $i$  and one of type  $j$  are linked is the coefficient  $(i, j)$  of a symmetric matrix  $R$  of dimension  $q \times q$ . Among other results, they located the birth of the complex component. We combine here the accuracy of the approach of [11] with the generality of the inhomogeneous random graph model.

Phase transitions for Boolean Satisfiability (SAT) and for Constraint Satisfaction Problems (CSP) are fundamental problems arising in different communities [1,5,9]. Several polynomial SAT and CSP problems can be encoded into the inhomogeneous (multi)graph model, and their probability of satisfiability linked to the phase transition corresponding to the birth of the complex component. In this paper, we derive a complete picture of the finite size scaling and the critical exponents associated to the birth of the complex component of inhomogeneous multigraphs, using analytic methods. Equivalent results can be obtained for graphs. As applications, we present a new proof of an already known result from [12] on the probability of 2-coloriability, and new results on the probability of satisfiability of quantified 2-XOR-formulas [4].

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## 1 Model

We consider a family of labelled multigraphs - loops and multiple edges are allowed - with colored vertices and weighted edges, close to the inhomogeneous random graph model [13]. Let  $R$  be a symmetric square matrix of dimension  $q \times q$  with non-negative coefficients and  $\sigma$  a fixed positive constant. Let  $\{c_1, \dots, c_q\}$  be a set of  $q$  distinct colors. A multigraph  $M$  is a  $(R, \sigma)$ -multigraph if each vertex  $x$  of  $M$  is colored with color  $c(x) \in \{c_1, \dots, c_q\}$  and each edge  $e = (x, y)$  of  $M$  is weighted with  $R_{c(x), c(y)}$ .

Following [11], the *compensation factor*  $\kappa(M)$  of a multigraph  $M$  with  $n$  vertices is defined as  $\prod_{x=1}^n (2^{m_{xx}} \prod_{y=x}^n m_{xy}!)^{-1}$  where  $m_{xy}$  is the number of edges binding  $x$  to  $y$  in  $M$  and  $m = \sum_{x=1}^n \sum_{y=x}^n m_{xy}$  is the total number of edges. For a multigraph  $M$  with  $m$  edges, the number of sequences of couples of vertices  $(x_1, y_1), \dots, (x_m, y_m)$  that lead to  $M$  is exactly  $2^m m! \kappa(M)$ .

Given a  $(R, \sigma)$ -multigraph  $M$ , we define its weight  $\omega(M)$  as

$$\omega(M) = \kappa(M) \times \sigma^{\#\text{cc}(M)} \times \prod_{(x, y) \in \{\text{edges of } M\}} R_{c(x), c(y)},$$

where  $\#\text{cc}(M)$  denotes the number of connected components of  $M$ . The parameter  $\sigma$  is not present in [13], but classic in statistic physics, for instance in the random cluster model. Finally,  $g_{R, \sigma}(n, m)$  denotes the sum of the weights of the  $(R, \sigma)$ -multigraphs built with  $n$  vertices and  $m$  edges

$$g_{R, \sigma}(n, m) = \sum_{|G|=n, \|G\|=m} \omega(G).$$

An edge-weighted multigraph is *vertex-transitive* if its automorphism group is transitive and also preserve the weights – see for instance Godsil and Gordon [10]. Intuitively, the vertex-transitive property means that, using only the topology of the multigraph, no vertex can be distinguished from another.

Let  $G$  be a multigraph with  $q$  vertices and weighted edges (loops are also allowed). The *weighted adjacency matrix*  $R = R(G)$  of  $G$  is built as follows:  $R$  is a  $q \times q$  matrix with entry  $R_{i, j}$  equal to the total weight of the edges between vertex  $i$  and vertex  $j$ . For simplicity, we say that a matrix  $R$  is *transitive* if the weighted multigraph associated is vertex-transitive. Due to size constraints, we only detail the results corresponding to transitive matrices  $R$ , although general theorems have also been derived and will be part of a longer version of the paper.

## 2 Main theorem

The asymptotic growth of  $g_{R,\sigma}(n, m)$  with respect to  $n$  is different whether  $m/n$  stays at a fixed distance below  $1/2$  or is close to it. We call this rupture the *phase transition* of inhomogeneous multigraphs. The evolution of the probability of satisfiability of several SAT and CSP problems can be explained by this phase transition: two examples are presented in the next section.

**Theorem 2.1.** *Let  $R$  be a transitive matrix with greatest eigenvalue  $\delta$  and  $\sigma$  a positive fixed constant. Let  $c$  be the number of connected components in the multigraph associated to  $R$ . Let  $\chi(X)$  be the polynomial  $\prod_{\lambda \in \text{Spec}(R) \setminus \delta} (1 - \frac{\lambda}{\delta} X)$ . For any small fixed  $\epsilon > 0$ , as  $n$  is large and  $m$  is such that  $\frac{2m}{n} < 1 - \epsilon$ , then*

$$g_{R,\sigma}(n, m) \sim \frac{\delta^m (\sigma q)^{n-m}}{\chi(\frac{2m}{n})^{\sigma/2}} \left(1 - \frac{2m}{n}\right)^{\frac{1-c\sigma}{2}} \sqrt{1 - \frac{m}{n}} \frac{n^{2m}}{2^m m!}.$$

As  $n$  is large and  $m = \frac{n}{2}(1 + \mu n^{-1/3})$  with  $\mu$  bounded,

$$g_{R,\sigma}(n, m) \sim \frac{\delta^m (\sigma q)^{n-m}}{\chi(1)^{\sigma/2}} n^{(c\sigma-1)/6} \frac{n^{2m}}{2^m m!} \phi_{c\sigma}(\mu)$$

where  $\phi_\sigma(\mu)$  is equal to  $\sqrt{2\pi} \sum_k e_k^{(\sigma)} \sigma^k A(3k + \frac{\sigma}{2}, \mu)$ ,  $e_k^{(\sigma)}$  is the  $(2k)$ -th coefficient of  $\left(\sum_n \frac{(6n)! z^{2n}}{(2n)!(3n)! 2^n (3!)^n}\right)^\sigma$  and  $A(y, \mu) = \frac{e^{-\mu^3/6}}{3^{(y+1)/3}} \sum_k \frac{(3^{2/3} \mu/2)^k}{k! \Gamma((y+1-2k)/3)}$ .

*Proof.* Let  $\delta$  denote the greatest eigenvalue of  $R$ . We can assume it to be equal to 1 without loss of generality, replacing  $R$  by  $\frac{1}{\delta} R$  and  $g(n, m)$  by  $\delta^m g(n, m)$ . We can also assume that the number  $c$  of connected components of the multigraph encoded by  $R$  is 1: the  $(R, \sigma)$ -multigraphs are in a one-to-one mapping with the  $(S, c\sigma)$ -multigraphs where  $S$  is the adjacency matrix of one of the connected components.

Since  $R$  is symmetric, there exist an orthogonal matrix  $Q$  and a diagonal matrix  $\Delta$  such that  $R = Q\Delta Q^T$ . Furthermore, the vector  $\vec{1}$  is an eigenvector for the greatest eigenvalue 1. We choose  $\Delta_{1,1} = 1$ .

A  $R$ -Cayley tree is a  $R$ -rooted non-planar labelled tree. Let  $T_i(z)$  denote its generating function when the color of the root is  $i$  and  $\vec{T}(z) = (T_1(z) \dots T_q(z))^T$ . The generating function of the  $R$ -unrooted trees and  $R$ -unicyclic graphs are denoted by  $U(z)$  and  $V(z)$ . A  $R$ -path of trees is a colored path that links two vertices (that may not be distinct) of color  $i$  and  $j$ . Each internal vertex of the path is the root of a colored  $R$ -Cayley

tree. Its generating function is  $P_{i,j}(z)$ . Using the analytic combinatorics tools (a good reference is [8]), the combinatorial specification of  $R$ -Cayley trees translates into the following equations:

$$\begin{aligned} \vec{T}(z) &= T(z)\vec{1} & V(z) &= -\frac{1}{2}\log(1 - T(z)) - \frac{1}{2}\log(\chi(T(z))) \\ U(z) &= q(T(z) - \frac{1}{2}T(z)^2) & P_{i,j}(z) &= \frac{1}{q(1-T(z))} + \sum_{l=2}^q Q_{i,l} Q_{j,l} \frac{\Delta_{l,l}}{1-\Delta_{l,l}T(z)} \end{aligned}$$

Remark that at the first order,  $U(z)$ ,  $V(z)$  and  $P_{i,j}(z)$  behave as their non-colored counterparts  $T(z)$ ,  $-\frac{1}{2}T(z)^2$ ,  $-\frac{1}{2}\log(1 - T(z))$  and  $\frac{1}{1-T(z)}$ . Furthermore, the first order of  $P_{i,j}(z)$  is independent of  $i$  and  $j$ .

When  $\frac{2m}{n} < 1 - \epsilon$ , with high probability a graph with  $n$  vertices and  $m$  edges contains only trees and unicyclic components (see [7]). Therefore,  $g_{R,\sigma}(n, m) \sim n![z^n] \frac{(\sigma U(z))^{n-m}}{(n-m)!} e^{\sigma V(z)}$ . The *Large Powers* Theorem of [8] ends the proof.

Let us remind that the *excess* of a graph is the difference between the number of edges and of vertices. The *complex* part of a graph is the set of its connected components that have positive excess. Deleting the vertices of degree one and fusing the vertices of degree two, each graph can be reduced to a simpler graph with same excess, called its *kernel*, with minimum degree at least three. Reciprocally, any such graph can be developed by replacing edges by paths and adding trees to the vertices.

When  $m = \frac{n}{2}(1 + \mu n^{-1/3})$ , [11] proved that the proportion of multigraphs with non-cubic kernel is negligible. This holds true for inhomogeneous graphs. The set  $\mathbb{K}_k$  of the cubic multigraphs of excess  $k$  is finite. The sum of their compensation factors with a weight  $\sigma$  for each connected component is  $(2k)!e_k^{(\sigma)}$  as defined in the theorem. The generating function of the  $R$ -developed cubic graphs of fixed excess  $k$  is

$$K_{k,\sigma}(z) = \sum_{K \in \mathbb{K}_k} \sum_{\vec{c} \in [1,q]^{2k}} \kappa(K)\sigma^{\#\text{cc}(K)} \frac{1}{(2k)!} \left( \prod_{i \in K} T_{c_i}(z) \right) \prod_{(i,j) \in K} P_{c_i,c_j}(z).$$

Replacing  $P_{i,j}(z)$  by its first order term, we obtain that near its dominant singularity,  $K_{k,\sigma}(z) \sim \frac{e_k^{(\sigma)} T(z)^{2k}}{q^k (1-T(z))^{3k}}$ . The number of  $(R, \sigma)$ -multigraphs with  $n$  vertices,  $m$  edges and kernel of excess  $k$  is

$$g(n, m, k) \sim n![z^n] \frac{(\sigma U(z))^{n-m+k}}{(n - m + k)!} e^{\sigma V(z)} K_{k,\sigma}(z).$$

The asymptotic is derived using the same tools as in Theorem 11 of [2] or Lemma 3 of [11]. Finally,  $g(n, m) = \sum_{k \geq 1} g(n, m, k)$ . When  $R$  is not transitive, but the multigraph encoded by  $R$  is connected, a similar result holds. □

### 3 Examples of application of the main theorem

The asymptotic probability for a multigraph with a large number  $n$  of vertices and  $m$  edges to be bipartite evolves from 1 to 0 when the quotient  $m/n$  increases from 0 to  $1/2$ . This phase transition can be precisely described using our main theorem. The bipartite multigraphs are in a one-to-one mapping with the  $\left(\binom{0 \ 1}{1 \ 0}, \frac{1}{2}\right)$ -multigraphs. Since the sum of the compensator factors of the multigraphs with  $n$  vertices and  $m$  edges is  $\frac{n^{2m}}{m!2^m}$ , the probability for a multigraph to be bipartite is  $\frac{2^m m!}{n^{2m}} g\left(\binom{0 \ 1}{1 \ 0}, \frac{1}{2}\right)(n, m)$ , given by the Theorem 2.1. This result has already been derived by Pittel and Yeum [12] when  $m$  is around  $n/2$ .

In [4], the authors analyse 2-QXOR-SAT formulas. Those are quantified conjunctions of  $m$  XOR-clauses with  $\beta$  universal and  $n$  existential variables

$$\forall x_1 \dots x_\beta \exists y_1 \dots y_n \bigwedge_{i=1}^m (y_{f_{i,1}} \oplus y_{f_{i,2}} = (e_{i(1)} \dots e_{i(\beta)}) \cdot (x_1 \dots x_\beta)^T)$$

where each  $e_i$  is a  $\beta$ -tuple of bits and  $(a_1 \dots a_k) \cdot (b_1 \dots b_k)^T = \bigoplus_{i=1}^k a_i \wedge b_i$ . A clause  $\varphi$  is characterized by a triplet  $(y_{\varphi,1}, y_{\varphi,2}, e_\varphi)$ . The authors study how the probability of satisfiability  $P_{\text{SAT}}(n, m)$  evolves with the number  $m$  of clauses when the number  $n$  of existential variables is large, and locate the value of  $m$  at which the phase transition occurs.

We consider clauses  $\varphi$  such that  $e_\varphi$  is in a certain fixed multiset  $E$  of  $\beta$ -tuples of bits. We call the formulas that contain only those clauses the  $E$ -formulas. Let  $x$  be an integer in  $[1, 2^\beta]$  and  $[x]_2$  denote the binary decomposition of  $x - 1$ , interpreted as a  $\beta$ -tuple of bits. To a multiset  $E$  of  $\beta$ -tuples of bits, we associate a square symmetric transitive matrix  $R^{(E)}$  of dimension  $2^\beta \times 2^\beta$  such that  $R_{x,y}^{(E)}$  is the number of occurrences of  $[x]_2 \oplus [y]_2$  in  $E$ :  $R_{x,y}^{(E)} = \#\{e \in E \mid [x]_2 \oplus [y]_2 = e\}$ . Using similar ideas as in [4], a one-to-one mapping is built between the satisfiable  $E$ -formulas with  $n$  existential variables and  $m$  clauses and the  $(R^{(E)}, 2^{-\beta})$ -multigraphs with  $n$  vertices and  $m$  edges. Therefore,  $P_{\text{SAT}}(n, m)$  is equal to  $\frac{2^m m!}{n^{2m} |E|^m} g_{R^{(E)}, 2^{-\beta}}(n, m)$  and its asymptotic is derived by the theorem 2.1, using the parameters  $q, \sigma = q^{-1}, \delta, c$  and  $\chi(X)$  of the matrix  $R^{(E)}$ . In [4], the authors located the phase transition and described it qualitatively - coarse or sharp - while our result is more precise and quantifies the evolution of  $P_{\text{SAT}}(n, m)$  from the subcritical to the critical range of  $m$ .

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Fixed-point



# On the Bruhat-Chevalley order on fixed-point-free involutions

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## 1 Introduction

The purpose of this paper is twofold. First is to prove that the Bruhat-Chevalley ordering restricted to fixed-point-free involutions is a lexicographically shellable poset. Second is to prove that the Deodhar-Srinivasan poset is a graded subposet of the Bruhat-Chevalley poset structure on fixed-point-free involutions.

In this work we are concerned with the interaction between two well known subgroups of the special linear group  $SL_{2n}$ , namely a Borel subgroup and a symplectic subgroup. Without loss of generality, we choose the Borel subgroup  $B$  to be the group of invertible upper triangular matrices, and define the *symplectic group*,  $Sp_{2n}$  as the subgroup of fixed elements of the involutory automorphism  $\theta : SL_{2n} \rightarrow SL_{2n}$ ,  $\theta(g) = J(g^{-1})^\top J^{-1}$ , where  $J$  denotes the skew form  $J = \begin{pmatrix} 0 & \omega_0 \\ -\omega_0 & 0 \end{pmatrix}$ , and  $\omega_0$  is the  $n \times n$ , 0/1 matrix with 1's on its main anti-diagonal.

It is clear that  $B$  acts by left-multiplication on  $SL_{2n}/Sp_{2n}$ . We investigate the covering relations of the poset  $F_{2n}$  of inclusion relations among the  $B$ -orbit closures. To further motivate our discussion and help the reader to place our work appropriately we look at a related situation. It is well known that the symmetric group of permutation matrices,  $S_m$  parameterizes the orbits of the Borel group of upper triangular matrices  $B \subset SL_m$  in the flag variety  $SL_m/B$ . For  $u \in S_m$ , let  $\dot{u}$  denote the right coset in  $SL_m/B$  represented by  $u$ . The classical *Bruhat-Chevalley ordering* is defined by  $u \leq_{S_m} v \iff B \cdot \dot{u} \subseteq \overline{B \cdot \dot{v}}$  for  $u, v \in S_m$ .

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A permutation  $u \in S_m$  is said to be an *involution*, if  $u^2 = id$ , or equivalently, its permutation matrix is a symmetric matrix. We denote by  $I_m$  the set of all involutions in  $S_m$ , and consider it as a subposet of the Bruhat-Chevalley poset  $(S_m, \leq_{S_m})$ . Let  $m$  be an even number,  $m = 2n$ . An element  $x \in I_{2n}$  is called *fixed-point-free*, if the matrix of  $x$  has no non-zero diagonal entries. In [15], Example 10.4, Richardson and Springer show that there exists a poset isomorphism between  $F_{2n}$  and a subposet of fixed-point-free involutions in  $I_{2n}$ . Unfortunately,  $F_{2n}$  does not form an interval in  $I_{2n}$ , hence it does not immediately inherit nice properties therein. Let  $\leq$  denote the restriction of the Bruhat-Chevalley ordering on  $F_{2n}$ . Our first main result is that  $(F_{2n}, \leq)$  is “*EL-shellable*,” which is a property that well known to be true for many other related posets. See [3–5, 8, 11, 12]. See [10], also. Notice that there are several versions of lexicographic shellability and *EL-shellability* is the strongest one. In [13] it is shown that  $F_{2n}$  is *CL-shellable*. So, our result is a strengthening of this result of [13] in the special case of the poset  $F_{2n}$ . Recall that a finite graded poset  $P$  with a maximum and a minimum element is called *EL-shellable*, if there exists a map  $f = f_\Gamma : C(P) \rightarrow \Gamma$  between the set of covering relations  $C(P)$  of  $P$  into a totally ordered set  $\Gamma$  satisfying: 1) In every interval  $[x, y] \subseteq P$  of length  $k > 0$  there exists a unique saturated chain  $\mathfrak{c} : x_0 = x < x_1 < \dots < x_{k-1} < x_k = y$  such that the entries of the sequence  $f(\mathfrak{c}) = (f(x_0, x_1), f(x_1, x_2), \dots, f(x_{k-1}, x_k))$  is weakly increasing. 2) The sequence  $f(\mathfrak{c})$  of the unique chain  $\mathfrak{c}$  from (1) is the smallest among all sequences of the form  $(f(x_0, x'_1), f(x'_1, x'_2), \dots, f(x'_{k-1}, x_k))$ , where  $x_0 \leq x'_1 \leq \dots \leq x'_{k-1} \leq x_k$ . Recall that the *order complex* of a poset  $P$  is the abstract simplicial complex  $\Delta(P)$  whose simplexes are the chains in  $P$ . For a lexicographically shellable poset the order complex is shellable, in particular it implies that  $\Delta(P)$  is Cohen-Macaulay [2]. These, of course, are among the most desirable properties of a topological space. One of the reasons the *EL-shellability* of  $F_{2n}$  is not considered before is that there is a closely related *EL-shellable* partial order studied by Deodhar and Srinivasan in [7] which we denote here as  $\leq_{DS}$ . By some authors the Deodhar-Srinivasan’s ordering is thought to be the same as Bruhat-Chevalley ordering on  $F_{2n}$ . A careful inspection of the Hasse diagrams of  $(F_{2n}, \leq)$  and  $(\tilde{F}_{2n}, \leq_{DS})$  reveals that these two posets are “almost” the same but different. Our second main result is that the rank functions of these posets are the same, and furthermore, the latter is a graded subposet of the former.



## 2 Preliminaries

We denote the set  $\{1, \dots, m\}$  by  $[m]$ . In this work, all posets are assumed to be finite and assumed to have a minimal and a maximal element, denoted by  $\hat{0}$  and  $\hat{1}$ , respectively. Recall that in a poset  $P$ , an element  $y$  is said to *cover* another element  $x$ , if  $x < y$  and if  $x \leq z \leq y$  for some  $z \in P$ , then either  $z = x$  or  $z = y$ . In this case, we write  $y \rightarrow x$ . Given  $P$ , we denote by  $C(P)$  the set of all covering relations of  $P$ . An (increasing) *chain* in  $P$  is a sequence of distinct elements such that  $x = x_1 < x_2 < \dots < x_{n-1} < x_n = y$ . A chain in a poset  $P$  is called *saturated* (or, *maximal*), if it is of the form  $x = x_1 \leftarrow x_2 \leftarrow \dots \leftarrow x_{n-1} \leftarrow x_n = y$ . Recall also that a poset is called *graded* if all maximal chains between any two comparable elements  $x \leq y$  have the same length. This amounts to the existence of an integer valued function  $\ell_P : P \rightarrow \mathbb{N}$  satisfying 1)  $\ell_P(\hat{0}) = 0$ , 2)  $\ell_P(y) = \ell_P(x) + 1$  whenever  $y$  covers  $x$  in  $P$ .  $\ell_P$  is called the *length function* of  $P$ .

Let  $Sym_n$  denote the affine space of symmetric matrices and let  $Sym_n^0$  denote its closed subset consisting of symmetric matrices with determinant 1. Similarly, let  $Skew_{2n}$  denote the affine space of skew-symmetric matrices, and let  $Skew_{2n}^0$  denote its closed subset consisting of elements with determinant 1. Let  $X$  denote any of the spaces  $Sym_n$ ,  $Sym_n^0$ ,  $Skew_{2n}$ , or  $Skew_{2n}^0$ . Then the special linear group of appropriate rank acts on  $X$  via  $g \cdot A = (g^{-1})^T A g^{-1}$ . Define  $SO_n := \{g \in SL_n : gg^T = id_n\}$ . The symmetric spaces  $SL_{2n}/Sp_{2n}$  and  $SL_n/SO_n$  can be canonically identified with the spaces  $Skew_{2n}^0$  and  $Sym_n^0$ , respectively (for details see [9]). Recall that an  $n \times n$  *partial permutation matrix* (or, a *rook matrix*) is a 0/1 matrix with at most one 1 in each row and each column. The set of all  $n \times n$  rook matrices is denoted by  $R_n$ . In [14], Renner shows that  $R_n$  parameterizes the  $B \times B$ -orbits on the monoid of  $n \times n$  matrices. It is known that the partial ordering on  $R_n$  induced from the containment relations among the  $B \times B$ -orbit closures is a lexicographically shellable poset (see [4]). On the other hand, for the purposes of this paper, it is more natural for us to look at the inclusion poset of  $B^T \times B$ -orbit closures in  $R_n$ , which we denote by  $(R_n, \leq_{Rook})$ . A symmetric rook matrix is called a *partial involution*. The set of all partial involutions in  $R_n$  is denoted by  $PI_n$ . It is known that each Borel orbit in  $Sym_n$  contains a unique element of  $PI_n$ . A rook matrix is called a *partial fixed-point-free involution*, if it is symmetric and does not have any non-zero entry on its main diagonal. We denote by  $PF_{2n}$  the set of all partial fixed-point-free involutions. It is known that  $PF_{2n}$  parameterizes the Borel orbits in  $Skew_{2n}$ .

Containment relations among the closures of Borel orbits in  $Skew_{2n}$  define a partial ordering on  $PF_{2n}$ . We denote its opposite by  $\leq_{Skew}$ . Similarly, on  $PI_n$  we have the opposite of the partial ordering induced from the containment relations among the Borel orbit closures in  $Sym_n$ . We denote this opposite partial ordering by  $\leq_{Sym}$ . In [11], Incitti, studying the restriction of the partial order  $\leq_{Sym}$  on  $I_n$ , finds an  $EL$ -labeling for  $I_n$ . Let us mention that in a recent preprint, Can and Twelbeck, using an extension of Incitti’s edge-labeling show that  $PI_n$  is  $EL$ -shellable. See [5].

There is a combinatorial method for deciding when two elements  $x$  and  $y$  from  $(R_n, \leq_{Rook})$  (respectively, from  $(PI_n, \leq_{Sym})$ , or from  $(PF_{2n}, \leq_{Skew})$ ) are comparable with respect to  $\leq_{Rook}$  (respectively, with respect to  $\leq_{Sym}$ , or  $\leq_{Skew}$ ). Denote by  $Rk(x)$  the matrix whose  $i, j$ -th entry is the rank of the upper left  $i \times j$  submatrix of  $x$ . We call  $Rk(x)$ , the *rank-control matrix* of  $x$ . Let  $A = (a_{i,j})$  and  $B = (b_{i,j})$  be two matrices of the same size with real number entries. We write  $A \leq B$  if  $a_{i,j} \leq b_{i,j}$  for all  $i$  and  $j$ . Then  $x \leq_{Rook} y \iff Rk(y) \leq Rk(x)$ . The same criterion holds for the posets  $\leq_{Sym}$  and  $\leq_{Skew}$ . Now, suppose  $x$  is an  $m \times m$  matrix with the rank-control matrix  $Rk(x) = (r_{i,j})_{i,j=1}^m$ . Set  $r_{0,i} = 0$  for  $i = 0, \dots, m$ , and define  $\rho_{\leq}(x) = \#\{(i, j) : 1 \leq i \leq j \leq 2n \text{ and } r_{i,j} = r_{i-1,j-1}\}$ ,  $\rho_{<}(x) = \#\{(i, j) : 1 \leq i < j \leq 2n \text{ and } r_{i,j} = r_{i-1,j-1}\}$ . Then the length function  $\ell_{PF_{2n}}$  of the poset  $PF_{2n}$  is equal to the restriction of  $\rho_{<}$  to  $PF_{2n}$ . Furthermore,  $y$  covers  $x$  if and only if  $Rk(y) \leq Rk(x)$  and  $\ell_{PF_{2n}}(y) - \ell_{PF_{2n}}(x) = 1$ . Similarly,  $\ell_{PI_{2n}}$  is the restriction of  $\rho_{\leq}$  to  $PI_{2n}$ , and that  $y$  covers  $x$  if and only if  $Rk(y) \leq Rk(x)$  and  $\ell_{I_{2n}}(y) - \ell_{I_{2n}}(x) = 1$ . For details, see [1].

### 3 Results

It turns out that the intersection  $PF_{2n} \cap I_{2n}$  is equal to  $F_{2n}$ , and furthermore,  $(F_{2n}, \leq_{Sym})$  and  $(F_{2n}, \leq_{Skew})$  are isomorphic. The relationships between the posets  $PI_{2n}$ ,  $PF_{2n}, I_{2n}$  and  $F_{2n}$  are as follows.

Let  $w_0 \in PI_{2n}$  denote the “longest permutation,” namely, the  $2n \times 2n$  anti-diagonal permutation matrix, and let  $j_{2n} \in F_{2n}$  denote the  $2n \times 2n$  fixed-point-free involution having non-zero entries at the positions

$$(1, 2), (2, 1), (3, 4), (4, 3), \dots, (2n - 1, 2n), (2n, 2n - 1), \text{ only .}$$

In other words,  $j_{2n}$  is the fixed-point-free involution with the only non-zero entries along its super-diagonals. Then  $I_{2n}$  is an interval in  $PI_{2n}$  with the smallest element  $id_{2n}$  and the largest element  $w_0$ . Similarly,  $F_{2n}$  is an interval in  $PF_{2n}$  with the smallest element  $j_{2n}$  and the largest element  $w_0$ .

Consider  $F_{2n}$  as a subposet of  $I_{2n}$  and let  $x, y \in F_{2n}$  be two elements such that  $x \leq y$ . It turns out there exists a saturated chain in  $I_{2n}$  from  $x$  to  $y$  consisting of fixed-point-free involutions only. With the help of this observation, we prove

**Theorem 3.1.**  $F_{2n}$  is an EL-shellable poset.

Sketch of the proof. Recall that  $F_{2n}$  is a connected graded subposet of  $I_{2n}$ . Therefore, its covering relations are among the covering relations of  $I_{2n}$  described in [11]. Let  $x$  and  $y$  be two fixed-point-free involutions. We know the existence of a saturated chain between  $x$  and  $y$  that is entirely contained in  $F_{2n}$ . Since lexicographic ordering is a total order on maximal chains, there exists a unique largest such chain, say  $c$ . The idea of the proof is showing that  $c$  is the unique decreasing chain. Once this is done, by switching the order of our totally ordered set  $\mathbb{Z}^2$ , we obtain the lexicographically smallest chain, which is the unique increasing chain.

As an important consequence of Theorem 3.1, we further show that

**Theorem 3.2.** The order complex  $\Delta(F_{2n})$  triangulates a ball of dimension  $n(n-1)-2$ .

As it is mentioned in the introduction, the posets  $(\tilde{F}_{2n}, \leq_{DS})$  and  $(F_{2n}, \leq)$  are different. Indeed, for  $2n = 6$  the Hasse diagrams of these two posets differ by an edge. Contrary to this observation, we have the following

**Theorem 3.3.** The length functions of  $(F_{2n}, \leq)$  and  $(\tilde{F}_{2n}, \leq_{DS})$  are the same. Covering relations of the poset  $\tilde{F}_{2n}$  are among the covering relations of  $F_{2n}$ .

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# A geometric approach to combinatorial fixed-point theorems: extended abstract

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**Abstract.** We develop a geometric framework that unifies several different combinatorial fixed-point theorems related to Tucker’s lemma and Sperner’s lemma, showing them to be different geometric manifestations of the same topological phenomena. In doing so, we obtain (1) new Tucker-like and Sperner-like fixed-point theorems involving an exponential-sized label set; (2) a generalization of Fan’s parity proof of Tucker’s Lemma to a much broader class of label sets; and (3) direct proofs of several Sperner-like lemmas from Tucker’s lemma via explicit geometric embeddings.

## 1 Introduction

Combinatorial fixed-point theorems such as the Sperner and Tucker lemmas have generated a wealth of interest in recent decades, in part due to the discovery of important new applications in economics and theoretical computer science (see [2, 12, 15]). Extensive research has examined the construction of *direct proofs* of the implications among these and other similar theorems (and generalizations), yielding many different proofs via a variety of methods (see [4, 6, 8]). Some of this work has succeeded in connecting fixed-point theorems in the (Brouwer, Sperner)-family to the seemingly unrelated antipodality theorems in the (Borsuk-Ulam, Tucker)-family; for example, Su has shown that it is possible to prove the Brouwer fixed-point theorem directly from the Borsuk-Ulam theorem via an explicit topological construction [13], and Živalcević [16] has shown how Ky Fan’s [3] generalization of Tucker’s lemma implies Sperner’s lemma. However, the construction of a direct proof that Tucker’s lemma implies Sperner’s lemma appears to remain an open question [11].

To shed some light on this question, we investigate the Tucker and Sperner lemmas from a *geometric* viewpoint. Cast in this light, it be-

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The full version of this paper is available on arXiv.org as [7].

comes apparent that the Tucker and Sperner lemmas are actually members of a much larger family of combinatorial fixed-point theorems sharing a common topological structure, but having different geometric manifestations. Our approach hence unifies many known combinatorial fixed-point theorems, and yields a new Tucker-type lemma and a new Sperner-type lemma, both with an exponential number of labels. In doing so, we generalize the technique in [3] to obtain a framework that proves our new Tucker-like theorem without any topological fixed-point theorems. As a bonus, our framework also permits us to prove some of the Sperner-like theorems directly from Tucker’s lemma via explicit geometric embeddings. Moreover, we derive some insight into why Sperner’s lemma may be difficult to prove directly from Tucker’s lemma—the analogy between Borsuk-Ulam and Tucker results is geometrically different from the analogy between Brouwer and Sperner results, and alternate Sperner-like theorems provide a more direct analogy.

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### Combinatorial fixed-point theorems

We omit definitions of standard terminology and notation; details can be found in the full version of the paper, or a standard text such as that of Matoušek [10]. We shall use the following conventions for embeddings:  $B^n$  is defined as  $\{x \in \mathbb{R}^n, \|x\|_2 \leq 1\}$ ;  $\Delta^{n-1}$  is defined as  $\text{conv}\{e_i : 1 \leq i \leq n\}$ ;  $\diamond^n$  is defined as  $\text{conv}\{\pm e_i\}$ ;  $\square^n$  is defined as  $\text{conv}\{(x_1, \dots, x_n) : x_i \in \{-1, 1\}\}$ ; here,  $\text{conv}(S)$  is the convex hull of the set  $S$ .

We define the *vertices*  $V(T)$  of a triangulation  $T$  to be the set of all 0-simplices in  $T$ . A *label function*  $\lambda$  is a mapping from  $V(T)$  to a finite *label set*  $L$ . In the case of the Tucker and Sperner lemmas, the sets  $\{1, \dots, n+1\}$  or  $\{1, -1, \dots, n, -n\}$  are typically used for  $L$ . However, in our paper, we will instead represent these labels as the sets of *extreme points*  $\text{ext}(\Delta^n) = \{e_1, \dots, e_{n+1}\}$  and  $\text{ext}(\diamond^n) = \{e_1, -e_1, \dots, e_n, -e_n\}$ . Cast in this framework, we shall state Sperner’s lemma as follows:

**Theorem 1.1 (Sperner’s lemma).** *Let  $T$  be a triangulation of  $\Delta^n$ . Let  $\lambda : V(T) \rightarrow \text{ext}(\Delta^n)$  be a label function with the property that for all  $x = (x_1, \dots, x_{n+1}) \in V(T)$ , for all  $1 \leq i \leq n+1$ , if  $x_i = 0$ , then  $\lambda(x) \neq e_i$  (such a  $\lambda$  is sometimes called a proper colouring). Then  $T$*

contains a panchromatic simplex—that is, a simplex  $\sigma$  such that  $\{\lambda(v) : v \in V(\sigma)\} = \{e_1, \dots, e_{n+1}\}$ .

We can establish results similar to Sperner’s lemma in other geometric spaces as long as we have suitable analogies of the notions of *proper colouring* and *panchromatic simplex*. The following are immediate:

**Proposition 1.2.** *Let  $\sigma$  be a simplex and let  $\lambda : V(\sigma) \rightarrow \text{ext}(\diamond^n)$  be a label function. Define a complementary edge to be two vertices  $v_1, v_2 \in V(\sigma)$  with  $\lambda(v_1) = -\lambda(v_2)$ . Then  $\text{conv}\{\lambda(v) : v \in V(\sigma)\}$  intersects the interior of  $\diamond^n$  if and only if  $\sigma$  contains a complementary edge.*

**Proposition 1.3.** *Let  $\sigma$  be a simplex and let  $\lambda : V(\sigma) \rightarrow \text{ext}(\square^n)$  be a label function. We say  $\sigma$  is a neutral simplex if for all  $1 \leq i \leq n$ , there exist vertices  $v_1, v_2 \in V(\sigma)$  such that  $\lambda_i(v_1) = -1$ ,  $\lambda_i(v_2) = +1$ , where  $\lambda_i(v)$  is the  $i^{\text{th}}$  coordinate of  $\lambda(v)$ . Then  $\text{conv}\{\lambda(v) : v \in V(\sigma)\}$  intersects the interior of  $\square^n$  if and only if  $\sigma$  is a neutral simplex.*

Two Sperner-like theorems can immediately be derived by imposing the right labelling constraints:

**Theorem 1.4 (Octahedral Sperner with octahedral labels).** *Let  $T$  be a triangulation of  $\diamond^n$ . Let  $\lambda : V(T) \rightarrow \text{ext}(\diamond^n)$  be a label function such that for all boundary vertices  $x = (x_1, \dots, x_n) \in V(T) \cap \partial(\diamond^n)$ , for all  $1 \leq i \leq n$ , if  $x_i \geq 0$  (respectively, if  $x_i \leq 0$ ), then  $\lambda(x) \neq -e_i$  (respectively,  $\lambda(x) \neq e_i$ ). Then  $T$  contains a complementary edge.*

**Theorem 1.5 (Cubical Sperner with cubical labels).** *Let  $T$  be a triangulation of  $\square^n$ . Let  $\lambda : V(T) \rightarrow \text{ext}(\square^n)$  be a label function such that for all vertices  $x = (x_1, \dots, x_n) \in V(T)$ , for all  $1 \leq i \leq n$ , if  $x_i \in \{-1, 1\}$ , then  $\lambda(x)_i = x_i$ . Then  $T$  contains a neutral simplex.*

Theorem 1.4 is a special case of a theorem originally conjectured by Atanassov and proven by De Loera et al. [1, 9]. Theorem 1.5 is implied by a result of Kuhn [8], in which the vertices of a triangulation of  $\square^n$  are labelled using only  $(n + 1)$  labels. Allowing the domain and codomain to differ yields several additional fixed-point theorems, such as the following:

**Theorem 1.6 (Cubical Sperner with octahedral labels).** *Let  $T$  be a triangulation of  $\square^n$ . Let  $\lambda : V(T) \rightarrow \text{ext}(\diamond^n)$  be a label function with the property that for all vertices  $x = (x_1, \dots, x_n) \in V(T)$ , for all  $1 \leq i \leq n$ , if  $x_i \in \{-1, 1\}$ , then  $\lambda_i(x) \neq -x_i e_i$ . Then  $T$  contains a complementary edge.*

**Theorem 1.7 (Octahedral Sperner with cubical labels).** *Let  $T$  be a triangulation of  $\diamond^n$ . Let  $\lambda : V(T) \rightarrow \text{ext}(\square^n)$  be a label function with the property that for all vertices  $x \in V(T)$ , for all  $v \in \text{ext}(\square^n)$ , if  $v^T x = 1$ , then  $\lambda(x) \neq -v$ . Then  $T$  contains a neutral simplex.*

Theorem 1.6 is due to Freund [4]. Theorem 1.7 appears to be novel, though it is related to a general result of Freund [5] that claims a conclusion too strong to hold here (details in full paper). Although the combinatorial methods of Freund and of Kuhn appear insufficient to prove Theorem 1.7, it can be proven using standard techniques via Brouwer's fixed-point theorem (see full version).

To state Tucker-like theorems, we require a notion of antipodality:

**Definition 1.8.** Let  $T$  be a topological triangulation of  $B^n$  with  $X = \bigcup_{\sigma \in T} \sigma$ .  $T$  is said to be *antipodally symmetric on the boundary* if, for all simplices  $\sigma \in \partial(T)$ , the reflected simplex  $-\sigma$  also lies in  $\partial(T)$ .

Tucker's lemma [14] can then be stated as follows:

**Theorem 1.9 (Tucker's lemma).** *Let  $T$  be a triangulation of  $B^n$  that is antipodally symmetric on the boundary of the domain  $X$ . Let  $\lambda : V(T) \rightarrow \text{ext}(\diamond^n)$  be a label function such that  $\lambda(v) = -\lambda(-v)$  for all  $v \in \partial(X)$ . Then  $T$  contains a complementary edge.*

We can establish similar theorems using labels from other codomains homeomorphic to  $B^n$ . By extending Ky Fan's theorem [3] to a broader class of label sets, we establish a combinatorial proof of the following in the full version:

**Theorem 1.10 (Tucker's lemma with cubical labels).** *Let  $T$  be a triangulation of  $B^n$  that is antipodally symmetric on the boundary of the domain  $X$ . Let  $\lambda : V(T) \rightarrow \text{ext}(\square^n)$  be a label function such that  $\lambda(v) = -\lambda(-v)$  for all  $v \in \partial(X)$ . Then  $T$  contains a neutral simplex.*

## 2 Geometric proofs of Sperner-like theorems

We now describe a technique that enables us to explicitly construct geometric reductions between some of combinatorial fixed-point theorems discussed above. We illustrate our technique through an example:

**Theorem 2.1.** *Tucker's theorem implies Theorem 1.4 (octahedral Sperner with octahedral labels).*

*Proof sketch.* Let  $T$  be a triangulation of  $\diamond^n$  with label function  $\lambda : V(T) \rightarrow \text{ext}(\diamond^n)$  satisfying the conditions of Theorem 1.4. Let  $X = 2\diamond^n$



be a dilated copy of the  $n$ -dimensional octahedron, so that  $\diamond^n$  lies entirely within the interior of  $X$ . Our key idea is to extend the triangulation  $T$  and label function  $\lambda$  to a triangulation  $T^*$  of  $X$  and a label function  $\lambda^* : V(T^*) \rightarrow \text{ext}(\diamond^n)$  so that the following properties hold:

1.  $T \subset T^*$ , and  $\lambda^*(v) = \lambda(v)$  for each vertex  $v$  in  $V(T)$ .
2.  $T^*$  is antipodally symmetric on the boundary, and  $\lambda^*(v) = -\lambda^*(-v)$  for each  $v \in V(T^*) \cap \partial(X)$ .
3. There are no complementary edges in  $T^* \setminus T$ .

If we can construct such a  $T^*$  and  $\lambda^*$ , then Theorem 1.4 immediately follows from Tucker's lemma, since  $T^*$  must contain a complementary edge if property (2) is true, and this complementary edge must then lie in  $T$  by properties (1) and (3). Further details are provided in the full version.  $\square$

A similar argument can also be used to show that Tucker's theorem implies Theorem 1.6 (Cubical Sperner with octahedral labels). Indeed, we can also use this technique to show the equivalence of Theorem 1.6 and Theorem 1.4. Unfortunately, this style of geometric argument relies crucially on a labelling scheme in which negations are permitted. Accordingly, it appears that additional insight is required in order for it to be possible to use a geometric construction of this nature to directly prove Sperner's lemma from a Tucker-like theorem.

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## Hamiltonicity



# Proof of a conjecture of Thomassen on Hamilton cycles in highly connected tournaments

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**Abstract.** A conjecture of Thomassen from 1982 states that for every  $k$  there is an  $f(k)$  so that every strongly  $f(k)$ -connected tournament contains  $k$  edge-disjoint Hamilton cycles. A classical theorem of Camion, that every strongly connected tournament contains a Hamilton cycle, implies that  $f(1) = 1$ . So far, even the existence of  $f(2)$  was open. In this paper, we prove Thomassen’s conjecture by showing that  $f(k) = O(k^2 \log^2 k)$ . This is best possible up to the logarithmic factor. As a tool, we show that every strongly  $10^4 k \log k$ -connected tournament is  $k$ -linked (which improves a previous exponential bound). The proof of the latter is based on a fundamental result of Ajtai, Komlós and Szemerédi on asymptotically optimal sorting networks.

**Main result.** A *tournament* is an orientation of a complete graph and a Hamilton cycle in a tournament  $T$  is a (consistently oriented) cycle which contains all the vertices of  $T$ .  $T$  is *strongly connected* if, for every pair of vertices  $x$  and  $y$  of  $T$ , there are directed paths from  $x$  to  $y$  and from  $y$  to  $x$ .  $T$  is *strongly  $k$ -connected* if  $|T| > k$  and  $T$  remains strongly connected after the removal of any  $k - 1$  vertices.

