Supplement[1]
Statistical structure of quantum theory and hidden variables

1. Introduction

By the end of XVIII century scientists developed the picture of the material world as a huge mechanism, the evolution of which is subject to a rigid dynamical laws and in principle can be predicted with arbitrary detail and accuracy. This system of conceptions which acquired the name "determinism" was progressive for that time and was based on a triumphant success of the Newtonian mechanics which allowed to give a rational explanation to a number of earlier inexplicable physical facts.

However, the development of natural sciences was leading to increase of the role of statistical concepts. The study of random phenomena started in XVII century with the simplest models of uncertainty emerging in hazard games. The notion of probability introduced in connection with such models was very successfully exploited in the second part of XIX century by creators of statistical mechanics. The laws of heat (thermodynamics) acquired explanation through statistical behavior of models of matter as a huge ensemble of identical interacting particles – molecules. Nevertheless by the beginning of XX century determinism continued to dominate at the foundations of natural sciences. The Nature was considered as fundamentally deterministic, and observed stochasticity was regarded as a secondary phenomenon reflecting incompleteness of our knowledge of the real state of the Nature.

Creation of basics of statistical physics marked a triumph of the ancient idea of atomism. But deeper study of the elementary components of the matter has led to a paradoxical conclusion – these components cannot be considered as particles in the proper, classical sense of the word. Depending on the conditions of observation, they can display

[1] This Supplement is self-contained and can be read independently of the main content of the book.

either corpuscular or wave properties. An electron interacting with vapor in Wilson chamber leaves a trace which is interpreted as a trajectory of a material particle. On the other hand, a beam of electrons passing through a crystal is diffracted making specific interference pattern similar to one produced by the light passing through a system of sufficiently small holes.

Attempts of theoretical explanation of such an unusual, "dualistic" behavior of microobjects has led in the beginning of XX century to the creation of quantum mechanics – the most revolutionary scientific theory of the recent age. Historically Heisenberg's "matrix mechanics" and Schrödinger's "wave mechanics" arose as result of ingenious insights in search of mathematical objects capable to combine discreteness and continuity in microprocesses. Soon after Born suggested statistical interpretation which organically fit the mathematical apparatus of quantum mechanics but excluded its deterministic explanation. A deep physico-philosophical analysis of the content of quantum mechanics developed by Bohr and Heisenberg has led to a conclusion that it represents a radically new type of theoretical model of the reality reflecting in its very structure the wholeness of the system in question and a complementarity between different aspects of its description. According to Bohr, the notion of complementarity is used to characterize the relation between the data obtained under different experimental conditions and may be interpreted only on the basis of mutually exclusive concepts. One might say that the quantum theory gives a "holographic" image of realities of the microworld. However nothing prevents from combining different projections of a three-dimensional object into one picture (remember the portraits of Picasso presenting the same person enface and in profile). But "elementarity" of the microobjects excludes a possibility of combining complementary measuring procedures which presuppose their own specific organization in space and time. For example, it makes no sense to speak about trajectory of electron diffracting on a crystal since any attempt to trace the trajectory changes the conditions of the experiment such that the interference becomes impossible.

From this point of view statistical nature of quantum mechanics becomes closely related to complementarity. The quantities that are measured in complementary conditions "cannot simultaneously be ascribed definite values. In this way, the statistical character of the formalism is displayed as a natural generalization of the description of classical physics" [155]. Thus the statistical nature of microprocesses acquires in quantum mechanics a primary importance. Not only "God does not play dice", but there is a fundamental source of uncertainty in the Nature which cannot be imitated by any classical mechanism of randomness. Of

course, the outcomes of any particular experiment are just a realization of a random variable, but the whole totality of experiments relevant to a given quantum system cannot be given a classical description. The classical method of description reducing in fact to listing the properties of the object turns out to be applicable to the objects of the surrounding macroscopical world to the extent that quantum uncertainties are negligible at the scale of this world.

In the dramatic discussion developed in 1930-s the main opponent of Bohr and other creators of the "orthodox" interpretation of quantum mechanics was Einstein whose views were shared by de Broglie, Schrödinger and some other scientists taking active part in establishing the basic principles of quantum theory. Einstein promoted the idea of incompleteness of quantum mechanics according to which its statistical nature is due to fluctuations of some "hidden variables" yet to be taken into account, and is to be replaced by a deterministic description in a future complete theory.

From that point of view an electron has an individual trajectory whether it is observed or not. The trajectory appears random since we do not know deeper principles governing the electron's motion. Having found these hidden principles, we recover determinism. Such a viewpoint might be appealing from the viewpoint of naive realism. However up to now all efforts to find alternative "deeper" interpretation of quantum mechanics turned out futile; moreover each such an effort led ultimately to strengthening of positions of the statistical interpretation shared by a majority of working physicists.

The hidden variable issue raised in this discussion can be thus formulated as the question: is it possible in principle to reduce the mathematical model of quantum mechanics to this or another form of classical probabilistic description? One should admit that the very analytical apparatus for describing uncertainty in quantum mechanics is so different from the language of probability theory that an idea of a mathematical proof of impossibility of introduction of hidden variables, which would stop once and forever all the controversy, appears naturally.

States and observables are described in quantum mechanics by matrices (operators) $\hat{S}, \hat{X}$; in probability theory the (statistical) states are described by probability distributions $S(d\omega)$ and the observable quantities – by functions $X(\omega)$ on the phase space $\Omega = \{\omega\}$ of the classical system. Thus the question is about possibility or impossibility of establishing a correspondence $S \to \hat{S}$, $X \to \hat{X}$ between classical and quantum states and observables, which would reproduce statistical prediction of quantum mechanics and, of course, satisfy certain important, physically motivated restrictions.

The first attempt of an impossibility proof was made in the famous von Neumann's book "Mathematical Foundations of Quantum Mechanics" which appeared in 1932. For some this was considered as a decisive argument against the hidden variables. However later it was understood that this argument does not solve the problem because it is based on a formal assumption lacking a physical motivation. During the last fifty years the issue was considerably clarified and even was made available for an experimental test. These investigations concerning basic understanding of the nature of physical reality are a substantial argument in favor of the viewpoint according to which introduction of physically meaningful hidden variables in quantum theory is not possible in principle.

In this essay we attempt to give a simple and self-consistent presentation of the hidden variable issue in a historical perspective.

In the first part we shall analyze general properties of description of any statistical experiment and outline the mathematical structures that arise from this description. These are convexity in the state space which is due to possibility of mixing ensembles, and partial order of observables reflecting the degree of informativity in measurement outcomes. Preservation of these structures is a minimal necessary condition for any hidden variables theory, due to their very universality. This general consideration of a statistical experiment will also allow us to distill the mathematical essence of the fundamental notion of complementarity.

The second part starts with the discussion of most important impossibility results for hidden variables. A number of such attempts, starting from von Neumann's theorem, descended from the belief rooted in the orthodox interpretation that it is the complementarity which is the main obstacle for a classical description in quantum mechanics. An important conclusion of our discussion is that complementarity excludes classical description only under additional assumption of one-to-one correspondence between quantum and classical entities. The "technical" condition of injectivity of the correspondence $S \to \hat{S}, X \to \hat{X}$ present in some modifications of von Neumann's theorem was recognized as the special property of "noncontextuality" of classical description. Thus a physically acceptable proof of impossibility can not be based on complementarity alone and requires appeal to other properties of quantum mechanical description. Such a property turns out to be the *quantum nonseparability* which is discussed in the last section. The argument based on the celebrated Bell's inequality shows impossibility of a classical description for a composite quantum system respecting separation into subsystems and hence, also the Einstein locality principle.

2. The structure of statistical theories

2.1. Axiomatic approaches in quantum mechanics

The story of von Neumann's theorem shows well that the hidden variables issue does not reduce to a merely mathematical problem. As far as the matter concerns interrelation between quantum mechanics and probability theory, the decisive role should be played by the choice of basic properties characterizing these structures. Therefore before proceeding to a mathematical consideration of the hidden variable issue we must carefully analyze and select basic postulates for both theories.

Here it is pertinent to remind that the problem of "mathematical formulation of the axioms of physics" was raised by Hilbert in his famous speech at the II-nd Mathematical Congress in 1900. Namely the question was about "axiomatic construction of those physical disciplines, in which mathematics already plays an outstanding role: these are in the first place probability theory and mechanics." Notably, in one place with the logical foundation of probability theory Hilbert put "development of the method of average values in mathematical physics, in particular, kinetic theory of gases," in this way pointing towards the most profound problems of mathematical physics, the investigation of which later led to the mathematical methods in statistical mechanics and theory of dynamical systems.