Hamilton cycles in tournaments have a long and rich history. For instance, one of the most basic results about tournaments is Camion’s theorem, which states that every strongly connected tournament has a Hamilton cycle [7]. This is strengthened by Moon’s theorem [16], which implies that such a tournament is also pancyclic, i.e. contains cycles of all possible lengths. Many related results have been proved; the monograph by Bang-Jensen and Gutin [4] gives an overview which also includes many recent results.

In 1982, Thomassen [18] made a very natural conjecture on how to guarantee not just one Hamilton cycle, but many edge-disjoint ones: he conjectured that for every  $k$  there is an  $f(k)$  so that every strongly  $f(k)$ -connected tournament contains  $k$  edge-disjoint Hamilton cycles (see also the recent surveys [3, 13]). This turned out to be surprisingly difficult: not even the existence of  $f(2)$  was known so far. Our main result shows that  $f(k) = O(k^2 \log^2 k)$ .

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**Theorem 1 ([12]).** *There exists  $C > 0$  such that for all  $k \in \mathbb{N}$  with  $k \geq 2$  every strongly  $Ck^2 \log^2 k$ -connected tournament contains  $k$  edge-disjoint Hamilton cycles.*

We also have a construction which shows that  $f(k) \geq (k - 1)^2/4$ , and so our bound on the connectivity is asymptotically close to best possible. Thomassen [18] observed that  $f(2) > 2$  and conjectured that  $f(2) = 3$ . He also observed that one cannot weaken the assumption in Theorem 1 by replacing strong connectivity with strong edge-connectivity.

We have made no attempt to optimize the value of the constant  $C$  in Theorem 1: one can take  $C := 10^{12}$  for  $k \geq 20$ . Rather than proving Theorem 1 directly, we deduce it as an immediate consequence of two further results, which are both of independent interest: we show that every sufficiently highly connected tournament is highly linked (see Theorem 3) and that every highly linked tournament contains many edge-disjoint Hamilton cycles (see Theorem 2).

**Linkedness in tournaments.** Given sets  $A, B$  of size  $k$  in a strongly  $k$ -connected digraph  $D$ , Menger's theorem implies that  $D$  contains  $k$  vertex-disjoint paths from  $A$  to  $B$ . A  $k$ -linked digraph is one in which we can even specify the initial and final vertex of each such path. More precisely a digraph  $D$  is  $k$ -linked if  $|D| \geq 2k$  and whenever  $x_1, \dots, x_k, y_1, \dots, y_k$  are  $2k$  distinct vertices of  $D$ , there exist vertex-disjoint paths  $P_1, \dots, P_k$  such that  $P_i$  is a path from  $x_i$  to  $y_i$ .

**Theorem 2 ([12]).** *There exists  $C' > 0$  such that for all  $k \in \mathbb{N}$  with  $k \geq 2$  every  $C'k^2 \log k$ -linked tournament contains  $k$  edge-disjoint Hamilton cycles.*

Similarly as for Theorem 1, the bound in Theorem 2 is asymptotically close to best possible. Moreover, one can take  $C' := 10^7$  for all  $k \geq 20$ . (As mentioned earlier, we have made no attempt to optimise the value of this constant.)

It is not clear from the definition that every (very) highly connected tournament is also highly linked. In fact, for general digraphs this is far from true: Thomassen [20] showed that for all  $k$  there are strongly  $k$ -connected digraphs which are not even 2-linked. On the other hand, he showed that there is an (exponential) function  $g(k)$  so that every strongly  $g(k)$ -connected tournament is  $k$ -linked [19]. Theorem 3 below shows that we can take  $g(k)$  to be almost linear in  $k$ . Note that Theorem 3 together with the construction mentioned earlier (showing that  $f(k) \geq (k - 1)^2/4$ ) shows that Theorem 2 is asymptotically best possible up to logarithmic terms.

**Theorem 3 ([12]).** *For all  $k \in \mathbb{N}$  with  $k \geq 2$  every tournament that is strongly  $10^4 k \log k$ -connected is  $k$ -linked.*

For small  $k$ , the constant  $10^4$  can easily be improved. The proof of Theorem 3 is based on a fundamental result of Ajtai, Komlós and Szemerédi [1,2] on the existence of asymptotically optimal sorting networks. Though their result is asymptotically optimal, it is not clear whether this is the case for Theorem 3. In fact, for the case of (undirected) graphs, a deep result of Bollobás and Thomason [6] states that every  $22k$ -connected graph is  $k$ -linked (this was improved to  $10k$  by Thomas and Wollan [17]). Thus one might believe that a similar relation also holds in the case of tournaments:

**Conjecture 4 ([12]).** *There exists  $C > 0$  such that for all  $k \in \mathbb{N}$  every strongly  $Ck$ -connected tournament is  $k$ -linked.*

Similarly, we believe that the logarithmic terms can also be removed in Theorems 1 and 2:

**Conjecture 5 ([12]).**

- (i) *There exists  $C' > 0$  such that for all  $k \in \mathbb{N}$  every  $C'k^2$ -linked tournament contains  $k$  edge-disjoint Hamilton cycles.*
- (ii) *There exists  $C'' > 0$  such that for all  $k \in \mathbb{N}$  every strongly  $C''k^2$ -connected tournament contains  $k$  edge-disjoint Hamilton cycles.*

Note that Conjectures 4 and 5(i) together imply Conjecture 5(ii).

**Algorithmic aspects.** Both Hamiltonicity and linkedness in tournaments have also been studied from an algorithmic perspective. Camion's theorem implies that the Hamilton cycle problem (though NP-complete in general) is solvable in polynomial time for tournaments. Chudnovsky, Scott and Seymour [8] solved a long-standing problem of Bang-Jensen and Thomassen [5] by showing that the linkedness problem is also solvable in polynomial time for tournaments. More precisely, for a given tournament on  $n$  vertices, one can determine in time polynomial in  $n$  whether it is  $k$ -linked and if yes, one can produce a corresponding set of  $k$  paths (also in polynomial time). Fortune, Hopcroft and Wyllie [10] showed that for general digraphs, the problem is NP-complete even for  $k = 2$ .

We can use the result in [8] to obtain an algorithmic version of Theorem 2. More precisely, given a  $C'k^2 \log k$ -linked tournament on  $n$  vertices, one can find  $k$  edge-disjoint Hamilton cycles in time polynomial in  $n$  (where  $k$  is fixed).

Note that this immediately results in an algorithmic version of Theorem 1.

**Related results and spanning regular subgraphs.** Our construction showing that  $f(k) \geq (k - 1)^2/4$  actually suggests that the 'bottleneck'

to finding  $k$  edge-disjoint Hamilton cycles is the existence of a  $k$ -regular subdigraph. Indeed, the construction shows that if the connectivity of a tournament  $T$  is significantly lower than in Theorem 1, then  $T$  may not even contain a spanning  $k$ -regular subdigraph. There are other results which exhibit this phenomenon: if  $T$  is itself regular, then Kelly's conjecture from 1968 states that  $T$  itself has a Hamilton decomposition. Kelly's conjecture was proved very recently (for large tournaments) by Kühn and Osthus [14].

Erdős raised a 'probabilistic' version of Kelly's conjecture: for a tournament  $T$ , let  $\delta^0(T)$  denote the minimum of the minimum out-degree and the minimum in-degree. He conjectured that for almost all tournaments  $T$ , the maximum number of edge-disjoint Hamilton cycles in  $T$  is exactly  $\delta^0(T)$ . In particular, this would imply that with high probability,  $\delta^0(T)$  is also the degree of a densest spanning regular subdigraph in a random tournament  $T$ . This conjecture of Erdős was proved by Kühn and Osthus [15], based on the main result in [14].

It would be interesting to obtain further conditions which relate the degree of the densest spanning regular subdigraph of a tournament  $T$  to the number of edge-disjoint Hamilton cycles in  $T$ . For undirected graphs, one such conjecture was made in [11]: it states that for any graph  $G$  satisfying the conditions of Dirac's theorem, the number of edge-disjoint Hamilton cycles in  $G$  is exactly half the degree of a densest spanning even-regular subgraph of  $G$ . An approximate version of this conjecture was proved by Ferber, Krivelevich and Sudakov [9], see e.g. [11, 15] for some related results.

The methods used in the current paper are quite different from those used e.g. in the above results. A crucial ingredient is the construction of highly structured dominating sets.

## 1 Sketch of the proof of Theorem 2

In this section, we give an outline of the proof of Theorem 2. An important idea is the notion of a 'covering edge'. Let  $T$  be a tournament, let  $x \in V(T)$ , and suppose  $C$  is a cycle in  $T$  covering  $T - x$ . If  $yz \in E(C)$  and  $yx, xz \in E(T)$ , then we can replace  $yz$  by  $yxz$  in  $C$  to turn  $C$  into a Hamilton cycle. We call  $yz$  a *covering edge* for  $x$ . More generally, if  $S \subseteq V(T)$  and  $C$  is a cycle in  $T$  spanning  $V(T) - S$ , then if  $C$  contains a covering edge for each  $x \in S$  then we can turn  $C$  into a Hamilton cycle by using all these covering edges. Note that this idea still works if  $C$  covers some part of  $S$ .

The following consequence of the Gallai-Milgram theorem is another important tool: suppose that  $G$  is an oriented graph on  $n$  vertices with

$\delta(G) \geq n - \ell$ . Then the vertices of  $G$  can be covered with  $\ell$  vertex-disjoint paths. We use this as follows: suppose we are given a highly linked tournament  $T$  and have already found  $i$  edge-disjoint Hamilton cycles in  $T$ . Then the Gallai-Milgram theorem implies that we can cover the vertices of the remaining oriented graph by a set of  $2i + 1$  vertex-disjoint paths. Very roughly, the aim is to link together these paths using the high linkedness of the original tournament  $T$ .

As above, suppose that we have already found  $i$  edge-disjoint Hamilton cycles in a highly linked tournament  $T$ . Let  $T'$  be the oriented subgraph of  $T$  obtained by removing the edges of these Hamilton cycles. Set  $t = 2i + 1$  and suppose that we also have the following ‘linked dominating structure’,  $L \subseteq T'$ , which is found at the outset of the proof using the assumption that  $T$  is highly linked.  $L$  consists of the vertices and edges of:

- small disjoint transitive out-dominating sets  $A_1, \dots, A_t$ ;
- small disjoint transitive in-dominating sets  $B_1, \dots, B_t$ ;
- a set of short vertex-disjoint paths  $P_1, \dots, P_t$ , where each  $P_\ell$  is a path from the sink of  $B_\ell$  to the source of  $A_\ell$ .

Here, we define  $A_\ell$  to be a transitive out-dominating set if  $T[A_\ell]$  is transitive and if every vertex of  $V(T) \setminus A_\ell$  receives an edge from  $A_\ell$ . Transitive in-dominating sets are similarly defined.

Since  $\delta(T' - V(L)) \geq n - 1 - 2i = n - t$ , the Gallai-Milgram theorem implies that we can cover the vertices of  $T' - V(L)$  with  $t$  vertex-disjoint paths  $Q_1, \dots, Q_t$ . Now, using the dominating sets in our ‘linked dominating structure’, we can extend  $Q_1, \dots, Q_t$  into  $L$ , and if this is done in the right way, then we can link up all the paths  $Q_1, \dots, Q_t$  and  $P_1, \dots, P_t$  into a single cycle  $C$  which covers all vertices outside  $V(L)$  (and some of the vertices inside  $V(L)$ ). In our construction, we will ensure that the paths  $P_\ell$  contain a set of covering edges for  $V(L)$ . So  $C$  also contains covering edges for  $V(L)$ , and so we can transform  $C$  into a Hamilton cycle as discussed earlier.

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# Proof of the 1-factorization and Hamilton decomposition conjectures

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and Andrew Treglown<sup>3</sup>

**Abstract.** We prove the following results (via a unified approach) for all sufficiently large  $n$ :

- (i) [*1-factorization conjecture*] Suppose that  $n$  is even and  $D \geq 2\lceil n/4 \rceil - 1$ . Then every  $D$ -regular graph  $G$  on  $n$  vertices has a decomposition into perfect matchings. Equivalently,  $\chi'(G) = D$ .
- (ii) [*Hamilton decomposition conjecture*] Suppose that  $D \geq \lfloor n/2 \rfloor$ . Then every  $D$ -regular graph  $G$  on  $n$  vertices has a decomposition into Hamilton cycles and at most one perfect matching.
- (iii) [*Optimal packings of Hamilton cycles*] Suppose that  $G$  is a graph on  $n$  vertices with minimum degree  $\delta \geq n/2$ . Then  $G$  contains at least  $(n - 2)/8$  edge-disjoint Hamilton cycles.

According to Dirac, (i) was first raised in the 1950's. (ii) and (iii) answer questions of Nash-Williams from 1970. All of the above bounds are best possible.

## 1 Introduction

In a sequence of four papers [5, 6, 11, 12], we provide a unified approach towards proving three long-standing conjectures for all sufficiently large graphs. Firstly, the 1-factorization conjecture, which can be formulated as an edge-colouring problem; secondly, the Hamilton decomposition conjecture, which provides a far-reaching generalization of Walecki's result [15] that every complete graph of odd order has a Hamilton decomposition and thirdly, a best possible result on packing edge-disjoint Hamilton cycles in Dirac graphs. The latter two were raised by Nash-Williams [17–19] in 1970. A key tool is the recent result of Kühn and Osthus [13] that every dense even-regular robustly expanding graph has a Hamilton decomposition.

### 1.1 The 1-factorization conjecture

Vizing's theorem states that for any graph  $G$  of maximum degree  $\Delta$ , its edge-chromatic number  $\chi'(G)$  is either  $\Delta$  or  $\Delta + 1$ . In general, it is a very difficult problem to determine which graphs  $G$  attain the (trivial) lower bound  $\Delta$  – much of the recent book [22] is devoted to the subject. For regular graphs  $G$ ,  $\chi'(G) = \Delta(G)$  is equivalent to the existence of a 1-factorization: a 1-factorization of a graph  $G$  consists of a set of edge-

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disjoint perfect matchings covering all edges of  $G$ . The long-standing 1-factorization conjecture states that every regular graph of sufficiently high degree has a 1-factorization. It was first stated explicitly by Chetwynd and Hilton [1, 3] (who also proved partial results). However, they state that according to Dirac, it was already discussed in the 1950's.

**Theorem 1.1.** *There exists an  $n_0 \in \mathbb{N}$  such that the following holds. Let  $n, D \in \mathbb{N}$  be such that  $n \geq n_0$  is even and  $D \geq 2\lceil n/4 \rceil - 1$ . Then every  $D$ -regular graph  $G$  on  $n$  vertices has a 1-factorization. Equivalently,  $\chi'(G) = D$ .*

The bound on the degree in Theorem 1.1 is best possible. To see this, suppose first that  $n \equiv 2 \pmod{4}$ . Consider the graph which is the disjoint union of two cliques of order  $n/2$  (which is odd). If  $n \equiv 0 \pmod{4}$ , consider the graph obtained from the disjoint union of cliques of orders  $n/2 - 1$  and  $n/2 + 1$  (both odd) by deleting a Hamilton cycle in the larger clique.

Note that Theorem 1.1 implies that for every regular graph  $G$  on an even number of vertices, either  $G$  or its complement has a 1-factorization. Also, Theorem 1.1 has an interpretation in terms of scheduling round-robin tournaments (where  $n$  players play all of each other in  $n - 1$  rounds): one can schedule the first half of the rounds arbitrarily before one needs to plan the remainder of the tournament.

The best previous result towards Theorem 1.1 is due to Perkovic and Reed [20], who proved an approximate version, *i.e.* they assumed that  $D \geq n/2 + \varepsilon n$ . This was generalized by Vaughan [23] to multigraphs of bounded multiplicity. Indeed, he proved an approximate version of the following multigraph version of the 1-factorization conjecture which was raised by Plantholt and Tipnis [21]: Let  $G$  be a regular multigraph of even order  $n$  with multiplicity at most  $r$ . If the degree of  $G$  is at least  $rn/2$  then  $G$  is 1-factorizable.

In 1986, Chetwynd and Hilton [2] made the following ‘overfull subgraph’ conjecture, which also generalizes the 1-factorization conjecture. Roughly speaking, this says that a dense graph satisfies  $\chi'(G) = \Delta(G)$  unless there is a trivial obstruction in the form of a dense subgraph  $H$  on an odd number of vertices. Formally, we say that a subgraph  $H$  of  $G$  is *overfull* if  $e(H) > \Delta(G)\lfloor |H|/2 \rfloor$  (note this requires  $|H|$  to be odd).

**Conjecture 1.2.** A graph  $G$  on  $n$  vertices with  $\Delta(G) \geq n/3$  satisfies  $\chi'(G) = \Delta(G)$  if and only if  $G$  contains no overfull subgraph.

This conjecture is still wide open – partial results are discussed in [22], which also discusses further results and questions related to the 1-factorization conjecture.

## 1.2 The Hamilton decomposition conjecture

Rather than asking for a 1-factorization, Nash-Williams [17, 19] raised the more difficult problem of finding a Hamilton decomposition in an even-

regular graph. Here, a *Hamilton decomposition* of a graph  $G$  consists of a set of edge-disjoint Hamilton cycles covering all edges of  $G$ . A natural extension of this to regular graphs  $G$  of odd degree is to ask for a decomposition into Hamilton cycles and one perfect matching (*i.e.* one perfect matching  $M$  in  $G$  together with a Hamilton decomposition of  $G - M$ ). The following result solves the problem of Nash-Williams for all large graphs.

**Theorem 1.3.** *There exists an  $n_0 \in \mathbb{N}$  such that the following holds. Let  $n, D \in \mathbb{N}$  be such that  $n \geq n_0$  and  $D \geq \lfloor n/2 \rfloor$ . Then every  $D$ -regular graph  $G$  on  $n$  vertices has a decomposition into Hamilton cycles and at most one perfect matching.*

Again, the bound on the degree in Theorem 1.3 is best possible. Previous results include the following: Nash-Williams [16] showed that the degree bound in Theorem 1.3 ensures a single Hamilton cycle. Jackson [8] showed that one can ensure close to  $D/2 - n/6$  edge-disjoint Hamilton cycles. Christofides, Kühn and Osthus [4] obtained an approximate decomposition under the assumption that  $D \geq n/2 + \varepsilon n$ . Under the same assumption, Kühn and Osthus [14] obtained an exact decomposition (as a consequence of their main result in [13] on Hamilton decompositions of robustly expanding graphs).

Note that Theorem 1.3 does not quite imply Theorem 1.1, as the degree threshold in the former result is slightly higher.

A natural question is whether one can extend Theorem 1.3 to sparser (quasi)-random graphs. Indeed, for random regular graphs of bounded degree this was proved by Kim and Wormald [9] and for (quasi)-random regular graphs of linear degree this was proved in [14] as a consequence of the main result in [13]. However, the intermediate range remains open.

### 1.3 Packing Hamilton cycles in graphs of large minimum degree

Although Dirac's theorem is best possible in the sense that the minimum degree condition  $\delta \geq n/2$  is best possible, the conclusion can be strengthened considerably: a remarkable result of Nash-Williams [18] states that every graph  $G$  on  $n$  vertices with minimum degree  $\delta(G) \geq n/2$  contains  $\lfloor 5n/224 \rfloor$  edge-disjoint Hamilton cycles. He raised the question of finding the best possible bound, which we answer below for all large graphs.

**Theorem 1.4.** *There exists an  $n_0 \in \mathbb{N}$  such that the following holds. Suppose that  $G$  is a graph on  $n \geq n_0$  vertices with minimum degree  $\delta \geq n/2$ . Then  $G$  contains at least  $(n - 2)/8$  edge-disjoint Hamilton cycles.*

The following construction (which is based on a construction of Babai, see [17]) shows that the bound is best possible for  $n = 8k + 2$ , where  $k \in \mathbb{N}$ . Consider the graph  $G$  consisting of one empty vertex class  $A$  of size  $4k$ , one vertex class  $B$  of size  $4k + 2$  containing a perfect matching

and no other edges, and all possible edges between  $A$  and  $B$ . Thus  $G$  has order  $n = 8k + 2$  and minimum degree  $4k + 1 = n/2$ . Any Hamilton cycle in  $G$  must contain at least two edges of the perfect matching in  $B$ , so  $G$  contains at most  $\lfloor |B|/4 \rfloor = k = (n - 2)/8$  edge-disjoint Hamilton cycles.

A more general question is to ask for the number of edge-disjoint Hamilton cycles one can guarantee in a graph  $G$  of minimum degree  $\delta$ . This number has been determined exactly by Kühn, Lapinskas and Osthus [10] unless  $G$  is close to one of the extremal graphs for Dirac's theorem (*i.e.* unless  $G$  is close to the complete balanced bipartite graph or close to the union of two disjoint copies of a clique). In particular, the number of edge-disjoint Hamilton cycles one can guarantee is known exactly whenever  $\delta \geq n/2 + \varepsilon n$ . This improves earlier results of Christofides, Kühn and Osthus [4] as well as Hartke and Seacrest [7]. Actually, our proof of Theorem 1.4 also settles the cases when  $G$  is close to the extremal graphs for Dirac's theorem. So altogether this solves the problem for all values of  $\delta$ .

## 2 Overview of the proofs of Theorems 1.1 and 1.3

The proofs develop methods established by Kühn and Osthus [13], who proved a generalization of Kelly's conjecture that every regular tournament has a Hamilton decomposition (for large tournaments). For all three of our main results, we split the argument according to the structure of the graph  $G$  under consideration:

- (i)  $G$  is close to the complete balanced bipartite graph  $K_{n/2, n/2}$ ;
- (ii)  $G$  is close to the union of two disjoint copies of a clique  $K_{n/2}$ ;
- (iii)  $G$  is a 'robust expander'.

Informally, a graph  $G$  is a robust expander if for every set  $S \subseteq V(G)$  which is not too large or too small, its neighbourhood is substantially larger than  $|S|$ , even if we delete a small proportion of the edges of  $G$ . In other words,  $G$  is an expander graph which is 'locally resilient'. The main result of [13] states that every dense regular robust expander has a Hamilton decomposition. This immediately implies Theorems 1.1 and 1.3 in Case (iii).

Suppose we are going to prove Theorem 1.3 in the case when  $D$  is even. So our aim is to decompose  $G$  into  $D/2$  edge-disjoint Hamilton cycles. As mentioned above, we may assume that  $G$  is in either Case (i) or Case (ii). In [6], we find an approximate Hamilton decomposition of  $G$  in both cases, *i.e.* a set of edge-disjoint Hamilton cycles covering almost all edges of  $G$ . However, one does not have any control over the 'leftover' graph  $H$ , which makes a complete decomposition seem infeasible. This problem was overcome in [13] by introducing the concept of

a ‘robustly decomposable graph’  $G^{\text{rob}}$ . Roughly speaking, this is a sparse regular graph with the following property: given *any* very sparse regular graph  $H$  with  $V(H) = V(G^{\text{rob}})$  which is edge-disjoint from  $G^{\text{rob}}$ , one can guarantee that  $G^{\text{rob}} \cup H$  has a Hamilton decomposition. This leads to a natural (and very general) strategy to obtain a decomposition of  $G$ :

- (1) find a (sparse) robustly decomposable graph  $G^{\text{rob}}$  in  $G$  and let  $G'$  denote the leftover;
- (2) find an approximate Hamilton decomposition of  $G'$  and let  $H$  denote the (very sparse) leftover;
- (3) find a Hamilton decomposition of  $G^{\text{rob}} \cup H$ .

It is of course far from clear that one can always find such a graph  $G^{\text{rob}}$ , especially in Case (ii) where  $G$  is close to being disconnected. In [5], we find  $G^{\text{rob}}$  for Case (i). In [11, 12], we find  $G^{\text{rob}}$  for Case (ii).

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# Regular hypergraphs: asymptotic counting and loose Hamilton cycles

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**Abstract.** We present results from two papers by the authors on analysis of  $d$ -regular  $k$ -uniform hypergraphs, when  $k$  is fixed and the number  $n$  of vertices tends to infinity. The first result is approximate enumeration of such hypergraphs, provided  $d = d(n) = o(n^\kappa)$ , where  $\kappa = \kappa(k) = 1$  for all  $k \geq 4$ , while  $\kappa(3) = 1/2$ . The second result is that a random  $d$ -regular hypergraph contains as a dense subgraph the uniform random hypergraph (a generalization of the Erdős-Rényi uniform graph), and, in view of known results, contains a loose Hamilton cycle with probability tending to one.

**1. Regular  $k$ -graphs and  $k$ -multigraphs.** We consider  $k$ -uniform hypergraphs (or  $k$ -graphs, for short) on the vertex set  $V = [n] := \{1, \dots, n\}$ , that is, families of  $k$ -element subsets of  $V$ . A  $k$ -graph  $H$  is  $d$ -regular, if the degree of every vertex  $v \in V$ ,  $\deg_H(v) := \deg(v) := |\{e \in H : v \in e\}|$  equals  $d$ .

Let  $\mathcal{H}^{(k)}(n, d)$  be the class of all  $d$ -regular  $k$ -graphs on  $[n]$ . Note that each  $H \in \mathcal{H}^{(k)}(n, d)$  has  $M := nd/k$  edges (throughout, we implicitly assume that  $k$  divides  $nd$ ). Let  $\mathbb{H}^{(k)}(n, d)$  be a  $k$ -graph chosen from  $\mathcal{H}^{(k)}(n, d)$  uniformly at random. We treat  $d$  as a function of  $n$  (possibly constant) and study  $\mathcal{H}^{(k)}(n, d)$  as well as  $\mathbb{H}^{(k)}(n, d)$  as  $n$  tends to infinity.

By a  $k$ -multigraph on the vertex set  $[n]$  we mean a multiset of  $k$ -element multisubsets of  $[n]$ . An edge is called a *loop* if it contains more than one copy of some vertex and otherwise it is called a *proper edge*. A  $k$ -multigraph is *simple*, if it is a  $k$ -graph.

A standard tool to study regular (hyper)graphs is the so called *configuration model* of a random  $k$ -multigraph (see [10] for  $k = 2$ ; its generalization to every  $k$  is straightforward). We use a slightly different model yielding the same distribution of  $k$ -multigraphs. Let  $\mathcal{S} \subset [n]^{nd}$  be the

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family of all sequences in which every value  $i \in [n]$  occurs precisely  $d$  times. Let  $\mathbf{Y} = (Y_1, \dots, Y_{nd})$  be a sequence chosen from  $\mathcal{S}$  uniformly at random, and define  $\mathbb{H}_*^{(k)}(n, d)$  as a  $k$ -multigraph with the edge set

$$\{Y_{ki+1} \dots Y_{ki+k} : i = 0, \dots, nd - 1\}.$$

**2. The switching.** The essential tool in both papers presented here is the so called *switching* technique, introduced by McKay [8] for asymptotic enumeration of regular graphs. McKay and Wormald [9] improved McKay’s result by applying a more advanced version of switching, which we extend to hypergraphs as follows.

Let us view sequences  $\mathcal{S}$  as ordered  $k$ -multigraphs (that is,  $k$ -multigraphs with an ordering of edges). Let  $\mathcal{E}_l \subset \mathcal{S}$  be the family of sequences with no multiple edges and exactly  $l$  loops, but no loops with less than  $k - 1$  distinct vertices. Thus, every loop in such a  $k$ -multigraph has only one multiple vertex, which has multiplicity two.

The *switching* is an operation which maps a sequence  $\mathbf{x} \in \mathcal{E}_l$  to  $\mathbf{y} \in \mathcal{E}_{l-1}$  as follows. Choose a loop  $f$  and two proper edges  $e_1, e_2$ . Select a vertex  $v \in e_1 \setminus e_2$  as well as a vertex  $w \in e_2 \setminus e_1$ . Suppose that  $u$  is the multiple vertex of  $f$ . Swap  $v$  with one copy of  $u$  and  $w$  with the other. The effect of this is that edges  $f, e_1, e_2$  in  $\mathbf{x}$  are replaced with the following three edges in  $\mathbf{y}$

$$f \setminus \{u, u\} \cup \{v, w\}, \quad e_1 \setminus \{v\} \cup \{u\}, \quad e_2 \setminus \{w\} \cup \{u\}.$$

**3. Counting Regular Hypergraphs.** In [5] we approximately count  $d$ -regular  $k$ -graphs. Since  $|\mathcal{S}| = (nd)!/(d!)^n$  and every simple  $k$ -graph is given by exactly  $M!(k!)^M$  sequences in  $\mathcal{S}$ , the number of  $d$ -regular  $k$ -graphs is precisely

$$|\mathcal{H}^{(k)}(n, d)| = \frac{(nd)!}{M!(k!)^M(d!)^n} \mathbb{P}(\mathbb{H}_*^{(k)}(n, d) \text{ is simple}).$$

Therefore the problem of asymptotic enumeration reduces to a the analysis of the probability. For graphs, that is,  $k = 2$ , this has been well studied (see [10]). For general  $k$  but fixed  $d$ , Cooper, Frieze, Molloy and Reed [1] showed that the probability converges to  $\exp\{-(k - 1)(d - 1)/2\}$ . In [5] we extend this to the following formula. Let  $\kappa(k) = 1$ , if  $k \geq 4$  and  $\kappa(3) = 1/2$ .

**Theorem 1.** For  $k \geq 3$  and  $1 \leq d = o(n^{\kappa(k)})$  we have

$$\mathbb{P}(\mathbb{H}_*^{(k)}(n, d) \text{ is simple}) = \exp \left\{ -\frac{(k - 1)(d - 1)}{2} + O \left( \frac{d^2}{n} + \sqrt{\frac{d}{n}} \right) \right\}$$

The proof of Theorem 1 is simplified by the fact (shown in [5]) that a randomly chosen sequence  $\mathbf{Y} \in \mathcal{S}$  with probability tending to one belongs to  $\mathcal{E}_l$  with a reasonably small  $l$ . This allows us to reduce the analysis of the probability to estimating the ratios  $|\mathcal{E}_l|/|\mathcal{E}_{l-1}|$ . This is done by bounding the number of ways one can apply switching to a sequence. The proof works because, as it turns out, the number of possible switchings depends essentially on the number of loops, but not on the structure of the sequence.

**4. Hamilton Cycles in Regular Hypergraphs.** Let us recall that for integer  $m \in [0, \binom{n}{k}]$ ,  $\mathbb{H}^{(k)}(n, m)$  is the random graph chosen uniformly at random among  $k$ -graph on  $[n]$  with precisely  $m$  edges.

Our main result in [6] is that we can couple  $\mathbb{H}^{(k)}(n, d)$  and  $\mathbb{H}^{(k)}(n, m)$  so that the latter is a subgraph of the former with probability tending to one.

**Theorem 2.** *For every  $k \geq 3$ , there are positive constants  $c$  and  $C$  such that if  $d \geq C \log n$ ,  $d = o(n^{1/2})$  and  $m = \lfloor cm \rfloor = \lfloor cnd/k \rfloor$ , then one can define a joint distribution of random graphs  $\mathbb{H}^{(k)}(n, d)$  and  $\mathbb{H}^{(k)}(n, m)$  in such a way that*

$$\mathbb{P}(\mathbb{H}^{(k)}(n, m) \subset \mathbb{H}^{(k)}(n, d)) \rightarrow 1, \quad n \rightarrow \infty.$$

The idea of the proof is as follows. Since  $m$  is a fraction of  $M$ , we are able to couple  $\mathbb{H}^{(k)}(n, m)$  with  $\mathbb{H}_*^{(k)}(n, d)$  (treated as an ordered  $k$ -multigraph) in such a way that with probability tending to one  $\mathbb{H}^{(k)}(n, m)$  is contained in an initial segment of  $\mathbb{H}_*^{(k)}(n, d)$ , which we colour *red*. Then we swap all red loops of  $\mathbb{H}_*^{(k)}(n, d)$  with randomly selected non-red (*green*) proper edges. Finally, we destroy the green loops of  $\mathbb{H}_*^{(k)}(n, d)$  one by one applying a randomly chosen switching which involves green edges only. This does not destroy the previously embedded copy of  $\mathbb{H}^{(k)}(n, m)$ . Moreover, it transforms  $\mathbb{H}_*^{(k)}(n, d)$  into a  $k$ -graph  $\tilde{\mathbb{H}}^{(k)}(n, d)$ , which is distributed approximately as  $\mathbb{H}^{(k)}(n, d)$ , that is, almost uniformly. Theorem 2 then follows by a (maximal) coupling of  $\tilde{\mathbb{H}}^{(k)}(n, d)$  and  $\mathbb{H}^{(k)}(n, d)$ .

A *loose Hamilton cycle* on a vertex set  $V$  is a set of edges  $e_1, \dots, e_s$  such that for some cyclic order of  $V$  each  $e_i$  consists of  $k$  consecutive vertices and  $|e_i \cap e_{i+1}| = 1$  for  $i = 1, \dots, s$ , with  $e_{s+1} = e_1$ . For  $k = 2$  this coincides with the standard notion of a Hamilton cycle.

Asymptotic hamiltonicity for graphs has been intensely investigated since 1978 and rather recently was established in full generality for every  $d \geq 3$ , both fixed and growing with  $n$  (see [2] and references in it).

As for general  $k$ , this has been known for some simpler models of random  $k$ -graphs. From results of Frieze [7], Dudek and Frieze [3] as well as Dudek, Frieze, Loh and Speiss [4], it follows that  $\mathbb{H}^{(k)}(n, m)$  contains a loose Hamilton cycle when the expected degree of a vertex grows faster than  $\log n$ . This, combined with Theorem 2, implies the following fact.

**Corollary 3.** *Suppose that  $d = o(n^{1/2})$ . If  $k = 3$  and  $d \geq C \log n$  for large constant  $C$  or  $k \geq 4$  and  $\log n = o(d)$ , then*

$$\mathbb{P}(\mathbb{H}^{(k)}(n, d) \text{ contains a loose Hamilton cycle}) \rightarrow 1, \quad n \rightarrow \infty.$$

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## Triangles

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# Dynamic concentration of the triangle-free process

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**Abstract.** The triangle-free process begins with an empty graph on  $n$  vertices and iteratively adds edges chosen uniformly at random subject to the constraint that no triangle is formed. We determine the asymptotic number of edges in the maximal triangle-free graph at which the triangle-free process terminates. We also bound the independence number of this graph, which gives an improved lower bound on Ramsey numbers: we show  $R(3, t) > (1/4 - o(1))t^2 / \log t$ , which is within a  $4 + o(1)$  factor of the best known upper bound. Furthermore, we determine which bounded size subgraphs are likely to appear in the maximal triangle-free graph produced by the triangle-free process: they are precisely those triangle-free graphs with maximal average density at most 2.

## 1 Introduction

Constrained random graph processes provide an interesting class of random graph models and a natural source for constructions in graph theory. Although the dependencies introduced by the constraints make such processes difficult to analyse, the evidence to date suggests that they are particularly useful for producing graphs of interest for certain extremal problems. Here we consider the triangle-free random graph process, which is defined by sequentially adding edges, starting with the empty graph, chosen uniformly at random subject to the constraint that no triangle is formed.

This process was introduced by Bollobás and Erdős (see [7]), and first analysed by Erdős, Suen and Winkler [10], using a differential equations method introduced by Ruciński and Wormald [17] for the analysis of the constrained graph process known as the ‘d-process’. One motivation for their work was that their analysis of the triangle-free process led to the best lower bound on the Ramsey number  $R(3, t)$  known at that time. The Ramsey number  $R(s, t)$  is the least number  $n$  such that any graph on

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$n$  vertices contains a complete graph with  $s$  vertices or an independent set with  $t$  vertices. In general, very little is known about these numbers, even approximately. The upper bound  $R(3, t) = O(t^2/\log t)$  was obtained by Ajtai, Komlós and Szemerédi [1], but for many years the best known lower bound, due to Erdős [9], was  $\Omega(t^2/\log^2 t)$ . The order of magnitude was finally determined by Kim [13], who showed that  $R(3, t) = \Omega(t^2/\log t)$ . He employed a semi-random construction that is loosely related to the triangle-free process, thus leaving open the question of whether the triangle-free process itself achieves this bound; this was conjectured by Spencer [19] and proved by Bohman [3]. There is now a large literature on the general  $H$ -free process, obtained by replacing ‘triangle’ by any fixed graph  $H$  in the definition; see [6, 8, 15, 16, 22–25]. However, the theory is still in its early stages: we conjectured that our lower bound for  $H$  strictly 2-balanced, given in [6], gives the correct order of magnitude for the length of the process, but so far this has only been proved for some special graphs.

In this paper we specialise to the triangle-free process, where we can now give an asymptotically optimal analysis. Our improvement on previous analyses of this process exploits the self-correcting nature of key statistics of the process.

Let  $G$  be the maximal triangle-free graph at which the triangle-free process terminates.

**Theorem 1.1.** *Whp every vertex of  $G$  has degree  $(1 + o(1))\sqrt{\frac{1}{2}n \log n}$ .*

We also obtain the following bound on the size of any independent set in  $G$ .

**Theorem 1.2.** *Whp  $G$  has independence number at most  $(1 + o(1))\sqrt{2n \log n}$ .*

An immediate consequence is the following new lower bound on Ramsey numbers. The best known upper bound is  $R(3, t) < (1 + o(1))\frac{t^2}{\log t}$ , due to Shearer [18].

**Theorem 1.3.**  $R(3, t) > \left(\frac{1}{4} - o(1)\right)\frac{t^2}{\log t}$ .

These results are predicted by a simple heuristic: the graph  $G(i)$  after  $i$  steps of the triangle-free process should resemble the Erdős-Rényi random graph  $G_{n,p}$  with  $i = n^2 p/2$ , with the exception that  $G_{n,p}$  has many triangles while  $G(i)$  has none. We also show that this heuristic extends to all small subgraph counts; in particular, we answer the question of which subgraphs appear in  $G$ . Suppose  $H$  is a graph with at least 3 vertices.

The *average density* of  $H$  is  $d(H) = \frac{|E_H|}{|V_H|}$ . The *maximum average density*  $m(H)$  of  $H$  is the maximum of  $d(H')$  over nonempty subgraphs  $H'$  of  $H$ .

**Theorem 1.4.** *Let  $H$  be a triangle-free graph with at least 3 vertices.*

- (i) *If  $m(H) \leq 2$  then  $\mathbb{P}(H \subseteq G) = 1 - o(1)$ .*
- (ii) *If  $m(H) > 2$  then  $\mathbb{P}(H \subseteq G) = o(1)$ .*

Thus, the small subgraphs that are likely to appear in  $G$  are exactly the same as the triangle-free subgraphs that appear in the corresponding  $G_{n,p}$ .

## 2 Overview of the proof

We are guided by the heuristic that  $G(i)$  resembles  $G_{n,p}$  with  $i = n^2 p/2$ . We introduce a continuous time that scales as  $t = in^{-3/2}$ . Note that  $p = 2tn^{-1/2}$ . We define  $Q(i)$  to be the number of open *ordered* pairs in  $G(i)$ . This variable is crucial to our understanding of the process: we have  $Q(0) = n^2 - n$ , and the process ends when  $Q(i) = 0$ . How do we expect  $Q(i)$  to evolve? If  $G(i)$  resembles  $G_{n,p}$  then for any pair  $uv$  we should have  $\mathbb{P}(uv \in O(i)) \approx (1 - p^2)^{n-2} \approx e^{-np^2} = e^{-4i^2}$ . We set  $q(t) = e^{-4t^2} n^2$  and expect to have  $Q(i) \approx q(t)$  for most of the evolution of the process. This is exactly what we prove.

### 2.1 Strategy

We use dynamic concentration inequalities for a carefully chosen ensemble of random variables associated with the process. We show  $V(i) \approx v(t)$  for all variables  $V$  in the ensemble, for some smooth function  $v(t)$ , which we refer to as the *scaling* of  $V$ . Here  $V(i)$  denotes the value of  $V$  after  $i$  steps of the process. For each  $V$  we define a *tracking variable*  $\mathcal{T}V(i)$  and show that  $\mathcal{D}V(i) = V(i) - \mathcal{T}V(i)$  satisfies  $|\mathcal{D}V(i)| < e_V(t)v(t)$ , for some error functions  $e_V(t)$ . We use  $\mathcal{T}V(i)$  rather than  $v(t)$  so that we can isolate variations in  $V$  from variations in other variables that have an impact on  $V$ .

The improvement to earlier analysis of the process comes from ‘self-correction’, *i.e.* the mean-reverting properties of the system of variables. We take  $e_V(t) = f_V(t) + 2g_V(t)$ , where we think of  $f_V(t)$  as the ‘main error term’ and  $g_V(t)$  as the ‘martingale deviation term’. We usually have  $g_V \ll f_V$ , but there are some exceptions when  $t$  is small and hence  $f_V(t)$  is too small. We require  $g_V(t)v(t)$  to be ‘approximately non-increasing’ in  $t$ , in that  $g_V(t')v(t') = O(g_V(t)v(t))$  for all  $t' \geq t$ . We define the *critical window*  $W_V(i) = [(f_V(t) + g_V(t))v(t), (f_V(t) + 2g_V(t))v(t)]$ .

We prove the *trend hypothesis*:  $\mathcal{Z}V(i) := |\mathcal{D}V(i)| - e_V(t)v(t)$  is a supermartingale when  $|\mathcal{D}V(i)| \in W_V(i)$ . The trend hypothesis will follow from the *variation equation* for  $e_V(t)$ , which balances the changes in  $\mathcal{D}V(i)$  and  $e_V(t)v(t)$ . Since errors can transfer from one variable to another, each variation equation is a differential inequality that can involve many of the error functions.

We track the process up to the time  $t_{max} = \frac{1}{2}\sqrt{(1/2 - \varepsilon)\log n}$ . If the tracking fails, then there is some  $i^* \leq i_{max}$  and a variable  $V$  such that  $\mathcal{D}V(i)$  enters  $W_V(i')$  from below at some step  $i' < i^*$ , stays in  $W_V(i)$  for  $i' \leq i \leq i^*$  then goes above  $W_V(i^*)$  at step  $i^*$ . During this time  $\mathcal{Z}V(i)$  is a supermartingale, with  $\mathcal{Z}V(i') \leq -g_V(t')v(t')$  and  $\mathcal{Z}V(i^*) \geq 0$ , so we have an increase of at least  $g_V(t')v(t')$  against the drift of the supermartingale. We can estimate the probability of this event using Freedman's martingale inequality [11], provided that we have good estimates on  $\text{Var}_V(t) = \text{Var}(\mathcal{Z}V(i) \mid \mathcal{F}_{i-1})$  and  $N_V(t) = |\mathcal{Z}V(i+1) - \mathcal{Z}V(i)|$ ; we refer to this as the *boundedness hypothesis*. Thus it suffices to verify the trend and boundedness hypotheses for all variables.

## 2.2 Variables

All definitions are with respect to the graph  $G(i)$ . Sometimes we use a variable name to also denote the set that it counts, *e.g.*  $Q(i)$  is the number of ordered open pairs, and also denotes the set of ordered open pairs. We usually omit  $(i)$  and  $(t)$  from our notation, *e.g.*  $Q$  means  $Q(i)$  and  $q$  means  $q(t)$ . We use capital letters for variable names and the corresponding lower case letter for the scaling. We express scalings using the (approximate) edge density and open pair density, namely  $p = 2in^{-2} = 2tn^{-1/2}$  and  $\hat{q} = e^{-4t^2}$ .