As it is well known, search for a mathematical basis for probability theory was completed with the publication in 1933 of the Kolmogorov's axiomatic system, giving a set of formally simple and intuitively appealing statements underlying the whole mathematical structure of the theory. Quantum mechanics could not be mentioned by Hilbert for the simple reason that the very physical notion of quantum did not yet exist – the famous Planck's report was made four months later in the same year 1900. The cornerstone for mathematization of quantum theory became the already mentioned monograph of von Neumann, who started investigations in this field in 1926-1927 while being Hilbert's collaborator [176].

Von Neumann's works initiated investigations in axiomatics of quantum theory. Considerable progress and clarification was achieved towards 1960-1970 along the following three mainstreams.

Algebraic approach [33, 127] takes as a basis the "algebra of observables" of the physical system. This approach turned out to be most productive from a mathematical viewpoint: together with group representations it served as a source for the modern highly refined structural theory of operator algebras. Physical applications of this approach concern mainly systems with infinite degrees of freedom – quantum fields and matter.

The quantum logic approach [71, 135, 153, 192] starts with the "lattice of propositions", *i.e.* observables taking only two values (0 and 1). Efforts in this direction culminated with construction of a certain axiom system characterizing the lattice of orthoprojections in a Hilbert space, *i.e.* "propositions" related to a quantum mechanical system. Introduction of certain algebraic structure (essentially the structure of Jordan algebra) in both approaches requires ultimately certain assumptions whose physical motivation is not so clear. In 1950-s American mathematician Mackey made an attempt to present the notions of quantum mechanics starting from some primary properties of statistical description of a physical system [92]. Although unfinished, this attempt had great influence on subsequent investigations.

In 1970-s the third approach to foundations of quantum mechanics appeared in which the notion of state plays a primary or equal role as compared to observables or measurements [27, 64, 171, 186]. The main component of the corresponding mathematical scheme is the convex set of states of the physical system. This approach, sometimes denoted as "convex" or "operational," could be also called "statistical" because it represents a far reaching development of statistical interpretation of quantum mechanics. In particular, it will be shown below that the "statistical ideology" provides also a natural basis for a discussion of hidden variables issue.

Notwithstanding apparent achievements, there is still no final form for a quantum axiomatics. Therefore the main attention in the first part of this essay will be paid to those basic assumptions which in this or the other form underlie any reasonable statistical theory. However first we must discuss the classical picture of statistical experiment which will serve a starting point for further discussion.

2.2. The classical picture of statistical experiment

In any experiment one can conventionally distinguish the two main stages. During the first stage of *preparation* a definite experimental arrangement is set to establish the "input data" of system or object under the experiment. At the subsequent stage of *measurement* the thus prepared system or object interacts with this or another measuring device, resulting in certain "output data" – the *outcome* of the measurement.

One of the most basic features of a scientific experiment is its *reproducibility i.e.* possibility of potentially indefinite independent repetitions of a given measurement in given conditions. Consider a sequence of such repetitions of certain experiment. As a rule, the outcomes of individual experiments will not be strictly the same and will be subject to

some random fluctuations the magnitude of which depend of the nature of the experiment and the system under investigation. Thus, although the outcomes of an experiments are conditioned by both preparation and measurement, this dependence usually is not deterministic but has statistical character. For classical systems described in terms of phase space, this can be expressed in the language of probability theory.

Denote by ω the complete set of variables characterizing the classical system or object. The space of all possible concrete values of ω makes the *phase space* of the object.

Preparation of any physical state of the object is made by certain device which, due to its imperfection or peculiarities of its construction may not be able to ensure exact reproduction of the values of all parameters for different individual representatives of this same object. Moreover, the object may be characterized by such an immense number of variables that there is no way to arrange complete control of all of them. It is assumed, however, that the variations of the values of ω being prepared are characterized by certain stability described by a probability distribution P. This probability distribution ascribing to an elementary phase space volume $d\omega$ its probability $P(d\omega)$ is called the *state* of the object.

Thus, this definition of state is essentially statistical one reflecting the possibility of fluctuations of the object's parameters. Its real content is that if one considers an ensemble, *i.e.* very large (potentially unbounded) collection of independent representatives of a given object, then the fraction (frequency) of those representatives for which the value of ω lies in a subset $B \subset \Omega$ is close to its theoretical value $P(B)$.