The next most important variable in our analysis, after the variable  $Q$  defined above, is the variable  $Y_{uv}$  which, for a fixed pair of vertices  $uv$ , is the number of vertices  $w$  such that  $uw$  is an open pair and  $vw$  is an edge. It is natural that  $Y_{uv}$  should play an important role in this analysis, as when the pair  $uv$  is added as an edge, the number of open edges that become closed is exactly  $Y_{uv} + Y_{vu}$ . The motivation for introducing the ensembles of variables defined below is as follows: control of the global variables is needed to get good control of  $Q$ , control of the stacking variables is needed to get good control of  $Y_{uv}$ , and controllable variables play a crucial role in our analysis of the stacking variables.

The *global variables* consist of  $Q$ ,  $R$  and  $S$ , where  $Q = 2|O(i)|$  is the number of ordered open pairs,  $R$  is the number of ordered triples with 3 open pairs, and  $S$  is the number of ordered triples  $abc$  where  $ab$  is an edge and  $ac, bc$  are open pairs.



The *stacking variables* are built from four basic building blocks:  $X_u$  is the number of vertices  $\omega$  such that  $u\omega$  is open,  $Y_u$  is the number of vertices  $\omega$  such that  $u\omega$  is an edge,  $X_{uv}$  is the number of vertices  $w$  such that  $uw$  and  $vw$  are open pairs,  $Y_{uv}$  is the number of vertices  $w$  such that  $uw$  is an open pair and  $vw$  is an edge. We defer the exact definition to the full version of the paper, but roughly speaking, the idea is that the relative errors in these variables decrease as the number of steps increases, so that after a large constant number of steps they are essentially global.

Finally, we formulate a very general condition under which we have some control on a variable. Suppose  $\Gamma$  is a graph,  $J$  is a spanning subgraph of  $\Gamma$  and  $A \subseteq V_\Gamma$ . We refer to  $(A, J, \Gamma)$  as an *extension*. Suppose that  $\phi : A \rightarrow [n]$  is an injective mapping. We define the *extension variables*  $X_{\phi, J, \Gamma}(i)$  to be the number of injective maps  $f : V_\Gamma \rightarrow [n]$  such that  $f$  restricts to  $\phi$  on  $A$ ,  $f(e) \in E(i)$  for every  $e \in E_J$  not contained in  $A$ , and  $f(e) \in O(i)$  for every  $e \in E_\Gamma \setminus E_J$  not contained in  $A$ . We introduce the abbreviations  $V = X_{\phi, J, \Gamma}$ ,  $n(V) = |V_\Gamma| - |A|$ ,  $e(V) = e_J - e_{J[A]}$ , and  $o(V) = (e_\Gamma - e_J) - (e_{\Gamma[A]} - e_{J[A]})$ . The *scaling* is  $v = x_{A, J, \Gamma} = n^{n(V)} p^{e(V)} \hat{q}^{o(V)}$ . We expect  $V \approx v$ , provided there is no subextension that is ‘sparse’, in that it has scaling much smaller than 1. Given  $A \subseteq B \subseteq B' \subseteq V_\Gamma$  we define  $S_B^{B'} = S_B^{B'}(J, \Gamma)$  to equal

$$n^{|B'| - |B|} p^{e_{J[B']} - e_{J[B]}} \hat{q}^{(e_{\Gamma[B']} - e_{J[B']}) - (e_{\Gamma[B]} - e_{J[B]})}.$$

Let  $t' \geq 1$ . We say that  $V$  is *controllable at time  $t'$*  if  $J \neq \Gamma$  (i.e. at least one pair is open) and for  $1 \leq t \leq t'$  and  $A \subsetneq B \subseteq V_\Gamma$  we have  $S_A^B(J, \Gamma) \geq n^\delta$ , where  $\delta > 0$  is a fixed global parameter that is sufficiently small given  $\varepsilon$ . (This condition is essentially identical to the condition needed to prove concentration of subgraphs counts in  $G_{n,p}$  using Kim-Vu polynomial concentration [14].)

### 3 Concluding remarks

We have determined  $R(3, t)$  to within a factor of  $4 + o(1)$ , so we should perhaps hazard a guess for its asymptotics: we are tempted to believe the construction rather than the bound, i.e. that  $R(3, t) \sim t^2/4 \log t$ . We should note that we only have an upper bound on the independence number of the graph  $G$  produced by the triangle-free process. So, formally speaking, the triangle-free process could produce a graph that gives a better lower bound on  $R(3, t)$ . But we believe that this is not the case; that is, we conjecture that the bound on the independence number in Theorem 1.2 is asymptotically best possible.

Our method for establishing self-correction builds on ideas used recently by Bohman, Frieze and Lubetzky [5] for an analysis of the triangle-removal process (see also [4] for a simpler context). Furthermore, the

results of this paper have also been obtained independently and simultaneously by Fiz Pontiveros, Griffiths and Morris; their proof also exploits self-correction, but is different to ours in some important ways.

Another natural direction for future research is to provide an asymptotically optimal analysis in greater generality for the  $H$ -free process. No doubt the technical challenges will be formidable, given the difficulties that arise in the case of triangles. But on an optimistic note, it is encouraging that one can build on two different proofs of this case.

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# Subcubic triangle-free graphs have fractional chromatic number at most $14/5$

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**Abstract.** We show that every subcubic triangle-free graph has fractional chromatic number at most  $14/5$ , thus confirming a conjecture of Heckman and Thomas [A new proof of the independence ratio of triangle-free cubic graphs. *Discrete Math.* 233 (2001), 233–237].

## 1 Introduction

One of the most celebrated results in Graph Theory is the Four-Color Theorem (4CT), which states that every planar graph is 4-colorable. It was proved by Appel and Haken [3,4] in 1977 and, about twenty years later, Robertson, Sanders, Seymour and Thomas [17] found a new (and much simpler) proof. However, both of the proofs require a computer assistance, and finding a fully human-checkable proof is still one of the main open problems in Graph Theory. An immediate corollary of the 4CT implies that every  $n$ -vertex planar graph contains an independent set of size  $n/4$  (this statement is sometimes called the Erdős-Vizing conjecture). Although this seems to be an easier problem than the 4CT itself, no proof without the 4CT is known. The best known result that does not use the 4CT is due to Albertson [1], who showed the existence of an independent set of size  $2n/9$ .

An intermediate step between the 4CT and the Erdős-Vizing conjecture is the fractional version of the 4CT — every planar graph is fractionally 4-

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colorable. In fact, fractional colorings were introduced in 1973 [12] as an approach for either disproving, or giving more evidence to the 4CT. For a real number  $k$ , a graph  $G$  is fractionally  $k$ -colorable, if for every assignment of weights to its vertices there is an independent set that contains at least  $(1/k)$ -fraction of the total weight. In particular, every fractionally  $k$ -colorable graph on  $n$  vertices contains an independent set of size at least  $n/k$ . The existence of independent sets of certain ratios in *subcubic* graphs, *i.e.*, graphs with maximum degree at most 3, led Heckman and Thomas to pose the following two conjectures.

**Conjecture 1.1 (Heckman and Thomas [10]).** Every subcubic triangle-free graph is fractionally  $14/5$ -colorable.

**Conjecture 1.2 (Heckman and Thomas [11]).** Every subcubic triangle-free planar graph is fractionally  $8/3$ -colorable.

Note that a graph is called *triangle-free* if it does not contain a triangle as a subgraph. Here we want to announce the confirmation of Conjecture 1.1. The manuscript of our result is available on arXiv [5].

Unlike for general planar graphs, colorings of triangle-free planar graphs are well understood. Already in 1959, Grötzsch [8] proved that every triangle-free planar graph is 3-colorable. Therefore, such a graph on  $n$  vertices has to contain an independent set of size  $n/3$ . In 1976, Albertson, Bollobás and Tucker [2] conjectured that a triangle-free planar graph also has to contain an independent set of size strictly larger than  $n/3$ .

Their conjecture was confirmed in 1993 by Steinberg and Tovey [19], even in a stronger sense: such a graph admits a 3-coloring where at least  $\lfloor n/3 \rfloor + 1$  vertices have the same color. On the other hand, Jones [13] found an infinite family of triangle-free planar graphs with maximum degree four and no independent set of size  $\lfloor n/3 \rfloor + 2$ . However, if the maximum degree is at most three, then Albertson *et al.* [2] conjectured that an independent set of size much larger than  $n/3$  exists. Specifically, they asked whether there is a constant  $s \in (\frac{1}{3}, \frac{3}{8}]$ , such that every subcubic triangle-free planar graph contains an independent set of size  $sn$ . We note that for  $s > 3/8$  the statement would not be true.

The strongest possible variant of this conjecture, *i.e.*, for  $s = 3/8$ , was finally confirmed by Heckman and Thomas [11]. However, for  $s = 5/14$ , it was implied by a much earlier result of Staton [18], who actually showed that every subcubic triangle-free (but not necessarily planar) graph contains an independent set of size  $5n/14$ . Jones [14] then found a simpler proof of this result; an even simpler one is due to Heckman and Thomas [10]. On the other hand, Fajtlowicz [6] observed that one cannot prove anything larger than  $5n/14$ . As we already mentioned, our main re-

sult is the strengthening of Staton's theorem to the fractional (weighted) version.

This conjecture attracted a considerable amount of attention and it spawned a number of interesting works in the last few years. In 2009, Hatami and Zhu [9] showed that for every graph that satisfies the assumptions of Conjecture 1.1, the fractional chromatic number is at most  $3 - 3/64 \approx 2.953$ . (The fractional chromatic number of a graph is the smallest number  $k$  such that the graph is fractionally  $k$ -colorable.) The result of Hatami and Zhu is the first to establish that the fractional chromatic number of every subcubic triangle-free graph is smaller than 3. In 2012, Lu and Peng [16] improved the bound to  $3 - 3/43 \approx 2.930$ . There are also two very recent improvements on the upper bound. The first one is due to Ferguson, Kaiser and Král' [7], who showed that the fractional chromatic number is at most  $32/11 \approx 2.909$ . The other one is due to Liu [15], who improved the upper bound to  $43/15 \approx 2.867$ .

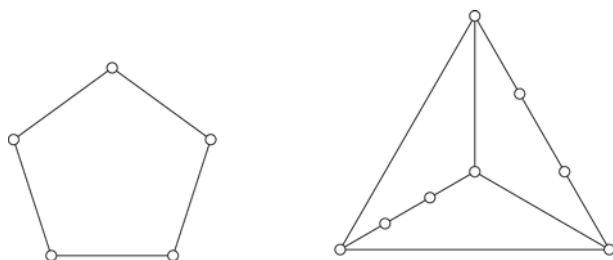
## 2 Main result

We start with another definition of a fractional coloring that will be used to state our main result. It is equivalent to the one mentioned in the previous section by Linear Programming Duality.

Let  $G$  be a graph. A *fractional  $k$ -coloring* is an assignment of measurable subsets of the interval  $[0, 1]$  to the vertices of  $G$  such that each vertex is assigned a subset of measure  $1/k$  and the subsets assigned to adjacent vertices are disjoint. The *fractional chromatic number of  $G$*  is the infimum over all positive real numbers  $k$  such that  $G$  admits a fractional  $k$ -coloring. Note that for finite graphs, such a real  $k$  always exists, the infimum is in fact a minimum, and its value is always rational. We let  $\chi_f(G)$  be this minimum.

A *demand function*  $f$  is a function from  $V(G)$  to  $[0, 1]$  with rational values. Let  $\mu$  be the Lebesgue measure on real numbers. An  *$f$ -coloring* of  $G$  is an assignment  $\varphi$  of measurable subsets of  $[0, 1]$  to the vertices of  $G$  such that  $\mu(\varphi(v)) \geq f(v)$  for every  $v \in V(G)$  and such that  $\varphi(u) \cap \varphi(v) = \emptyset$  whenever  $u$  and  $v$  are two adjacent vertices of  $G$ . Note that for a rational number  $r$ , the graph  $G$  has fractional chromatic number at most  $r$  if and only if it has an  $f_r$ -coloring for the function  $f_r$  that assigns  $1/r$  to every vertex of  $G$ .

A graph  $H$  is *dangerous* if  $H$  is either a 5-cycle or the graph  $K'_4$  obtained from  $K_4$  by subdividing both edges of its perfect matching twice, see Figure 2.1. The vertices of degree two of a dangerous graph are called *special*. Let  $G$  be a subcubic graph and let  $B$  be a subset of its vertices. Let  $H$  be a dangerous induced subgraph of  $G$ . A special vertex  $v$  of  $H$



**Figure 2.1.** Dangerous graphs.

is  $B$ -safe if either  $v \in B$  or  $v$  has degree three in  $G$ . If  $G$  is a subcubic graph, a set  $B \subseteq V(G)$  is called a *nail* if every vertex in  $B$  has degree at most two and every dangerous induced subgraph of  $G$  contains at least two  $B$ -safe special vertices. For a subcubic graph  $G$  and its nail  $B$ , let  $f_B^G$  be the demand function defined as follows: if  $v \in B$ , then  $f_B^G(v) = (7 - \deg_G(v))/14$ ; otherwise  $f_B^G(v) = (8 - \deg_G(v))/14$ .

In order to show that every subcubic triangle-free graph has fractional chromatic number at most  $14/5$ , we show the following stronger statement.

**Theorem 2.1.** *If  $G$  is a subcubic triangle-free graph and  $B \subseteq V(G)$  is a nail, then  $G$  has an  $f_B^G$ -coloring.*

We point out that the motivation for the formulation of Theorem 2.1 as well as for some parts of its proof comes from the work of Heckman and Thomas [10], in which an analogous strengthening is used to prove the existence of an independent set of size  $5n/14$ .

### 3 Conclusion

In order to prove Theorem 2.1, we used several equivalent definitions of fractional colorings. As a consequence, unlike the result of Heckman and Thomas [10], our proof is not constructive and the following question remains open.

**Problem 3.1.** Does there exist a polynomial-time algorithm to find a fractional  $14/5$ -coloring of a given input subcubic triangle-free graph?

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# Henneberg steps for triangle representations

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**Abstract.** Which plane graphs admit a straight line representation such that all faces have the shape of a triangle? In previous work we have studied necessary and sufficient conditions based on flat angle assignments, *i.e.*, selections of angles of the graph that have size  $\pi$  in the representation. A flat angle assignment that fulfills these conditions is called good. The complexity for checking whether a graph has a good flat angle assignment remains unknown.

In this paper we deal with extensions of good flat angle assignments. We show that if  $G$  has a good flat angle assignment and  $G^+$  is obtained via a planar Henneberg step of type 2, then  $G^+$  also admits a good flat angle assignment. A similar result holds for certain combinations of Henneberg type 1 steps followed by a type 2 step. As a consequence we obtain a large class of pseudo-triangulations that admit drawings such that all faces have the shape of a triangle. In particular, every 3-connected, plane generic circuit admits a good flat angle assignment.

## 1 Introduction

In this paper we study a representation of planar graphs in the classical setting, *i.e.*, vertices are presented as points in the Euclidean plane and edges as straight line segments. We are interested in the class of planar graphs that admit a representation in which all faces are triangles. Note that in such a representation each face  $f$  has exactly  $\deg(f) - 3$  incident vertices that have an angle of size  $\pi$  in  $f$ . Conversely each vertex has at most one angle of size  $\pi$ . In [2] we have studied necessary and sufficient conditions based on flat angle assignments, *i.e.*, selections of angles of the graph that have size  $\pi$  in the representation. Flat angle assignments that fulfill these conditions are called *good*. The complexity for checking whether a graph has a good flat angle assignment remains unknown.

A *pseudo-triangle* is a simple polygon with precisely three convex angles, all other vertices of the polygon admit a concave angle at the inte-

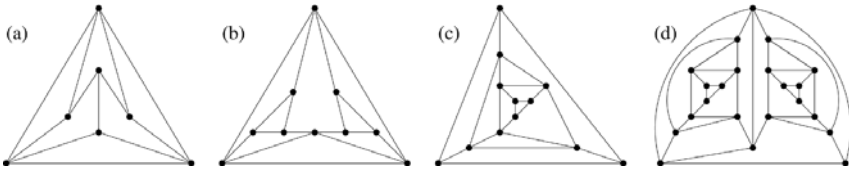
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The full version of this paper can be found online [1].

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**Figure 1.1.** (a) A pseudo-triangulation that does not induce an SLTR, (b) a Laman graph that has an SLTR but can not be constructed using only the two steps we give in Section 2, (c) a planar Laman graph that has no SLTR for this embedding, (d) a planar Laman graph that has no SLTR.

rior of the polygon. A *pseudo-triangulation* (PT) is a planar graph with a drawing such that all faces are pseudo-triangles. An example of a PT is given in Figure 1.1 (a).

A pseudo-triangulation is *pointed* if each vertex has an angle of size  $> \pi$ . A pointed pseudotriangulation with  $n$  vertices must have exactly  $2n - 3$  edges. Indeed pointed pseudotriangulations have the Laman property: they have  $2n - 3$  edges, and subgraphs induced by  $k$  vertices have at most  $2k - 3$  edges. Laman graphs, and hence also pointed pseudotriangulations, are minimally rigid graphs. A detailed survey on pseudo-triangulations has been given by Rote et al. [8].

Pseudotriangulations induce an assignment of big angles to vertices. This assignment is closely related to a flat angle assignment.

A **Straight Line Triangle Representation** (SLTR) of a graph  $G$  is a plane drawing of  $G$  such that all edges are straight line segments and all faces are triangles (*e.g.* Figure 1.1 (b)). Throughout this paper  $G = (V, E)$  will be a plane, internally 3-connected graph. Three vertices which are the corners of the outer face in an SLT Representation of the graph are given and we call these vertices suspensions. A plane graph  $G$  with suspensions  $s_1, s_2, s_3$  is said to be *internally 3-connected* when the addition of a new vertex  $v_\infty$  in the outer face, that is made adjacent to the three suspension vertices, yields a 3-connected plane graph.

A *flat angle assignment* (FAA) of a graph is a mapping from a subset  $U$  of the non-suspension vertices to faces such that, the vertex is incident to the face and,

[C<sub>v</sub>] Every vertex of  $U$  is assigned to at most one face,

[C<sub>f</sub>] For every face  $f$ , precisely  $|f| - 3$  vertices are assigned to  $f$ .

An FAA is called *good* (GFAA) when it induces an SLTR. In [2] we have shown that an FAA is good if and only if it induces a contact family of pseudosegments  $\Sigma$  which has the following property:

[C<sub>P</sub>] Every subset  $S$  of  $\Sigma$  with  $|S| \geq 2$  has at least three free points.

Informally, pseudosegments arise from merging the edges that are incident to an assigned angle of a vertex, the vertex will be an interior

point of the pseudosegment. Since a vertex is assigned at most once, the pseudosegments do not cross. The pseudosegments will be stretched to straight line segments to obtain an SLTR. Let  $p$  be an endpoint of a pseudosegment in  $S \subseteq \Sigma$ . If  $p$  is a suspension vertex, then  $p$  is a free point for  $S$ . When  $p$  is not a suspension then it is a free point for  $S$  if:  $p$  is incident to the unbounded region of  $S$ , it has at least one neighbor not in  $S$  and it is not an interior point for a pseudosegment in  $S$ .

The drawback of this characterization is that we are not aware of an efficient way to test whether a given graph has an FAA that is good.

A *combinatorial pseudo-triangulation* (CPT) is an assignment of the labels *big* and *small* to the angles around each vertex. Each vertex has at most one angle labeled *big* and each inner face has precisely three incident angles labeled *small*, the outer face has all angles labeled *big*. For an interior angle labeled *big*, let the incident vertex be assigned to the incident face, and a vertex is not assigned if it has no angle labeled *big*. Three vertices of the outer face are chosen to be the suspensions, the other vertices are assigned to the outer face. Hence a CPT induces an FAA and the similarly an FAA induces a CPT.

A CPT does not always induce a PT, Orden et al. have shown that the *generalized Laman condition* is necessary and sufficient for a CPT to induce a PT [7].

**Lemma 1.1 (Generalized Laman Condition).** *Let  $G$  be the graph of a pseudo-triangulation of a planar point set in general position. Every subset of  $x$  not assigned vertices plus  $y$  assigned vertices of  $G$ , with  $x + y \geq 2$  spans a subgraph with at most  $3x + 2y - 3$  edges.*

**Proposition 1.2.** *A GFAA of an internally 3-connected plane graph satisfies the generalized Laman Condition.*

An FAA that is *not* good may also satisfy the generalized Laman Condition (e.g. Figure 1.1 (a)), therefore this condition is necessary but not sufficient. Every Laman graph can be constructed from an edge by Henneberg steps [6,9]. A graph  $G = (V, E)$ , with  $|E| = 2|V| - 3$  that has an SLTR, has a CPT that induces a PT by Prop. 1.2 and by the result of Haas et al. it must be a Laman graph. Therefore it must have a Henneberg construction. In the next section we will investigate how to use this construction such that a GFAA can be extended along the steps.

## 2 Construction steps

It has been shown that planar Laman graphs admit a planar Henneberg construction [5]. Since we consider plane graphs (with a given set of

suspension vertices), we consider the Henneberg steps in a plane setting. Recall that the graph must be internally 3-connected.

**Henneberg type 2 step**

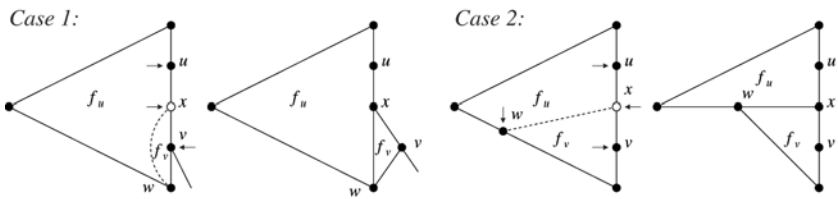
Given a graph  $G$  and a GFAA  $\psi$  of  $G$ . A Henneberg Type 2 step (HEN<sub>2</sub>) subdivides an edge  $uv$  and connects the new vertex  $x$  to a third vertex  $w$  (see Figure 2.1). The face  $f$ , incident to  $uv$  and  $w$  is splitted into  $f_u$  (the face incident to  $u$ ) and  $f_v$ . The other face incident to  $uv$  is denoted with  $f_x$ . The resulting graph is denoted  $G^+$ . We will construct an assignment  $\psi^+$  for  $G^+$  and proof that  $\psi^+$  is a GFAA.

There are three vertices not assigned to  $f$  under  $\psi$ , we will call them *corners* of  $f$ . We consider two cases, firstly  $f_u$  is incident to all corners of  $f$ , secondly,  $f_u$  is incident to precisely two corners of  $f$ . Note that if  $w$  is a corner of  $f$  it will be a corner for both  $f_u$  and  $f_v$ . The vertices different from  $u, v, w, x$ , that are assigned to  $f$  under  $\psi$ , will be assigned in the trivial way under  $\psi^+$ , i.e., such a vertex is assigned to  $f_u$  resp.  $f_v$ , if in  $G^+$  it is incident to  $f_u$  resp.  $f_v$ .

**Case 1:**  $f_u$  is incident to all corners of  $f$ . If  $u$  or  $w$  is assigned to  $f$  under  $\psi$ , it is assigned to  $f_u$  under  $\psi^+$ . The vertex  $v$  is assigned to  $f_x$  and  $x$  to  $f_u$  under  $\psi^+$ .

**Case 2:**  $f_u$  is incident to precisely two corners of  $f$ . If  $u$  or  $w$  is assigned to  $f$  under  $\psi$ , it is assigned to  $f_u$  under  $\psi^+$ , if  $v$  was assigned to  $f$  it is assigned to  $f_v$  under  $\psi^+$  and  $x$  is assigned to  $f_x$ .

This yields an assignment  $\psi^+$  for  $G^+$ .



**Figure 2.1.** Updating the assignment after a HEN<sub>2</sub> step. The white vertex and dashed edge represent the step.

**Theorem 2.1.** *Given a 3-connected, plane graph  $G$  with a GFAA  $\psi$ . Let  $G^+$  be the result of a HEN<sub>2</sub> step applied to  $G$  and let  $\psi^+$  be the updated assignment. Then  $\psi^+$  is a GFAA and  $G^+$  admits an SLTR.*

*Proof.* It is easy to see that  $\psi^+$  satisfies  $C_v$  and  $C_f$  and hence is an FAA.

We consider the induced families of pseudosegments,  $\Sigma$  and  $\Sigma^+$  of  $\psi$  and  $\psi^+$  respectively. Since  $\psi$  is a Good FAA, we know that every subset

$S$  of  $\Sigma$  has at least three free points or  $S$  has cardinality at most one. Let  $S \subseteq \Sigma^+$  of cardinality at least two, and for every pseudosegment  $p$  of  $\Sigma^+$  which is not a pseudosegment of  $\Sigma$  consider  $p \in S$  and show that  $S$  has at least three free points by using that there is an equivalent set under  $\Sigma$  which has at least three free points.

In both cases there are three pseudosegments that have changed, we will only discuss Case 1 here. Let  $s_x$  resp.  $s_v$  be the pseudosegment that has  $x$  resp.  $v$  as interior point and let  $s_w$  the pseudosegment containing the edge  $vw$ .

- If  $s_x \in S$  then replace  $s_x$  by the pseudosegment  $s'_x$  of  $\Sigma$  that has  $u$  and  $v$  as interior points.
- If  $s_v \in S$  then replace  $s_v$  by the pseudosegment  $s'_v$  of  $\Sigma$  that ends in  $v$  and contains all the edges, except  $vx$ , of  $s_v$ .
- If  $s_w \in S$  then delete  $s_w$ .

Now we have a set  $S' \in \Sigma$ , thus  $S'$  has three free points unless  $|S'| = 1$ .

- If  $s_x \in S$  then  $s_x$  contributes the same free points to  $S$  as  $s'_x$  to  $S'$ .
- If  $s_v \in S$  then if  $v$  was a free point for  $S'$ ,  $x$  is for  $S$ . Hence  $s_v$  contributes the same number of free points to  $S$  as  $s'_v$  to  $S'$ .
- If  $s_w \in S$  then if  $|S'| = 1$ ,  $s_w$  contributes at least one free point to  $S$  and it covers no other points, thus  $S$  has three free points, or, if  $|S'| > 1$  then  $S'$  has at least three free points, adding  $s_w$  does not cover any of them and therefore  $S$  has at least three free points.

We conclude that in Case 1 every set  $S$  of cardinality at least two has at least three free points. The argumentation for Case 2 is similar and it follows that  $\psi^+$  is a GFAA.  $\square$

A graph  $G = (V, E)$  is a *generic circuit* if  $|E| = 2|V| - 2$  and subgraphs induced by  $k$  vertices have at most  $2k - 3$  edges. The generic circuit with the least number of vertices is  $K_4$ .

**Theorem 2.2.** *Every 3-connected, plane, generic circuit admits an SLTR.*

*Proof.* A 3-connected, generic circuit can be constructed with  $\text{HEN}_2$  steps from  $K_4$  (Berg and Jordán [3]) and  $K_4$  admits an SLTR. Every plane 3-connected generic circuit can be constructed with  $\text{HEN}_2$  steps from  $K_4$  such that all intermediate graphs are plane. By Theorem 2.1 we have that every 3-connected, plane generic circuit admits an SLTR.  $\square$

### Henneberg combination step

For connectivity reasons single  $\text{HEN}_1$  steps are not compatible with SLTRs. However certain sequences of  $\text{HEN}_1$  steps followed by a  $\text{HEN}_2$  step allow

for the extension of a GFAA. Next we will describe such a combination step denoted  $\text{HEN}_{1n_2}$ .

Let  $f$  be a face with  $n + 1$  vertices. The first  $\text{HEN}_1$  step stacks a new vertex  $v_0$  over an edge of  $f$ . The vertices  $v_0, \dots, v_{n-1}$  are introduced by the  $n$   $\text{HEN}_1$  steps in such a way that  $v_0, \dots, v_{n-1}$  is a path and each vertex of  $f$  is a neighbor of some  $v_i$ . The final  $\text{HEN}_2$  step subdivides an edge which is incident to some  $v_i$  (not  $v_{n-1}$ ) and connects the new vertex  $v_n$  to  $v_{n-1}$ . Of course the construction has to maintain planarity.

**Theorem 2.3.** *Given a 3-connected, plane graph  $G$  with a GFAA  $\psi$ . Let  $G_n$  be the result of a  $\text{HEN}_{1n_2}$  step applied to  $G$  and  $\psi_n$  be the updated assignment. Then  $\psi_n$  is a GFAA and  $G_n$  admits an SLTR.*

Due to the lack of space we have left out the algorithm that decides how to update the assignment, this and the proof of Theorem 2.3 can be found in the full version.

### 3 Conclusion and open problems

We have given two construction steps such that a GFAA can be extended along these steps and the extended assignment is also a GFAA. However, this does not define the class of Laman graphs that have an SLTR. Therefore the problem: Is the recognition of graphs that have an SLTR (GFAA) in  $P$ ? is still open, even for graphs in which all non-suspension vertices have to be assigned.

The class of 3-connected quadrangulations is well-defined, *e.g.* Brinkmann et al. give a characterization using two expansion steps [4]. Adding a diagonal edge in the outer face of a plane, 3-connected quadrangulation yields a Laman graph. One of the expansion steps (denoted  $P_3$  in [4]) is a Henneberg Combination step, hence a GFAA can be extended along this step. It would be interesting to know if a GFAA could also be extended along the other expansion step (denoted  $P_1$  in [4]).

Adding an edge in a plane graph that has a GFAA requires only minor changes to the GFAA of the original graph to obtain a GFAA for the resulting graph. An interesting question arises: Does every graph that admits an SLTR in which not every non-suspension vertex admits a straight angle, have a spanning Laman subgraph that admits an SLTR?

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## Cycles and Girth of Graphs

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# Cycle-continuous mappings – order structure

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Robert Šámal<sup>1</sup>

**Abstract.** Given two graphs, a mapping between their edge-sets is *cycle-continuous*, if the preimage of every cycle is a cycle. Answering a question of DeVos, Nešetřil, and Raspaud, we prove that there exists an infinite set of graphs with no cycle-continuous mapping between them. Further extending this result, we show that every countable poset can be represented by graphs and existence of cycle-continuous mappings between them.

## 1 Introduction

Many questions at the core of graph theory can be formulated as questions about cycles or more generally about flows on graphs. Examples are the cycle double cover conjecture, the Berge-Fulkerson conjecture, and Tutte’s 3-flow, 4-flow, and 5-flow conjectures. For a detailed treatment of this area the reader may refer to [7] or [8].

As an approach to these problems Jaeger [4] and DeVos, Nešetřil, and Raspaud [2] defined a notion of graph morphism continuous with respect to group-valued flows. In this paper we restrict ourselves to the case of  $\mathbb{Z}_2$ -flows, that is to cycles. Thus, the following is the principal notion we study in this paper:

Given graphs (parallel edges or loops allowed)  $G$  and  $H$ , a mapping  $f : E(G) \rightarrow E(H)$  is called *cycle-continuous*, if for every cycle  $C \subseteq E(H)$ , the preimage  $f^{-1}(C)$  is a cycle in  $G$ . We emphasize, that by a *cycle* we understand (as is common in this area) a set of edges such that every vertex is adjacent with an even number of them. For shortness we sometimes call cycle-continuous mappings just *cc* mappings.

The fact that  $f$  is a *cc* mapping from  $G$  to  $H$  is denoted by  $f : G \xrightarrow{cc} H$ . If we just need to say that there exists a *cc* mapping from  $G$  to  $H$ , we write  $G \xrightarrow{cc} H$ . With the definition covered, we mention the main conjecture describing the properties of *cc* mappings.

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**Conjecture 1.1 (Jaeger).** For every bridgeless graph  $G$  we have  $G \xrightarrow{cc} \text{Pt}$ , where Pt denotes the Petersen graph.

If true, this would imply many conjectures in the area. To illustrate this, suppose we want to find a 5-tuple of cycles in a graph  $G$  covering each of its edges twice (this is conjectured to exist by 5-Cycle double cover conjecture). Further, suppose  $f : G \xrightarrow{cc} \text{Pt}$ . We can use  $C_1, \dots, C_5$  — a 5-tuple of cycles in the Petersen graph double-covering its edges — and then it is easy to check that  $f^{-1}(C_1), \dots, f^{-1}(C_5)$  have the same property in  $G$ .

DeVos *et al.* [2] study this notion further and ask the following question about the structure of cycle-continuous mappings. We say that graphs  $G, G'$  are *cc*-incomparable if there is no *cc* mapping between them, that is  $G \not\xrightarrow{cc} G'$  and  $G' \not\xrightarrow{cc} G$ .

**Question 1.2 ([2]).** Is there an infinite set  $\mathcal{G}$  of bridgeless graphs such that every two of them are *cc*-incomparable?

DeVos *et al.* [2] also show that arbitrary large sets of *cc*-incomparable graphs exist. Their proof is based on the notion of critical snarks and on Lemma 3.1; these will be crucial also for our proof. We will show, that the answer to Conjecture 1.2 is positive, even in a stronger form. Thus, the following are our main results.

**Theorem 1.3.** *There is an infinite set  $\mathcal{G}$  of cubic bridgeless graphs such that every two of them are *cc*-incomparable.*

**Theorem 1.4.** *Every countable (finite or infinite) poset can be represented by a set of graphs and existence of cycle-continuous mappings between them.*

## 2 Properties of cycle-continuous mappings

### 2.1 Properties of a 2-join

In this and the next section we will describe two common constructions of *snarks* (*i.e.*, 3-regular bridgeless graphs, that are not 3-edge-colorable). While the constructions are known (see, *e.g.*, [8]), the relation to cycle-continuous mappings has not been investigated elsewhere, and is crucial to our result. The first construction can be informally described as adding a “gadget” on an edge of a graph. Formally, let  $G_1, G_2$  be graphs, and let  $e_i = x_i y_i$  be an edge of  $G_i$ . We delete edge  $e_i$  from  $G_i$  (for  $i = 1, 2$ ), and connect the two graphs by adding two new edges  $x_1 x_2$  and  $y_1 y_2$ . The resulting graph will be called the *2-join* of the graphs  $G_1, G_2$  (some authors call this a 2-sum); it will be denoted by  $G_1 \oplus_2 G_2$ . We note that

the resulting graph depends on our choice of the edges  $x_i y_i$ , but for our purposes this coarse description will suffice.

**Lemma 2.1.** *For every graphs  $G_1, G_2$  we have  $G_i \xrightarrow{cc} G_1 \oplus_2 G_2$  for  $i \in \{1, 2\}$ .*

**Lemma 2.2.** *Let  $G_1, G_2$  be any graphs. Let  $K$  be an edge-transitive graph. Then  $G_1 \oplus_2 G_2 \xrightarrow{cc} K$  if and only if  $G_1 \xrightarrow{cc} K$  and  $G_2 \xrightarrow{cc} K$ .*

As an immediate corollary we get the following classical result about snarks and 2-joins:

**Corollary 2.3.** *If  $G_1, G_2$  are bridgeless cubic. Then  $G_1 \oplus_2 G_2$  is a snark whenever at least one of  $G_1, G_2$  is a snark.*

Another easy corollary of Lemma 2.2 is that minimal counterexample (if it exists) to Conjecture 1.1 does not contain a nontrivial 2-edge-cut.

**Corollary 2.4.** *Let  $G_1, G_2$  be cubic bridgeless graphs. If  $G_1 \oplus_2 G_2 \not\xrightarrow{cc} \text{Pt}$  then  $G_i \not\xrightarrow{cc} \text{Pt}$  for some  $i \in \{1, 2\}$ .*

## 2.2 Properties of a 3-join

A 3-join (also called 3-sum) is a method to create new snarks – ones that contain nontrivial 3-edge cuts. One way to view this is that we replace a vertex in a graph by a “gadget” created from another graph. To be more precise, we consider graphs  $G_1$  and  $G_2$ , delete a vertex  $u_i$  of each  $G_i$ , and add a matching between neighbors of former vertices  $u_1$  and  $u_2$ . The resulting (cubic) graph in general depends on our choice of  $u_i$ ’s, and of the matching. We use  $G_1 \oplus_3 G_2$  to denote (any of) the resulting graph(s); we call in the 3-join of  $G_1$  and  $G_2$ . Connecting edges of the 3-join are the three edges we added to connect  $G_1$  and  $G_2$ .

**Lemma 2.5.** *For any graphs  $G_1, G_2$  we have  $G_i \xrightarrow{cc} G_1 \oplus_3 G_2$  for  $i = 1, 2$ . We shall call the cycle-continuous mapping from  $G_i$  to  $G_1 \oplus_3 G_2$  a natural inclusion.*

**Lemma 2.6.** *Let  $G_1, G_2$  be any graphs. Let  $K$  be a cyclically 4-edge-connected cubic graph that is 2-transitive. Then  $G_1 \oplus_3 G_2 \xrightarrow{cc} K$  if and only if  $G_1 \xrightarrow{cc} K$  and  $G_2 \xrightarrow{cc} K$ .*

As an immediate corollary we get the following classical result about snarks and 3-joins:

**Corollary 2.7.** *Let  $G_1, G_2$  be cubic bridgeless graphs. Then  $G_1 \oplus_3 G_2$  is a snark, iff at least one of  $G_1, G_2$  is a snark.*

As another easy application, we observe that minimal counterexample (if it exists) to Conjecture 1.1 does not contain a nontrivial 3-edge-cut.

**Corollary 2.8.** *Let  $G_1, G_2$  be cubic bridgeless graphs. If  $G_1 \oplus_3 G_2 \not\stackrel{cc}{\rightarrow} \text{Pt}$  then  $G_i \not\stackrel{cc}{\rightarrow} \text{Pt}$  for some  $i \in \{1, 2\}$ .*

The above notwithstanding, we proceed to study the structure of cycle-continuous mapping in graphs with 3-edge-cuts, for two reasons: first we believe, it provides insights that might be useful in further progress towards solving Conjecture 1.1, second, we find it has an independent interest.

**Lemma 2.9.** *Let  $G_1, G_2$  be cc-incomparable snarks. Then  $G_1 \oplus_3 G_2 \stackrel{cc}{\rightarrow} G_i$  for each  $i \in \{1, 2\}$ .*

### 3 The proof

#### 3.1 Critical snarks

For our construction we will need the following notion of criticality of snarks. It appears in Neděla *et al.* [5] and in DeVos *et al.* [2]; see also [1], where these graphs are called flow-critical snarks.

Recall a graph  $G$  is a snark if  $G \not\stackrel{cc}{\rightarrow} K_2^3$ , where  $K_2^3$  is a graph formed by two vertices and three parallel edges. We say  $G$  is a *critical snark* if for every edge  $e$  of  $G$  we have  $G - e \stackrel{cc}{\rightarrow} K_2^3$ . (Equivalently [1],  $G/e \stackrel{cc}{\rightarrow} K_2^3$ .)

The following lemma is the basis of our control over cycle-continuous mappings between graphs in our construction.

**Lemma 3.1 ([2]).** *Let  $G, H$  be cyclically 4-edge-connected cubic graphs, both of which are critical snarks, suppose that  $|E(G)| = |E(H)|$ . Then  $G \stackrel{cc}{\rightarrow} H$  iff  $G \cong H$ . Moreover, every cycle-continuous mapping is a bijection that is induced by an isomorphism of  $G$  and  $H$ .*

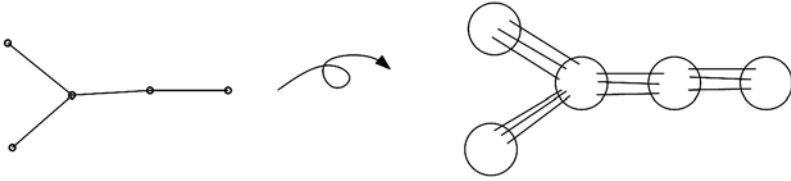
**Lemma 3.2.** *There are two snarks  $B_1, B_2$  with 18 vertices, that are critical and nonisomorphic. Moreover, none of  $B_1, B_2$  is vertex transitive.*

*Proof.* It is well-known that the two Blanuša snarks on 18 vertices satisfy these requirements.  $\square$

#### 3.2 Tree of snarks

Let  $\mathcal{G} = \{G_1, \dots, G_n\}$  be a family of critical snarks of the same size, so that for  $i \neq j$  graphs  $G_i$  and  $G_j$  are not isomorphic (equivalently:  $G_i \not\stackrel{cc}{\rightarrow} G_j$  and  $G_j \not\stackrel{cc}{\rightarrow} G_i$ ).

Let  $T$  be a tree with a vertex coloring (not necessarily proper)  $c : V(T) \rightarrow [n]$ . We denote by  $T(\mathcal{G})$  a family of graphs that can be obtained by replacing each  $v \in V(T)$  by a copy of  $G_{c(v)}$  and performing a 3-join on each edge; see Fig. 3.1 for an illustration. There are in general many graphs that can be constructed in this way, depending on which vertices one chooses for the 3-join operations.



**Figure 3.1.** Illustration of the “tree-snark” construction.

More precisely, for each  $v \in V(T)$  we fix a bijection  $r_v$  from  $N_T(v)$  to an independent set  $A_v$  in  $G_{c(v)}$ , we also specify an ordering of edges going out of vertices of  $A_v$ . Next, we split each vertex  $w$  in  $A_v$  into three degree 1 vertices; these will be denoted by  $w_1, w_2, w_3$ . For each edge  $uv$  of  $T$  we identify vertices  $r_u(v)_i$  with  $r_v(u)_i$  for  $i = 1, 2, 3$ . Finally, we suppress all vertices of degree 2.

If  $H$  is a graph in  $T(\mathcal{G})$  and  $v$  a vertex of  $T$ , we let  $H_v$  denote a “copy” of  $G_{c(v)}$ : subgraph of  $H$  consisting of a copy of  $G_{c(v)} - A_v$  together with the incident edges and neighboring vertices in  $H$ . Further, we let  $\iota_v$  denote the natural inclusion of  $G_{c(v)}$  into  $H$ , which maps bijectively on  $H_v$ .

We define  $\bar{H}_v$  to be the graph  $H$  with all edges outside of  $H_v$  contracted. In other words,  $\bar{H}_v$  is truly an isomorphic copy of  $G_{c(v)}$ . Further,  $H_{u,v}$  will denote the three edges in the intersection  $H_u \cap H_v$ .

The following theorem is the key to our construction.