Consider two ensembles corresponding to the states P_1 and P_2 consisting of N representatives each and let us form the new ensemble by taking pN representatives of the first ensemble and $(1-p)N$ representatives of the second one, where $0 \leq p \leq 1$. In accordance with the frequency interpretation the new ensemble will be described by the state $pP_1 + (1-p)P_2$, which is called the *mixture* of the states P_1 and P_2 in the proportion $p : (1-p)$. Mixtures $\sum_j p_j P_j$ of arbitrary finite collections of states are introduced similarly. Moreover, one can consider continuous mixtures $\int p(d\alpha) P_\alpha$ where $p(d\alpha)$ is a probability distribution. Such mixtures may describe states prepared by a device with fluctuating parameter α. If the states are represented as elements (points) of a certain set, then arbitrary mixtures of P_1 and P_2 fill out the segment connecting the points P_1 and P_2. Such a set, which contains the segment connecting arbitrary two points of the set, is called *convex*. Thus the set of classical states which we shall denote $\mathfrak{S}(\Omega)$ is convex. A point in a convex set is called *extreme* if it is not an internal point of a segment belonging entirely to $\mathfrak{S}(\Omega)$. Extreme points describe *pure* states which can not

be represented as a mixture of other states. In the classical picture the pure states are just degenerate probability distributions concentrated at the points ω of the phase space.

To make this clear consider the simplest case where Ω consists of finite number of points: $\Omega = \{\omega_1, \ldots, \omega_N\}$. In this case state P is given by the finite probability distribution $[p_1, \ldots, p_N]$, where $p_j \geq 0, \sum_j p_j = 1$. Pure states are the degenerate distributions $[1, 0, \ldots, 0], \ldots, [0, \ldots, 0, 1]$.

An important theorem (proved by Caratheodory in the finite-dimensional case and by Krein and Milman in infinite dimensions) says that in a compact convex set an arbitrary point can be represented as a mixture of extreme points. If such a representation is unique, then this special convex set is called *simplex*. In particular, this is the case in the example above. In the case of arbitrary phase space Ω one should admit continuous mixtures of states. Thus in the classical picture of statistical experiments the state space $\mathfrak{S}(\Omega)$ makes simplex in which every state is a unique mixture of pure states, corresponding to exact values of all system's parameters.

The second, conclusive, stage of the experiment consists of measurement of a certain quantity X. Classically, in the ideal case the measurement does not introduce any changes in the system, *i.e.* amounts to an observation. In such a case *observable* X is described by a function which assigns to every possible $\omega \in \Omega$ its objective value $X(\omega)$. Having thus observed X, one can compute outcomes of observations of quantities $f(X)$, where f is a function, without direct observation of these quantities.

Let for simplicity observable X take finite number of real values $\{x_j\}$. Then

$$X(\omega) = \sum_i x_i E_i(\omega), \tag{2.1}$$

where $E_i(\omega)$ is the indicator function of the subset $\Omega_i \subset \Omega$ on which $X(\omega)$ takes the value x_i, *i.e.* the function equal to 1 on Ω_i and 0 outside Ω_i. The subsets Ω_i are disjoint and form a decomposition of the space Ω, while the family of their indicator functions $E = \{E_i(\omega)\}$ makes an *orthogonal resolution of identity*:

$$\sum_i E_i(\omega) = 1; \quad E_i(\omega)E_j(\omega) = 0 \quad \text{for} \quad i \neq j; \quad E_i(\omega)^2 = E_i(\omega)$$

for all $\omega \in \Omega$. Now consider observable $f(X(\omega))$, where f is a real function. Clearly,

$$f(X(\omega)) = \sum_i f(x_i) E_i(\omega). \tag{2.2}$$

Even if all x_i were different, some of the values $f(x_i)$ may coincide. Therefore, to enable treatment of (2.1) and (2.2) on equal footing, it is convenient to admit from the very beginning that some of x_i may coincide. Then *observation* (measurement without errors) will be described by the resolution of identity E, and to any observable X there correspond a variety of methods of observation, differing by detail of decomposition of the phase space Ω.