**Theorem 3.3.** *Let  $T_1, T_2$  be two trees and let  $c_i : V(T_i) \rightarrow [n]$  be arbitrary colorings. Let  $\mathcal{G}$  be as above.*

*Suppose  $H_i \in T_i(\mathcal{G})$  for  $i = 1, 2$ . Every  $cc$  mapping  $g : H_1 \xrightarrow{cc} H_2$  is guided by a homomorphism  $f : T_1 \rightarrow T_2$  of reflexive colored graphs: There is a mapping  $f : V(T_1) \rightarrow V(T_2)$  such that*

- $c_2(f(v)) = c_1(v)$  ( $f$  respects colors), and
- if  $uv$  is an edge of  $T_1$ , then  $f(u)f(v)$  is an edge of  $T_2$  or  $f(u) = f(v)$ .  
*In the first case,  $g$  maps  $H_{u,v}$  to  $H_{f(u),f(v)}$ . In the second one,  $H_{u,v}$  is mapped to some  $H_{f(u),v'}$ .*

*Moreover,  $g$  induces a mapping  $(\bar{H}_1)_v$  to  $(\bar{H}_2)_{f(v)}$  that is cycle-continuous.*

As a corollary we obtain our first result, that already answers Question 1.2.

**Corollary 3.4.** *There is an infinite set of cc-incomparable graphs.*

*Proof.* Let  $T_n$  be a path with vertices  $\{0, 1, \dots, n\}$  colored as  $1(2)^{n-1}1$ . We let  $\mathcal{G} = \{B_1, B_2\}$ , where as in Lemma 3.2,  $B_i$ 's denote the Blanuša snarks on 18 vertices. We fix vertices  $a, b$  of  $B_2$  so that no automorphism of  $B_2$  sends  $a$  to  $b$ . For all vertices  $v \in V(T_n)$  of degree 2 we create  $r_v$  so, that  $r_v(v-1) = a$  and  $r_v(v+1) = b$ . We do not specify  $A_0$  nor  $A_n$ , neither the order of edges adjacent to  $a$  or  $b$ . We let  $H_n$  denote any of  $T_n(\mathcal{G})$ .

Consider  $H_m, H_n$ , suppose that  $g : H_m \xrightarrow{cc} H_n$  is cc mapping. Let  $f : V(T_m) \rightarrow V(T_n)$  be the mapping guaranteed by Theorem 3.3. As  $f$  respects colors, we have  $\{f(0), f(m)\} \subseteq \{0, n\}$ . Next, define  $G_i = (H_m)_i$ , and  $G'_j = (H_n)_j$ . By Theorem 3.3 again,  $g$  is cc mapping  $G_i \xrightarrow{cc} G'_{f(i)}$ . As  $G_i$  and  $G'_{f(i)}$  are isomorphic to  $B_2$ , Lemma 3.2 implies that  $f(i+1) = f(i) + 1$ . It follows that  $m = n$ , which finishes the proof.  $\square$

### 3.3 Representing posets by cycle-continuous mappings

Question 1.2 should be understood as a question about how complicated is the structure of cc mappings. Next, we provide even further indication, that the structure is complicated indeed.

**Corollary 3.5.** *Every countable (finite or infinite) poset can be represented by a set of graphs and existence of cc mappings between them.*

*Proof.* (Sketch) We use the result of Hubička and Nešetřil [3], claiming that arbitrary countable posets can be represented by finite directed paths and existence of homomorphisms between them.

Thus, we only need to find a mapping  $m$  that to directed paths assigns cubic bridgeless graphs, so that  $P_1 \text{ hom } P_2$  iff  $m(P_1) \xrightarrow{cc} m(P_2)$ . We choose vertices  $a, b$  in the Blanuša snark  $B_2$  so that no automorphism of  $B_2$  maps  $a$  to  $b$ . Then we replace each directed edge by a copy of  $B_2$  “from  $a$  to  $b$ ” and perform a 3-join operation in-between each pair of adjacent edges. Formally, let  $P$  be a path with edges (from one end to the other)  $e_1, \dots, e_m$ . We let  $t(i)$  be the index of the edge at the tail of  $e_i$  – that is  $t(i)$  is either  $i-1$  (if  $e_i$  goes forward with respect to our labeling) or  $i+1$ . Note that  $t(i)$  may be undefined for  $i \in \{1, m\}$ . Similarly, we define  $h(i)$  to be the index of the edge adjacent to  $e_i$  at its head. We will use the construction from Section 3.2. Our tree  $T$  will be a path with vertices  $1, \dots, m$  all colored by 1, our set of snarks will consist just of the second Blanuša snark,  $\mathcal{G} = \{G_1 = B_2\}$ . We define  $r_i(t(i)) = a$ ,

and  $r_i(h(i)) = b$ , whenever  $t(i)$  ( $h(i)$ , resp.) are defined. We choose an ordering of edges going out of  $a$ , and  $b$ ; we keep this fixed for all vertices of all paths. Then we let  $m(P)$  be the graph in  $T(\mathcal{G})$  determined by the above described choices. (The rest of the proof is omitted in the extended abstract.)  $\square$

#### 4 Concluding remarks

While being a resolution to Question 1.2, none of the family of examples we gave does violate Conjecture 1.1:

**Theorem 4.1.** *If  $H \in T(\mathcal{G})$  and for every  $G \in \mathcal{G}$  we have  $G \xrightarrow{cc} \text{Pt}$  then  $H \xrightarrow{cc} \text{Pt}$ .*

Still, the presented results illustrate the complexity of  $cc$  mappings. To better understand their structure, we suggest the following questions:

**Question 4.2.** Does the poset of cubic cyclically 4-edge-connected graphs and  $cc$  mappings between them have infinite antichains? Does it contain every countable poset as a subposet? How about cyclically 5-edge-connected graphs?

For the next question, recall that in a poset  $(X, \leq)$  an interval  $(a, b)$  is the set  $\{x \in X : a < x < b\}$  (we must have  $a < b$  for this definition to make sense, otherwise we call  $(a, b)$  degenerate interval).

**Question 4.3.** In the poset of graphs and  $cc$  mappings between them, is every non-degenerate interval nonempty? Does every non-degenerate interval contain infinite antichain? Does every non-degenerate interval contain every countable poset?

Note, that if Conjecture 1.1 is true, then  $(Pt, K_2)$  is an empty but non-degenerated interval. Is there some other?

We also briefly note the more general definition of flow-continuous mappings, that extends the notion of cycle-continuous mappings: a mapping  $f : E(G) \rightarrow E(H)$  is called  $M$ -flow-continuous (for an abelian group  $M$ ) if for every  $M$ -flow  $\varphi$  on  $H$ , the composition  $\varphi \circ f$  is an  $M$ -flow on  $G$ . For detailed discussion, see [2] or [6]. We only mention here, that cycle-continuous mappings are exactly  $\mathbb{Z}_2$ -flow-continuous ones, and that Corollaries 3.4 and 3.5 extend trivially to  $\mathbb{Z}$ -flow-continuous mappings.

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# On the structure of graphs with given odd girth and large minimum degree

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**Abstract.** We study the structure of graphs with high minimum degree conditions and given odd girth. For example, the classical work of Andrásfai, Erdős, and Sós implies that every  $n$ -vertex graph with odd girth  $2k+1$  and minimum degree bigger than  $\frac{2n}{2k+1}$  must be bipartite. We consider graphs with a weaker condition on the minimum degree. Generalizing results of Häggkvist and of Häggkvist and Jin for the cases  $k = 2$  and  $3$ , we show that every  $n$ -vertex graph with odd girth  $2k + 1$  and minimum degree bigger than  $\frac{3n}{4k}$  is homomorphic to the cycle of length  $2k + 1$ . This is best possible in the sense that there are graphs with minimum degree  $\frac{3n}{4k}$  and odd girth  $2k + 1$  which are not homomorphic to the cycle of length  $2k + 1$ . Similar results were obtained by Brandt and Ribe-Baumann.

## 1 Introduction

We consider finite and simple graphs without loops and for any notation not defined here we refer to the standard textbooks. In particular, we denote by  $K_r$  the complete graph on  $r$  vertices, by  $C_r$  a cycle of length  $r$ , where the length of a cycle or of a path denotes its number of edges. A *homomorphism* from a graph  $G$  into a graph  $H$  is a mapping  $\varphi: V(G) \rightarrow V(H)$  with the property that  $\{\varphi(u), \varphi(w)\} \in E(H)$  whenever  $\{u, w\} \in E(G)$ . We say that  $G$  is *homomorphic* to  $H$  if there exists a homomorphism from  $G$  into  $H$ . Furthermore, a graph  $G$  is a *blow-up* of a graph  $H$ , if there exists a surjective homomorphism  $\varphi$  from  $G$  into  $H$ , but for any supergraph of  $G$  on the same vertex set the mapping  $\varphi$  is not a homomorphism into  $H$  anymore. In particular, a graph  $G$  is homomorphic to  $H$  if and only if it is a subgraph of a suitable blow-up of  $H$ . Moreover, we say a blow-up  $G$  of  $H$  is *balanced* if the homomorphism  $\varphi$  signifying

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that  $G$  is a blow-up has the additional property that  $|\varphi^{-1}(u)| = |\varphi^{-1}(u')|$  for all vertices  $u$  and  $u'$  of  $H$ .

Homomorphisms can be used to capture structural properties of graphs. For example, a graph is  $k$ -colourable if and only if it is homomorphic to  $K_k$ . Many results in extremal graph theory establish relationships between the minimum degree of a graph and the existence of a given subgraph. The following theorem of Andrásfai, Erdős and Sós [2] is a classical result of that type.

**Theorem 1.1 (Andrásfai, Erdős & Sós).** *For every integer  $r \geq 3$  and for every  $n$ -vertex graph  $G$  the following holds. If  $G$  has minimum degree  $\delta(G) > \frac{3r-7}{3r-4}n$  and  $G$  contains no copy of  $K_r$ , then  $G$  is  $(r-1)$ -colourable.*

In the special case  $r = 3$ , Theorem 1.1 states that every triangle-free  $n$ -vertex graph with minimum degree greater than  $2n/5$  is bipartite, *i.e.*, it is homomorphic to  $K_2$ . Several extensions of this result and related questions were studied. One line of research (see, *e.g.*, [4, 7, 9, 10]) concerned the question for which minimum degree condition a triangle-free graph  $G$  is homomorphic to a graph  $H$  of bounded size, which is triangle-free itself. In particular, Häggkvist [7] showed that triangle-free graphs  $G = (V, E)$  with  $\delta(G) > 3|V|/8$  are homomorphic to  $C_5$ . In other words, such a graph  $G$  is a subgraph of suitable blow-up of  $C_5$ . This can be viewed as an extension of Theorem 1.1 for  $r = 3$ , since balanced blow-ups of  $C_5$  show that the degree condition  $\delta(G) > 2|V|/5$  is sharp there. Strengthening the assumption of triangle-freeness to graphs of higher odd girth, allows us to consider graphs with a more relaxed minimum degree condition. In this direction Häggkvist and Jin [8] showed that graphs  $G = (V, E)$  which contain no odd cycle of length three and five and with minimum degree  $\delta(G) > |V|/4$  are homomorphic to  $C_7$ .

We generalize those results to arbitrary odd girth. We say a graph  $G$  has *odd girth* at least  $g$ , if the shortest cycle with odd length has length at least  $g$ .

**Theorem 1.2.** *For every integer  $k \geq 2$  and for every  $n$ -vertex graph  $G$  the following holds. If  $G$  has minimum degree  $\delta(G) > \frac{3n}{4k}$  and  $G$  has odd girth at least  $2k + 1$ , then  $G$  is homomorphic to  $C_{2k+1}$ .*

Note that the degree condition given in Theorem 1.2 is best possible as the following example shows. For an even integer  $r \geq 6$  we denote by  $M_r$  the so-called *Möbius ladder* (see, *e.g.*, [6]), *i.e.*, the graph obtained by adding all diagonals to a cycle of length  $r$ , where a diagonal connects vertices of distance  $r/2$  in the cycle. One may check that  $M_{4k}$  has

odd girth  $2k + 1$ , but it is not homomorphic to  $C_{2k+1}$ . Moreover,  $M_{4k}$  is 3-regular and, consequently, balanced blow-ups of  $M_{4k}$  show that the degree condition in Theorem 1.2 is best possible when  $n$  is divisible by  $4k$ .

We also remark that Theorem 1.2 implies that every graph with odd girth at least  $2k + 1$  and minimum degree bigger than  $\frac{3n}{4k}$  contains an independent set of size at least  $\frac{kn}{2k+1}$ . This answers affirmatively a question of Albertson, Chan, and Haas [1]. Similar results were obtained by Brandt and Ribe-Baumann.

## 2 Sketch of the proof

In the proof of Theorem 1.2 we consider an edge-maximal graph and show that it is either a bipartite graph or a blow-up of a  $(2k+1)$ -cycle. We say that a graph  $G$  with odd girth at least  $2k + 1$  is *edge-maximal* if adding any edge to  $G$  yields an odd cycle of length at most  $2k - 1$ . We denote by  $\mathcal{G}_{n,k}$  all edge-maximal  $n$ -vertex graphs satisfying the assumptions of the main theorem, *i.e.*, for integers  $k \geq 2$  and  $n$  we set

$$\mathcal{G}_{n,k} = \{G = (V, E) : |V| = n, \delta(G) > \frac{3n}{4k}, \text{ and } G \text{ is edge-maximal with odd girth } 2k + 1\}.$$

The proof of the theorem relies on two lemmas, Lemmas 2.1 and 2.3 below, which state that certain configurations cannot occur in such edge-maximal graphs.

**Lemma 2.1.** *Let  $\Phi$  denote the graph obtained from  $C_6$  by adding one diagonal. For all  $k \geq 2$  and  $n$ , every  $G \in \mathcal{G}_{n,k}$  does not contain an induced copy of  $\Phi$ .*

*Proof (sketch).* Suppose, contrary to the assertion, that  $G = (V, E)$  contains  $\Phi$  in an induced way. Since  $G$  is edge-maximal, the non-existence of a diagonal must be forced by the existence of an even path which, together with the missing diagonal, would yield an odd cycle of length at most  $2k - 1$ . One can show that such a path must have length exactly  $2k - 2$  and that it must be edge-disjoint from  $\Phi$ . Since there are two missing diagonals and since one can show that the related paths are also disjoint, the resulting configuration  $\Phi'$  has  $4k$  vertices. Finally one shows that no vertex in  $G$  can be joined to four vertices of  $\Phi'$ , which leads to a contradiction to the minimum degree condition of  $G$ .  $\square$

We remark that the above lemma can also be deduced from [8, Lemma 2], where is shown that  $G \in \mathcal{G}_{n,k}$  cannot contain a cycle of length  $4k$  with

two consecutive diagonals. The next lemma states that graphs  $G \in \mathcal{G}_{n,k}$  contain no graph from the following family, which can be viewed as tetrahedra with three faces formed by cycles of length  $2k + 1$ .

**Definition 2.2.** For  $k \geq 2$  we denote by  $\mathcal{T}_k$  those subdivisions  $T$  of  $K_4$  satisfying

- (i) three triangles of  $K_4$  are subdivided into cycles of length  $2k + 1$
- (ii) two of the three edges contained in two of those triangles are subdivided into paths of length at least two.

We remark that in similar context “odd subdivisions” of  $K_4$  appeared in [5].

**Lemma 2.3.** For all integers  $k \geq 2$  and  $n$  and for every  $G \in \mathcal{G}_{n,k}$  we have that  $G$  does not contain any  $T \in \mathcal{T}_k$  as a (not necessarily induced) subgraph.

*Proof (sketch).* Similarly to the previous lemma, one can show that if such a  $T \in \mathcal{T}_k$  is contained in  $G$ , then we get a contradiction to the minimum degree condition. In fact, (i) in Definition 2.2 implies that all four triangles of  $K_4$  must be subdivided into a cycle of odd length in  $T$ . Since all these cycles must have length at least  $2k + 1$  it follows that  $T$  consists of at least  $4k$  vertices. Then some case analysis shows that any vertex in  $G$  can be joined to at most three vertices in  $T$ , contradicting the assumption on the minimum degree of  $G$ .  $\square$

In the proof of the main theorem, we assume that  $G$  is not bipartite and show that  $G$  is a blow-up of a  $(2k + 1)$ -cycle. In particular, we show that if a vertex of  $G$  is not contained in a maximal blow-up, then it gives rise to one of the forbidden configurations of Lemmas 2.1 and 2.3.

*Proof of Theorem 1.2 (sketch).* Suppose  $G$  is not bipartite. The edge-maximality of  $G$  implies that it contains a cycle of length  $C_{2k+1}$ . Let  $B$  be a vertex-maximal blow-up of a  $(2k + 1)$ -cycle contained in  $G$  with vertex classes  $A_0, \dots, A_{2k}$ . We show  $B = G$ . Suppose that there exists a vertex  $x \in V \setminus V(B)$ . Owing to the odd girth assumption on  $G$ , the vertex  $x$  can have neighbours in at most two of the vertex classes of  $B$  and if there are two such classes, then within  $B$  each vertex in one class has distance two from the vertices in the other class.

Suppose first that  $x$  has neighbours in two classes  $A_{i-1}$  and  $A_{i+1}$ . If we are able to prove that  $x$  is adjacent to all the vertices in the two classes, then  $x$  can be included in  $A_i$ . If this is not the case, then by symmetry we may assume that there exists some vertex  $b_{i-1} \in A_{i-1}$  which is not

a neighbour of  $x$ . Fix vertices  $a_{i-2} \in A_{i-2}$  and  $a_i \in A_i$  arbitrarily and let  $a_{i-1} \in A_{i-1}$  and  $a_{i+1} \in A_{i+1}$  be neighbours of  $x$ . This fixes a cycle of length six in  $G$ , namely  $xa_{i+1}a_ib_{i-1}a_{i-2}a_{i-1}x$ , with one diagonal  $\{a_{i-1}, a_i\}$ . Due to Lemma 2.1, there must be at least one more diagonal and one can easily show that this diagonal must be  $\{b_{i-1}, x\}$ , since  $\{a_{i+1}, a_{i-2}\}$  is a “shortcut” in the blow-up which would create a cycle of length  $2k - 1$  in  $G$ .

Suppose that all neighbours of  $x$  in  $B$  are in  $A_i$ . Let  $a_i \in A_i$  be a neighbour of  $x$  and fix a cycle  $a_0a_1 \dots a_{2k}$  in  $B$  containing  $a_i$ . Due to the edge-maximality, the non-existence of the edges  $\{x, a_{i-2}\}$  and  $\{x, a_{i+2}\}$  is forced by two paths which, together with the missing edges, would create short odd cycles. One can check that such paths have length exactly  $2k - 2$  and, together with the fixed cycle, they form a graph  $T \in \mathcal{T}_k$ , contradicting Lemma 2.3. Since  $G$  is connected due to its edge-maximality, this concludes the proof of Theorem 1.2.  $\square$

### 3 Concluding remarks

**Extremal case in Theorem 1.2.** A more careful analysis yields that the unique  $n$ -vertex graph with odd girth at least  $2k + 1$  and minimum degree exactly  $\frac{3n}{4k}$ , which is not homomorphic to  $C_{2k+1}$ , is the balanced blow-up of the Möbius ladder  $M_{4k}$ . In fact, the proofs of Lemmas 2.1 and 2.3 can be adjusted such that they either exclude the existence of  $\Phi$  resp.  $T$  in  $G$  or they yield  $M_{4k} \subseteq G$ . In the former case, one can repeat the proof of Theorem 1.2 based on those lemmas and obtains that  $G$  is homomorphic to  $C_{2k+1}$ . In the latter case, one uses the degree assumption to deduce that  $G$  is isomorphic to a balanced blow-up of  $M_{4k}$ .

**Open questions.** It would be interesting to study the situation, when we further relax the degree condition in Theorem 1.2. It seems plausible that if  $G$  has odd girth at least  $2k + 1$  and  $\delta(G) \geq (\frac{3}{4k} - \varepsilon)n$  for sufficiently small  $\varepsilon > 0$ , then the graph  $G$  is homomorphic to  $M_{4k}$ . In fact, this could be true until  $\delta(G) > \frac{4n}{6k-1}$ . At this point blow-ups of the  $(6k - 1)$ -cycle with all chords connecting two vertices of distance  $2k$  in the cycle added show that this is best possible. For  $k = 2$  such a result was proved in [4] and for  $k = 3$  it appeared in [3].

For  $\ell \geq 2$  and  $k \geq 3$  let  $F_{\ell,k}$  be the graph obtained from a cycle of length  $(2k - 1)(\ell - 1) + 2$  by adding all chords which connect vertices with distance of the form  $j(2k - 1) + 1$  in the cycle for some  $j = 1, \dots, \lfloor (\ell - 1)/2 \rfloor$ . Note that  $F_{2,k} = C_{2k+1}$  and  $F_{3,k} = M_{4k}$ . For every  $\ell \geq 2$  the graph  $F_{\ell,k}$  is  $\ell$ -regular, has odd girth  $2k + 1$ , and it has chromatic number three. Moreover,  $F_{\ell+1,k}$  is not homomorphic to  $F_{\ell,k}$ , but contains it as a subgraph.

A possible generalization of the known results would be the following: if an  $n$ -vertex graph  $G$  has odd girth at least  $2k + 1$  and minimum degree bigger than  $\frac{\ell n}{(2k-1)(\ell-1)+2}$ , then it is homomorphic to  $F_{\ell-1,k}$ . This is known to be false for  $k = 2$  and  $\ell > 10$ , since such a graph  $G$  may contain a copy of the Grötzsch graph which (due to having chromatic number four) is not homomorphically embeddable into any  $F_{\ell,k}$ . In some sense this is the only exception for  $k = 2$  and  $\ell > 10$ , since adding the condition  $\chi(G) \leq 3$  makes the statement true for  $k = 2$  (see, e.g., [4]). It is not known what happens for  $k > 2$ .

This discussion motivates the following extension of a result of Łuczak for triangle-free graphs from [10]. For fixed  $k$  the density of  $F_{\ell,k}$  tends to  $\frac{1}{2k-1}$  as  $\ell \rightarrow \infty$ . Is it true that every  $n$ -vertex graph with odd girth at least  $2k + 1$  and minimum degree at least  $(\frac{1}{2k-1} + \varepsilon)n$  can be mapped homomorphically into a graph  $H$  which also has odd girth at least  $2k + 1$  and  $V(H)$  is bounded by a constant  $C = C(\varepsilon)$  independent of  $n$ ? In [10] Łuczak proved this for  $k = 2$ .

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# On the order of cages with a given girth pair

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## 1 Cages with a given girth pair

The cage problem asks for the construction of regular simple graphs with specified degree  $k$ , girth  $g$  and minimum order  $n(k; g)$ , (see [5] for a complete survey). This problem was first considered by Tutte [10]. In 1963, Erdős and Sachs [4] proved that  $(k; g)$ -cages exist for any given values of  $k$  and  $g$ .

Counting the numbers of vertices in the distance partition with respect to a vertex when  $g$  is odd, and with respect to an edge when  $g$  is even, yields the lower bound  $n_0(k; g)$  on the order of a  $(k; g)$ -cage. For  $k \geq 3$  and  $g \geq 5$  the order  $n(k; g)$  of a cage is bounded by

$$n(k; g) \geq n_0(k; g) = \begin{cases} 1 + k \sum_{i=0}^{(g-3)/2} (k-1)^i & g \text{ odd} \\ 2 \sum_{i=0}^{(g-2)/2} (k-1)^i & g \text{ even} \end{cases} \quad (1.1)$$

This bound is called the Moore bound, it is known that the order of a  $(k; g)$ -cage  $n(k; g) = n_0(k; g)$  only for  $g = 6, 8, 12$  and  $k = q + 1$  with  $q$  a prime power; and for  $g = 5$  and  $k = 3, 7, 57$  (*cf.* [1, 3]). Therefore

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Biggs and Ito [2] defined the excess of a cage to be the number  $n(k; g) - n_0(k; g)$  and proved the following theorem.

**Theorem 1.1 ([2]).** [2] *Let  $G$  be a  $(k; g)$ -cage of girth  $g = 2m \leq 6$  and excess  $e$ . If  $e \leq k - 2$  then  $G$  is bipartite and its diameter is  $m + 1$ .*

By allowing a *girth pair*  $g < h$  (one even and the other odd), in [6], Harary and Kovács introduced the concept of a  $(k; g, h)$ -cage as the smallest  $k$ -regular graph with girth pair  $g, h$ . They obtained the bound  $n(k; g, h) \leq 2n(k; h)$  that relates the order of a cage with girth pair with the order of a cage, and showed that  $n(k; h - 1, h) \leq n(k; h)$ , i.e. in general the bound  $n(k; g, h) \leq 2n(k; h)$  is not the best and stated the following conjecture.

**Conjecture 1.2 ([6]).**  $n(k; g, h) \leq n(k; h)$ , for all  $k, g \geq 3$ .

Xu, Wang, Wang [11] proved the strict inequality,  $n(k; h - 1, h) < n(k; h)$ . Kovács proved that the Möbius ladder of order  $2(h - 1)$  is the unique minimal  $(3; 4, h)$ -graph [7]. Campbell [8] studied the size of smallest cubic graphs with girth pair  $(6, b)$  and constructed the cages for the exact values  $(3; 6, 7)$ ,  $(3; 6, 9)$  and  $(3; 6, 11)$ .

We obtain that the conjecture  $n(k; g, h) < n(k; h)$  holds for all  $(k; g, h)$ -cages when  $g$  is odd. For  $g$  even, we settle the conjecture for cages of small excess, i.e. such that  $n(k; g) - n_0(k; g) \leq k - 1$ , also we prove that  $n(k; g, h) < n(k; h)$  for  $h$  sufficiently large, in both cases, for  $g$  even and  $g$  odd, under the assumption that  $(k; g)$ -cages are bipartite for  $g$  even. Notice that the cages of even girth and small excess are known to be bipartite [2], furthermore it is conjectured that all cages with even girth are bipartite [9, 12].

## 2 Notation

For any graph of girth  $g \geq 4$  even,  $uv \in E(G)$  and  $0 \leq l \leq \frac{g}{2} - 1$ , let us denote the sets

$$B_{uv}^l = \{x \in V(G) : d(x, u) = l \text{ and } d(x, v) = l + 1\} \text{ and } \overline{B}_{uv}^l = \bigcup_{i=0}^l B_{uv}^i.$$

Observe that  $B_{uv}^0 = \{u\} = \overline{B}_{uv}^0$  and  $B_{uv}^1 = N(u) - v$  while  $\overline{B}_{uv}^1 = (N(u) - v) \cup \{u\}$ . Moreover, note that  $B_{uv}^l \neq B_{vu}^l$  and  $\overline{B}_{uv}^l \neq \overline{B}_{vu}^l$ . Denote  $T_{uv}^l = G[\overline{B}_{uv}^l \cup \overline{B}_{vu}^l]$  and observe that if  $l \leq \frac{g}{2} - 2$ , where  $g$  is the girth of  $G$ , then  $T_{uv}^l$  is the tree rooted in the edge  $uv$  of depth  $l$ . When  $l = \frac{g}{2} - 1$  the subgraph  $T_{uv}^l$  may not be a tree, it can contain edges between vertices in  $B_{uv}^l$  and vertices in  $B_{vu}^l$ .

We will denote the set of cycles in  $G$  by  $\mathcal{C}(G) = \{\alpha : \alpha \text{ is a cycle in } G\}$ .

Let  $G$  be a  $(k; g)$ -cage of even girth  $g = 2m$ . The *excess*  $e$  of  $G$  with respect to an edge  $uv \in E(G)$  is the cardinality of the set  $X = V(G) \setminus T_{uv}^{m-1}$ . Note that the order of  $T_{uv}^{m-1}$  is the same for every edge  $uv \in E(G)$ . Thus,  $e = |X| = n(k; g) - n_0(k; g)$ .

### 3 Constructions for $g$ odd

Deleting from  $G$  the sets  $\overline{B}_{uv}^{\binom{h-g-1}{2}-1} \cup \overline{B}_{vu}^{\binom{h-g+1}{2}-1}$  and completing the degrees of the remaining vertices we obtain Theorem 3.2. In order to keep cycles of length  $h$  after the deletion we proved the following lemma.

**Lemma 3.1.** *Let  $G$  be a  $(k; h)$ -cage with  $k \geq 3$  and even girth  $h \geq 6$ . Then  $G$  contains a cycle  $\beta$  of length  $h$  such that  $V(\beta) \cap B_{uv}^{\frac{h}{4}-1} = \emptyset$  or  $V(\beta) \cap B_{vu}^{\frac{h}{4}-1} = \emptyset$ .*

**Theorem 3.2.** *Let  $h \geq 6$  even and  $k \geq 3$ . Suppose that there is a bipartite  $(k; h)$ -cage. If  $g \geq 5$  is an odd number such that  $\frac{h}{2} + 1 \leq g < h$ , then*

$$n(k; g, h) \leq n(k; h) - 2 \sum_{i=0}^{\frac{h-g-3}{2}} (k-1)^i - (k-1)^{\frac{h-g-1}{2}}.$$

**Theorem 3.3.** *Let  $h \geq 6$  even and  $k \geq 3$ . Suppose that there is a bipartite  $(k; h)$ -cage. If  $g$  is an odd number such that  $g < h$ , then  $n(k; g, h) < n(k; h)$ .*

### 4 Constructions for $g$ even and $h$ odd

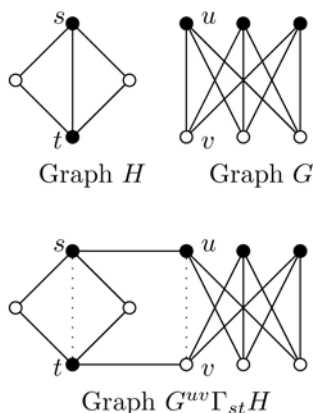
First of all we introduce a construction that we will use later for breaking short odd cycles while preserving the regularity and the even girth.

**Definition 4.1.** Let  $G, H$  be two vertex-disjoint graphs,  $uv \in E(G)$  and  $st \in E(H)$ . We will define a new graph  $G^{uv} \Gamma_{st} H$ , that we will call the *insertion* of  $(G, uv)$  into  $(H, st)$  by letting:

- $V(G^{uv} \Gamma_{st} H) = V(G) \cup V(H)$
- $E(G^{uv} \Gamma_{st} H) = (E(G) \setminus \{uv\}) \cup (E(H) \setminus \{st\}) \cup \{us, vt\}$ .

See Figure 4.1, for an example illustrating this definition.

The first basic result with respect to  $g$  even and  $h$  odd is the following theorem, the bound is proved inserting a graph  $(G, uv)$  into a copy  $(G', u'v')$ , and performing some edge operations in order to obtain cycles of length  $2g - 1$ .



**Figure 4.1.** The insertion  $G^{uv}\Gamma_{st}H$ .

**Theorem 4.2.** *Let  $k \geq 3$  and  $g \geq 6$  even. Then  $n(k; g, 2g - 1) \leq 2n(k; g)$  provided that there is a bipartite  $(k; g)$ -cage.*

To prove the corresponding result for  $n(k; g, g + r)$  we will use the following remark.

**Remark 4.3.** Let  $G, H$  be graphs with girths  $g, h$ , respectively, such that  $g \leq h$ , and let  $G^{uv}\Gamma_{st}H$  be the insertion of  $(G, uv)$  into  $(H, st)$ . Then the set of cycles in  $G^{uv}\Gamma_{st}H$  is:

$$\begin{aligned} \mathcal{C}(G^{uv}\Gamma_{st}H) &= (\mathcal{C}(G) \setminus \{\alpha \in \mathcal{C}(G) : uv \in E(\alpha)\}) \\ &\cup (\mathcal{C}(H) \setminus \{\beta \in \mathcal{C}(H) : st \in E(\beta)\}) \\ &\cup \{\gamma = P_1vtP_2su : P_1 \text{ is a } uv\text{-path in } G - uv \\ &\quad \text{and } P_2 \text{ is a } ts\text{-path in } H - st\}. \end{aligned}$$

This means that if there were cycles of lengths  $c_1$  and  $c_2$  in graphs  $G$  and  $H$  that used the edges  $uv$  and  $st$ , respectively, they are removed in the new graph  $G^{uv}\Gamma_{st}H$  and new cycles of length  $c_1 + c_2$  are created.

**Theorem 4.4.** *Let  $k \geq 3$ ,  $g$  even such that  $6 \leq g$  and  $r$  an odd number such that  $1 \leq r \leq g - 3$ . Then  $n(k; g, g + r) \leq 4n(k; g)$ , provided that there is a bipartite  $(k; g)$ -cage.*

By applying the insertion to the graph obtained in Theorem 4.4 on specific edges, we obtain the following lemma.

**Lemma 4.5.** *Let  $k \geq 3$ ,  $g \geq 6$  even and suppose that there is a bipartite  $(k; g)$ -cage. Then  $n(k; g, mg + r) \leq 4n(k; g) + k(m - 1)n(k; g)$ , for  $m \geq 1$  and  $r$  any odd number such that  $1 \leq r \leq g - 1$ . In particular when  $r = g - 1$ , from Theorem 4.2, we have  $n(k; g, (m + 1)g - 1) \leq 2mn(k; g)$ .*

**Theorem 4.6.** *Suppose there is a bipartite  $(k; g)$ -cage with degree  $k \geq 3$  and even girth  $g \geq 6$ . Then  $n(k; g, h) < n(k, h)$ , for  $h$  sufficiently large.*

So, we have proved Conjecture 1.2 for even girth  $g$  in general but asymptotically. For specific values of  $k$  and  $g$ , the Conjecture 1.2 can be completely settled, as we will show next.

**Corollary 4.7.** *For every  $(k; g)$ -cage of even girth  $g$ , degree  $k \geq 3$  and excess  $e \leq k - 2$ , we have  $n(k + 1; g, h) < n(k + 1; h)$ .*

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## Enumerations, Lattices and Tableaux



# Directed and multi-directed animals on the King's lattice

Axel Bacher<sup>1</sup>

**Abstract.** We define the *directed King's lattice* to be the square lattice with diagonal (next nearest neighbor) bonds and with the preferred directions  $\{\leftarrow, \nearrow, \uparrow, \searrow, \rightarrow\}$ . We enumerate directed animals on this lattice using a bijection with Viennot's heaps of pieces. We also define and enumerate a superclass of directed animals, the elements of which are called multi-directed animals. This follows Bousquet-Mélou and Rechnitzer's work on the directed triangular and square lattices. Our final results show that directed and multi-directed animals asymptotically behave similarly to the ones on the triangular and square lattices.

## 1 Introduction

An animal on a lattice is a finite and connected set of vertices. The enumeration of animals (up to a translation) is a longstanding problem in statistical physics and combinatorics. The problem, however, is extremely difficult, and little progress has been made [12,16]. A more realistic goal, therefore, is to enumerate natural subclasses of animals.

The class of *directed animals* is one of the most classical of these subclasses. Directed animals have been enumerated in a variety of lattices; let us cite, non-exhaustively, the square and triangular lattices [2,9,11,14,18], Bousquet-Mélou and Conway's lattices  $\mathcal{L}_n$  [4,8], and the "strange" or  $n$ -decorated lattices [3,7] (Figure 1.1). Unsolved lattices include, notably, the honeycomb lattice [13].

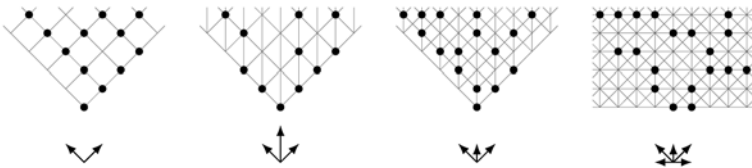
The class of *multi-directed animals* is a superclass of directed animals, first introduced by Klarner [15] on the square and triangular lattices. Bousquet-Mélou and Rechnitzer [5] clarified Klarner's definition and introduced a variant class on the square lattice. Moreover, they gave closed expressions for the generating function of multi-directed animals and showed that it is not D-finite.

The goal of this paper is to enumerate directed and multi-directed animals on a new lattice. We call *King's lattice* the square lattice with added

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diagonal bonds, or next nearest neighbor bonds. We also consider the preferred orientations  $\{\leftarrow, \nearrow, \uparrow, \searrow, \rightarrow\}$  (Figure 1.1, right). Directed animals on the King's lattice are a superclass of directed animals on Bousquet-Mélou and Conway's lattice  $\mathcal{L}_3$ , which has arcs  $\{\nearrow, \uparrow, \searrow\}$  [4].



**Figure 1.1.** Directed animals on a selection of lattices. From left to right: the square lattice, the triangular lattice, the lattice  $\mathcal{L}_3$ , and the King's lattice.

Several techniques have been used to enumerate directed animals on the various lattices. Among them are direct bijections with other combinatorial objects [11], comparison with gas models [1, 3, 9, 17] and the use of Viennot's theory of heaps of pieces [2, 5, 8, 20, 21]. In this paper, we use the last method; we show that directed animals on the King's lattice are in bijection with heaps of segments, already defined in [6].

## 2 Animals on the King's lattice and heaps of segments

### 2.1 Definitions

**Definition 2.1.** We call *segment* a closed real interval of the form  $[i, j]$ , where  $i$  and  $j$  are integers such that  $j > i$ . Two segments are called *concurrent* if they intersect, even by a point. A *heap of segments* is a finite sequence of segments, considered up to commutation of non-concurrent segments.

The heaps of segments described here are the same as in [6], except that the segment reduced to a point is not allowed. More information on heaps of pieces in general can be found in [20]. Graphically, a heap is built by dropping segments in succession; a segment either falls on the ground or on another segment concurrent to it. Examples are shown in Figures 2.1 and 2.2.

### 2.2 Directed animals and pyramids of segments

Let  $A$  be an animal; we say that a site  $t$  of  $A$  is *connected* to another site  $s$  if there exists a directed path (*i.e.* respecting the preferred directions of the lattice) from  $s$  to  $t$  visiting only sites of  $A$ . We say that the animal  $A$  is *directed* of source  $s$  if every site  $t$  of  $A$  is connected to  $s$ . The source  $s$

is not unique; it may be any of the bottommost sites of  $A$  (see Figure 2.1, left). By convention, we call *source* of  $A$  the leftmost bottom site.

In Figure 2.1 is illustrated a bijection between directed animals and *pyramids of segments* (or heaps with only one segment lying on the ground). This bijection works identically to the classical bijection between directed animals on the square lattice and strict pyramids of dimers [2,20].



**Figure 2.1.** Left: a directed animal on the King's lattice (represented, for clarity, as a polyomino on the dual lattice) with its source circled. Right: the pyramid of segments obtained by replacing each maximal sequence of  $\ell$  consecutive sites by a segment of length  $\ell$ .

### 2.3 Multi-directed animals and connected heaps of segments

Let  $A$  be an animal. For any abscissa  $i$ , we denote by  $b(i)$  the ordinate of the bottommost site of  $A$  at abscissa  $i$  (or  $b(i) = +\infty$  if there is no site of  $A$  at abscissa  $i$ ). We call *source* of  $A$  a site that realizes a local minimum of  $b$  and *keystone* of  $A$  a site that realizes a local maximum. In case several consecutive sites realize a minimum or maximum, the source or keystone is the leftmost one (Figure 2.2, left). This is a purely arbitrary choice that does not alter the definition.

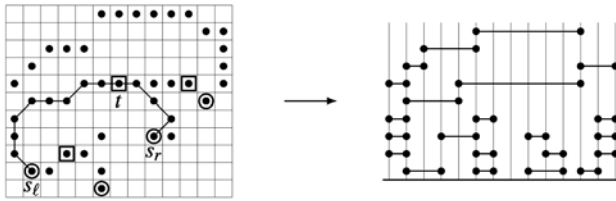
**Definition 2.2.** Let  $A$  be an animal. The animal  $A$  is said *multi-directed* if it satisfies the two conditions:

- for every site  $t$  of  $A$ , there exists a source  $s$  such that  $t$  is connected to  $s$ ;
- for every keystone  $t$  of  $A$ , there exist two sources  $s_\ell$  and  $s_r$ , to the left and to the right of  $t$  respectively, such that  $t$  is connected to both  $s_\ell$  and  $s_r$ . Moreover, the directed paths connecting  $t$  to  $s_\ell$  and  $s_r$  do not go through a keystone at the same height as  $t$ .

As a directed animal has only one source and no keystone, every directed animal is multi-directed. Multi-directed animals are in bijection with *connected heaps of segments* (or heaps without an empty column). A multi-directed animal and its corresponding heap are depicted in Figure 2.2.

Definition 2.2 can be adapted in the directed square and triangular lattices; the animals thus defined are in bijection with *connected heaps of*





**Figure 2.2.** Left: a multi-directed animal with four sources (circled) and three keystones (boxed). The directed paths connecting one keystone, denoted by  $t$ , to the sources  $s_\ell$  and  $s_r$  are shown. Right: the corresponding connected heap of segments, with has four minimal pieces (one for each source of the animal).

*dimers*. Bousquet-Mélou and Rechnitzer also defined multi-directed animals in bijection with connected heaps of dimers in [5], in a slightly different way. Our definition of multi-directed animals has the advantages of being more intrinsic and of having a vertical symmetry.

### 3 Enumeration of directed animals

In this section, we use the bijection with pyramids of segments to enumerate directed animals on the King's lattice. We call *half-animal* a directed animal with no site on the left side of its source. The associated pyramids are called *half-pyramids*. We adapt Bétéma and Penaud's methods [2] to decompose the pyramids of segments, which yields the following result.

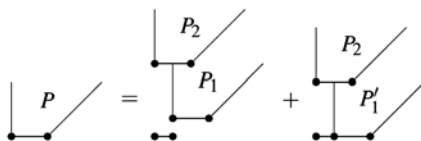
**Theorem 3.1.** *The generating functions  $S(t)$  and  $D(t)$  of half-animals and animals are:*

$$S(t) = \frac{1 - 3t - \sqrt{1 - 6t + t^2}}{4t}; \quad D(t) = \frac{1}{4} \left( \frac{1 + t}{\sqrt{1 - 6t + t^2}} - 1 \right).$$

The decomposition of the half-pyramids is sketched in Figure 3.1. Interestingly, the generating function  $S(t)$  is already known in combinatorics. Its coefficients are the little Schröder numbers, **A001003** in the OEIS [19]. The coefficients of  $D(t)$  also appear as **A047781**. This is reminiscent of the triangular lattice, where the half-animals are enumerated by the Catalan numbers [2].

### 4 Enumeration of multi-directed animals

In this section, we enumerate multi-directed animals, or, equivalently, connected heaps of segments. To do this, we adapt the *Nordic decomposition*, invented by Viennot to enumerate connected heaps of dimers [21].



**Figure 3.1.** Sketch of the two cases in the decomposition of half-pyramids. The generating function of the possible heaps  $P_1$  and  $P_2$  is  $1 + S(t)$ , while the generating function of the possible heaps  $P'_1$  is  $S(t)$ . This shows the identity  $S = t(1 + S)^2 + tS(1 + S)$ , from which we derive the value of  $S(t)$ .

**Theorem 4.1.** Let  $M = M(t)$  be the generating function of multi-directed animals. Let  $S = S(t)$ ,  $D = D(t)$  be the power series defined in Theorem 3.1,  $R = S + t(1 + S)$  and  $Q = 2(1 - t)S - t$ . The generating function  $M$  is given by:

$$M = \frac{D}{1 - \sum_{k \geq 0} S(1 + S)^k \frac{QR^k}{1 - QR^k}}.$$

## 5 Asymptotic results

Finally, we derive asymptotic results from Theorems 3.1 and 4.1.