From the point of view of statistics, the resolution of the identity $E = \{E_i(\omega)\}$ bears all essential information about the measurement: probability of the $i-$th outcome in the state P is equal to

$$\mu_P^E(i) = P(\Omega_i) = \int_\Omega P(d\omega) E_i(\omega). \tag{2.3}$$

Operationally, E decomposes initial statistical ensemble into subensembles characterized by the properties $\omega \in \Omega_i$.

Hence the *mean value* of the observable (2.1) in the state P (the expectation) is equal to $M_P\{X\} = \int_\Omega P(d\omega) X(\omega)$.

The measurements described by the orthogonal resolutions of the identity are deterministic in the sense that they sharply classify representatives of an ensemble. A nondeterministic (unsharp, fuzzy) measurement provides probabilities $M_i(\omega)$ for i-th outcomes for ensemble representative characterized by the phase space point ω, so that

$$\sum_i M_i(\omega) = 1, \quad M_i(\omega) \geq 0.$$

The collection of operators $M = \{M_i(\omega)\}$ is a *resolution of the identity* in Ω, which is in general, nonorthogonal in the sense that $M_i(\omega) M_j(\omega) \not\equiv 0$ for $i \neq j$. Moreover, $M_i(\omega)^2 \leq M_i(\omega)$. Probability of the i-th outcome in the state P for such a measurement is equal to

$$\mu_P^M(i) = \int_\Omega P(d\omega) M_i(\omega). \tag{2.4}$$

This formula shows how the uncertainty of the outcome in the classical picture arises from two sources: from the uncertainty in the state preparation P and in the measurement procedure M.

The resolution of the identity M gives just the probabilities of the outcomes for an unsharp measurement, but knowing these probabilities, one can model statistical realization of the measurement involving generator of random numbers. Assume that there is such a device allowing to obtain values of random variable λ uniformly distributed in $[0, 1]$ (say, a properly calibrated roulette). Let us describe deterministic measurement

$E = \{E_i(\omega, \lambda)\}$ over the system which consists of the given object and a generator of random numbers, which is statistically equivalent to the measurement $M = \{M_i(\omega)\}$ in the sense that for any state P probabilities of all outcomes for the measurements M and E are the same. For this decompose the phase space of the composite system $\Omega \times \Lambda$ into the subsets $\Omega'_i = \left\{(\omega, \lambda) : \sum_{k=1}^{i-1} M_k(\omega) < \lambda \le \sum_{k=1}^{i} M_k(\omega)\right\}$ and denote by $E_i(\omega, \lambda)$ the indicator function of the subset Ω'_i.

By the construction we have

$$\int d\lambda E_i(\omega, \lambda) = M_i(\omega).$$

Indeed, for a given ω the integral is simply the integral with respect to $d\lambda$ from $\sum_{k=1}^{i-1} M_k(\omega)$ to $\sum_{k=1}^{i} M_k(\omega)$, *i.e.* $M_k(\omega)$. Integrating this identity with respect to $P(d\omega)$ we get

$$\begin{aligned} \mu_P^M(i) &= \int_\Omega P(d\omega) M_i(\omega) \\ &= \int_\Omega \int_\Lambda P(d\omega) d\lambda E_i(\omega, \lambda) = \mu_{P\times d\lambda}^M(i), \end{aligned} \tag{2.5}$$

which means the statistical equivalence of the measurements M and E.

The procedure of obtaining the outcome by using a generator of random numbers, introduced into statistics by Wald, is called randomization [22] and the corresponding measurements can be also called randomized.

From the viewpoint of the statistics, the results of experiment consisting of preparation of the state P and subsequent measurement M are completely described by the probability distribution of measurement outcomes $\mu_P^M = \left\{\mu_P^M(i)\right\}$. Notice that the correspondence $P \to \mu_P^M$ has the characteristic property of *affinity*: if the state P is mixture of states P_1 and P_2 in the proportion $p : (1 - p)$ then the probability distribution μ_P^M is mixture of the probability distributions $\mu_{P_1}^M$ and $\mu_{P_2}^M$ in the same proportion, $\mu_{pP_1+(1-p)P_2}^M = p\mu_{P_1}^M + (1 - p)\mu_{P_2}^M$.

2.3. The main features of statistical description

Having in mind the passage to quantum mechanics, let us try to formulate axiomatically the main features of statistical description of an experiment without using assumption of classicality, *i.e.* without introducing a phase space.