**Theorem 5.1.** Let  $D_n$  and  $M_n$  be the number of directed and multi-directed animals of area  $n$ , respectively. As  $n$  tends to infinity, we have:

$$D_n \sim \kappa(3 + \sqrt{8})^n n^{-1/2}; \quad M_n \sim \lambda \mu^n,$$

with  $\mu = 6.475\dots$ . The average width of directed animals grows like  $\sqrt{n}$ , while the average width of multi-directed animals grows like  $n$ . Finally, the series  $M(t)$  is not  $D$ -finite.

The results on directed animals are a straightforward application of singularity analysis [10, Theorem VI.4]. The results on multi-directed animals are more involved. Similar results already exist on the square and triangular lattices, including the non- $D$ -finiteness of the series  $M(t)$  [5].

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# Results and conjectures on the number of standard strong marked tableaux

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**Abstract.** Standard strong marked tableaux play a role for  $k$ -Schur functions similar to the role standard Young tableaux play for Schur functions. We discuss results and conjectures toward an analogue of the hook length formula.

## 1 Introduction

In 1988, Macdonald [5] introduced a new class of polynomials and conjectured that they expand positively in terms of Schur functions. This conjecture, proved by Haiman [2], has led to an enormous amount of work, including the development of the  $k$ -Schur functions. These were defined in [3] and led to a refinement of the Macdonald conjecture. The  $k$ -Schur functions have since been found to arise in other contexts; for example, as the Schubert cells of the cohomology of affine Grassmannian permutations, and they are related to the quantum cohomology of the affine permutations.

One of the intriguing features of standard Young tableaux is the hook-length formula, which enumerates them. It has many different proofs and generalizations, see *e.g.* [6, Chapter 7] and [1].

In this extended abstract, we partially succeed in finding an analogue of the hook-length formula for standard strong marked tableaux (or starred tableaux for short), which are a natural generalization of standard Young tableaux in the context of  $k$ -Schur functions. For a fixed  $n$ , the shape of a starred tableau is necessarily an  $n$ -core, a partition for which all hook-lengths are different from  $n$ . In [4, Proposition 9.17], a formula is given for the number of starred tableaux for  $n = 3$ , which can be rewritten as

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$m! / \prod_{\substack{i,j \in \lambda \\ h_{ij} < 3}} h_{ij}$ , where  $m$  is the number of boxes of  $\lambda$  with hook-length  $< 3$ .

3. This is reminiscent of the classical hook-length formula. The authors left the enumeration for  $n > 3$  as an open problem.

The main result (Theorem 3.1) of this extended abstract implies that for each  $n$ , there exist  $(n - 1)!$  rational numbers (which we call *correction factors*) so that the number of starred tableaux of shape  $\lambda$  for any  $n$ -core  $\lambda$  can be easily computed. In fact, Theorem 3.1 gives a  $t$ -analogue of this result. The theorem is “incomplete” in the sense that we were not able to find explicit formulas for the (weighted) correction factors. We have, however, been able to state some of their properties (some conjecturally), the most interesting of them being the unimodality conjecture (Conjecture 3.4). Another result of interest is a new, alternative description of strong marked covers via simple triangular arrays of integers which we call *residue tables* and *quotient tables* (Theorem 4.2). We think these tables are of great importance in the theory of  $k$ -Schur functions.

## 2 Preliminaries

Here we introduce notation and review some constructions. Please see [6] for the definitions of integer partitions, ribbons, hook lengths, the hook-length formula, standard and semistandard Young tableaux, etc., which we omit in this extended abstract.

### 2.1 Cores and bounded partitions

Let  $n$  be a positive integer. An  $n$ -core is a partition  $\lambda$  such that  $h_{ij}^\lambda \neq n$  for all  $(i, j) \in \lambda$ . Core partitions were introduced to describe when two ordinary irreducible representations of the symmetric group belong to the same block. There is a simple bijection between  $(k + 1)$ -cores and  $k$ -bounded partitions. Given a  $(k + 1)$ -core  $\lambda$ , let  $\pi_i$  be the number of boxes in row  $i$  of  $\lambda$  with hook-length  $\leq k$ . The resulting  $\pi = (\pi_1, \dots, \pi_\ell)$  is a  $k$ -bounded partition, we denote it  $\mathfrak{b}(\lambda)$ . Conversely, given a  $k$ -bounded partition  $\pi$ , move from the last row of  $\pi$  upwards, and in row  $i$ , shift the  $\pi_i$  boxes of the diagram of  $\pi$  to the right until their hook-lengths are  $\leq k$ . The resulting  $(k + 1)$ -core is denoted  $\mathfrak{c}(\pi)$ . In this extended abstract,  $n = k + 1$ .

**Example 2.1.** The reader can check that for  $k = 5$  and  $\lambda = 953211$ ,  $\mathfrak{b}(\lambda) = 432211$ , and  $\mathfrak{c}(\pi) = 75221$  for  $k = 6$  and  $\pi = 54221$ .

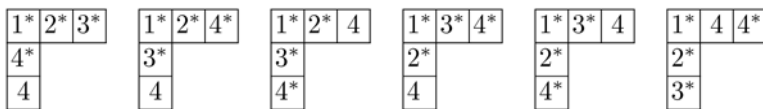
Of particular importance are  $k$ -bounded partitions  $\pi$  for which  $m_i(\pi) \leq k - i$  for all  $i = 1, \dots, k$ . We call such partitions  *$k$ -irreducible partitions*, see [3]. The number of  $k$ -irreducible partitions is  $k!$ .

### 2.2 Strong marked and starred tableaux

The Frame-Thrall-Robinson hook-length formula shows how to compute  $f_\lambda$ , the number of standard Young tableaux of shape  $\lambda$ : we have  $f_\lambda = \frac{|\lambda|!}{\prod_{(i,j) \in \lambda} h_{ij}^\lambda}$ . There exists a well-known weighted version of this formula, see e.g. [6, Corollary 7.21.5]. Our goal is to find an analogue of this form in the setting of  $k$ -bounded partitions and  $k$ -Schur functions.

The *strong  $n$ -core poset*  $\mathcal{C}_n$  is the subposet of  $\mathcal{Y}$  induced by the set of all  $n$ -core partitions. That is, its vertices are  $n$ -core partitions and  $\lambda \leq \mu$  in  $\mathcal{C}_n$  if  $\lambda \subseteq \mu$ . The cover relations are trickier to describe in  $\mathcal{C}_n$  than in  $\mathcal{Y}$ , see e.g. [4, Proposition 9.5]

The rank of an  $n$ -core is the number of boxes of its diagram with hook-length  $< n$ . If  $\lambda < \mu$  and  $\mu/\lambda$  consists of  $m$  ribbons, we say that  $\mu$  covers  $\lambda$  in the strong order with multiplicity  $m$ . A *strong marked cover* is a triple  $(\lambda, \mu, c)$  such that  $\lambda < \mu$  and that  $c$  is the content of the head of one of the ribbons. We call  $c$  the *marking* of the strong marked cover. A *strong marked horizontal strip of size  $r$  and shape  $\mu/\lambda$*  is a sequence  $(v^{(i)}, v^{(i+1)}, c_i)_{i=0}^{r-1}$  of strong marked covers such that  $c_i < c_{i+1}$ ,  $v^{(0)} = \lambda$ ,  $v^{(r)} = \mu$ . If  $\lambda$  is an  $n$ -core, a *strong marked tableau  $T$*  of shape  $\lambda$  is a sequence of strong marked horizontal strips of shapes  $\mu^{(i+1)}/\mu^{(i)}$ ,  $i = 0, \dots, m - 1$ , such that  $\mu^{(0)} = \emptyset$  and  $\mu^{(m)} = \lambda$ . The *weight* of  $T$  is the composition  $(r_1, \dots, r_m)$ , where  $r_i$  is the size of the strong marked horizontal strip  $\mu^{(i)}/\mu^{(i-1)}$ . If all strong marked horizontal strips are of size 1, we call  $T$  a *standard strong marked tableau* or a *starred tableau* for short. For a  $k$ -bounded partition  $\pi$ , denote the number of starred tableaux of shape  $\mathfrak{c}(\pi)$  by  $F_\pi^{(k)}$ . The next figure illustrates  $F_{211}^{(3)} = 6$ .



If  $\lambda$  is a  $k$ -bounded partition that is also a  $k + 1$ -core ( $\lambda_1 + \ell(\lambda) \leq k + 1$ ), then strong marked covers on the interval  $[\emptyset, \lambda]$  are equivalent to the covers in the Young lattice, strong marked tableaux of shape  $\lambda$  are equivalent to semistandard Young tableaux of shape  $\lambda$ , and starred tableaux of shape  $\lambda$  are equivalent to standard Young tableaux of shape  $\lambda$ .

### 3 Main results and conjectures

For a starred tableau  $T$ , define the *descent set* of  $T$ ,  $D(T)$ , as the set of all  $i$  for which the marked box at  $i$  is strictly above the marked box at  $i + 1$ . Define the *major index* of  $T$ ,  $\text{maj}(T)$ , by  $\sum_{i \in D(T)} i$ . For a  $k$ -bounded

partition  $\pi$ , define the polynomial  $F_\pi^{(k)}(t) = \sum_T t^{\text{maj}(T)}$ , where the sum is over all starred tableaux of shape  $\mathfrak{c}(\pi)$ . Clearly  $F_\pi^{(k)} = F_\pi^{(k)}(1)$ .

**Theorem 3.1.** *Let  $\pi$  be  $k$ -bounded,  $\pi = (k^{a_1+1}w_1, \dots, 1^{a_k+k \cdot w_k})$  for  $0 \leq a_i < i$ . If  $\sigma = \langle k^{a_1}, \dots, 1^{a_k} \rangle$  and  $(\mathbf{j}) = 1 + \dots + t^{j-1}$ , then*

$$F_\pi^{(k)}(t) = \frac{t^{\sum_{i=1}^k w_i \binom{i}{2}(k-i+1)} (|\boldsymbol{\pi}|)! F_\sigma^{(k)}(t)}{(|\boldsymbol{\sigma}|)! \prod_{j=1}^k (\mathbf{j})^{\sum_{i=1}^k w_i \min\{i, j, k+1-i, k+1-j\}}}$$

The theorem implies that in order to compute  $F_\pi^{(k)}(t)$  (and, by plugging in  $t = 1$ ,  $F_\pi^{(k)}$ ) for all  $k$ -bounded partitions  $\pi$ , it suffices to compute  $F_\sigma^{(k)}(t)$  (resp.,  $F_\sigma^{(k)}$ ) only for  $k$ -irreducible partitions  $\sigma$ .

**Example 3.2.** For  $k = 3$ , we have, among other formulas,

$$F_{3^{w_1}2^{1+2w_2}1^{1+3w_3}}^{(3)}(t) = \frac{t^{2w_2+3w_3+1} \cdot (\mathbf{3}w_3 + \mathbf{4}w_2 + \mathbf{3}w_1 + \mathbf{3})!}{(\mathbf{2})^{w_1+2w_2+w_3} \cdot (\mathbf{3})^{w_1+w_2+w_3+1}}$$

For a  $k$ -bounded partition  $\pi$ , let  $H_\pi^{(k)}(t) = \prod (\mathbf{h}_{ij})$ , where the product is over all boxes  $(i, j)$  of the  $(k + 1)$ -core  $\mathfrak{c}(\pi)$  with hook-lengths at most  $k$ , and let  $H_\pi^{(k)} = H_\pi^{(k)}(1)$  be the product of all hook-lengths  $\leq k$  of  $\mathfrak{c}(\pi)$ . Furthermore, if  $b_j$  is the number of boxes in the  $j$ -column of  $\mathfrak{c}(\pi)$  with hook-length at most  $k$ , write  $b_\pi^{(k)} = \sum_j \binom{b_j}{2}$ .

Let  $C_\sigma^{(k)}(t) = F_\sigma^{(k)}(t)H_\sigma^{(k)}(t)/(t^{b_\sigma^{(k)}}(|\boldsymbol{\sigma}|)!C_\sigma^{(k)}(t))$  be the *weighted correction factors* of a  $k$ -irreducible partition  $\sigma$ . By (and in the notation of) Theorem 3.1, we can express  $F_\pi^{(k)}(t)$  (for all  $k$ -bounded partitions  $\pi$ ) in another way which is reminiscent of the classical hook-length formula:

$$F_\pi^{(k)}(t) = \frac{t^{b_\sigma^{(k)} + \sum_{i=1}^k w_i \binom{i}{2}(k+1-i)} (|\boldsymbol{\pi}|)! C_\sigma^{(k)}(t)}{H_\sigma^{(k)}(t) \cdot \prod_{j=1}^k (\mathbf{j})^{\sum_{i=1}^k w_i \min\{i, j, k+1-i, k+1-j\}}}$$

For  $k \leq 3$ , all weighted correction factors are 1. For  $k = 4$ , all but four of the 24 weighted correction factors are 1, and the ones different from 1 are  $\frac{1+2t+t^2+t^3}{(\mathbf{2})(\mathbf{3})}$ ,  $\frac{1+t+2t^2+t^3}{(\mathbf{2})(\mathbf{3})}$ ,  $\frac{1+2t+2t^2+2t^3+t^4}{(\mathbf{3})^2}$ ,  $\frac{1+t+3t^2+t^3+t^4}{(\mathbf{3})^2}$  for partitions 2211, 321, 3211 and 32211.

For a  $k$ -bounded partition  $\pi$ , denote by  $\partial_k(\pi)$  the boxes of  $\mathfrak{c}(\pi)$  with hook-length  $\leq k$ . If  $\partial_k(\pi)$  is not connected, we say that  $\pi$  *splits* into  $k$ -bounded partitions  $(\pi^i)_{i=1}^m$ , where the connected components of  $\partial_k(\pi)$  are horizontal translates of  $\partial_k(\pi^i)$ . It turns out that  $C_\pi^{(k)}(t) = \prod_{i=1}^m C_{\pi^i}^{(k)}(t)$ .

**Conjecture 3.3.** For a  $k$ -irreducible partition  $\pi$ , the weighted correction factor is 1 if and only if  $\pi$  splits into  $\pi^1, \dots, \pi^l$ , where each  $\pi^i$  is a  $k$ -bounded partition that is also a  $(k + 1)$ -core.



**Conjecture 3.4.** For a  $k$ -irreducible partition  $\sigma$ , we can write  $1 - C_\sigma^{(k)}(t)$  as  $\frac{P_1(t)}{P_2(t)}$ , where  $P_1(t)$  is a unimodal polynomial with non-negative integer coefficients and  $P_2(t) = \prod_{i=1}^{k-1} (j)^{w_j}$  for some integers  $w_j \geq 0$ .

### 4 Strong covers and $k$ -bounded partitions

Our proof of Theorem 3.1, omitted in the extended abstract, closely follows one of the possible proofs (via quasisymmetric functions) of the classical (non-weighted and weighted) hook-length formula, see e.g. [6, Section 7.21]. Note, however, that the truly elegant proofs (e.g. [1]) are via induction. In this section, we show the first steps toward such a proof. In the process, we present a new description of strong marked covers in terms of bounded partitions (previous descriptions included cores—at least implicitly, via  $k$ -conjugation—affine permutations and abacuses). See the definition of residue and quotient tables below, and Theorem 4.2.

Identify  $\pi = \langle k^{p_1}, \dots, 1^{p_k} \rangle$  with  $p = (p_1, \dots, p_k)$ . Given  $i, j, m$ ,  $0 \leq m < i \leq j \leq k$ , define  $p^{i,j,m}$  as follows:  $p_{i-1}^{i,j,m} = p_{i-1} + m$ ,  $p_i^{i,j,m} = p_i - m$  for  $i \neq j$ ,  $p_j^{i,j,m} = p_j - m - 1$  for  $i \neq j$ ,  $p_i^{i,i,m} = p_i - 2m - 1$ ,  $p_{j+1}^{i,j,m} = p_{j+1} + m + 1$ ,  $p_h^{i,j,m} = p_h$  for  $h \neq i - 1, i, j, j + 1$ . See Example 4.3.

Define upper-triangular arrays  $\mathcal{R} = (r_{ij})_{1 \leq i \leq j \leq k}$  (*residue table*),  $\mathcal{Q} = (q_{ij})_{1 \leq i \leq j \leq k}$  (*quotient table*) by

- $r_{jj} = p_j \bmod j, r_{ij} = (p_i + r_{i+1,j}) \bmod i$  for  $i < j$ ,
- $q_{jj} = p_j \operatorname{div} j, q_{ij} = (p_i + r_{i+1,j}) \operatorname{div} i$  for  $j < i$ .

**Example 4.1.** Take  $k = 4, p = (1, 3, 2, 5)$ . Then  $\mathcal{R} = \begin{matrix} & & 0000 & & 1222 \\ & & 111 & & 121 \\ & & 20 & & 01 \\ & & 1 & & 1 \end{matrix}, \mathcal{Q} = \begin{matrix} & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{matrix}$ .

**Theorem 4.2.** Take  $p = (p_1, \dots, p_k)$  and  $1 \leq i \leq j \leq k$ . If  $r_{ij} < r_{i+1,j}, \dots, r_{jj}$ , then  $p$  covers  $p^{i,j,r_{ij}}$  in the strong order with multiplicity  $q_{ij} + \dots + q_{jj}$ . Furthermore, these are precisely all strong covers.

**Example 4.3.** For  $k = 4, p = (1, 3, 2, 5)$ , we have  $r_{ij} < r_{i+1,j}, \dots, r_{jj}$  for all  $(i, j) \neq (1, 4), (2, 4)$ . Therefore  $p$  covers, for example,  $p^{1,2,0} = (1, 2, 3, 5)$  with multiplicity 3, and  $p^{2,3,1} = (2, 2, 0, 7)$  with multiplicity 2.

For a  $k$ -bounded partition  $\pi$ , we clearly have  $F_\pi^{(k)} = \sum_\tau m_{\tau\pi} F_\tau^{(k)}$ , where the sum is over all  $k$ -bounded  $\tau$  that are covered by  $\pi$ , and  $m_{\tau\pi}$  is the multiplicity of the cover. Therefore the theorem can be used to prove Theorem 3.1 for  $t = 1$  for small values of  $k$  by induction: all we have to do is check  $k!$  equalities. The authors did that for  $k \leq 8$ .

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# On independent transversals in matroidal Latin rectangles

Ron Aharoni<sup>1</sup>, Daniel Kotlar<sup>2</sup> and Ran Ziv<sup>2</sup>

A *Latin rectangle* is an  $m \times n$  matrix in which the entries of each row and each column are all distinct numbers. In a row-Latin rectangle only the rows are required to consist of distinct entries. A *partial transversal* in a Latin rectangle is a set of distinct numbers such that no two of them are in the same row or column. A well-known conjecture of Ryser, Brualdi and Stein [2,8] asserts that every  $n \times n$  Latin rectangle has a partial transversal of size  $n - 1$ . Woolbright [9], and independently and Brouwer, de Vries and Wieringa [1], proved the existence of a partial transversal of size  $n - \sqrt{n}$  and Shor and Hatami [5,7] improved the bound to  $n - O(\log^2 n)$ .

Chappell [3], and later on the current authors [6], generalized the notion of a Latin rectangle, or Latin square, to matroids, by replacing “distinct” with “independent”. Thus, a matroidal Latin rectangle is an  $m \times n$  array over the ground set of a matroid  $\mathcal{M}$ , whose rows and columns are independent sets. A matroidal row-Latin rectangle is defined analogously. An *independent partial transversal* is an independent set whose elements belong to different rows and columns.

Chappell [3] proved that every  $(2n - 1) \times n$  matroidal row-Latin rectangle has an independent partial transversal of size  $n$ . This generalized an earlier result of Drisko [4]. Drisko gave an example where  $(2n - 2)$  rows are not enough to ensure an independent transversal of size  $n$ .

We suggest dealing with the more general problem of finding a rainbow set in the intersection of two matroids: given two matroids  $\mathcal{M}$  and  $\mathcal{N}$  on the same ground set  $S$ , an *independent matching* is a subset of  $S$  in  $\mathcal{M} \cap \mathcal{N}$ . A partial rainbow set for a family of sets  $\mathcal{F}$  is a set of representatives for some sub-family of  $\mathcal{F}$ . We conjecture that any  $n$  independent matchings, each of size  $n$ , have a partial rainbow independent matching of size  $n - 1$ .

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In the current paper we present the results for two matroids  $\mathcal{M}$  and  $\mathcal{N}$ , defined on the same ground set  $S$ . The first theorem generalizes the result mentioned above of Woolbright [9] and Brouwer, de Vries and Wieringa [1]:

**Theorem 1.** *Any  $n$  independent matchings, each of size  $n$ , have a partial rainbow independent matching of size  $n - \sqrt{n}$ .*

The next theorem generalizes the results mentioned above of Drisko [4] and Chappell [3]:

**Theorem 2.** *Any  $2n - 1$  independent matchings, each of size  $n$ , have a partial rainbow independent matching of size  $n$ .*

In both proofs we assume that a maximal rainbow matching has size smaller than the one we claim to exist, and we generate a larger rainbow matching using the notion of *colorful alternating path*. In the process of constructing the path we use some new basis exchange properties of matroids.

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# Multivariate Lagrange inversion formula and the cycle lemma

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**Abstract.** We give a multitype extension of the cycle lemma of (Dvoretzky and Motzkin 1947). This allows us to obtain a combinatorial proof of the multivariate Lagrange inversion formula that generalizes the celebrated proof of (Raney 1963) in the univariate case, and its extension in (Chottin 1981) to the two variable case.

Until now, only the alternative approach of (Joyal 1981) and (Labelle 1981) via labelled arborescences and endofunctions had been successfully extended to the multivariate case in (Gessel 1983), (Goulden and Kulkarni 1996), (Bousquet et al. 2003), and the extension of the cycle lemma to more than 2 variables was elusive.

The cycle lemma has found a lot of applications in combinatorics, so we expect our multivariate extension to be quite fruitful: as a first application we mention economical linear time exact random sampling for multispecies trees.

## 1 Introduction

For any power series  $g(x)$  with  $g(0) \neq 0$ , there exists a unique power series  $f(t)$  solution of the equation  $f = tg(f)$ . The Lagrange inversion formula says that the  $n$ th coefficient of  $f(t)$  is  $\frac{1}{n}[x^{n-1}]g(x)^n$ . This formula is now known as a fundamental tool to derive tree enumeration results. The two simpler and most classical examples are:

- if  $g(x) = \exp(x)$  then  $f(t)$  is the exponential generating function of Cayley trees, so that the number of rooted Cayley trees with  $n$  nodes is equal to  $\frac{1}{n} \left[ \frac{x^{n-1}}{(n-1)!} \right] \exp(nx) = n^{n-2}$ ;
- if  $g(x) = (1+x)^2$  then  $f(t)$  is the ordinary generating function of binary trees, so that the number of rooted binary trees with  $n$  nodes is equal to  $\frac{1}{n}[x^{n-1}](1+x)^{2n} = \frac{1}{n} \binom{2n}{n-1} = \frac{1}{n+1} \binom{2n}{n}$ .

These examples are in a sense generic: the bijection between doubly rooted Cayley trees and endofunctions underlies Labelle's proof of the Lagrange inversion formula [11], while the cyclic lemma used in [7] to count ballot numbers underlies Raney's proof [16].

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Our interest is in the multivariate extension of the Lagrange inversion formula. Fix an integer  $k \geq 1$ . Let bold letters denote  $k$ -dimensional vectors; write  $\mathbf{x}^n = x_1^{n_1} \cdots x_k^{n_k}$  and  $\mathbf{x}^{n-1} = x_1^{n_1-1} \cdots x_k^{n_k-1}$ . Let  $h(\mathbf{x})$  and  $g_1(\mathbf{x}), \dots, g_k(\mathbf{x})$  be power series in  $\mathbf{x}$  such that for  $i = 1, \dots, k$ ,  $g_i(\mathbf{0}) \neq 0$ . Again there is a unique family of power series  $\mathbf{f}(\mathbf{t})$  solution of the system of equations  $f_i = t_i g_i(f_i)$  for  $i = 1, \dots, k$  and the multivariate Lagrange inversion formula admits the two equivalent following formulations (among several others):

$$[\mathbf{t}^n]h(\mathbf{f}(\mathbf{t})) = [\mathbf{x}^{n-1}]h(\mathbf{x})\mathbf{g}(\mathbf{x})^n \det \left( \delta_{i,j} - \frac{x_i}{g_j(\mathbf{x})} \frac{\partial g_j(\mathbf{x})}{\partial x_i} \right) \quad (1.1)$$

$$= \frac{1}{n_1 \cdots n_k} [\mathbf{x}^{n-1}] \sum_T \partial_T(h, g_1^{n_1}, \dots, g_k^{n_k}) \quad (1.2)$$

where the sum is over oriented 0-rooted Cayley trees (non-plane trees with arcs going toward 0) with vertices  $\{0, \dots, k\}$  and the derivative  $\partial_G$  with respect to a directed graph  $G$  with vertex set  $V = \{0, \dots, k\}$  and edge set  $E$  is defined as

$$\partial_G(f_0(\mathbf{x}), \dots, f_k(\mathbf{x})) = \prod_{j=0}^k \left( \left( \prod_{(i,j) \in E} \frac{\partial}{\partial x_i} \right) f_j(\mathbf{x}) \right).$$

Several variants of (1.1) are given in [8] but (1.2) appeared more recently, implicitly in [9] and explicitly in [1]. As far as we know all combinatorial proofs of the multivariate Lagrange inversion extend Joyal and Labelle's approach [10, 11]: [8] proves another variant of (1.1), [3, 9] prove (1.2). A completely different approach was recently proposed in [2].

Chottin [4, 5] instead proposed a remarkable extension of Raney's strategy to prove the two variable Lagrange inversion formula. Yet he failed to move to three variables and the problem of proving the multivariate Lagrange inversion formula with the cycle lemma was considered as difficult. Apart from the theoretical interest of such a proof, an extension of the cycle lemma is desirable in view of its numerous applications, to tree and map enumeration [12, Chapter 11], [13, Chapter 9], probability [14], and random sampling [6]. We present such an extension in this paper.

## 2 Generalized cycle lemma

Following [5] and the modern accounts of Raney's proof, our combinatorial construction is in terms of encodings of rooted plane trees by sequences of nodes. To deal with the multivariate case, we introduce colored trees.

A *colored tree* is a plane, rooted tree in which all edges have a color in the set  $\{1, \dots, k\}$ . A *colored bush* is a colored tree that can have “pending” edges with no node attached. We call such edges *free edges*; an edge is *occupied* if a node is attached to it.

Let  $\ell$  be an integer with  $0 \leq \ell \leq k$ . An  $\ell$ -*bush* is a colored bush such that occupied edges have color more than  $\ell$  and free edges have color at most  $\ell$ . In particular, a  $k$ -bush only has a single node, while a 0-bush is a colored tree. Therefore,  $\ell$ -bushes can be seen as intermediate objects between colored nodes and colored trees.

**Definition 2.1.** Let  $0 \leq \ell \leq k$ . We denote by  $\mathcal{S}_\ell$  the set of tuples of the form  $S = (S_0, \dots, S_\ell, e_1, \dots, e_k)$ , where  $S_0, \dots, S_\ell$  are sequences of  $\ell$ -bushes and  $e_i$  is an edge of color  $i$  in  $S$  (referred to as a *marked edge*), satisfying the following conditions.

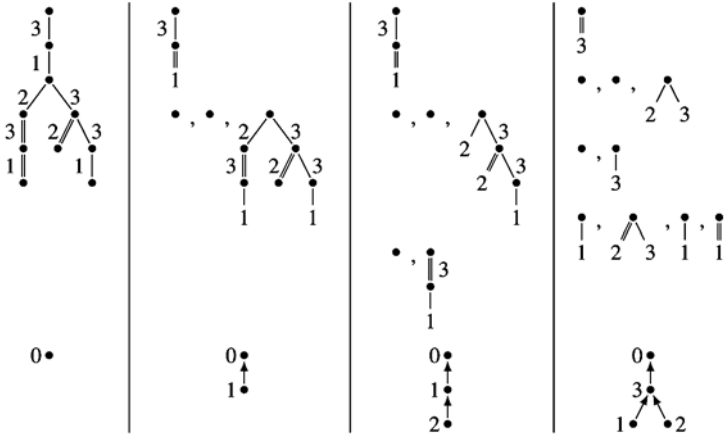
1. The sequence  $S_0$  has only one element; for  $i = 1, \dots, \ell$ , the number of elements of the sequence  $S_i$  is equal to the number of edges of color  $i$  in  $S$ .
2. Let  $T$  be the graph with vertices  $0, \dots, \ell$  and an arc  $i \rightarrow j$  if the edge  $e_j$  is in the sequence  $S_i$ . The graph  $T$  is a 0-rooted Cayley tree.

According to the previous remark, the objects of  $\mathcal{S}_\ell$  may also be seen as intermediates between two objects. If  $k = \ell$ , then  $S_0, \dots, S_k$  are simply sequences of nodes. If  $\ell = 0$ , the unique element of the sequence  $S_0$  is a colored tree.

**Theorem 2.2 (generalized cycle lemma).** *There is a bijection between the sets  $\mathcal{S}_\ell$  and  $\mathcal{S}_{\ell-1}$  that works by attaching the elements of the sequence  $S_\ell$  to the edges of color  $\ell$ .*

The actual description of the bijection is given in Section 3; an example is given in Figure 2.1. Observe that for  $\ell = k = 1$ , the statement is equivalent to the standard Cycle Lemma: there is a bijection between  $p$ -tuples of rooted plane trees with one pointed node (represented here as a unique tree with an extra node of degree  $p$  at the root) and pairs formed of a node of degree  $p$  having a marked free edge and a sequence of nodes such that the number of nodes in the sequence equals the total number of free edges.

Iterating the bijection yields a bijection between  $\mathcal{S}_0$  and  $\mathcal{S}_k$ . Given a Cayley tree  $T$  with vertices  $\{0, \dots, k\}$ , the generating function of the associated subset of  $\mathcal{S}_k$  is  $hg_1^{n_1} \cdots g_k^{n_k}$  with the proper derivatives to mark edges (taking a derivative of  $g_j^{n_j}$  with respect to  $x_i$  amounts to marking an edge of color  $i$  in the sequence  $S_j$ ). It can thus be seen that the formula (1.2) is a corollary of the generalized Cycle Lemma.



**Figure 2.1.** An example with  $k = 3$ , a tree of  $S_0$  and the corresponding elements of  $S_1, S_2$  and  $S_3$  (from left to right), with their associated Cayley trees (bottom). Marked edges are represented as double lines.

### 3 The bijection

We now describe without proof our bijection between the sets  $S_\ell$  and  $S_{\ell-1}$ , for  $1 \leq \ell \leq k$ , illustrated in Figure 2.1. This bijection uses *Prüfer codes* of Cayley trees [15]. The Prüfer code of a 0-rooted Cayley tree  $T$  with vertices  $0, \dots, s$  is a sequence  $p_1, \dots, p_s$  with  $p_s = 0$ ; there exists a permutation  $\sigma$  such that the parent vertex of  $j$  in  $T$  is  $p_{\sigma(j)}$  for  $j = 1, \dots, s$ . Moreover, every sequence corresponds to a unique tree.

Let  $S = (S_0, \dots, S_\ell, e_1, \dots, e_k)$  be an element of  $S_\ell$ ; let  $T$  be the associated Cayley tree. In the following, we call  $\ell$ -edge an edge of color  $\ell$ . Since we are dealing with plane trees, there is a natural order on the set of  $\ell$ -edges of  $S$  given by depth-first traversals of the  $\ell$ -bushes.

Let  $r$  be the number of  $\ell$ -edges in the sequences  $S_0, \dots, S_{\ell-1}$ . Since  $T$  is a Cayley tree, it does not contain the arc  $\ell \rightarrow \ell$ , which means that the marked edge  $e_\ell$  is not in the sequence  $S_\ell$ . Thus, we have  $r \geq 1$ . Let  $u$  be the root of the first element of  $S_\ell$ . By Definition 2.1, the sequence  $S_\ell$  has exactly  $r$  more elements than it has  $\ell$ -edges. We combine the elements of  $S_\ell$  using the Cycle Lemma; we denote by  $b_1, \dots, b_r$  the resulting  $\ell - 1$ -bushes, so that the node  $u$  is in  $b_1$ .

**First case: the vertex  $\ell$  is a leaf of  $T$ .** In this case, we attach the bushes  $b_1, \dots, b_r$  to the remaining free  $\ell$ -edges in order, with a cyclic permutation chosen so that the bush  $b_1$  is attached to the marked edge  $e_\ell$ .

**Second case: the vertex  $\ell$  is not a leaf of  $T$ .** In the Cayley tree  $T$ , all arcs going toward  $\ell$  correspond to marked edges in the sequence  $S_\ell$ . Let



$m_1, \dots, m_s$  be the  $\ell - 1$ -bushes that we just formed containing at least one marked edge.

We break up the Cayley tree  $T$  in the following manner. For  $i = 1, \dots, s$ , let  $T_i$  be the forest composed of the colors of the marked edges in  $m_i$  and their descendants in  $T$ . Let  $T_0$  be the tree composed of the remaining vertices of  $T$  with the vertex  $\ell$  deleted; in other words, the vertices of  $T_0$  are exactly the non-descendants of  $\ell$ . We also assume that the order of the  $m_i$ 's was chosen so that the  $T_i$ 's are ordered according to their lowest label.

Now, attach the bushes  $b_1, \dots, b_r$  to the free  $\ell$ -edges in order, with a cyclic permutation chosen so that  $b_s$  is attached to the marked edge  $e_\ell$ . For  $j = 1, \dots, s$ , let  $p_j = i$  if  $m_j$  is attached to an edge in the sequence with an index in  $T_i$ . As the edge  $e_\ell$  is in the sequence corresponding to the parent of  $\ell$  in  $T$ , which is in  $T_0$ , we have  $p_s = 0$ .

We can therefore regard the sequence  $p_1, \dots, p_s$  as the Prüfer code of a 0-rooted Cayley tree  $\tilde{T}$  with labels  $\{0, \dots, s\}$ . Let  $\sigma$  be the permutation associated to this tree and swap the bushes  $m_1, \dots, m_s$  according to the permutation  $\sigma$ .

Let  $S'_0, \dots, S'_{\ell-1}$  be the sequences resulting from this procedure. As no free  $\ell$ -edges remain, every element of the sequences  $S'_0, \dots, S'_{\ell-1}$  is an  $\ell - 1$ -bush. Finally, let  $e'_\ell$  be the parent edge of the node  $u$  defined at the beginning. Let  $T'$  be the graph associated with the marked edges  $e_1, \dots, e_{\ell-1}$ . In the first case above, this is the tree  $T$  with the leaf  $\ell$  deleted; in the second, it is a compound of the tree  $T_0$  and the forests  $T_1, \dots, T_s$ , arranged according to the tree  $\tilde{T}$ ; it is therefore a Cayley tree. This shows that the tuple  $(S'_0, \dots, S'_{\ell-1}, e_1, \dots, e'_\ell, \dots, e_k)$  is an element of  $\mathcal{S}_{\ell-1}$ .

We conclude with some comments. We use Cayley trees and Prüfer codes in a manner that may seem needlessly complicated; however, in the first stage, we see how important the condition that the graph  $T$  contains no edge  $\ell \rightarrow \ell$  is, which is implied by the fact that  $T$  is a Cayley tree. The construction aims at ensuring that the graph  $T'$  describing the marked edges  $e_1, \dots, e_{\ell-1}$  remains a Cayley tree.

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## Combinatorics and algorithms

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# Simplifying inclusion - exclusion formulas

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and Martin Tancer<sup>5</sup>

**Abstract.** Let  $\mathcal{F} = \{F_1, F_2, \dots, F_n\}$  be a family of  $n$  sets on a ground set  $S$ , such as a family of balls in  $\mathbb{R}^d$ . For every finite measure  $\mu$  on  $S$ , such that the sets of  $\mathcal{F}$  are measurable, the classical *inclusion-exclusion formula* asserts that  $\mu(F_1 \cup F_2 \cup \dots \cup F_n) = \sum_{I: \emptyset \neq I \subseteq [n]} (-1)^{|I|+1} \mu\left(\bigcap_{i \in I} F_i\right)$ ; that is, the measure of the union is expressed using measures of various intersections. The number of terms in this formula is exponential in  $n$ , and a significant amount of research, originating in applied areas, has been devoted to constructing simpler formulas for particular families  $\mathcal{F}$ . We provide an upper bound valid for an arbitrary  $\mathcal{F}$ : we show that every system  $\mathcal{F}$  of  $n$  sets with  $m$  nonempty fields in the Venn diagram admits an inclusion-exclusion formula with  $m^{O(\log^2 n)}$  terms and with  $\pm 1$  coefficients, and that such a formula can be computed in  $m^{O(\log^2 n)}$  expected time.

## 1 Introduction

One of the basic topics in introductory courses of discrete mathematics is the *inclusion-exclusion principle* (also called the *sieve formula*), which allows one to compute the number of elements of a union  $F_1 \cup F_2 \cup \dots \cup F_n$  of  $n$  sets from the knowledge of the sizes of all intersections of the  $F_i$ 's.

We will consider a slightly more general setting, where we have a ground set  $S$  and a (finite) *measure*  $\mu$  on  $S$ ; then the inclusion-exclusion

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principle asserts that for every collection  $F_1, F_2, \dots, F_n$  of  $\mu$ -measurable sets, we have

$$\mu\left(\bigcup_{i=1}^n F_i\right) = \sum_{I:\emptyset \neq I \subseteq [n]} (-1)^{|I|+1} \mu\left(\bigcap_{i \in I} F_i\right). \quad (1.1)$$

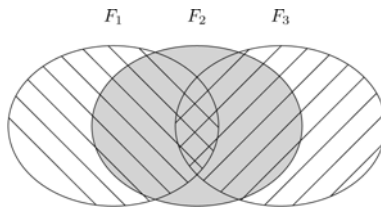
(Here, as usual,  $[n] = \{1, 2, \dots, n\}$  and  $|I|$  denotes the cardinality of the set  $I$ .) This principle not only plays a fundamental role in various areas of mathematics such as probability theory or combinatorics, but it also has important algorithmic applications, e.g., the best known algorithms for several NP-hard problems including graph  $k$ -coloring [2], travelling salesman problem on bounded-degree graphs [3], dominating set [11], or partial dominating set and set splitting [8].

The inclusion-exclusion principle involves a number of summands that is exponential in  $n$ , the number of sets. In general this cannot be avoided if one wants an *exact* formula valid for *every* family  $\mathcal{F} = \{F_1, F_2, \dots, F_n\}$ . Yet, since this is a serious obstacle to efficient uses of inclusion-exclusion, much effort has been devoted to finding “smaller” formulas. These efforts essentially organize along two lines of research.

The first approach gives up on exactness and tries to *approximate* efficiently the measure of the union using the measure of only *some* of the intersections. See, e.g., classical *Bonferroni inequalities* [4]. We give a short overview of this line in the full version of this paper [7].

The second line of research looks for “small” inclusion-exclusion formulas valid for *specific* families of sets. To illustrate the type of simplifications afforded by fixing the sets, consider the family  $\mathcal{F} = \{F_1, F_2, F_3\}$  of Figure 1.1. As  $F_1 \cap F_3 = F_1 \cap F_2 \cap F_3$ , Formula (1.1) can be simplified to

$$\mu(F_1 \cup F_2 \cup F_3) = \mu(F_1) + \mu(F_2) + \mu(F_3) - \mu(F_1 \cap F_2) - \mu(F_2 \cap F_3).$$



**Figure 1.1.** Three subsets of  $\mathbb{R}^2$  admitting a simpler inclusion-exclusion formula. The ground set  $F_1 \cup F_2 \cup F_3$  splits into six nonempty regions recognizable by the filling pattern.

More generally, let us consider a family  $\mathcal{F} = \{F_1, F_2, \dots, F_n\}$ , and let us say that a coefficient vector

$$\alpha = (\alpha_I)_{\emptyset \neq I \subseteq [n]} \in \mathbb{R}^{2^n - 1}$$

is an *IE-vector* for  $\mathcal{F}$  if we have

$$\mu\left(\bigcup_{i=1}^n F_i\right) = \sum_{I: \emptyset \neq I \subseteq [n]} \alpha_I \mu\left(\bigcap_{i \in I} F_i\right) \quad (1.2)$$

for every finite measure  $\mu$  on the ground set of  $\mathcal{F}$  (with all the  $F_i$ 's measurable). Given  $\mathcal{F}$ , we would like to find an IE-vector for  $\mathcal{F}$ , such that both the number of nonzero coefficients is small, and the coefficients themselves are not too large. We learned this idea from [1] and refer to the monograph of Dohmen [5] for an overview of this line of research.

Given a specific family  $\mathcal{F} = \{F_1, F_2, \dots, F_n\}$  of sets, how small can we expect an inclusion-exclusion formula to be? To formalize the problem, we should specify how  $\mathcal{F}$  is given. Let us consider the *Venn diagram* of  $\mathcal{F}$ . For each nonempty index set  $\tau \subseteq [n]$ , we define the *region* of  $\tau$ , denoted by  $\text{reg}(\tau)$ , as the set of all points that belong to the sets  $F_i$  with  $i \in \tau$  and no others (see Figure 1.1);

$$\text{reg}(\tau) = \left(\bigcap_{i \in \tau} F_i\right) \setminus \left(\bigcup_{i \notin \tau} F_i\right).$$

The *Venn diagram* of  $\mathcal{F}$  is then the collection of all subsets of  $[n]$  with non-empty regions; that is,

$$\mathcal{V} = \mathcal{V}(\mathcal{F}) := \{\tau \subseteq [n]: \text{reg}(\tau) \neq \emptyset\}.$$

We regard the Venn diagram as a set system on the ground set  $[n]$ ; it is a “dual” of the set system  $\mathcal{F}$ .

As far as inclusion-exclusion formulas are concerned, all points in a single region are equivalent; it only matters which of the regions are nonempty. Thus, in order to simplify our formulations, we can assume that  $\mathcal{F}$  is *standardized*, meaning that the ground set equals the union of the  $F_i$ 's and each nonempty region has exactly one point. From an algorithmic point of view, this amounts to a preprocessing step for  $\mathcal{F}$ , in which the part of the ground set  $S$  in each nonempty region is contracted to a single point.

Let  $\mathcal{F} = \{F_1, F_2, \dots, F_n\}$  be a family of sets and let  $m$  denote the size of  $\mathcal{V}$  (which equals the size of the ground set for  $\mathcal{F}$  standardized).

A linear-algebraic argument shows that every (finite) family  $\mathcal{F}$  has an inclusion-exclusion formula with at most  $m$  terms [7, Corollary 2.4]. However, the coefficients might be exponentially large (see [7, Example 2.5]). This requires computing the measure of the intersections with enormous precision to obtain meaningful results. Thus, we prefer inclusion-exclusion formulas where not only the number of terms is small, but the coefficients are also small.

Our main result is the following general upper bound; to our knowledge, it is the first upper bound applicable for an arbitrary family.

**Theorem 1.1.** *Let  $n$  and  $m$  be integers and let  $k = \lceil 2e \ln m \rceil \lceil 1 + \ln \frac{n}{\ln m} \rceil$ . Then for every family  $\mathcal{F}$  of  $n$  sets with Venn diagram of size  $m$ , there is an IE-vector  $\alpha$  for  $\mathcal{F}$  that has at most  $\sum_{i=1}^k \binom{n}{i} \leq m^{O(\ln^2 n)}$  nonzero coefficients, and in which all nonzero coefficients are  $\pm 1$ 's. Such an  $\alpha$  can be computed in  $m^{O(\ln^2 n)}$  expected time if  $\mathcal{F}$  is standardized.*

The bound in this theorem is quasi-polynomial, but not polynomial, in  $m$  and  $n$ . We do not know if a polynomial bound can be achieved with  $\pm 1$  coefficients but in the full version we show that inclusion-exclusion formulas of linear size are impossible in general. Specifically, we show [7, Theorem 1.2] that for infinitely many values of  $n$ , there are families of  $n$  sets on  $n$  points, for which every IE-vector has  $\ell_1$ -norm at least  $(n/2)^{3/2}$ .

We only sketch proofs of the theorems above. In particular, we skip proofs of auxiliary lemmas. They are proved in the full version [7].

## 2 Sketch of a proof of Theorem 1.1

We consider a family  $\mathcal{F} = \{F_1, F_2, \dots, F_n\}$  of sets on a ground set  $S$ , and assume that the  $F_i$  are all distinct. Besides the Venn diagram  $\mathcal{V}$ , we associate yet another set system with  $\mathcal{F}$ , namely, the nerve  $\mathcal{N}$  of  $\mathcal{F}$ :

$$\mathcal{N} = \mathcal{N}(\mathcal{F}) := \left\{ \sigma \subseteq [n] : \sigma \neq \emptyset, \bigcap_{i \in \sigma} F_i \neq \emptyset \right\}.$$

So both of  $\mathcal{N}$  and  $\mathcal{V}$  have ground set  $[n]$ , and we have  $\mathcal{V} \subseteq \mathcal{N}$ .

Let us enumerate the elements of  $\mathcal{V}$  as  $\mathcal{V} = \{\tau_1, \tau_2, \dots, \tau_m\}$  in such a way that  $|\tau_i| \leq |\tau_j|$  for  $i < j$ , and let us enumerate  $\mathcal{N} = \{\sigma_1, \sigma_2, \dots, \sigma_{|\mathcal{N}|}\}$  so that the sets of  $\mathcal{V}$  come first, i.e.,  $\sigma_i = \tau_i$  for  $i = 1, 2, \dots, m$ .

In the introduction, we were indexing IE-vectors for  $\mathcal{F}$  by all possible subsets  $I \subseteq [n]$ . But if  $I$  is not in the nerve, the corresponding intersection is empty, and thus w.l.o.g. we may assume that its coefficient is zero. Thus, from now on, we will index IE-vectors  $x$  as  $(x_1, \dots, x_{|\mathcal{N}|})$ , where  $x_j$  is the coefficient of  $\mu(\bigcap_{i \in \sigma_j} F_i)$ .

**Abstract tubes.** An (*abstract*) *simplicial complex* with vertex set  $[n]$  is a hereditary system of nonempty subsets of  $[n]$ .<sup>6</sup> An *abstract tube* is a pair  $(\mathcal{F}, \mathcal{K})$ , where  $\mathcal{F} = \{F_1, F_2, \dots, F_n\}$  is a family of sets and  $\mathcal{K}$  is a simplicial complex with vertex set  $[n]$ , such that for every nonempty region  $\tau$  of the Venn diagram of  $\mathcal{F}$ , the subcomplex *induced* on  $\mathcal{K}$  by  $\tau$ ,  $\mathcal{K}[\tau] := \{\vartheta \in \mathcal{K} : \vartheta \subseteq \tau\}$ , is contractible.<sup>7</sup>

As first noted by Naiman and Wynn [9, 10], if  $(\mathcal{F}, \mathcal{K})$  is an abstract tube, then

$$\mu \left( \bigcup_{i=1}^n F_i \right) = \sum_{I \in \mathcal{K}} (-1)^{|I|+1} \mu \left( \bigcap_{i \in I} F_i \right). \quad (2.1)$$

Moreover, truncating the sum yields upper and lower bounds in the spirit of the Bonferroni inequalities ([10]; also see [5, Theorem 3.1.9]).

Small abstract tubes have been identified for families of balls [1, 9, 10] or halfspaces [10] in  $\mathbb{R}^d$ , and similar structures were found for families of pseudodisks [6]. We establish Theorem 1.1 by proving that for every family of sets there exists an abstract tube with “small” size that, in addition, can be computed efficiently. We will use the following sufficient condition guaranteeing that  $(\mathcal{F}, \mathcal{K})$  is an abstract tube; it is a reformulation of [5, Theorem 4.2.5]. Let  $\text{MNF}(\mathcal{K})$  denote the system of all inclusion-minimal non-faces of  $\mathcal{K}$ , i.e., of all nonempty sets  $I \subseteq [n]$  with  $I \notin \mathcal{K}$  but with  $I' \in \mathcal{K}$  for every proper subset  $I' \subset I$ .

**Proposition 2.1.** *Let  $\mathcal{F} = \{F_1, F_2, \dots, F_n\}$  be a family of sets with Venn diagram  $\mathcal{V}$  and let  $\mathcal{K}$  be a simplicial complex with vertex set  $[n]$ . If no set of  $\mathcal{V}$  can be expressed as a union of sets in  $\text{MNF}(\mathcal{K})$ , then  $(\mathcal{F}, \mathcal{K})$  is an abstract tube.*

**Abstract tubes from selectors.** Let  $\mathcal{F} = \{F_1, F_2, \dots, F_n\}$  be a family of sets, and let  $\mathcal{V}$  be the Venn diagram of  $\mathcal{F}$ . A *selector* for  $\mathcal{V}$  is a map  $w: \mathcal{V} \rightarrow [n]$  such that  $w(\tau) \in \tau$  for every  $\tau \in \mathcal{V}$ . We observe that each selector for  $\mathcal{V}$  provides an abstract tube for  $\mathcal{F}$  (which satisfies the sufficient condition of Proposition 2.1).

**Lemma 2.2.** *Let  $\mathcal{F} = \{F_1, F_2, \dots, F_n\}$ ,  $\mathcal{V} = \mathcal{V}(\mathcal{F})$ , and let  $w$  be a selector for  $\mathcal{V}$ . We define the simplicial complex  $\mathcal{K}_w = \{\sigma \in \mathcal{N}(\mathcal{F}) : \text{for all nonempty } \vartheta \subseteq \sigma \text{ there is } \tau \in \mathcal{V} \text{ such that } w(\tau) \in \vartheta \subseteq \tau\}$ . Then  $(\mathcal{F}, \mathcal{K}_w)$  is an abstract tube.*

<sup>6</sup> We emphasize that we exclude an empty set from the definition of a simplicial complex. This is non-standard definition; however, it is convenient for our purposes.

<sup>7</sup> By *contractible* we mean contractibility in the sense of topology; roughly speaking, the topological space defined by  $\mathcal{K}[\tau]$  can be continuously shrunk to a point.



**Large simplices in random  $\mathcal{K}_w$ .** Let  $\rho$  be a permutation of  $[n]$ . We define a selector  $w_\rho$  for  $\mathcal{V}$  by taking  $w(\tau)$  as the smallest element of  $\tau$  in the linear ordering  $<$  on  $[n]$  given by  $\rho(1) < \rho(2) < \dots < \rho(n)$ .

For better readability we write  $\mathcal{K}_\rho$  instead of  $\mathcal{K}_{w_\rho}$ . We want to show that for random  $\rho$ ,  $\mathcal{K}_\rho$  is unlikely to contain too large simplices, and thus leads to a small inclusion-exclusion formula.

Let  $\Gamma$  denote the incidence matrix of  $\mathcal{V}$ , that is, the 0-1 matrix with  $m$  rows and  $n$  columns where  $\Gamma_{ij} = 1$  if and only if  $j \in \tau_i$  (if the original system  $\mathcal{F}$  was standardized, then  $\Gamma$  is the transposition of the usual incidence matrix of  $\mathcal{F}$ ). We also denote by  $\Gamma_\rho$  the matrix obtained by applying the permutation  $\rho$  to the columns of  $\Gamma$ : the  $\rho(i)$ th column of  $\Gamma_\rho$  is the  $i$ th column of  $\Gamma$  and represents the incidences between permuted  $[n]$  and  $\mathcal{V}$ . The lemma below says that if  $\mathcal{K}_\rho$  contains a large simplex, then  $\Gamma_\rho$  contains a particular substructure.

We say that a row  $R$  of  $\Gamma_\rho$  is *compatible* with a subset  $I \subseteq [n]$  if  $R$  contains 1's in all columns with index in  $I$  and 0's in all columns with index smaller than  $\min(I)$ .

**Lemma 2.3.** *If  $\rho(\tau) = \{i_1, i_2, \dots, i_k\}$  for a simplex  $\tau$  in  $\mathcal{K}_\rho$ , with  $i_1 < i_2 < \dots < i_k$ , then for every  $s \in \{1, 2, \dots, k\}$  the matrix  $\Gamma_\rho$  contains a row compatible with  $\{i_s, i_{s+1}, \dots, i_k\}$ .*

*Sketch of a proof of Theorem 1.1.* Let  $n$  and  $m \geq 3$  be integers. Let  $\mathcal{F} = \{F_1, F_2, \dots, F_n\}$  be a family of  $n$  sets whose Venn diagram  $\mathcal{V}$  has size  $m$ . We argue that if  $\rho$  is a permutation of  $[n]$  chosen uniformly at random, the probability that all compatibility conditions of Lemma 2.3 are satisfied for some  $\{i_1, \dots, i_k\}$  is at most  $\frac{1}{2}$  for  $k := \lceil 2e \ln m \rceil \lceil 1 + \ln \frac{n}{\ln m} \rceil$ . In particular, there exists a permutation  $\rho^*$  such that  $\mathcal{K}_{\rho^*}$  contains no simplex of size  $k$  (or larger). Lemma 2.2 concludes the proof of Equation (2.1).

In order to actually compute a suitable coefficient vector, we choose a random permutation  $\rho$  and compute  $\mathcal{K}_\rho$ .

The choice of a random permutation  $\rho$  takes  $O(n \ln n)$  time and  $n$  random bits. Accepting or rejecting a new simplex by brute-force testing takes  $O(mn)$  time. The expected number of times we have to start over with a new permutation  $\rho$  is  $O(1)$ . Altogether, the expected running time of this algorithm is  $O\left(\binom{n}{k} mn\right) = m^{O(\ln^2 n)}$ . □

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# Majority and plurality problems

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**Abstract.** Given a set of  $n$  balls each colored with a color, a ball is said to be majority,  $k$ -majority, plurality if its color class has size larger than half of the number of balls, has size at least  $k$ , has size larger than any other color class; respectively. We address the problem of finding the minimum number of queries (a comparison of a pair of balls if they have the same color or not) that is needed to decide whether a majority,  $k$ -majority or plurality ball exists and if so then show one such ball. We consider both adaptive and non-adaptive strategies and in certain cases, we also address weighted versions of the problems.

## 1 Introduction

Two very much investigated problems in combinatorial search theory are the so-called majority and plurality problems. In this context, we are given  $n$  balls in an urn, each colored with one color. A majority ball is one such that its color class has size strictly larger than  $n/2$ . A plurality ball is one such that its color class is strictly larger than any other color class. The aim is either to decide whether there exists a majority/plurality ball or even to show one (if there exists one). Note that if the number of colors is two, then the majority and the plurality problems coincide. Although there are other models (*e.g.* [6]), in the original settings a query is a pair of balls and the answer to the query tells us whether the two balls have the same color or not. Throughout the paper we consider queries of this sort.

We distinguish two types of algorithms for each problem we consider. An algorithm is *adaptive* if the  $i$ th query might depend on the answers received for the first  $i - 1$  queries. A *non-adaptive* algorithm is simply a set of queries that should be answered at the same time. Clearly, any non-adaptive algorithm can be viewed as an adaptive one and therefore for any kind of combinatorial search problem, the minimum number of queries required in an adaptive algorithm is not more than the minimum number of queries required in a non-adaptive algorithm.

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The first results concerning plurality and majority problems are due to Fisher and Salzberg [7] and Saks and Werman [8]. In [7] it is proved that if the number of possible colors is unknown, then the minimum number of queries in an adaptive search for a majority ball is  $\lfloor 3n/2 \rfloor - 2$ , while [8] contains the result that if the number of colors is two, then the minimum number of queries needed to find a majority ball is  $n - b(n)$ , where  $b(n)$  is the number of 1's in the binary representation of  $n$ . The latter result was later reproved in a simpler way by Alonso, Reingold, and Schott [3] and Wiener [10].

The adaptive version of the plurality problem was first considered by Aigner, De Marco, and Montanero in [2], where they showed that for any fixed positive integer  $c$ , if the number of possible colors is at most  $c$ , then the minimum number of queries needed in an adaptive search for a plurality ball is of linear order, and the constants depend on  $c$ . Non-adaptive and other versions of the plurality problem were considered in [1].

Non-adaptive strategies were also studied by Chung, Graham, Mao and Yao [4,5]. They showed a linear upper bound for the majority problem in case the existence of a majority color is assumed. They mention a quadratic lower bound without this extra assumption. We precisely determine the minimum number of queries needed. They also obtain lower and upper bounds on the plurality problem in the non-adaptive case. We improve those bounds and find the correct asymptotics of the minimum number of queries.

## 1.1 Preliminaries and notation and main results

To state our results we introduce some notations.  $M_c(n)$  denotes the minimum number of queries that is needed to determine if there exists a majority color and if so, then to show one ball of that color and  $P_c(n)$  denotes the minimum number of queries that is needed to determine if there exists a plurality color and if so, then to show one ball of that color. In both cases the subscript  $c$  stands for the number of possible colors. The corresponding non-adaptive parameters are denoted by  $M_c^*(n)$  and  $P_c^*(n)$ . A ball is said to be  $k$ -majority if its color class contains at least  $k$  balls.  $M_c(n, k)$  denotes the minimum number of queries that is needed to determine if there exists a  $k$ -majority color and if so, then to show one ball of that color and  $M_c^*(n, k)$  denotes the parameter of the non-adaptive variant.

We also consider weighted problems. Let  $S = \{w(1), \dots, w(n)\}$  be a multiset of positive numbers, where  $w(i)$  is considered to be the weight of the  $i$ th ball. For all weighted problems considered in the paper, we

assume that the weights are known to all participants. The total weight  $w = w(S)$  of the balls is  $\sum_{i=1}^n w(i)$ . The weight  $w(T)$  of a subset  $T \subseteq [n]$  is  $\sum_{i \in T} w(i)$ . A color is majority if its color class  $C$  satisfies  $w(C) > w/2$  and  $k$ -majority if  $w(C) \geq k$  holds. A color is said to be plurality if the weight  $w(C)$  of its color class  $C$  is strictly greater than the weights of all the other color classes. The appropriate parameters are denoted by  $M_c(S)$ ,  $M_c(S, k)$ ,  $M_c^*(S)$ ,  $M_c^*(S, k)$  and  $P_c(S)$ ,  $P_c^*(S)$ .

For a set  $Q$  of queries we define the *query graph*  $G_Q$  to be the graph where the vertices correspond to balls and two vertices are joined by an edge if and only if there exists a query in  $Q$  that asks for the comparison of the two corresponding balls. Our main results are the following two theorems.

**Theorem 1.1.** *Suppose  $n \geq c > 2$ . Then  $M_c^*(n) = \lceil \lceil n/2 \rceil n/2 \rceil$ .*

**Theorem 1.2.** *For any pair of integers  $n$  and  $c$ , the following holds:*

$$\left\lceil \frac{1}{2} \left( n - 1 - \frac{n - 1}{c - 1} \right) n \right\rceil \leq P_c^*(n) \leq \frac{c - 2}{2(c - 1)} n^2 + n.$$

## 2 Sketches of proofs and additional results

Instead of proving Theorem 1.1 directly, let us address the more general, weighted  $k$ -majority model. The next theorem characterizes the query graphs that solve the weighted  $k$ -majority problem provided some extra assumptions are satisfied. For simplicity, we will assume that the vertex set of the query graph is  $[n]$ . Given a multiset  $S = \{w_1, w_2, \dots, w_n\}$  of weights let  $\mathcal{F} = \{F \subset [n] : w(F) \geq k\}$  be the family of the  $k$ -majority sets. Let  $\mathcal{F}_0$  denote the subfamily of minimal sets in  $\mathcal{F}$ .

**Theorem 2.1.** *Suppose there are no 1-element sets in  $\mathcal{F}$ . Then*

- (i) *If each member of  $\mathcal{F}_0$  induces a connected subgraph of the query graph  $G_Q$ , then  $G_Q$  solves the weighted  $k$ -majority problem.*
- (ii) *If  $2w([n]) < (k + 1)(c + 1) - 2$ ,  $c > 2$  and  $G_Q$  solves the weighted  $k$ -majority problem, then each member of  $\mathcal{F}_0$  induces a connected subgraph of  $G_Q$ .*
- (iii) *Considering the non-weighted version, suppose  $k$  is an integer. If  $n \leq ck - k - c + 2$ ,  $c > 2$  and  $G_Q$  solves the  $k$ -majority problem, then each member of  $\mathcal{F}$ , i.e. any set with at least  $k$  elements, induces a connected subgraph of  $G_Q$ .*

**Corollary 2.2.** *Suppose  $c > 2$ ,  $n > k > n/2$  and  $n > 1$ . Then a query graph  $G_Q$  solves the  $k$ -majority problem if and only if  $G_Q$  is  $(n - k + 1)$ -connected.*

Theorem 1.1 can be easily deduced from Corollary 2.2. Let us note here that the upper bound of Theorem 1.1 holds also in the weighted case, but such general lower bound cannot be found without extra assumptions on the multiset  $S$  of weights. Indeed, if  $w_1 > \sum_{i=2}^n w_i$ , holds, then without any query one knows that the ball with weight  $w_1$  is a majority ball.

Let us now turn our attention to adaptive majority problems. We will only address problems where the number of colors is two. Let  $\mu(n)$  denote the largest integer  $l$  such that  $2^l$  divides  $n$ .

**Proposition 2.3.** *Let  $k > n/2$ . Then*

$$M_2(n, k) \geq n - 1 - \mu \left( \sum_{i=k}^n \binom{n}{i} \right).$$

**Proposition 2.4.** *Let  $k > n/2$ . Then*

- (i) *Let us fix an arbitrary ball. The number  $\text{Fix}_2(n, k)$  of questions needed to determine if the fixed ball is a  $k$ -majority ball is at least  $n - 1 - \mu(\sum_{i=k}^n \binom{n-1}{i-1})$ .*
- (ii)  $M_2(n, k) \geq n - 2 - \mu(\sum_{i=k}^n \binom{n-1}{i-1})$ .

Let us now consider the weighted (adaptive) majority problem with two colors. Suppose there are  $p \neq 0$  ways to partition the multiset  $S$  into two parts of equal weight. Then

**Proposition 2.5.**

- (i) *At least  $n - 1 - \mu(p)$  questions are needed.*
- (ii) *In case  $p$  is even,  $n - 2$  questions are enough.*

Note that this means  $M_2(S) = n - 1$  iff  $\mu(p) = 0$ . If  $\mu(p) = 1$ , then  $M_2(S) = n - 2$ , but the opposite direction is not true.

Another possible assumption about the multiset  $S$  of weights is that “every element matters”, *i.e.* for every  $s \in S$  there exists a coloring of  $S \setminus \{s\}$  with two colors (red and blue) such that the majority color is different if we extend this coloring by giving  $s$  color red or blue. We say that a multiset  $S$  of weights is *non-slavery* if the above condition is satisfied.

**Proposition 2.6.** *For every non-slavery multiset  $S$  of weights the inequality  $M_2(S) \geq \lfloor n/2 \rfloor$  holds.*

We now address plurality problems. The lower bound of Theorem 1.2 immediately follows from the following lemma.

**Lemma 2.7.** *If  $Q$  is a set of queries that solve the problem, then the minimum degree in  $G_Q$  is larger than  $n - 1 - \lceil \frac{n-1}{c-1} \rceil$ . Furthermore, if  $n - 1 \equiv 1 \pmod{c - 1}$ , then the minimum degree in  $G_Q$  is larger than  $n - 1 - \lfloor \frac{n-1}{c-1} \rfloor$ .*

To obtain the upper bound of Theorem 1.2 it is enough to show a graph  $G_{c,n}$  with  $\frac{c-2}{2(c-1)}n^2 + n$  edges such that no matter what colors the balls have, we are able to solve the problem after receiving the answers to queries corresponding to edges of  $G_{c,n}$ . Let  $G_{c,n}$  be the  $(c - 1)$ -partite Turán graph on  $n$  vertices with a spanning cycle added to each partite set  $V_1, \dots, V_{c-1}$ .

One can improve the upper bound of Theorem 1.2 for  $c = 3$ .

**Theorem 2.8.**

- (i)  $P_3^*(2k) = k(k + 1)$ ,
- (ii)  $\frac{1}{2}(k + 1)(2k + 1) \leq P_3^*(2k + 1) \leq \frac{1}{2}(k + 1)(2k + 1) + k - 1$ .

Finally, we turn our attention to the non-adaptive weighted plurality problem, *i.e.* determining  $P_c^*(S)$  for a multiset  $S$  of weights. Theorem 1.2 shows that in general we cannot hope for anything better than the number of edges of the balanced complete  $(c - 1)$ -partite graph on  $n$  vertices. Our last theorem states that for any multiset of weights the number of edges of the balanced complete  $c$ -partite graph on  $n$  vertices and a linear number of additional queries can solve the problem.

**Theorem 2.9.** *For any multiset  $S$  of  $n$  weights the inequality  $P_c^*(S) \leq \frac{c-1}{2c}n^2 + n - c$  holds.*

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# Combinatorial bounds on relational complexity

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## 1 Introduction

An *ultrahomogeneous* structure is a (finite or countable) relational structure for which every partial isomorphism between finite substructures can be extended to a global isomorphism. This very strong symmetry condition implies that there are just a few ultrahomogeneous structures. For example, by [14], there are just countably many ultrahomogeneous undirected graphs. The *classification program* is one of the celebrated lines of research in the model theory, see [4, 15]. Various measures were introduced in order to modify a structure to an ultrahomogeneous one. A particularly interesting measure is the minimal arity of added relations (*i.e.* the minimal arity of an extension or lift) which suffice to produce an ultrahomogeneous structure. If these added relations are not changing the automorphism group then the problem is called the *relational complexity* and this is the subject of this paper. In the context of permutation groups, the relational complexity was defined in [5] and was recently popularized by Cherlin [2, 3]. We determine the relational complexity of one of the most natural class of structures (the class of structures defined by forbidden homomorphisms). This class has a (countably) universal structure [6]. As a consequence of our main result (Theorem 3.1) we strengthen this by determining its relational complexity. Although formulated in the context of model theory this result has a combinatorial character. Full details will appear in [9].

## 2 Preliminaries

A *relational structure* (or simply *structure*)  $\mathbf{A}$  is a pair  $(A, (R_{\mathbf{A}}^i : i \in I))$ , where  $R_{\mathbf{A}}^i \subseteq A^{\delta_i}$  (*i.e.*,  $R_{\mathbf{A}}^i$  is a  $\delta_i$ -ary relation on  $A$ ). The family  $(\delta_i : i \in I)$  is called the *type*  $\Delta$ . The type is assumed to be fixed and understood from the context thorough this paper. The class of all (countable) relational structures of type  $\Delta$  will be denoted by  $\text{Rel}(\Delta)$ . If the set  $A$  is finite we call  $\mathbf{A}$  a *finite structure*. We consider only countable

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or finite structures. We see relational structures as a generalization of digraphs and adopt standard graph theoretic terms (such as isomorphism, homomorphism or connected structures).

Let  $\Delta' = (\delta'_i; i \in I')$  be a type containing type  $\Delta$ . (That is  $I \subseteq I'$  and  $\delta'_i = \delta_i$  for  $i \in I$ .) Then every structure  $\mathbf{X} \in \text{Rel}(\Delta')$  may be viewed as a structure  $\mathbf{A} = (A, (R_A^i; i \in I)) \in \text{Rel}(\Delta)$  together with some additional relations for  $i \in I' \setminus I$ . We will thus also write  $\mathbf{X} = (A, (R_A^i; i \in I), (R_X^i; i \in I' \setminus I))$ .

We call  $\mathbf{X}$  a *lift* of  $\mathbf{A}$ . Note that a lift is also in the model-theoretic setting called an *expansion* (as we are expanding our relational language).

For a class  $\mathcal{K}$  of relational structures, we denote by  $\text{Age}(\mathcal{K})$  the class of all finite structures isomorphic to an (induced) substructure of some  $\mathbf{A} \in \mathcal{K}$  and call it the *age of  $\mathcal{K}$* . For a structure  $\mathbf{A}$ , the age of  $\mathbf{A}$ ,  $\text{Age}(\mathbf{A})$ , is  $\text{Age}(\{\mathbf{A}\})$ .

The classical result of Fraïssé characterize ultrahomogeneous structures in terms of their age and it can be seen as “zero instance” of problems considered in this paper. Ages of ultrahomogeneous have amalgamation property and there is 1-1 correspondence in between ultrahomogeneous structures and their ages. See *e.g.* [10] for details.

### 3 Relational complexity

Let  $\mathbf{A}$  be a relational structure and let  $\text{Aut}(\mathbf{A})$  be the automorphism group of  $\mathbf{A}$ . A  $k$ -ary relation  $\rho \subseteq A^k$  is an *invariant* of  $\text{Aut}(\mathbf{A})$  if  $(\alpha(x_1), \dots, \alpha(x_k)) \in \rho$  for all  $\alpha \in \text{Aut}(\mathbf{A})$  and all  $(x_1, \dots, x_k) \in \rho$ . Let  $\text{Inv}_k(\mathbf{A})$  denote the set of all  $k$ -ary invariants of  $\text{Aut}(\mathbf{A})$  and let  $\text{Inv}(\mathbf{A}) = \bigcup_{k \geq 1} \text{Inv}_k(\mathbf{A})$ ,  $\text{Inv}_{\leq k} = \bigcup_{1 \leq k' \leq k} \text{Inv}_{k'}(\mathbf{A})$ .

It easily follows that lift  $(A, (R_A^i; i \in I), \text{Inv}(\mathbf{A}))$  (possibly of infinite type) is an ultrahomogeneous structure for every structure  $\mathbf{A} = (A, (R_A^i; i \in I))$ . For a structure  $\mathbf{A}$  the *relational complexity*,  $\text{rc}(\mathbf{A})$ , of a  $\mathbf{A}$  is the least  $k$  such that  $(A, (R_A^i; i \in I), \text{Inv}_{\leq k}(\mathbf{A}))$  is ultrahomogeneous, if such a  $k$  exist. If no such  $k$  exists, we say that the relational complexity of  $\mathbf{A}$  is not finite and write  $\text{rc}(\mathbf{A}) = \infty$ . Note that if  $\text{rc}(\mathbf{A})$  is less than the arity of some relation in  $\Delta$ , then  $\text{rc}(\mathbf{A})$  may be lower than the relational complexity of  $\text{Aut}(\mathbf{A})$  as defined in [2].

By Fraïssé Theorem the amalgamation property can be seen as the critical property of age  $\mathcal{K}$  such that there exists structure  $\mathbf{U}$ ,  $\text{Age}(\mathbf{U}) = \mathcal{K}$ , satisfying  $\text{rc}(\mathbf{A}) = 0$ . We seek, for given  $n$ , the structural properties of age  $\mathcal{K}$  such that there exists structure  $\mathbf{U}$ ,  $\text{Age}(\mathbf{U}) = \mathcal{K}$ ,  $\text{rc}(\mathbf{A}) = n$ .

Relational complexity is not interesting for rigid structures (with trivial automorphism group), where it is always 1. Such a structure exists for almost every age. We thus restrict our attention to  $\omega$ -categorical structures. Recall that structure is  $\omega$ -categorical if and only if it has only

finitely many orbits on  $n$ -tuples, for every  $n$ , and thus also there are only finitely many invariant relations of arity  $n$ , see [10]. Moreover countable  $\omega$ -categorical structure  $\mathbf{U}$  contains as an induced substructure every countable structure  $\mathbf{A}$ ,  $\text{Age}(\mathbf{A}) \subseteq \text{Age}(\mathbf{U})$ , see [1]. We say that  $\mathbf{U}$  is *universal* for the class of all structures of age at most  $\text{Age}(\mathbf{A})$  (also called structures *younger* than  $\mathbf{U}$ ).

There is no 1-1 correspondence in between  $\omega$ -categorical structures and their ages. Consider Rado graph  $\mathbf{R}$ , graph  $\mathbf{R}'$  created as a disjoint union of two Rado graphs, and graph  $\mathbf{R}''$  created from  $\mathbf{R}'$  by adding a vertex of degree 1 connected to one of vertices of  $\mathbf{R}'$ . It is not difficult to see that all three graphs are  $\omega$ -categorical and their age is the class of finite graphs. The relational complexities are different:  $\text{rc}(\mathbf{R}) = 0$ ,  $\text{rc}(\mathbf{R}') = 2$ , and  $\text{rc}(\mathbf{R}'') = 1$ .

Relational complexity of a structure differs from the minimal arity of lifted relation needed to turn the structure into an ultrahomogeneous one (studied *i.e.* in [11]).  $\mathbf{R}'$  can be homogenized by adding unary relation distinguishing vertices of the first copy of Rado graph; this lift is however not invariant.

Among all  $\omega$ -categorical structures with a given age we can turn our attention to the “most ultrahomogeneous like” in the following sense. Structure  $\mathbf{A}$  with  $\text{Age}(\mathbf{A}) = \mathcal{K}$  is *existentially complete* if for every structure  $\mathbf{B}$ , such that  $\text{Age}(\mathbf{B}) = \mathcal{K}$  and the identity mapping (of  $A$ ) is an embedding  $\mathbf{A} \rightarrow \mathbf{B}$ , every existential statement  $\psi$  which is defined in  $\mathbf{A}$  and true in  $\mathbf{B}$  is also true in  $\mathbf{A}$ . By [6] for every age  $\mathcal{K}$  defined by forbidden monomorphisms with  $\omega$ -categorical universal structure there is also up to isomorphism unique  $\omega$ -categorical, existentially complete, and  $\omega$ -saturated universal structure. This in fact holds more generally. In such cases, for a given age  $\mathcal{K}$ , the *canonical universal structure* of age  $\mathcal{K}$  is the unique  $\omega$ -categorical, existentially complete, and  $\omega$ -saturated structure  $\mathbf{U}$  such that  $\text{Age}(\mathbf{U}) = \mathcal{K}$ . Given an age  $\mathcal{K}$  we can thus ask:

- I. What is the minimal relational complexity of an  $\omega$ -categorical structure  $\mathbf{U}$  such that  $\text{Age}(\mathbf{U}) = \mathcal{K}$ ?
- II. What is the relational complexity of the canonical universal structure of age  $\mathcal{K}$ ?

We consider universal structures for class  $\text{Forb}_h(\mathcal{F})$  where  $\mathcal{F}$  is a family of connected structures.  $\text{Forb}_h(\mathcal{F})$  denotes the class of all structures  $\mathbf{A}$  for which there is no homomorphism  $\mathbf{F} \rightarrow \mathbf{A}$ ,  $\mathbf{F} \in \mathcal{F}$ . Classes  $\text{Forb}_h(\mathcal{F})$  are among the most natural ones where the existence of a universal structure is guaranteed for every finite  $\mathcal{F}$ , see [6]. For such  $\mathcal{F}$  we can fully answer the questions above.

For a structure  $\mathbf{A} = (A, (R_A^i, i \in I))$ , the *Gaifman graph* is the graph  $G_{\mathbf{A}}$  with vertices  $A$  and all those edges which are a subset of a tuple of a

relation of  $\mathbf{A}$ , i.e.,  $G = (A, E)$ , where the neighborhood of  $\{x, y\} \in E$  if and only if  $x \neq y$  and there exists a tuple  $\vec{v} \in R_{\mathbf{A}}^i, i \in I$ , such that  $x, y \in \vec{v}$ . For a structure  $\mathbf{A}$  and a subset of its vertices  $B \subseteq A$ , the *neighborhood* of set  $B$  is the set of all vertices of  $A \setminus B$  connected in  $G_{\mathbf{A}}$  by an edge to a vertex of  $B$ . We denote by  $G_{\mathbf{A}} \setminus B$  the graph created from  $G_{\mathbf{A}}$  by removing the vertices in  $B$ .

A *g-cut* in  $\mathbf{A}$  is a subset  $C$  of  $A$  that is a vertex cut of  $G_{\mathbf{A}}$ . A g-cut  $C$  is *minimal g-separating* in  $\mathbf{A}$  if there exists structures  $\mathbf{A}_1 \neq \mathbf{A}_2$  induced by  $\mathbf{A}$  on two connected components of  $G_{\mathbf{A}} \setminus C$  such that  $C$  is the intersection of the neighborhood of  $A_1$  and the neighborhood of  $A_2$  in  $\mathbf{A}$ .

A family of structures is called *minimal* if and only if all structures in  $\mathcal{F}$  are cores and there is no homomorphism between two structures in  $\mathcal{F}$ .

**Theorem 3.1.** *Let  $\mathcal{F}$  be a finite minimal family of finite connected relational structures and  $\mathbf{U}$  an  $\omega$ -categorical universal structure for  $\text{Forb}_h(\mathcal{F})$ . Denote by  $n$  the size of the largest minimal g-separating g-cut in  $\mathcal{F}$ . Then (a)  $\text{rc}(\mathbf{U}) \geq n$ ; (b) if  $\mathbf{U}$  is the canonical universal structure for  $\text{Forb}_h(\mathcal{F})$ , then  $\text{rc}(\mathbf{U}) = n$ .*

In the rest of the paper we outline the bounds given by this theorem.

### 3.1 Upper bounds on relational complexity

It appears that relational complexity is closely related to the homogenization method of constructing universal structures as used in [7]. The main result of [7] is in fact a variant of Fraïssé Theorem with the amalgamation reduced to so-called local failure of amalgamation.

*Amalgamation failure* of a given age  $\mathcal{K}$  is a triple  $(\mathbf{A}, \mathbf{B}, \mathbf{C})$  such that  $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}$ , the identity mapping (of  $C$ ) is an embedding  $\mathbf{C} \rightarrow \mathbf{A}$  and  $\mathbf{C} \rightarrow \mathbf{B}$ , and there is no amalgamation of  $\mathbf{A}$  and  $\mathbf{B}$  over  $\mathbf{C}$  in  $\mathcal{K}$ . (i.e.  $(\mathbf{A}, \mathbf{B}, \mathbf{C})$  shows that  $\mathcal{K}$  has no amalgamation property). Amalgamation failure is *minimal* if there is no another amalgamation failure  $(\mathbf{A}', \mathbf{B}', \mathbf{C}')$  such that identity mappings are embeddings  $\mathbf{A}' \rightarrow \mathbf{A}$ ,  $\mathbf{B}' \rightarrow \mathbf{B}$  and  $\mathbf{C}' \rightarrow \mathbf{C}$ . By techniques of [7] we can show:

**Theorem 3.2.** *Let  $\mathbf{U}$  be the canonical universal structure for age  $\mathcal{K}$  and  $S$  the set of isomorphism types of minimal amalgamation failures of  $\mathbf{U}$ . If  $S$  is finite then  $\text{rc}(\mathbf{U})$  is bounded from above by the largest size of  $\mathbf{C}$  such that  $(\mathbf{A}, \mathbf{B}, \mathbf{C}) \in S$ .*

In the special case of  $\mathcal{K} = \text{Age}(\text{Forb}_h(\mathcal{F}))$  one can prove a stronger result. This is a consequence of [12].

**Theorem 3.3.** *Let  $\mathcal{F}$  be a (finite or infinite) family of connected structures such that there exists  $\mathbf{U}$ , the canonical universal structure for  $\text{Age}(\text{Forb}_h(\mathcal{F}))$ . Then  $\text{rc}(\mathbf{U})$  is bounded from above by the size of the largest minimal g-separating g-cut in  $\mathcal{F}$ .*

**Examples.** Theorem 3.3 can be easily applied to many families  $\mathcal{F}$ . For example:

1. Let  $\mathcal{F}$  be family of relational trees and  $\mathbf{U}$  the canonical universal structure for  $\text{Forb}_h(\mathcal{F})$  (if it exists). By Theorem 3.3  $\text{rc}(\mathbf{U}) \leq 1$ . In fact  $\mathbf{U}$  can be seen as a “blown up” core of a homomorphism dual  $\mathbf{D}$  (given by [8] even for some infinite families  $\mathcal{F}$ ) where each vertex is replaced by infinitely many vertices and each edge by a random bipartite graph. In this case the bound given by Theorem 3.2 is not tight even for  $\mathcal{F}$  consisting of an oriented path on 4 vertices.

2. Let  $\mathcal{F}_{C_n}$  contain a single odd graph cycle on  $n$  vertices. The relational complexity the canonical universal structure for  $\text{Forb}_h(\mathcal{F}_{C_n})$  is at most 2.

3. Let  $\mathcal{F}_{\text{odd}}$  be class of all odd graph cycles. The canonical universal structure for  $\text{Forb}_h(\mathcal{F}_{\text{odd}})$  is the random bipartite graph  $\mathbf{B}$ . By Theorem 3.3  $\text{rc}(\mathbf{B}) \leq 2$ .

### 3.2 Lower bounds on relational complexity

We obtain the following bound:

**Theorem 3.4.** *Let  $\mathcal{F}$  be a finite minimal family of finite connected structures and  $\mathbf{U}$  the  $\omega$ -categorical universal structure for  $\text{Forb}_h(\mathcal{F})$ . Then  $\text{rc}(\mathbf{U})$  is bounded from below by the size of largest minimal  $g$ -separating  $g$ -cut in  $\mathcal{F}$ .*

We use of the following result proved by a special Ramsey-type construction. This is not a technical finesse but this is in a way necessary. It has been shown by [16,17] that Ramsey classes are related to ultrahomogeneous structures. This connection has been elaborated in the context of topological dynamics in [13].

**Theorem 3.5** ([11]). *Let  $\mathcal{F}$  be a finite minimal family of finite connected relational structures and  $\mathcal{K}$  a lift of class  $\text{Forb}_h(\mathcal{F})$  adding finitely many new relations of arity at most  $r$ . If  $\mathcal{K}$  contains ultrahomogeneous lift  $\mathbf{U}$  that is universal for  $\mathcal{K}$  then the size of minimal  $g$ -separating  $g$ -cuts of  $\mathbf{F} \in \mathcal{F}$  is bounded by  $r$ .*

**Examples.** The complexity of a  $\omega$ -categorical graph universal for  $\text{Forb}_h(\mathcal{F}_{C_n})$ ,  $n \geq 5$ , (of graphs without odd cycles of length at most  $n$ ) is at least 2. Combining with Theorem 3.3 we know that relational complexity of the canonical universal structure for the class  $\text{Forb}_h(\mathcal{F}_{C_n})$  is 2. On the other hand, however, this does not hold for the class  $\mathcal{F}_{\text{odd}}$ . There exists universal bipartite graphs of relational complexity 1. Finiteness and minimality assumptions are thus needed in Theorem 3.4.

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# A combinatorial approach to colourful simplicial depth

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**Abstract.** The colourful simplicial depth conjecture states that any point in the convex hull of each of  $d + 1$  sets, or colours, of  $d + 1$  points in general position in  $\mathbb{R}^d$  is contained in at least  $d^2 + 1$  simplices with one vertex from each set. We verify the conjecture in dimension 4 and strengthen the known lower bounds in higher dimensions. These results are obtained using a combinatorial generalization of colourful point configurations called octahedral systems, which was suggested by Imre Bárány. We present properties of octahedral systems generalizing earlier results on colourful point configurations and exhibit an octahedral system which cannot arise from a colourful point configuration. The number of octahedral systems is also given.

## 1 Colourful simplicial depth

Given three blue points, three red points, and three green points in the plane such that the convex hull of each of those three monochromatic sets contains the origin  $\mathbf{0}$ , there exists a blue point, a red point, and a green point forming a triangle containing  $\mathbf{0}$ .

Generally, a *colourful point configuration* in  $\mathbb{R}^d$  is a collection of  $d + 1$  sets of points, or colours,  $\mathbf{S}_1, \dots, \mathbf{S}_{d+1}$ . A *colourful simplex* is defined as the convex hull of a subset  $S$  of  $\bigcup_{i=1}^{d+1} \mathbf{S}_i$  with  $|S \cap \mathbf{S}_i| \leq 1$  for  $i = 1, \dots, d + 1$ . The Colourful Carathéodory Theorem proven by Bárány in 1982 states that, if the origin  $\mathbf{0}$  is in the convex hull of each set of a colourful point configuration, there is a colourful simplex containing  $\mathbf{0}$ .

**Theorem 1.1 (Colourful Carathéodory's theorem [1]).** *Let  $\mathbf{S}_1, \dots, \mathbf{S}_{d+1}$  be a colourful point configuration. If  $\mathbf{0} \in \bigcap_{i=1}^{d+1} \text{conv}(\mathbf{S}_i)$ , then there is a subset  $S$  of  $\bigcup_{i=1}^{d+1} \mathbf{S}_i$  with  $|S \cap \mathbf{S}_i| \leq 1$  for  $i = 1, \dots, d + 1$ , containing  $\mathbf{0}$  in its convex hull.*

Assuming that all points are in general position, we define  $\mu(d)$  to be the minimum number of colourful simplices containing  $\mathbf{0}$  over all colourful point configurations with  $\mathbf{0} \in \bigcap_{i=1}^{d+1} \text{conv}(\mathbf{S}_i)$ . It has been recently

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investigated by Bárány and Matoušek [3], by Stephen and Thomas [4] and by Deza, Stephen and Xie [7]. In particular, it has been proven that  $\mu(d) \leq d^2 + 1$  [2] and, in the same paper, this inequality is conjectured to be an equality. This conjecture has been proven for  $d = 1, 2, 3$ . It is known that  $\mu(d) \geq \left\lceil \frac{(d+1)^2}{2} \right\rceil$  for  $d \geq 1$ , see [7].

This quantity  $\mu(d)$  has been used to obtain a lower bound for the minimum number of simplices containing  $\mathbf{0}$  drawn from a set of points in  $\mathbb{R}^d$ . We refer to [5] for a recent breakthrough on this number by Gromov.

Our main result is Theorem 1.2 which improves the known lower bounds of  $\mu(d)$ . The proof uses a combinatorial generalization of the colourful point configurations, called octahedral systems, defined in Section 2.