Axiom 2.1. Let there be given a set $\mathfrak{S}$ whose elements are called states and a set $\mathfrak{M}$ whose elements are called measurements. With a measurement $M \in \mathfrak{M}$ is associated the space U of its possible outcomes. For any

pair $S \in \mathfrak{S}$, $M \in \mathfrak{M}$ there is a probability distribution μ_S^M on U called the probability distribution of the measurement M in the state S.

Intuitively S represents a more or less detailed description of the preparation of a "statistical ensemble" and M – a measurement in this ensemble. An attempt to interpret these notions by introducing a phase space leads in fact to "hidden variables", but we shall discuss this later. Here S and M are considered as primary notions. For any (measurable) subset $B \subset U$ the quantity $\mu_S^M(B)$ is interpreted as theoretical value for the fraction of representatives of the ensemble prepared in the state S for which the outcome of the measurement M lies in the subset B.

The first axiom thus formalizes the requirement of reproducibility of the individual experiments and the stability of frequencies. The following axiom says that mixing of ensembles is an admissible way of state preparation.

Axiom 2.2. For arbitrary states S_1, S_2 and an arbitrary number $p, 0 < p < 1$, there exists the state S called mixture of the states S_1 and S_2 in the proportion $p : (1 - p)$ such that $\mu_S^M = p\mu_{S_1}^M + (1 - p)\mu_{S_2}^M$ for all measurements $M \in \mathfrak{M}$.

An outcome of a measurement can be the data obtained from one or several measuring devices as well as any other method of representing the information – for example, a picture on a computer display. Quite often the information obtained as a result of a measurement should be processed in this or another way. The result of such a processing can be regarded as an outcome of a complex measuring procedure including the given transformation. If M_1 is a measurement with the values in U_1 and M_2 – a measurement with the values in U_2, such that there is a (measurable) function $f : U_2 \to U_1$ satisfying the relation

$$\mu_S^{M_1}(B) = \mu_S^{M_2}(f^{-1}(B)); \quad B \subset U_1,$$

for all $S \in \mathfrak{S}$ then this means that the outcomes of the measurement M_1 are obtained from the outcomes of measurement M_2 by a functional transformation f. (Recall that $f^{-1}(B)$ denotes the inverse image of B *i.e.* the set of all such $u_2 \in U_2$, that $f(u_2) \in B$). In this case we say that M_1 is *subordinated* to the measurement M_2. If U_1 and U_2 are finite sets, this means that

$$\mu_S^{M_1}(u_1) = \sum_{u_2 : f(u_2) = u_1} \mu_S^{M_2}(u_2),$$

i.e. subordination means coarse-graining of the measurement outcomes.

Axiom 2.3. For any measurement M, the set $\mathfrak{M}$ contains all the measurements subordinated to M.

A pair of sets $(\mathfrak{S}, \mathfrak{M})$ satisfying the axioms 2.1 - 2.3 will be called *statistical model*. The statistical model is said to be *separated* if

Axiom 2.4. From $\mu_{S_1}^M = \mu_{S_2}^M$ for all $M \in \mathfrak{M}$ it follows that $S_1 = S_2$ and from $\mu_S^{M_1} = \mu_S^{M_2}$ for all $S \in \mathfrak{S}$ it follows that $M_1 = M_2$.

For a separated model both the operation of mixing in $\mathfrak{S}$ and the coarse-graining in $\mathfrak{M}$ are uniquely defined. Then the set of states $\mathfrak{S}$ obtains a convex structure while the set of measurements $\mathfrak{M}$ – a structure of partial order.

To illustrate the general notion of statistical model, let us come back to the classical picture where statistical states of an object are described by probability distributions on a phase space Ω so that $\mathfrak{S} = \mathfrak{S}(\Omega)$. If we consider measurements without errors, which correspond to random variables (see the relation (2.1)), then $\mathfrak{M}$ consists of the orthogonal resolutions of the identity on Ω. Then the probabilities of measurement outcomes in a given state are determined by the formula (2.3). If, on the other hand, we include measurements with random errors, then $\mathfrak{M}$ will consist of all, not only orthogonal, resolutions of the identity on Ω, and the probabilities of measurement outcomes will be determined by the formula (2.4). In this way arise the two basic classical models which differ in the structure of the set $\mathfrak{M}$. The first is the Kolmogorov model while the second may be called the Wald model. Both these models are separated.