**Theorem 1.2.**

$$\mu(d) \geq \frac{1}{2}d^2 + \frac{7}{2}d - 8 \text{ for } d \geq 4$$

Furthermore, we show that  $\mu(4) = 17$  proving the conjecture for  $d = 4$ .

## 2 Combinatorial approach: octahedral systems

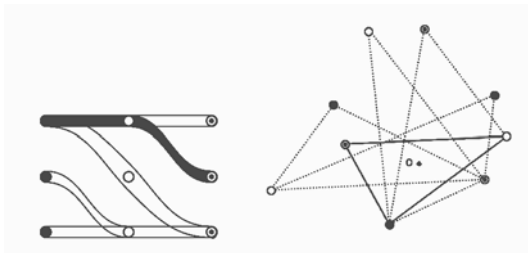
An  $n$ -uniform hypergraph is said to be  $n$ -partite if its vertex set is the disjoint union of  $n$  sets  $V_1, \dots, V_n$  and each edge intersects each  $V_i$  at exactly one vertex. Such a hypergraph is an  $(n+1)$ -tuple  $(V_1, \dots, V_n, E)$  where  $E$  is the set of edges. We consider the following combinatorial generalization suggested by Bárány to study  $\mu(d)$ . An *octahedral system*  $\Omega$  is an  $n$ -uniform  $n$ -partite hypergraph  $(V_1, \dots, V_n, E)$  with  $|V_i| \geq 2$  for  $i = 1, \dots, n$  and satisfying the following *parity condition*: the number of edges of  $\Omega$  induced by  $X \subseteq \bigcup_{i=1}^n V_i$  is even if  $|X \cap V_i| = 2$  for  $i = 1, \dots, n$ .

The Octahedral Lemma [6] states that, given a colourful point configuration  $\mathbf{S}_1, \dots, \mathbf{S}_{d+1}$  and a subset  $X \subseteq \bigcup_{i=1}^{d+1} \mathbf{S}_i$  of points such that  $|X \cap \mathbf{S}_i| = 2$  for  $i = 1, \dots, d+1$ , there is an even number of colourful simplices generated by points of  $X$  and containing the origin  $\mathbf{0}$ . It shows that the hypergraph  $\Omega = (V_1, \dots, V_{d+1}, E)$ , with  $V_i = \mathbf{S}_i$  for  $i = 1, \dots, d+1$ , and where the edges in  $E$  correspond to the colourful simplices containing  $\mathbf{0}$ , is an octahedral system.

An octahedral system arising from a colourful point configuration  $\mathbf{S}_1, \dots, \mathbf{S}_{d+1}$  such that  $\mathbf{0} \in \bigcap_{i=1}^{d+1} \text{conv}(\mathbf{S}_i)$  is without isolated vertex, *i.e.* each vertex belongs to at least one edge. Indeed, an improved version of the Colourful Carathéodory Theorem, given by Bárány [1], states that, any point of such a colourful point configuration is the vertex of at least



one colourful simplex containing  $\mathbf{0}$ . Such a colourful point configuration and its corresponding octahedral system are shown in Figure 2.1.



**Figure 2.1.** An octahedral system arising from a colourful configuration in  $\mathbb{R}^2$ .

Theorem 2.1 provides a bound for the number of edges of an octahedral system without isolated vertex.

**Theorem 2.1.** *Any octahedral system without isolated vertex and with  $|V_1| = \dots = |V_n| = m$  has at least  $\frac{1}{2}m^2 + \frac{5}{2}m - 11$  edges, for  $4 \leq m \leq n$ .*

This theorem is proven by induction on the size  $\sum_{i=1}^n |V_i|$  of octahedral systems. Throughout the induction, either a vertex and its incident edges can be deleted, resulting in a smaller octahedral system without isolated vertex, or the number of edges can be estimated via combinatorial arguments. Theorem 1.2 is the special case of Theorem 2.1 where  $m = n = d + 1$ .

We exhibit octahedral systems without isolated vertex with exactly  $\sum_{i=1}^n (|V_i| - 2) + 2$  edges, proving that the lower bound of Theorem 2.1 cannot be improved beyond  $n(m - 2) + 2$  for  $m \leq n$ .

### 3 Additional properties of octahedral systems

Proposition 3.1 generalizes a parity property given in [2], Proposition 3.2 generalizes the bound  $\mu(d) \geq d + 1$  due to the improved version of Colourful Carathéodory's Theorem by Bárány [1], and Proposition 3.3 improves the result of Theorem 2.1 for  $m = n = 5$  refining the same arguments.

**Proposition 3.1.** *An octahedral system  $\Omega = (V_1, \dots, V_n, E)$  with even  $|V_i|$  for  $i = 1, \dots, n$  has an even number of edges.*

**Proposition 3.2.** *A non-trivial octahedral system  $\Omega = (V_1, \dots, V_n, E)$  has at least  $\min_i |V_i|$  edges, and this bound is tight.*

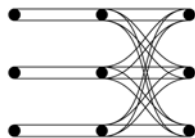
**Proposition 3.3.** *An octahedral system without isolated vertex  $\Omega = (V_1, \dots, V_5, E)$  with  $|V_1| = \dots = |V_5| = 5$  has at least 17 edges.*

Proposition 3.3 gives  $\mu(4) = 17$ , since  $\mu(d) \leq d^2 + 1$ . It shows that the conjecture  $\mu(d) = d^2 + 1$  holds for  $d = 4$ .

Theorem 3.4 determines the number of distinct octahedral systems, solving an open question raised in [6].

**Theorem 3.4.** *Given  $n$  disjoint finite vertex sets  $V_1, \dots, V_n$ , the number of octahedral systems on  $V_1, \dots, V_n$  is  $2^{\prod_{i=1}^n |V_i| - \prod_{i=1}^n (|V_i| - 1)}$ .*

The proof uses the fact that the octahedral systems form the kernel of some linear map between  $\mathbb{F}_2$ -vector spaces. Thus the octahedral systems form a  $\mathbb{F}_2$ -vector space, whose dimension is computed via the rank-nullity theorem.



**Figure 3.1.** A non-realizable  $(3, 3, 3)$ -octahedral system with 9 edges

Finally, we provide an octahedral system without isolated vertex that cannot arise from a colourful point configuration  $S_1, \dots, S_{d+1}$  in  $\mathbb{R}^d$ , answering Question 6 of [6], see Figure 3.1.

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# Complexity and approximation of the smallest $k$ -enclosing ball problem

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**Abstract.** Given an  $n$ -point set in Euclidean space  $\mathbb{R}^d$  and an integer  $k$ , consider the problem of finding the smallest ball enclosing at least  $k$  of the points. In the case of a fixed dimension the problem is polynomial-time solvable but in the general case, when  $d$  is not fixed, the complexity status of the problem was not yet known. We prove that the problem is strongly NP-hard and describe an idea of PTAS.

The Smallest  $k$ -Enclosing Ball problem is considered:

**Problem  $Sk$ -EB** Given a set  $X$  of  $n$  points in Euclidean space  $\mathbb{R}^d$  and an integer  $k$ . Find the smallest ball enclosing at least  $k$  of the points.

The problem has a lot of interesting interpretations of life, due to simplicity of the formulation. One of them is in the area of military affairs: given coordinates of  $n$  purposes, hit  $k$  of them in one gulp with a minimum charge.

## Related work

The earliest reference of a special case of this problem occurs in the middle of the 19th century [9]. In the case of a fixed dimension, particularly Euclidean plane (the most studied case), the problem is polynomially solvable [2]. However, the running time of the best known algorithms depends exponentially on dimension [4,6]. In the general case, when  $d$  is not fixed (belongs to a problem instance), the complexity status of the problem was not yet known. Agarwal *et al.* [1] study a very related problem “smallest enclosing ball with outliers” and present an approximation scheme (PTAS) for high dimensions based on coresets.

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## Our results

We prove that the problem is strongly NP-hard and unless  $P = NP$  there is no fully polynomial-time approximation scheme (FPTAS) (that does not follow from the strong NP-hardness when solution values are not integer). Also we describe a straightforward idea of PTAS that computes a  $(1 + \varepsilon)$ -approximation in  $O(n^{1/\varepsilon^2+1}d)$  time for any  $\varepsilon > 0$ .

## 1 Hardness results

We formulate a special case of  $Sk$ -EB in the form of the problem of verification of properties on a set of Boolean vectors:

**Special Case** Given a set  $X$  of  $n$  Boolean points,  $X \subseteq \{0, 1\}^d$ , an integer  $k \in [1, n]$ , and a real value  $R > 0$ . Determine whether there is a Euclidean ball of radius  $R$  enclosing at least  $k$  of the points.

Using ideas from the work of [7], we give a reduction to this special case from the following strongly NP-complete problem [8]:

**Clique in Regular Graph** Given a regular graph  $G$  and an integer  $k$ . Determine whether there is a complete  $k$ -vertex subgraph of  $G$ .

**Theorem 1.1.** *Clique in Regular Graph can be reduced to Special Case in polynomial time.*

*Reduction.* Let  $G$  be a regular graph on  $n$  vertices and  $m$  edges, and  $\Delta$  be the degree of the vertices of  $G$ . Define a set  $X$  as the set of  $m$ -dimensional rows of the incidence matrix of  $G$ . Observe that any two points  $x$  and  $y$  of  $X$  have a distance  $\sqrt{2\Delta - 2}$  if the corresponding vertices are adjacent, and  $\sqrt{2\Delta}$  otherwise.

For any  $k$ -point set  $K$  in  $\mathbb{R}^d$ , define an average of  $K$  as  $\bar{c}(K) = \sum_{x \in K} x / k$ , and for any  $y \in \mathbb{R}^d$ , define a value  $f(y, K) = \sum_{x \in K} \|x - y\|^2$ .

**Lemma 1.2.**  $f(y, K) = f(\bar{c}(K), K) + k \|y - \bar{c}(K)\|^2$ .

*Proof.* Indeed,  $f(y, K) = \sum_{x \in K} \|x - \bar{c}(K)\|^2 - \sum_{x \in K} 2 \langle x - \bar{c}(K), y - \bar{c}(K) \rangle + \sum_{x \in K} \|y - \bar{c}(K)\|^2$ , where  $\langle \cdot, \cdot \rangle$  is scalar product of vectors. The first term in this expression is equal to  $f(\bar{c}(K), K)$ , the latest is equal to  $k \|y - \bar{c}(K)\|^2$ , and the second is equal to zero, since  $\sum_{x \in K} x = k \bar{c}(K)$ .  $\square$

For any  $k$ -point set  $K$  in  $\mathbb{R}^d$ , define a value  $g(K) = \sum_{x \in K} \sum_{y \in K} \|x - y\|^2$ .

**Lemma 1.3.**  $g(K) = 2k f(\bar{c}(K), K)$ .

*Proof.* This follows from Lemma 1.2. □

**Lemma 1.4.** *Let  $R$  be the radius of the smallest ball enclosing at least  $k$  of the points of  $X$ , and  $A = (1 - 1/k)(\Delta - 1)$ . Then  $R^2 \leq A$ , if there is a  $k$ -clique in the graph  $G$ , and  $R^2 \geq A + 2/k^2$  otherwise.*

*Proof.* Suppose that a  $k$ -clique exists. Consider the point  $\bar{c}(K)$ , where  $K$  is the set of points corresponding to the vertices of the clique. Using Lemma 1.3, we have  $f(\bar{c}(K), K) = g(K)/2k = (k^2 - k)(2\Delta - 2)/2k = kA$ . By symmetry, distances from  $\bar{c}(K)$  to all the points of  $K$  are the same. It follows that the squares of these distances are equal to the value  $f(\bar{c}(K), K)/k = A$ . Thus, the ball of radius  $\sqrt{A}$  centered at the point  $\bar{c}(K)$  covers all the points of  $K$ .

Suppose that there is no  $k$ -clique. Let  $c$  be the center of the smallest ball enclosing  $k$  points of  $X$ , and  $K$  be the  $k$ -point subset it covers. Then  $R$  is equal to the maximal distance between  $c$  and the points of  $K$ . Since the maximum of squares of distances is at least its average, we have  $R^2 \geq f(c, K)/k$ . On the other hand, from Lemma 1.2 it follows that  $f(c, K) \geq f(\bar{c}(K), K)$ . Therefore,  $R^2 \geq f(\bar{c}(K), K)/k = g(K)/2k^2$ . But any of  $k^2 - k$  summands in the definition of  $g(K)$  corresponding to the pairs of distinct points is either  $2\Delta$  or  $2\Delta - 2$ . And, by the assumption of nonexistence of  $k$ -clique, at least two of them equals  $2\Delta$ . Therefore,  $g(K) \geq (k^2 - k)(2\Delta - 2) + 4$ , and then  $R^2 \geq (1 - 1/k)(\Delta - 1) + 2/k^2 = A + 2/k^2$ . □

*Proof of Theorem 1.1.* Lemma 1.4 implies that existence of a  $k$ -clique in graph  $G$  is equivalent to existence of a ball of radius  $\sqrt{A}$  enclosing  $k$  points of  $X$ . This completes the proof of Theorem 1.1. □

**Corollary 1.5.** *The Smallest  $k$ -Enclosing Ball problem is strongly NP-hard.*

In the case of optimization problems with integer-valued solutions the strong NP-hardness implies that there is no fully polynomial-time approximation scheme (FPTAS) [5, 10]. Unfortunately, the radius of the smallest  $k$ -enclosing ball is not integer in general, and so that fact needs proof.

**Theorem 1.6.** *For the Smallest  $k$ -Enclosing Ball problem, there is no fully polynomial-time approximation scheme (FPTAS) unless  $P = NP$ .*

*Proof.* By Lemma 1.4, if there is a  $k$ -clique in a graph  $G$  then the radius of the smallest ball enclosing  $k$  of the points of  $X$  is bounded by  $\sqrt{A}$ , and otherwise it is at least  $\sqrt{A + 2/k^2} > \sqrt{A}(1 + 1/2Ak^2) > \sqrt{A}(1 + 1/2n^3)$ . It follows that there is no polynomial-time algorithm that computes a  $(1 + 1/2n^3)$ -approximation unless  $\mathbf{P} = \mathbf{NP}$ . On the other hand, for an arbitrary polynomial  $p(n)$ , any FPTAS allows to compute a  $(1 + 1/p(n))$ -approximation in polynomial time. Therefore, existence of FPTAS is impossible.  $\square$

## 2 Approximation scheme

We describe an idea of a polynomial-time approximation scheme (PTAS) for  $Sk$ -EB based on the simple gradient descent type algorithm [3] for the Small Enclosing Ball problem:

**Problem SEB** Given a set  $X$  of  $n$  points in Euclidean space  $\mathbb{R}^d$ . Find the smallest ball enclosing all the points.

Observe that this is a special case of  $Sk$ -EB:  $k = n$ . The following approximation algorithm is considered in [3]:

**Algorithm for SEB** Let  $i$  be an arbitrary positive integer.

**Step 1:** Choose any point  $c_1 \in K$ .

**Step  $j$ ,  $j = 2, 3, \dots, i$ :** Take a point  $p_j \in K$ , which is furthest away from  $c_{j-1}$ , and define  $c_j = c_{j-1} + (p_j - c_{j-1})/j$ .

**Output:** The ball of radius  $R(c_i, K) = \max_{x \in K} \|x - c_i\|$  centered in  $c_i$ .

Observe that the points  $p_1, p_2, \dots, p_i$ , where  $p_1 = c_1$ , are not necessary distinct. And the point  $c_i$  is equal to the average of the points  $p_1, \dots, p_i$ :  $c_i = \sum_{j=1}^i p_j / i$ .

**Proposition 2.1 ([3]).** Let  $c^*$  and  $R^*$  be the center and the radius of the smallest ball enclosing all the points of  $K$ . Then  $\|c_i - c^*\| \leq R^*/\sqrt{i}$ .

Proposition 2.1 and the triangle inequality imply that the ball of radius  $R(c_i, K)$  centered in  $c_i$  is a  $(1 + 1/\sqrt{i})$ -approximation for the problem SEB. In fact, the above algorithm is a fully polynomial-time approximation scheme (FPTAS) that computes a  $(1 + \varepsilon)$ -approximate solution of SEB in  $O(nd/\varepsilon^2)$  time for any  $\varepsilon > 0$ .

Describe an algorithm for the original problem  $Sk$ -EB. Let  $c^*$  and  $R^*$  be the center and the radius of the smallest ball enclosing at least  $k$  of the points. Clearly, this ball is an optimal solution of the problem SEB on the  $k$ -point set  $K^*$  the ball covers. Then by Proposition 2.1, the average of some points  $p_1, \dots, p_i \in K^*$  is at distance  $R^*/\sqrt{i}$  from the point  $c^*$ .

Initially, we have no any information about the points  $p_1, \dots, p_i$  (and about the whole set  $K^*$ ), but we know that these points are in the set  $X$ . The idea of the algorithm is brute-force searching for all the sequences of length  $i$  in the set  $X$  to find that sequence  $p_1, \dots, p_i$ , whose average is close to  $c^*$ .

**Algorithm for  $Sk$ -EB** Let  $i$  be an arbitrary positive integer and  $t^i : \{1, \dots, n^i\} \rightarrow X^i$  be an enumeration of all the sequences of length  $i$  in the set  $X$ .

**Step  $s$ ,  $s = 1, \dots, n^i$ :** Consider the sequence  $t^i(s)$ , say  $t^i(s) = p_1, \dots, p_i$ . Define a point  $c^i(s) = \sum_{j=1}^i p_j / i$ , find a set  $K^i(s)$  of the  $k$  points of  $X$  nearest to  $c^i(s)$ , and obtain the radius  $R(c^i(s), K^i(s)) = \max_{x \in K^i(s)} \|x - c^i(s)\|$ .

**Output:** The ball of radius  $R^i$  centered in  $c^i$  corresponding to the minimal radius  $R(c^i(s), K^i(s))$  obtained at steps  $s = 1, \dots, n^i$ .

**Theorem 2.2.** *Let  $R^*$  be the radius of the smallest ball enclosing  $k$  of the points of  $X$ . Then  $R^i / R^* \leq 1 + 1/\sqrt{i}$ .*

*Proof.* As mentioned above, an optimal solution of the problem  $Sk$ -EB is also optimal for the problem SEB on the  $k$ -point set  $K^*$  this solution covers. Suppose that points  $p_1, \dots, p_i$  are chosen at steps  $1, \dots, i$  of the algorithm for SEB on the set  $K^*$ ,  $c_i = \sum_{j=1}^i p_j / i$  and  $R(c_i, K^*) = \max_{x \in K^*} \|x - c_i\|$ . By Proposition 2.1 and the triangle inequality, it follows that  $R(c_i, K^*) / R^* \leq 1 + 1/\sqrt{i}$ .

On the other hand, the sequence  $p_1, \dots, p_i$  is equal to some sequence  $t^i(s)$  in the algorithm for  $Sk$ -EB. Then  $c^i(s) = c_i$  and  $R(c^i(s), K^i(s)) = R(c_i, K^i(s)) \leq R(c_i, K^*)$  since the set  $K^i(s)$  consists of the nearest points to  $c_i$ . Therefore,  $R^i \leq R(c_i, K^*)$  and we have  $R^i / R^* \leq 1 + 1/\sqrt{i}$ . □

Estimate the running time of the algorithm. Since a choice of the  $k$  points of  $X$  nearest to  $c^i(s)$  takes at most  $O(n)$  operations (e.g. using the algorithm for the  $k$ th smallest number from  $n$  [11]), and all the operations over  $d$ -dimensional points take a time  $O(d)$ , the running time of the algorithm for  $Sk$ -EB is bounded by  $O(n^{i+1}d)$ .

Observe that for any  $\varepsilon > 0$ , we can take the parameter  $i = 1/\varepsilon^2$  to compute a  $(1 + \varepsilon)$ -approximation for  $Sk$ -EB. In this case the running time is bounded by  $O(n^{1/\varepsilon^2+1}d)$ . Thus, this algorithm is actually a polynomial-time approximation scheme (PTAS) for the Smallest  $k$ -Enclosing Ball problem.

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# Testing uniformity of stationary distribution

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**Abstract.** In this paper, we prove that for a regular directed graph whether the uniform distribution on the vertices of the graph is a stationary distribution, depends on a local property of the graph, namely if  $(u, v)$  is an directed edge then outdegree  $(u)$  is equal to indegree  $(v)$ . This result also has an application to the problem of testing, whether the stationary distribution obtained by random walk on a directed graph is uniform or “far” from being uniform. We reduce this problem to testing Eulerianity in the orientation model.

## 1 Introduction

Markov chains are one of the most important and most studied structures in Theoretical Computer Science. The most important characteristics of a Markov chain are its stationary distribution and its mixing time. In particular, one often wants to know if a given distribution is a stationary distribution of a given Markov chain. In this paper, we focus on the Markov chain obtained by a random walk on a directed graph. Stationary distribution of a Markov chain is a global property of the graph, hence whether a particular distribution is a stationary distribution of a Markov chain depends on the global structure of that Markov chain. We prove that contrary to normal perception, if the graph is regular then whether the uniform distribution on the vertices of the graph is a stationary distribution depends on a local property of the graph. The following theorem, which is the main result of this paper, is a statement about that local property. (See [3] for full version of this paper.)

**Theorem 1.1.** *If  $\vec{G} = (V, \vec{E})$  is a digraph such that the total degree (that is  $\text{Indegree}(v) + \text{Outdegree}(v)$ ) for every vertex  $v \in V$  is the same, then the uniform distribution on the vertices of  $\vec{G}$  is a stationary distribution (for the Markov chain generated by a random walk on  $\vec{G}$ ) if and only if the graph have the following properties:*

1. *For all  $v \in V$ ,  $\text{Indegree}(v) \neq 0$  and  $\text{Outdegree}(v) \neq 0$ ,*
2. *For every edge  $(u, v) \in \vec{E}$ ,  $\text{Outdegree}(u) = \text{Indegree}(v)$ .*

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## 2 Preliminaries

### 2.1 Graph notations

Throughout this paper, we will be dealing with directed graphs (possibly with multiple edges between any two vertices) in which each edge is directed only in one direction. We will call them **oriented graphs**. We will denote the oriented graph by  $\vec{G} = (V, \vec{E})$  and the underlying undirected graph (that is when the direction on the edges are removed) by  $G = (V, E)$ . For a vertex  $v \in V$ , the in-degree and the out-degree of  $v$  in  $\vec{G}$  are denoted by  $d^-(v)$  and  $d^+(v)$  respectively. An oriented graph  $\vec{G} = (V, \vec{E})$  is called a degree- $\Delta$  oriented graph if for all  $v \in V$ ,  $d^-(v) + d^+(v) = \Delta$ . In this paper, we will be focusing on degree- $\Delta$  oriented graphs.

### 2.2 Markov chains

A Markov chain is a stochastic process on a set of states given by a transition matrix. Let  $S$  be the set of states with  $|S| = n$ . Then, the transition matrix  $T$  is a  $n \times n$  matrix with entries from positive real; the rows and columns are indexed by the states; the  $u, v$ -th entry  $T_{u,v}$  of the matrix denotes the probability of transition from state  $u$  to state  $v$ . Since  $T$  is stochastic,  $\sum_v T_{u,v}$  must be 1. A distribution  $\mu : S \rightarrow \mathbb{R}^+$  on the vertices is said to be stationary if for all vertices  $v$ ,

$$\sum_u \mu(u) T_{u,v} = \mu(v).$$

If  $\vec{G}$  is an oriented graph then a random walk on  $\vec{G}$  defines a Markov chain, where, the states are the vertices of the graph; the probability to traverse an edge is given by the quantity  $p_{u,v} = \frac{1}{d^+(u)}$ ; and hence, the transition probability  $T_{u,v}$  from vertex  $u$  to vertex  $v$  is  $p_{u,v}$  times the number of edges between  $u$  and  $v$ . The uniform distribution on the vertices of  $\vec{G}$  is a stationary distribution for this Markov chain if and only if for all  $v \in V$ ,

$$\sum_{u:(u,v) \in \vec{E}} p_{u,v} = 1 = \sum_{w:(v,w) \in \vec{E}} p_{v,w}.$$

## 3 Structure of graphs with uniform stationary distribution

The following Theorem is a rephrasing of Theorem 1.1.

**Theorem 3.1.** *Let  $\vec{G} = (V, \vec{E})$  be a degree- $\Delta$  oriented graph, then the uniform distribution on the vertices of  $\vec{G}$  is a stationary distribution (for*

the Markov chain generated by a random walk on  $\vec{G}$ ) if and only if for all  $v \in V$ , both  $d^-(v), d^+(v) \neq 0$  and for all  $(u, v) \in \vec{E}$ ,

$$d^+(u) = d^-(v)$$

*Proof.* First of all, note that the uniform distribution is a stationary distribution for  $\vec{G}$ , iff for all  $v \in V$

$$\sum_{u:(u,v) \in \vec{E}} p_{u,v} = 1 = \sum_{w:(v,w) \in \vec{E}} p_{v,w},$$

where  $p_{u,v}$  is the transition probability (defined in subsection 2.2) from vertex  $u$  to vertex  $v$ . Thus, if the graph  $\vec{G}$  has the property that for all  $(u, v) \in \vec{E}$ ,  $d^+(u) = d^-(v)$ , then note that

$$\sum_{u:(u,v) \in \vec{E}} p_{u,v} = \sum_{u:(u,v) \in \vec{E}} \frac{1}{d^+(u)} = \sum_{u:(u,v) \in \vec{E}} \frac{1}{d^-(v)} = 1,$$

the last equality holds because the summation is over all the edges entering  $v$  (which is non-empty) and thus have  $d^-(v)$  number of items in the summation.

Similarly, we can also prove that  $\sum_{w:(v,w) \in \vec{E}} p_{v,w} = 1$ . Thus, we have proved this direction.

Now let us prove the other direction, that is, let us assume that the uniform distribution is a stationary distribution for the Markov chain. Note that, if the uniform distribution is a stationary distribution then there is a path from  $u$  to  $v$  if and only if  $u$  and  $v$  are in the same strongly connected component of  $\vec{G}$ . This is because the uniform distribution is a stationary distribution if and only if for every cut  $C = V_1 \cup V_2$  where  $V_2 = (V \setminus V_1)$ , we have

$$\sum_{(u,v) \in \vec{E}, \text{ and } u \in V_1, v \in V_2} p_{u,v} = \sum_{(u,v) \in \vec{E}, \text{ and } u \in V_2, v \in V_1} p_{u,v}.$$

In other words, if a stationary distribution is uniform then it implies that every connected component in the undirected graph is strongly connected in the directed graph.

Let  $v_0, v_1, v_2, \dots, v_t$  be a sequence of vertices such that the following conditions are satisfied. We call such a sequence as “degree-alternating” sequence of vertices.

- For all  $i \geq 0$ ,  $(v_{i+1}, v_i) \in \vec{E}$
- For all  $i \geq 0$ ,  $d^+(v_{2i+1}) = \min \left\{ d^+(w) : (w, v_{2i}) \in \vec{E} \right\}$  and
- For all  $i > 0$ ,  $d^+(v_{2i}) = \max \left\{ d^+(w) : (w, v_{2i-1}) \in \vec{E} \right\}$ .

**Claim 3.2.** Let  $\{v_i\}$  be a “degree-alternating” sequence of vertices. If we define a new sequence  $\{S\}$  of positive integers as: for all  $k \geq 0$ ,  $s_{2k} = d^-(v_{2k})$  and  $s_{2k+1} = d^+(v_{2k+1})$ , then this sequence of positive integers is a non-increasing sequence. Moreover, if  $v_i$  and  $v_{i+1}$  are two consecutive vertices in the sequence such that  $d^-(v_{i+1}) \neq d^+(v_i)$  then  $s_{i+1} < s_i$ .

Using this claim, we would finish the proof of Theorem 3.1. (Due to limitation of space we are unable to give the proof of this claim in this abstract.) Let there be one vertex  $w \in V$  such that  $d^+(u) \neq d^-(w)$  for some edge  $(u, w) \in \vec{E}$ . Let  $w'$  be the vertex such that  $(w', w) \in \vec{E}$  and  $d^+(w') = \min\{d^+(u) : (u, w) \in \vec{E}\}$ .

Since we have already argued that in the graph every connected component has to be strongly connected, we can create an infinite sequence of vertices such that  $w$  and  $w'$  appears consecutively and infinitely often. Now by Claim 3.2, it means that the sequence  $\{S\}$  is a non-increasing sequence that decreases infinitely many times. But this cannot happen as all the numbers in the sequence  $\{S\}$  represent in-degree or out-degree of vertices and hence, are always finite integers and can never be negative. Thus, if one vertex  $w \in V$  such that  $d^+(u) \neq d^-(w)$  for some edge  $(u, w) \in \vec{E}$ , then we hit a contradiction.

Thus, for all edges  $(u, v) \in \vec{E}$ ,  $d^+(u) = d^-(v)$ . □

From Theorem 3.1 we can also obtain the following corollary. Both Theorem and Corollary has an application to property testing. We briefly present this application in the next section.

**Corollary 3.3.** Let  $\vec{G} = (V, \vec{E})$  be a connected degree- $\Delta$  oriented graph. Then the uniform distribution of vertices is a stationary distribution for the random walk markov chain on  $\vec{G}$ , if and only if the following conditions apply:

1. If  $G = (V, E)$  is non-bipartite, then the graph  $\vec{G}$  is Eulerian.
2. If  $G$  is bipartite with bipartition  $V_1 \cup V_2 = V$  then  $|V_1| = |V_2|$  and in-degree of all vertices in one partition will be same and it will be equal to out-degree of all vertices in other partition.

## 4 Application to property testing

In property testing, the goal is to look at a very small fraction of the input and tell whether the input has a certain property or it is “far” from satisfying the property. Here “far” means that one has to change at least  $\epsilon$  fraction of the input to make the input satisfy the property. Theorem 1.1 also has an application to the problem of testing whether a given distribution is uniform or “far” from being uniform. More precisely, if the

distribution is the stationary distribution of the *lazy random walk*<sup>2</sup> on a directed graph and the graph is given as an input, then how many bits of the input graph do one need to query in order to decide whether the distribution is uniform or “far” from it? We consider this problem in the **orientation model** (see [2]). In the orientation model, the underlying graph  $G = (V, E)$  is known in advance. Each edge in  $E$  is oriented (that is directed in exactly one direction). The orientation of the edges has to be queried. The graph is said to be “ $\epsilon$ -far” from satisfying the property  $\mathcal{P}$  if one has to reorient at least  $\epsilon$  fraction of the edges to make the graph have the property.

We reduced this problem to testing Eulerianity in the orientation model. And using result from [1] on query complexity of testing Eulerianity, we obtain bounds on the query complexity for testing whether the stationary distribution is uniform. We briefly discuss this as follows:

Given a degree- $\Delta$  oriented graph  $\vec{G} = (V, \vec{E})$ , we say that the graph has the property  $\mathcal{P}$  if for all  $(u, v) \in \vec{E}$ , we have  $d^+(u) = d^-(v)$ .

Since the underlying undirected graph is known in advance, we have the connected components. If the graph  $\vec{G}$  is “ $\epsilon$ -far” from satisfying the property  $\mathcal{P}$ , then there is at least one connected component of  $\vec{G}$  that is also “ $\epsilon$ -far” from satisfying the property  $\mathcal{P}$ . Thus, we can do testing connected-component wise and *w.l.o.g.*, we can assume that the graph  $\vec{G}$  is connected.

From Corollary 3.3, if  $\vec{G}$  is non-bipartite then we have to test whether  $\vec{G}$  is Eulerian. Since we can determine whether graph is bipartite or not just by looking at the underlying undirected graph, if  $\vec{G}$  is non-bipartite then we use the Eulerianity testing algorithm from [1].

Now let  $\vec{G}$  be bipartite. Let the bipartition be  $V_L$  and  $V_R$ . If  $|V_L| \neq |V_R|$  then the graph surely does not satisfies property  $\mathcal{P}$ . From Corollary 3.3, if  $|V_L| = |V_R|$  then the graph must have the property that the out-degree of all vertices in  $V_L$  must be equal to the in-degree of all vertices in  $V_R$  and vice versa. Let  $v$  be a vertex in  $V_L$  and  $d^-(v) = k_1$  and  $d^+(v) = k_2$ . Now consider any bipartite directed graph  $\vec{G}^* = (V, \vec{E}^*)$  with bipartition  $V_L$  and  $V_R$  that satisfies the following conditions:

- The underlying undirected graphs of  $\vec{G}$  and  $\vec{G}^*$  are exactly same
- $\forall v \in V_L, d_{\vec{G}^*}^-(v) = k_2, d_{\vec{G}^*}^+(v) = k_1$  and  $\forall v \in V_R, d_{\vec{G}^*}^-(v) = k_1, d_{\vec{G}^*}^+(v) = k_2$ .

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<sup>2</sup> A lazy random walk always converges to a unique stationary distribution.

Now consider the graph  $\vec{G}^{\oplus} = (V, \vec{E} + \vec{E}^*)$  obtained by superimposing  $\vec{G}$  and  $\vec{G}^*$ . Clearly, if  $\vec{G}$  has the property  $\mathcal{P}$  then  $\vec{G}^{\oplus}$  is Eulerian, and farness from having property  $\mathcal{P}$  is also true by following lemma:

**Lemma 4.1.** *If  $\vec{G}$  is “ $\epsilon$ -far” from having property  $\mathcal{P}$  then  $\vec{G}^{\oplus}$  is “ $\frac{\epsilon}{2}$ -far” from being Eulerian.*

Now, all we have to test is whether the new graph  $\vec{G}^{\oplus}$  is Eulerian or “ $\frac{\epsilon}{2}$ -far” from being Eulerian. Note that every query to  $\vec{G}^{\oplus}$  can be simulated by a single query to  $G$ . Thus, we can now use the Eulerian testing algorithm from [1].

## 5 Conclusion

The result holds only for graphs where the in-degree plus out-degree of all the vertices are the same. It would be interesting to see if one can make a similar statement for general graphs.

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# On a covering problem in the hypercube

Lale Özkahya<sup>1</sup> and Brendon Stanton<sup>2</sup>

**Abstract.** In this paper, we address a particular variation of the Turán problem for the hypercube. Alon, Krech and Szabó (2007) asked “In an  $n$ -dimensional hypercube,  $Q_n$ , and for  $\ell < d < n$ , what is the size of a smallest set,  $S$ , of  $Q_\ell$ 's so that every  $Q_d$  contains at least one member of  $S$ ?” Likewise, they asked a similar Ramsey type question: “What is the largest number of colors that we can use to color the copies of  $Q_\ell$  in  $Q_n$  such that each  $Q_d$  has all the colors represented on the copies of  $Q_\ell$ 's.” We find upper and lower bounds for each of these questions and provide constructions of the set  $S$  above for some specific cases.

## 1 Introduction

For graphs  $Q$  and  $P$ , let  $\text{ex}(Q, P)$  denote the *generalized Turán number*, *i.e.*, the maximum number of edges in a  $P$ -free subgraph of  $Q$ . The  $n$ -dimensional hypercube,  $Q_n$ , is the graph whose vertex set is  $\{0, 1\}^n$  and whose edge set is the set of pairs that differ in exactly one coordinate. For a graph  $G$ , we use  $n(G)$  and  $e(G)$  to denote the number of vertices and the number of edges of  $G$ , respectively.

In 1984, Erdős [9] conjectured that

$$\lim_{n \rightarrow \infty} \frac{\text{ex}(Q_n, C_4)}{e(Q_n)} = \frac{1}{2}.$$

Note that this limit exists, because the function above is non-increasing for  $n$  and bounded. The best upper bound  $\text{ex}(Q_n, C_4)/e(Q_n) \leq 0.6068$  was recently obtained by Balogh, Hu, Lidický and Liu [2] by improving the bound 0.62256 given by Thomason and Wagner [17]. Brass, Harborth and Nienborg [4] showed that the lower bound is  $\frac{1}{2}(1 + 1/\sqrt{n})$ , when  $n = 4^r$  for integer  $r$ , and  $\frac{1}{2}(1 + 0.9/\sqrt{n})$ , when  $n \geq 9$ .

Erdős [9] also asked whether  $o(e(Q_n))$  edges in a subgraph of  $Q_n$  would be sufficient for the existence of a cycle  $C_{2k}$  for  $k > 2$ . The value

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of  $\text{ex}(Q_n, C_6)/e(Q_n)$  is between  $1/3$  and  $0.3755$  given by Conder [7] and Balogh et al. [2], respectively. On the other hand, nothing is known for the cycle of length 10. Except  $C_{10}$ , the question of Erdős is answered positively by showing that  $\text{ex}(Q_n, C_{2k}) = o(e(Q_n))$  for  $k \geq 4$  in [5, 8] and [11].

A generalization of Erdős’ conjecture above is the problem of determining  $\text{ex}(Q_n, Q_d)$  for  $d \geq 3$ . As for  $d = 2$ , the exact value of  $\text{ex}(Q_n, Q_3)$  is still not known. The best lower bound for  $\text{ex}(Q_n, Q_3)/e(Q_n)$  has been  $1 - (5/8)^{0.25} \approx 0.11086$  due to Graham, Harary, Livingston and Stout [12] until Offner [15] improved it to  $0.1165$ . The best upper bound is  $\text{ex}(Q_n, Q_3)/e(Q_n) \leq 0.25$  due to Alon, Krech and Szabó [1]. They also gave the best bounds for  $\text{ex}(Q_n, Q_d), d \geq 4$ , as

$$\Omega\left(\frac{\log d}{d^{2d}}\right) = 1 - \frac{\text{ex}(Q_n, Q_d)}{e(Q_n)} \leq \begin{cases} \frac{4}{(d+1)^2} & \text{if } d \text{ is odd,} \\ \frac{4}{d(d+2)} & \text{if } d \text{ is even.} \end{cases} \tag{1.1}$$

These Turán problems are also asked when vertices are removed instead of edges and most of these problems are also still open. In a recent paper, Bollobás, Leader and Malvenuto [3] discuss open problems on the vertex-version and their relation to Turán problems on hypergraphs.

Here, we present results on a similar dual version of the hypercube Turán problem that is asked by Alon, Krech and Szabó in [1]. For  $\ell < d$ , we call a collection of  $Q_\ell$ ’s a  $(d, \ell)$ -covering set if removing this collection leaves  $Q_n$   $Q_d$ -free. Let  $f^{(\ell)}(n, d)$  denote the minimum size of a  $(d, \ell)$ -covering set of  $Q_n$ . Determining this function when  $\ell = 1$  is equivalent to the determination of  $\text{ex}(Q_n, Q_d)$ , since  $\text{ex}(Q_n, Q_d) + f^{(1)}(n, d) = e(Q_n)$  and the best bounds for  $f^{(1)}(n, d)$  are given in [1] as (1.1). In [1], also the Ramsey version of this problem is asked as follows. For  $\ell < d$ , a coloring of the copies of  $Q_\ell$ ’s is called  $d, \ell$ -polychromatic if each  $Q_d$  has all the colors represented on the copies of  $Q_\ell$ ’s. Let  $pc^{(\ell)}(n, d)$  be the largest number of colors for which there exists a  $d, \ell$ -polychromatic coloring of  $Q_n$ . Trivially,  $pc^{(\ell)}(n, d) \leq \binom{d}{\ell} 2^{d-\ell}$ .

We define  $c^{(\ell)}(n, d)$  as

$$c^{(\ell)}(n, d) = \frac{f^{(\ell)}(n, d)}{2^{n-\ell} \binom{n}{\ell}}. \tag{1.2}$$

One can observe that

$$c^{(\ell)}(n, d) \leq \frac{1}{pc^{(\ell)}(n, d)}, \tag{1.3}$$

since any color class used in a  $d, \ell$ -polychromatic coloring is a  $(d, \ell)$ -covering set of  $Q_n$ . Note that the following limits exist, since  $c^{(\ell)}(n, d)$



is non-decreasing,  $pc^{(\ell)}(n, d)$  is non-increasing and both are bounded.

$$c_d^{(\ell)} = \lim_{n \rightarrow \infty} c^{(\ell)}(n, d), \quad p_d^{(\ell)} = \lim_{n \rightarrow \infty} pc^{(\ell)}(n, d).$$

We obtain bounds on  $p_d^{(\ell)}$  as follows.

**Theorem 1.1.** *For integers  $n > d > \ell$ , let  $0 < r \leq \ell + 1$  such that  $r = d + 1 \pmod{\ell + 1}$ . Then*

$$\begin{aligned} e^{\ell+1} \left(\frac{d+1}{\ell+1}\right)^{\ell+1} &\geq \binom{d+1}{\ell+1} \geq p_d^{(\ell)} \\ &\geq \left\lceil \frac{d+1}{\ell+1} \right\rceil^r \left\lfloor \frac{d+1}{\ell+1} \right\rfloor^{\ell+1-r} \approx \left(\frac{d+1}{\ell+1}\right)^{\ell+1}. \end{aligned} \tag{1.4}$$

Note that a trivial lower bound on  $f^{(\ell)}(n, d)$  is given by dividing the number of  $Q_d$ 's in  $Q_n$  to the number of  $Q_d$ 's a single  $Q_\ell$  can cover at most. Thus, by (1.2), for all  $n$ ,

$$c^{(\ell)}(n, d) \geq \left\lfloor \frac{2^{n-d} \binom{n}{d}}{\binom{n-\ell}{n-d}} \right\rfloor \cdot \frac{1}{2^{n-\ell} \binom{n}{\ell}} = \left(2^{d-\ell} \binom{d}{\ell}\right)^{-1}. \tag{1.5}$$

By (1.5), we have the lower bound in the following corollary. The upper bound in Corollary 1.2 is implied by (1.3) and Theorem 1.1.

**Corollary 1.2.** *For integers  $n > d > \ell$  and  $r = d - \ell \pmod{\ell + 1}$ ,*

$$\left(2^{d-\ell} \binom{d}{\ell}\right)^{-1} \leq c_d^{(\ell)} \leq \left\lceil \frac{d+1}{\ell+1} \right\rceil^{-r} \left\lfloor \frac{d+1}{\ell+1} \right\rfloor^{-(\ell+1-r)}.$$

The determination of the exact values of  $p_d^\ell$  and  $c_d^\ell$  remains open. When  $d$  and  $\ell$  have a bounded difference from  $n$ , we obtain the upper bound in Theorem 1.3, which is a constant factor of the lower bound in (1.5).

**Theorem 1.3.** *Let  $n - d$  and  $n - \ell$  be fixed finite integers, where  $d > \ell$ . Then, for sufficiently large  $n$ ,*

$$c^{(\ell)}(n, d) \leq \left\lceil \frac{r \log(n - \ell)}{\log \binom{r^r}{r^r - r!}} \right\rceil \frac{1 + o(1)}{2^{d-\ell} \binom{d}{\ell}},$$

where  $r = n - d$ .

Finally, we show an exact result for  $c^{(\ell)}(n, d)$  when  $d = n - 1$ .

**Theorem 1.4.** *For integers  $n - 1 > \ell$ ,*

$$c^{(\ell)}(n, n - 1) = \frac{\left\lceil \frac{2n}{n-\ell} \right\rceil}{2^{n-\ell} \binom{n}{\ell}}.$$

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# A classification of positive posets using isotropy groups of Dynkin diagrams

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**Abstract.** We continue our study of positive one-peak posets (presented in our talk in EuroComb 2011). Here we present a more general approach to the classification of arbitrary positive posets  $J$ . In particular we show that the Coxeter spectral classification of positive posets can be effectively solved using the right action  $*$ :  $\mathbb{M}_n(\mathbb{Z}) \times \text{Gl}(n, \mathbb{Z})_D \rightarrow \mathbb{M}_n(\mathbb{Z})$ ,  $A \mapsto A * B := B^{tr} \cdot A \cdot B$ , of isotropy groups  $\text{Gl}(n, \mathbb{Z})_D$  of simply-laced Dynkin diagrams  $D$ . By applying recent results of the second named author in [SIAM J. Discrete Math. 27(2013)] we are able to show that, given two connected positive posets  $I$  and  $J$  with at most 8 points: (i) the incidence matrices  $C_I$  and  $C_J$  of  $I$  and  $J$  are  $\mathbb{Z}$ -congruent if and only if the Coxeter spectra of  $I$  and  $J$  coincide, and (ii) the matrix  $C_I$  is  $\mathbb{Z}$ -congruent with its transpose  $C_I^{tr}$ .

## 1 Preliminaries and main results

We continue our Coxeter spectral study of positive posets we started in [3–5] in relation with a Coxeter spectral classification of loop-free edge-bipartite graphs developed in [12–14] and combinatorial properties of root systems of simply-laced Dynkin diagrams, that is, the graphs:

$$\begin{array}{l}
 \mathbb{A}_n: \bullet \text{---} \bullet \text{---} \dots \text{---} \bullet \text{---} \bullet \quad (n \geq 1); \\
 \mathbb{D}_n: \bullet \text{---} \bullet \text{---} \dots \text{---} \bullet \text{---} \bullet \quad (n \geq 3); \quad \mathbb{E}_6: \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \\
 \mathbb{E}_7: \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \quad \mathbb{E}_8: \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet
 \end{array}$$

We mainly study some of the Coxeter spectral analysis problems stated in [13] for loop-free edge-bipartite graphs  $D$  and in [15] for finite posets. We use the terminology and notation introduced there. In particular, by  $\mathbb{N}$  we denote the set of non-negative integers, by  $\mathbb{Z}$  the ring of integers, and by  $\mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$  the rational, the real and the complex number field, respectively. We view  $\mathbb{Z}^n$ , with  $n \geq 1$ , as a free abelian group, and we denote by  $e_1, \dots, e_n$  the standard  $\mathbb{Z}$ -basis of  $\mathbb{Z}^n$ . We denote by  $\mathbb{M}_n(\mathbb{Z})$  the  $\mathbb{Z}$ -algebra of all square  $n$  by  $n$  matrices, by  $E \in \mathbb{M}_n(\mathbb{Z})$  the identity matrix, and by  $\text{Gl}(n, \mathbb{Z}) := \{A \in \mathbb{M}_n(\mathbb{Z}), \det A \in \{-1, 1\}\}$  the general

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$\mathbb{Z}$ -linear group. By a finite poset  $I \equiv (I, \leq)$  we mean a partially ordered set  $I$ , with respect to a partial order relation  $\leq$ . We say that  $I$  is a **one-peak poset** if it has a unique maximal element  $*$ . Every poset  $I$  is uniquely determined by its **incidence matrix**  $C_I \in \mathbb{M}_m(\mathbb{Z})$ ,  $m = |I|$ , that is, the integer square matrix

$$C_I = [c_{ij}]_{i,j \in I} \in \mathbb{M}_m(\mathbb{Z}) \equiv \mathbb{M}_I(\mathbb{Z})$$

with  $c_{ij} = 1$ , for  $i \leq j$ , and  $c_{ij} = 0$ , for  $i \not\leq j$ .

The matrix  $G_I := \frac{1}{2}(C_I + C_I^{tr}) \in \mathbb{M}_I(\mathbb{Q})$  is called the **symmetric Gram matrix** of a poset  $I$ . A poset  $I$  is said to be **positive** (resp. **non-negative**), if the symmetric Gram matrix is positive definite (resp. positive semi-definite), see [15].

Following the main idea of the Coxeter spectral analysis of loop-free edge-bipartite graphs (signed graphs [17]) presented in [13, 14], we study finite posets  $I$  by means of the **Coxeter spectrum**  $\text{spec}_I \subseteq \mathbb{C}$ , that is, the set  $\text{spec}_I$  of all  $m = |I|$  eigenvalues of the **Coxeter matrix**  $\text{Cox}_I := -C_I \cdot C_I^{-tr} \in \mathbb{M}_m(\mathbb{Z}) \equiv \mathbb{M}_I(\mathbb{Z})$  of  $I$ , or equivalently, the set  $\text{spec}_I$  of all  $m = |I|$  roots of the **Coxeter polynomial**  $\text{cox}_I(t) := \det(t \cdot E - \text{Cox}_I) \in \mathbb{Z}[t]$ , introduced in [11].

We study finite posets  $I, J$ , up to two  $\mathbb{Z}$ -congruences  $\sim_{\mathbb{Z}}$  and  $\approx_{\mathbb{Z}}$ , where  $I \sim_{\mathbb{Z}} J$  iff the symmetric Gram matrices  $G_I$  and  $G_J$  are  $\mathbb{Z}$ -congruent, and  $I \approx_{\mathbb{Z}} J$  iff the incidence matrices  $C_I$  and  $C_J$  are  $\mathbb{Z}$ -congruent, that is,  $C_J = C_I * B := B^{tr} \cdot C_I \cdot B$ , for some  $B \in \text{Gl}(m, \mathbb{Z})$ . We recall from [11, 15] that if  $I$  is non-negative, the Coxeter spectrum  $\text{spec}_I$  lies on the unit circle  $S^1 := \{z \in \mathbb{C}; |z| = 1\}$  and all points  $z \in \text{spec}_I$  are roots of unity. Moreover, non-negative  $I$  is positive if and only if  $1 \notin \text{spec}_I$ . By [10, 11, 15], if  $I \approx_{\mathbb{Z}} J$  then  $I \sim_{\mathbb{Z}} J$  and  $\text{spec}_I = \text{spec}_J$ , but the converse implication does not hold in general.

One of the main questions of the Coxeter spectral analysis of connected positive posets is if the congruence  $I \approx_{\mathbb{Z}} J$  holds if and only if  $\text{spec}_I = \text{spec}_J$ . We have proved in [3–5] that this is the case for positive one-peak posets. It is done by computing a complete list of positive one-peak posets and then a case by case inspection.

In the present notes we give an alternative proof of this fact for positive one-peak posets. Moreover, we show that this is also the case for a class of arbitrary (not necessarily one-peak) positive posets  $I$  and  $J$ . Here we do it by a reduction to a combinatorial problem for  $\text{Gl}(m, \mathbb{Z})_D$ -orbits in the set  $\text{Mor}_D \subset \text{Gl}(m, \mathbb{Z})$  of matrix morsifications of a simply-laced Dynkin diagram  $D$ , where  $\text{Gl}(m, \mathbb{Z})_D$  is the isotropy group of  $D$  studied in [13, 14]. Our main results are the following theorems.

**Theorem 1.1.** *Assume that  $I$  and  $J$  are connected finite posets.*

- (a)  *$I$  is positive if and only if  $I \sim_{\mathbb{Z}} DI$ , where  $DI$  is a simply-laced Dynkin diagram uniquely determined by  $I$ .*
- (b) *If  $I$  and  $J$  are positive, with  $|I| \leq 8$  and  $|J| \leq 8$ , then the congruence  $I \approx_{\mathbb{Z}} J$  holds if and only if  $\mathbf{specc}_I = \mathbf{specc}_J$ .*
- (c) *If  $I$  is positive and  $m := |I| \leq 8$  then there exists a  $\mathbb{Z}$ -invertible matrix  $B \in \mathbb{M}_m(\mathbb{Z})$  such that  $C_I^{tr} = B^{tr} \cdot C_J \cdot B$  and  $B^2 = E$ .*

*Outline of proof of (a).* The “only if” part follows by applying definitions. The “if” part is a consequence of the inflation algorithm  $I \mapsto DI$  [13, Algorithm 3.5] (see also [8, 15]) that reduces (in a finite number of steps) a positive connected poset  $I$  to a simply-laced Dynkin diagram  $D := DI$  such that  $G_D = B^{tr} \cdot G_I \cdot B$ , for some  $B \in \mathrm{Gl}(m, \mathbb{Z})$  and  $G_D$  is the symmetric Gram matrix of the Dynkin diagram  $D$ , viewed as a poset with a fixed orientation of edges.

The proof of (b) and (c) is outlined in the following section. Its main idea is similar to that one used in the proof of [13, Theorem 2.7].

Our Coxeter spectral study of finite posets, edge-bipartite graphs and matrix morsifications is inspired by their important application in the representation theory of posets, finite groups, finite-dimensional algebras over a field  $K$ , and cluster  $K$ -algebras, see [1, 8, 13, 14]. We also use ideas of the spectral graph theory, a graph coloring technique, and algebraic methods in graph theory, see [2].

## 2 Outline of the proof via matrix morsifications

In the proof of Theorem 1.1, given a simply-laced Dynkin diagram  $D$ , with  $n \geq 2$  vertices, we use the isotropy group of  $D$

$$\mathrm{Gl}(n, \mathbb{Z})_D = \{B \in \mathrm{Gl}(n, \mathbb{Z}); G_D * B = G_D\} \subseteq \mathbb{M}_n(\mathbb{Z})$$

and its action  $*$ :  $\mathbb{M}_n(\mathbb{Q}) \times \mathrm{Gl}(n, \mathbb{Z})_D \rightarrow \mathbb{M}_n(\mathbb{Q})$ ,  $(A, B) \mapsto A * B := B^{tr} \cdot A \cdot B$  on  $\mathbb{M}_n(\mathbb{Q})$  and on  $\mathbb{M}_n(\mathbb{Z})$ . By [14], the matrix Weyl group  $\mathbb{W}_D$  of  $D$  is a subgroup of  $\mathrm{Gl}(n, \mathbb{Z})_D$ . Our main aim in this section is to show that the proof of Theorem 1.1 reduces to a computation of some properties of  $\mathrm{Gl}(n, \mathbb{Z})_D$ -orbits on the set  $\mathbf{Mor}_D$  of matrix morsifications for simply-laced Dynkin diagrams  $D$ . Here we mainly apply the technique and results given in [9], and [12, 13].

Following [10]–[14], an integral (resp. rational) **matrix morsification** of  $D$  with  $n \geq 2$  vertices and the symmetric Gram matrix  $G_D \in \mathbb{M}_n(\mathbb{Q})$ , is a non-singular matrix  $A \in \mathbb{M}_n(\mathbb{Z})$  (resp.  $A \in \mathbb{M}_n(\mathbb{Q})$ ) such that  $A + A^{tr} = 2 \cdot G_D$  and the  $\mathbb{Z}$ -invertible **Coxeter matrix**  $\mathrm{Cox}_A := -A \cdot A^{-tr}$

has integer coefficients. By  $\mathbf{Mor}_D \subseteq \widehat{\mathbf{Mor}}_D \subseteq \mathbb{M}_n(\mathbb{Q})$  we denote the sets of all integral and all rational morsifications of  $D$ . By [14], the map

$$* : \widehat{\mathbf{Mor}}_D \times \mathrm{Gl}(n, \mathbb{Z})_D \rightarrow \widehat{\mathbf{Mor}}_D, \quad (A, B) \mapsto A * B := B^{tr} \cdot A \cdot B,$$

is an action of the isotropy group  $\mathrm{Gl}(n, \mathbb{Z})_D$  of  $D$ , and the subset  $\mathbf{Mor}_D \subseteq \widehat{\mathbf{Mor}}_D$  is  $\mathrm{Gl}(n, \mathbb{Z})_D$ -invariant. Moreover, for  $A \in \widehat{\mathbf{Mor}}_D$ ,  $\det A \in \mathbb{Q}$ , the **Coxeter number**  $\mathbf{c}_A \geq 2$  (i.e., a minimal integer  $r \geq 2$  such that  $\mathrm{Cox}_A^r = E$ ), and the **Coxeter polynomial**  $\mathrm{cox}_A(t) := \det(t \cdot E - \mathrm{Cox}_A) \in \mathbb{Z}[t]$  of  $A$  are  $\mathrm{Gl}(n, \mathbb{Z})_D$ -invariant.

Given a positive finite poset  $I$ , with  $n = |I|$  and  $DI$  the unique simply-laced Dynkin diagram such that  $I \sim_{\mathbb{Z}} DI$  as in Theorem 1.1 (a), we fix a matrix  $M_I \in \mathrm{Gl}(n, \mathbb{Z})$  defining the congruence  $DI \sim_{\mathbb{Z}} I$ , that is, the equality  $G_{DI} = M_I^{tr} \cdot G_I \cdot M_I$  holds. By [13, 14], the matrix  $C_I * M_I := M_I^{tr} \cdot C_I \cdot M_I$  lies in  $\mathbf{Mor}_{DI}$  and we have  $\det C_I * M_I = \det C_I = 1$ ,  $\mathrm{cox}_{C_I * M_I}(t) = \mathrm{cox}_{C_I}(t)$ , and the Coxeter numbers  $\mathbf{c}_I$  and  $\mathbf{c}_{C_I * M_I}$  of  $I$  and of the morsification  $C_I * M_I$  coincide.

Assume that  $I$  and  $J$  are connected positive posets such that  $\mathrm{spec}c_I = \mathrm{spec}c_J$ . Then  $n = |I| = |J|$ ,  $C_I * M_I \in \mathbf{Mor}_{DI}$ ,  $C_J * M_J \in \mathbf{Mor}_{DJ}$ ,  $\mathrm{cox}_{C_I * M_I}(t) = \mathrm{cox}_I(t) = \mathrm{cox}_J(t) = \mathrm{cox}_{C_J * M_J}(t)$  and  $\det C_I * M_I = \det C_I = 1 = \det C_J = \det C_J * M_J$ . Hence we conclude, as in [13], that  $DI = DJ$ .

Now we show that  $I \approx_{\mathbb{Z}} J$ . We set  $D := DI = DJ$  and we use the isotropy group  $\mathrm{Gl}(n, \mathbb{Z})_D$  of the Dynkin diagram  $D$ . Consider the right action of  $\mathrm{Gl}(n, \mathbb{Z})_D$  on  $\widehat{\mathbf{Mor}}_D$ . Denote by  $F(t) \in \mathbb{Z}[t]$  the Coxeter polynomial  $F(t) := \mathrm{cox}_I(t) = \mathrm{cox}_J(t) = \mathrm{cox}_{C_I * M_I}(t) = \mathrm{cox}_{C_J * M_J}(t)$ . By rather lengthy computer calculation, we compute the set  $\widehat{\mathbf{Mor}}_D$ , the Coxeter polynomials  $\mathrm{cox}_A(t)$ , with  $A \in \widehat{\mathbf{Mor}}_D$ , the isotropy group  $\mathrm{Gl}(n, \mathbb{Z})_D$ , and we show that there exists a matrix  $A_F \in \mathbf{Mor}_D^F := \{A \in \mathbf{Mor}_D; \mathrm{cox}_A(t) = F(t)\}$ , such that  $\mathbf{Mor}_D^F = A_F * \mathrm{Gl}(n, \mathbb{Z})_D$  if  $D$  has at most 8 vertices (see Section 3).

Since the matrices  $C_I * M_I$  and  $C_J * M_J$  lie in  $\mathbf{Mor}_D^F = A_F * \mathrm{Gl}(n, \mathbb{Z})_D$  then there exists a matrix  $B \in \mathrm{Gl}(n, \mathbb{Z})_D$  such that  $C_I * M_I = B^{tr} \cdot (C_J * M_J) \cdot B$ , that is,  $C_I * M_I = (C_J * M_J) * B$ . It follows that  $C_I = C_J * (M_J B M_I^{-1})$ , that is,  $I \approx_{\mathbb{Z}} J$ . Hence we conclude that the implications in (b) of Theorem 1.1 hold.

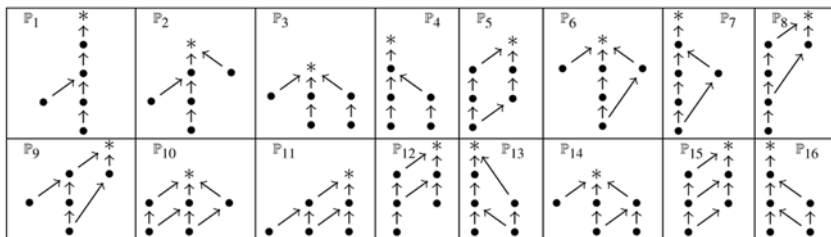
To prove (c), we note that  $C_I^{tr} = C_J$ , where  $J = I^{op}$  is the poset opposite to  $I$ . Since  $\mathrm{cox}_{I^{op}}(t) = \mathrm{cox}_I(t)$ , then (b) applies and there is a  $\mathbb{Z}$ -invertible matrix  $B$  such that  $C_I^{tr} = B^{tr} \cdot C_I \cdot B$ . It remains to show that the matrix  $B$  can be chosen such that  $B^2 = E$ . Fortunately, the technique presented earlier allows us to reduce the problem to an analogous problem for matrix morsifications of  $D = DI$ . However, a

complete proof is rather lengthy and combinatorial (it is similar to that one in [14], see also [3]- [6]). Details will be presented in a subsequent paper.

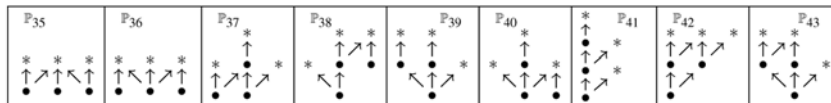
### 3 Our calculation technique and concluding remarks

In this section we give few concluding remarks on the proof of Theorem 1.1 and algorithmic techniques used to calculate the isotropy groups of Dynkin diagrams, see also [7].

**Remark 3.1.** Our technique discussed in Section 2 gives an alternative proof of the classification of one-peak positive posets presented in [4], where a complete list of such a posets  $I$  with  $|I| \leq 8$  is given. We recall a complete list of one-peak positive posets of type  $\mathbb{E}_6$ , see [3,4].



**Remark 3.2.** By applying our technique, we have computed a complete list of all (not necessarily one-peak) posets of type  $\mathbb{E}_6$ , consisting of 43 posets: 16 posets with one maximal element, 18 posets with two maximal elements and 9 posets with three maximal elements. The last part of this list looks as follows:



**Remark 3.3.** Our approach requires a numeric computation of the isotropy group  $Gl(n, \mathbb{Z})_D$  of any Dynkin diagram  $D$  with  $n$  vertices. This is a challenging computational task as the number of matrices in the  $Gl(n, \mathbb{Z})_D$  grows quickly.

$D$	$ D $	$ Gl(n, \mathbb{Z})_D $	$D$	$ D $	$ Gl(n, \mathbb{Z})_D $	$D$	$ D $	$ Gl(n, \mathbb{Z})_D $
$\mathbb{A}_6$	6	10080	$\mathbb{D}_6$	6	46080	$\mathbb{E}_6$	6	103680
$\mathbb{A}_7$	7	80640	$\mathbb{D}_7$	7	645120	$\mathbb{E}_7$	7	2903040
$\mathbb{A}_8$	8	725760	$\mathbb{D}_8$	8	10321920	$\mathbb{E}_8$	8	696729600

It is difficult not only to compute the  $\mathrm{Gl}(n, \mathbb{Z})_D$  group but also to choose a good storage schema of such a big set of data. Here we present the results of our calculations. We will discuss the technical details in the subsequent paper.

To prove Theorem 1.1 we compute the set  $\widehat{\mathbf{Mor}}_D \subset \mathbb{M}_n(\mathbb{Q})$  of all rational morsifications of Dynkin diagram  $D$  with at  $n \leq 8$  vertices (see [14, Note added in proof]) and the isotropy group  $\mathrm{Gl}(n, \mathbb{Z})_D$ . As a result of computations we prove the following key part of our technique.

**Corollary 3.4.** *If  $D$  is a Dynkin diagram with  $n \leq 8$  vertices, then there exists a matrix  $A_F \in \mathbf{Mor}_D^F := \{A \in \mathbf{Mor}_D; \mathrm{cox}_A(t) = F(t)\} \subseteq \mathbb{M}_n(\mathbb{Z})$  such that  $\mathbf{Mor}_D^F = A_F * \mathrm{Gl}(n, \mathbb{Z})_D$ .*

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## Posters



# Enumeration and classification of self-orthogonal partial Latin rectangles by using the polynomial method

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The current paper deals with the enumeration and classification of the set  $\text{SOPLR}_{r,n}$  of self-orthogonal  $r \times r$  partial Latin rectangles based on  $n$  symbols. It can be identified with the set of zeros of the zero-dimensional ideal  $I_{r,n} = \langle x_{ijk} \cdot (1 - x_{ijk}), x_{ijk} \cdot x_{i'jk}, x_{ijk} \cdot x_{ij'k}, x_{ijk} \cdot x_{ijk'} : i \in [r], j \in [s], k \in [n], i' \in [r] \setminus [i], j' \in [s] \setminus [j], k' \in [n] \setminus [k] \rangle \cup \langle x_{ijp} \cdot x_{klp} \cdot x_{jiq} \cdot x_{lkq} : i, j, k, l \in [r], p, q \in [n], (i, j) \neq (k, l) \rangle \subseteq \mathbb{Q}[x_{111}, \dots, x_{rrn}]$ . Moreover,  $|\text{SOPLR}_{r,n}| = \dim_{\mathbb{Q}}(\mathbb{Q}[x_{111}, \dots, x_{rrn}]/I_{r,n})$ . In particular, we obtain the following data:

$n$	$ \text{SOPLR}_{r,n} $				
	$r$				
	1	2	3	4	5
1	2	5	24	147	1,050
2	3	21	407	13,701	660,447
3	4	73	5,086	850,567	256,344,232
4	5	209	47,373	35,805,129	*
5	6	501	333,236	1,035,763,371	*
6	7	1,045	1,826,659	21,134,413,357	*
7	8	1,961	8,103,642	314,221,824,351	*
8	9	3,393	30,148,121	*	*
9	10	5,509	96,972,688	*	*

Given  $s \in [n] \cup \{0\}$ , let  $\sigma_{r,s}$  be the number of partial Latin rectangles of  $\text{SOPLR}_{r,n}$  which contain exactly  $s$  symbols in their cells. In particular,  $|\text{SOPLR}_{r,n}| = \sum_{s=0}^n \binom{n}{s} \cdot \sigma_{r,s}$ . If  $\mathfrak{P}_{s,P}$  denote the main class of  $P \in \text{SOPLR}_{r,s}$ , then  $\sigma_{r,s} = \sum_{P \in \mathfrak{P}_{r,s;s}} |\mathfrak{P}_{s,P}| = \sum_{P \in \mathfrak{P}_{r,s;s}} \frac{2 \cdot r! \cdot s!}{|\mathcal{I}_s(P,P)| + |\mathcal{I}_s(P,P^t)|}$ , where given two partial Latin rectangles  $P, Q \in \text{SOPLR}_{r,s}$ , the set  $\mathcal{I}_s(P, Q)$  denotes the set of isotopisms which transform  $P$  into  $Q$ , which can be identified with the set of zeros of  $I_{\mathcal{I}_s(P,Q)} = \langle 1 - \sum_{j \in [r]} x_{ij} : i \in [r] \rangle + \langle 1 - \sum_{j \in [n]} y_{ij} : i \in [s] \rangle + \langle 1 - \sum_{i \in [r]} x_{ij} : j \in [r] \rangle + \langle 1 -$

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$\sum_{i \in [n]} y_{ij} : j \in [s] \rangle + \langle x_{ij} \cdot (1 - x_{ij}) : i, j \in [r] \rangle + \langle y_{ij} \cdot (1 - y_{ij}) : i, j \in [s] \rangle + \langle x_{ik} \cdot x_{jl} \cdot (y_{p_{ij}q_{kl}} - 1) : i, j, k, l \in [r], \text{ such that } p_{ij}, q_{kl} \in [s] \rangle + \langle x_{ik} \cdot x_{jl} : i, j, k, l \in [r], \text{ such that } (p_{ij} = \emptyset \text{ and } q_{kl} \in [s]) \text{ or } (p_{ij} \in [s] \text{ and } q_{kl} = \emptyset) \rangle$ . It is then verified that:

- i)  $|\text{SOPLR}_{1,n}| = n + 1$ .
- ii)  $|\text{SOPLR}_{2,n}| = n^4 - 2n^3 + 5n^2 + 1$ .
- ii)  $|\text{SOPLR}_{3,n}| = n^9 - 15n^8 + 122n^7 - 604n^6 + 1973n^5 - 4201n^4 + 5640n^3 - 4240n^2 + 1347n + 1$ .

# Polynomial graph invariants from homomorphism numbers

Delia Garijo<sup>1</sup>, Andrew J. Goodall<sup>2</sup> and Jaroslav Nešetřil<sup>2</sup>

The number of homomorphisms  $\text{hom}(G, K_k)$  from a graph  $G$  to the complete graph  $K_k$  is the value of the chromatic polynomial of  $G$  at a positive integer  $k$ . This motivates the following:

**Definition.** A sequence of graphs  $(H_{\mathbf{k}})$ ,  $\mathbf{k} = (k_1, \dots, k_h) \in \mathbb{N}^h$ , is *strongly polynomial* if for every graph  $G$  there is a polynomial  $p(G; x_1, \dots, x_h)$  such that  $\text{hom}(G, H_{\mathbf{k}}) = p(G; k_1, \dots, k_h)$  for every  $\mathbf{k} \in \mathbb{N}^h$ .

Many important graph polynomials  $p(G)$  are determined by strongly polynomial sequences of graphs  $(H_{\mathbf{k}})$ : e.g. the Tutte polynomial, the Averbouch-Godlin-Makowsky polynomial (which includes the matching polynomial) and the Tittmann-Averbouch-Godlin polynomial (which includes the independence polynomial).

We give a new construction of strongly polynomial sequences, which among other things offers a natural generalization of the above polynomials. We start with a simple graph  $H$  given as a spanning subgraph of the closure of a rooted tree  $T$ . For each  $\mathbf{k} = (k_s : s \in V(T)) \in \mathbb{N}^{|V(T)|}$  we use the tree  $T$  to recursively construct a graph  $T^{\mathbf{k}}(H)$ , in which, for each  $s \in V(T)$ , we create  $k_s$  isomorphic copies of the subtree  $T_s$  of  $T$  rooted at  $s$ , all pendant from the same vertex as  $T_s$ , while propagating adjacencies of  $H$  in the closure of  $T$  to these copies of  $T_s$ .

**Theorem 1.** *The sequence  $(T^{\mathbf{k}}(H))$  is strongly polynomial.*

Define  $\beta(H)$  to be the minimum value of  $|V(T)|$  such that  $H$  is a subgraph of the closure of  $T^{\mathbf{k}}(T)$ . For example,  $\beta(K_{1,\ell}) = 2$ ,  $\beta(P_{2\ell}) =$

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$2\ell$ ,  $\beta(P_{2\ell-1}) = \ell$ , and  $\beta(K_\ell) = \ell$ . We have tree-depth  $\text{td}(H) \leq \beta(H)$  and  $\beta(H) = |V(H)|$  if  $H$  has no involutive automorphisms.

**Theorem 2.** *Let  $\mathcal{H}$  be a family of simple graphs such that  $\{\beta(H) : H \in \mathcal{H}\}$  is bounded. Then  $\mathcal{H}$  can be partitioned into a finite number of subsequences of strongly polynomial sequences of graphs.*

# An Erdős–Ko–Rado theorem for matchings in the complete graph

Vikram Kamat<sup>1</sup> and Neeldhara Misra<sup>1</sup>

We consider the following higher-order analog of the Erdős–Ko–Rado theorem [1]. For positive integers  $r$  and  $n$  with  $r \leq n$ , let  $\mathcal{M}_n^r$  be the family of all matchings of size  $r$  in the complete graph  $K_{2n}$ . For any edge  $e \in E(K_{2n})$ , the family  $\mathcal{M}_n^r(e)$ , which consists of all sets in  $\mathcal{M}_n^r$  containing  $e$  is called the star centered at  $e$ . We prove the following result:

**Theorem 1.** *For  $r < n$ , if  $\mathcal{A} \subseteq \mathcal{M}_n^r$  is an intersecting family of  $r$ -matchings, then  $|\mathcal{A}| \leq \phi(n, r)$  with equality holding if and only if  $\mathcal{A} = \mathcal{M}_n^r(e)$  for some  $e \in E$ .*

We note that the case  $r = n$  is settled (as part of a stronger theorem for uniform set partitions) by Meagher and Moura [3]. To prove Theorem 1, we use an analog of Katona’s cycle method [2]. As in Katona’s original proof of the Erdős–Ko–Rado theorem, the main challenge is to come up with a class of objects over which to carry out the double counting argument. We use the notion of Baranyai partitions to construct these objects.

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# A constrained path decomposition of cubic graphs and the path number of cacti

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Fábio Botler<sup>1</sup> and Yoshiko Wakabayashi<sup>1</sup>

Kotzig (1957) proved that a cubic graph has a perfect matching if and only if it has a 3-path decomposition (that is, a partition of the edge set into paths of length 3). This result was generalized by Jaeger, Payan, and Kouider (1983), who proved that a  $(2k + 1)$ -regular graph with a perfect matching can be decomposed into bistars. (A bistar is a graph obtained from two disjoint stars by joining their centers with an edge.) In another direction, Heinrich, Liu and Yu (1999) proved that a  $3m$ -regular graph  $G$  admits a balanced 3-path decomposition if and only if  $G$  contains an  $m$ -factor.

We generalize the result of Kotzig by proving the following result.

**Theorem 1.** *A cubic  $n$ -vertex graph  $G$  has a matching of size  $k$  if and only if  $G$  has a minimum path decomposition into paths of length 2, 3 or 4, with  $2k - \frac{n}{2}$  paths of length 3.*

The *path number* of a graph  $G$ , denoted by  $\text{pn}(G)$ , is the minimum number of edge-disjoint paths needed to partition the edge set of  $G$ . According to Lovász (1968), Erdős asked about this parameter, and Gallai conjectured that  $\text{pn}(G) \leq (n + 1)/2$  for every  $n$ -vertex connected graph.

This parameter is not known for most of the graphs. Lovász (1968) proved that  $\text{pn}(G) = n/2$  for an  $n$ -vertex graph  $G$  without even degree vertices. We present here a formula for the path number of cacti, which generalizes the path number of trees. We recall that a cactus is a connected graph in which any two simple cycles have at most one vertex in common.

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**Theorem 2.** *Let  $G$  be a union of vertex-disjoint cacti. Let  $o$  be the number of odd-degree vertices of  $G$ , let  $c_\ell$  be the number of cycles of  $G$  that contain exactly one vertex of degree greater than 2, and let  $c_i$  be the number of cycles of  $G$  that contain no vertex of degree greater than 2. Then the path number of  $G$  is given by*

$$\text{pn}(G) = o/2 + c_\ell + 2c_i.$$

# On push chromatic number of planar graphs and planar $p$ -cliques

Sagnik Sen<sup>1</sup>

An *oriented graph*  $\vec{G}$  is a directed graph without cycles of length 1 or 2. *Pushing* a vertex  $v$  of an oriented graph  $\vec{G}$  is to change the orientation of all its arcs (replacing the arc  $\vec{xy}$  by  $\vec{yx}$ ) incident to  $v$ . If we can obtain  $\vec{G}_2$  by pushing some vertices of  $\vec{G}_1$  then, the two graphs are in an equivalence relation called *push relation*. A *push graph*  $[\vec{G}]$  is an equivalence class of oriented graphs ( $\vec{G}$  is an element of the class) with respect to push relation. A *homomorphism* of an oriented graph  $\vec{G}_1$  to another oriented graph  $\vec{G}_2$  is a mapping  $f$  from  $V(\vec{G}_1)$  to  $V(\vec{G}_2)$  such that, if  $\vec{uv}$  is an arc of  $\vec{G}_1$ , then  $f(u)f(v)$  is an arc of  $\vec{G}_2$ . The *Push chromatic number*  $\chi_p(\vec{G})$  of an oriented graph  $\vec{G}$  is the minimum order of an oriented graph  $\vec{H}$  such that, some element of  $[\vec{G}]$  admits homomorphism to some element of  $[\vec{H}]$ . Push graph and the notion of homomorphism and chromatic number of push graph has been introduced in [1]. We define a *push clique* or simply  *$p$ -clique* to be an oriented graph  $\vec{G}$  such that  $\chi_p(\vec{G}) = |V(\vec{G})|$ .

## Theorem 1.

- (i) For an oriented planar graph  $\vec{H}$ ,  $9 \leq \chi_p(\vec{H}) \leq 40$ .
- (ii) For a girth 4, oriented planar graph  $\vec{H}$ ,  $6 \leq \chi_p(\vec{H}) \leq 20$ .
- (iii) For a girth 5, oriented planar graph  $\vec{H}$ ,  $4 \leq \chi_p(\vec{H}) \leq 8$ .
- (iv) For a girth 6, oriented planar graph  $\vec{H}$ ,  $4 \leq \chi_p(\vec{H}) \leq 7$ .
- (v) For a girth 8, oriented planar graph  $\vec{H}$ ,  $\chi_p(\vec{H}) = 4$ .

**Theorem 2.** The maximum order of a planar  $p$ -clique is 8.

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# Firefighting with general weights

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Vitor Costa<sup>1</sup>, Simone Dantas<sup>1</sup>, Mitre C. Dourado<sup>2</sup>,  
Lucia D. Penso<sup>3</sup> and Dieter Rautenbach<sup>3</sup>

At the 25th Manitoba Conference on Combinatorial Mathematics and Computing in Winnipeg 1995 Hartnell introduced the firefighter game modelling the containment of the spreading of an undesired property within a network. An initial configuration of the game consists of a pair  $(G, r)$  where  $G$  is a finite, simple, and undirected graph and  $r$  is a burned vertex of  $G$ . The game proceeds in rounds. In each round, first at most one vertex of  $G$  that is not burned is defended and then all vertices of  $G$  that are neither burned nor defended and have a burned neighbor are burned. Once a vertex is burned or defended, it remains so for the rest of the game. The game ends with the first round, in which no further vertex is burned. All vertices of  $G$  that are not burned at the end of the game are saved.

Here we study a generalization for weighted graphs, where the weights can be positive as well as negative. The objective of the player is to maximize the total weight of the saved vertices of positive weight minus the total weight of the burned vertices of negative weight, that is, the player should save vertices of positive weight and let vertices of negative weight burn. Allowing negative weights drastically changes the character of the game.

Our contributions are two hardness results and two greedy approximation algorithms for trees. We prove that weighted firefighter is hard al-

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ready for binary trees, which stands in contrast to the fact that unweighted firefighter is easy for binary trees. Furthermore, we show that weighted firefighter remains hard even if we allow arbitrarily many defended vertices per round. Our two greedy algorithms achieve approximation factors of  $\frac{1}{3}$  and  $\frac{1}{2}$ .

# Nowhere-zero flows on signed regular graphs

Eckhard Steffen<sup>1</sup> and Michael Schubert<sup>1</sup>

A signed graph  $(G, \sigma)$  is a graph  $G$  together with a function  $\sigma : E(G) \rightarrow \{\pm 1\}$ , which is called a signature of  $G$ . The set  $N_\sigma = \{e : \sigma(e) = -1\}$  is the set of negative edges of  $(G, \sigma)$  and  $E(G) - N_\sigma$  the set of positive edges. We study flows on signed graphs, and  $F_c((G, \sigma))$  ( $F((G, \sigma))$ ) denotes the circular (integer) flow number of  $(G, \sigma)$ .

Bouchet [1] conjectured that  $F((G, \sigma)) \leq 6$  for every flow-admissible signed graph. This conjecture is equivalent to its restriction on cubic graphs. We prove this conjecture for flow-admissible cubic graphs that have three 1-factors such that any two of them induce a hamiltonian circuit of  $G$ . In particular, every flow-admissible uniquely 3-edge-colorable cubic graph has a nowhere-zero 6-flow.

For a graph  $G$  and  $X \subseteq E(G)$  let  $\Sigma_X(G)$  be the set of signatures  $\sigma$  of  $G$ , for which  $(G, \sigma)$  is flow-admissible and  $N_\sigma \subseteq X$ . Define  $\mathcal{S}_X(G) = \{r : \text{there is a signature } \sigma \in \Sigma_X(G) \text{ such that } F_c((G, \sigma)) = r\}$  to be the  $X$ -flow spectrum of  $G$ . The  $E(G)$ -flow spectrum is the flow spectrum of  $G$  and it is denoted by  $\mathcal{S}(G)$ . If we restrict our studies on integer-valued flows, then  $\overline{\mathcal{S}}_X(G)$  denotes the integer  $X$ -flow spectrum of  $G$ .

We study the integer flow spectrum of signed cubic graphs  $G$ . There are cubic graphs whose integer flow spectrum does not contain 5 or 6. But we show, that  $\{3, 4\} \subseteq \overline{\mathcal{S}}(G)$ , for every bridgeless cubic graph  $G \neq K_2^3$ , where  $K_2^3$  is the unique cubic graph on two vertices. We construct an infinite family of bridgeless cubic graphs with integer flow spectrum  $\{3, 4, 6\}$ .

We further study the flow spectrum of  $(2t + 1)$ -regular graphs ( $t \geq 1$ ). In [2] is proven that a  $(2t + 1)$ -regular graph  $G$  is bipartite if and only if  $F_c((G, \emptyset)) = 2 + \frac{1}{t}$ . Furthermore, if  $G$  is not bipartite, then  $F_c((G, \emptyset)) \geq 2 + \frac{2}{2t-1}$ . We extend this kind of result to signed  $(2t + 1)$ -regular graphs.

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Let  $r \geq 2$  be a real number and  $G$  be a graph. A set  $X \subseteq E(G)$  is  $r$ -minimal if (1) there is a signature  $\sigma$  of  $G$  such that  $F_c((G, \sigma)) = r$  and  $N_\sigma = X$ , and (2)  $F_c((G, \sigma')) \neq r$  for every signature  $\sigma'$  of  $G$  with  $N_{\sigma'} \subset X$ .

We show for  $(2t + 1)$ -regular graphs  $G$ , which have a  $t$ -factor: A set  $X \subseteq E(G)$  is  $(2 + \frac{1}{t})$ -minimal if and only if  $X$  is a minimal set such that  $G - X$  is bipartite.

Furthermore, if  $X \subseteq E(G)$  is a  $(2 + \frac{1}{t})$ -minimal set and  $r \in \mathcal{S}_X(G)$ , then  $r = 2 + \frac{1}{t}$  or  $r \geq 2 + \frac{2}{2t-1}$ .

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# New transience bounds for long walks in weighted digraphs

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Bernadette Charron-Bost<sup>1</sup>, Matthias Függer<sup>2</sup> and Thomas Nowak<sup>1</sup>

Fix two nodes  $i$  and  $j$  in an edge-weighted digraph and form the following sequence: Let  $a(n)$  be the maximum weight of walks from  $i$  to  $j$  of length  $n$ ; if no such walk exists,  $a(n) = -\infty$ . It is known that, if  $G$  is strongly connected, the sequence  $a(n)$  is always eventually periodic with linear defect, *i.e.*, after the *transient*,  $a(n + p) = a(n) + p \cdot \lambda$ . In fact, the ratio  $\lambda$  is the largest mean weight of cycles in  $G$ . We call these cycles *critical*. Periodicity stems from the fact that the weights of critical cycles eventually dominate the maximum weight walks.

In this paper, we show two new asymptotically tight upper bounds on transients in weighted digraphs, taking into account the graph parameters cyclicity and girth. The previously best bound of Hartmann and Arguelles (Math. Oper. Res. 24, 1999) is, in general, incomparable with both our bounds. The significant benefit of our two new bounds is that each of them turns out to be linear in the number of nodes in various classes of weighted digraphs, whereas Hartmann and Arguelles' bound is intrinsically at least quadratic. In particular, our bounds are linear for (bi-directional) trees.

We hence prove the following two upper bounds on the transient in strongly connected digraphs with  $N$  nodes. They both contain the (un-weighted) index of convergence  $\text{ind}(G)$ , *i.e.*, the transient of  $G$  when choosing all weights to be equal. The term  $\|G\|$  denotes the difference of the maximum and minimum weight of edges in  $G$ . The mean weight of critical cycles is denoted by  $\lambda$ , and  $\lambda_{nc}$  denotes the largest mean weight of cycles that have no node on critical cycles. Denote by  $G_c$  the subgraph induced by the critical cycles.

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**Theorem 1 (Repetitive bound).** Denoting by  $\hat{g}$  the maximum girth of strongly connected components of  $G_c$ , the transient between any two nodes is at most

$$\max \left\{ \frac{\|G\| \cdot (3N - 2 + \text{ind}(G))}{\lambda - \lambda_{\text{nc}}}, (\hat{g} - 1) + 2\hat{g} \cdot (N - 1) \right\} .$$

**Theorem 2 (Explorative bound).** Denoting by  $\hat{\gamma}$  and  $\hat{\text{ind}}$  the maximum cyclicity and maximum index of strongly connected components of  $G_c$ , respectively, the transient between any two nodes is at most

$$\max \left\{ \frac{\|G\| \cdot (3N - 2 + \text{ind}(G))}{\lambda - \lambda_{\text{nc}}}, (\hat{\gamma} - 1) + 2\hat{\gamma} \cdot (N - 1) + \hat{\text{ind}} \right\} .$$



# Complexity of determining the irregular chromatic index of a graph

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Julien Bensmail<sup>1</sup>

An edge colouring  $\phi$  of a graph  $G$  is *locally irregular* if each colour class of  $\phi$  induces a graph whose every adjacent vertices have distinct degrees. The least number  $\chi'_{irr}(G)$  of colours used by a locally irregular edge colouring of  $G$  (if any) is referred to as the *irregular chromatic index* of  $G$ .

Locally irregular edge colouring was introduced as another type of edge colouring permitting to distinguish the adjacent vertices of a graph. This notion is thus related to several other notions of this field, like *vertex-colouring edge-weighting* [1] and *detectable colouring* of graphs [2]. In particular, it was shown that a locally irregular edge colouring is also a vertex-colouring edge-weighting or detectable colouring in some situations [3]. As for these two types of edge colouring, it is conjectured that three colours suffice to obtain a locally irregular edge colouring of any graph whose irregular chromatic index is defined [3].

We here focus on the complexity of the following decision problem.

$$k\text{-LIEC} = \{A \text{ graph } G : \text{is it true that } \chi'_{irr}(G) \leq k?\}$$

The problem 1-LIEC is in P, while the relevance of studying  $k$ -LIEC for any  $k \geq 3$  depends on the correctness of the conjecture mentioned above. We show that 2-LIEC is NP-hard, even when restricted to planar graphs with maximum degree at most 6. This result is proved by reduction from 1-IN-3 SAT.

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