2.4. Statistical model of quantum mechanics

In the previous section we have seen that quite general properties of statistical description lead to emergence of the two main mathematical structures: convexity (mixing) in the state space and subordination in the set of measurements. Importance of these structures in the context of quantum mechanics was stressed already in the book of von Neumann. Let us consider this in more detail.

In quantum mechanics to a system is related a complex Hilbert space $\mathcal{H}$. For simplicity we take it finite-dimensional. Then $\mathcal{H}$ consists of column vectors $|\psi\rangle = \begin{bmatrix} \psi_1 \\ \dots \\ \psi_n \end{bmatrix}$, where ψ_j are complex numbers. Denoting $\langle\psi| = \begin{bmatrix} \bar{\psi}_1 & \dots & \bar{\psi}_n \end{bmatrix}$ the Hermitian conjugate row-vector, we can write the inner product in $\mathcal{H}$ in the form $\langle\varphi|\,\psi\rangle$. Any ensemble of quantum systems defining a quantum state is described by a density matrix $\hat{S} = \left[s_{ij}\right]$ satisfying

$$\hat{S} \geq 0, \quad \mathrm{Tr}\hat{S} = 1.$$

The first relation means that the matrix $\hat{S}$ is Hermitian positive semidefinite while $\mathrm{Tr}\hat{S} = \sum_i s_{ii}$ denotes the trace of the matrix. If the ensem-

bles described by the density matrices $\hat{S}_1, \hat{S}_2$ are mixed in the proportion $p : (1-p)$ then the mixed ensemble is described by the matrix $p\hat{S}_1 + (1-p)\hat{S}_2$ which is again a density matrix. Thus the set $\mathfrak{S} = \mathfrak{S}(\mathcal{H})$ of all density matrices is convex, the convex combinations describing mixing of the ensembles. The extreme points of the convex set $\mathfrak{S}(\mathcal{H})$ – pure quantum states – are given by the density matrices of the form $\hat{S}_\psi = |\psi\rangle\langle\psi|$, with ψ a unit vector, $\langle\psi|\psi\rangle = 1$. It is difficult to describe the set $\mathfrak{S}(\mathcal{H})$ in pictorial way, except for the case $n = 2$, when it turns out to be isomorphic to the unit ball in the real 3-dimensional space. Any density matrix can be represented as a mixture of extreme points

$$\hat{S} = \sum_j p_j \hat{S}_{\psi_j}; \quad p_j \geq 0, \sum_j p_j = 1,$$

but contrary to the classical picture such a representation is highly nonunique.

A real-valued quantum observable is described by a Hermitian matrix $\hat{X} = \left[x_{ij}\right]$ with its eigenvalues x_j as possible outcomes. Let us write the spectral decomposition of the matrix $\hat{X}$:

$$\hat{X} = \sum_{j=1}^{n} x_j \hat{E}_j, \tag{2.6}$$

where E_j is the orthogonal projector onto the subspace of eigenvectors corresponding to the eigenvalue x_j. The family $\hat{\boldsymbol{E}} = \{\hat{E}_j\}$ forms an orthogonal resolution of identity in $\mathcal{H}$:

$$\sum_{j=1}^{n} \hat{E}_j = \hat{I}; \quad \hat{E}_j \hat{E}_k = 0 \quad \text{if } j \neq k; \quad \hat{E}_j^2 = \hat{E}_j,$$

where $\hat{I}$ is the unit matrix. The space $\mathcal{H}$ is then decomposed into the direct orthogonal sum of the subspaces $\hat{E}_j(\mathcal{H})$.

According to the rules of linear algebra, one has for a real function f

$$f(\hat{X}) = \sum_{j=1}^{n} f(x_j)\hat{E}_j. \tag{2.7}$$

The numbers $f(x_j)$ are not all necessarily different, so that some terms in (2.7) may be joined in groups. Similarly to the classical picture, it is convenient from the beginning to consider representations of an observable in a form (2.6) where x_j are not necessarily different. Then the resolution of the identity $\boldsymbol{E}$ may be defined not uniquely and can be more or

less "detailed". The spectral representation gives the unique least detailed resolution of the identity. By making more and more detailed decompositions we finally arrive to a "maximal" decomposition which is defined by a basis of eigenvectors of $\hat{X}$. Such a maximal resolution of the identity will be nonunique if $\hat{X}$ has eigenvalues of multiplicity greater than 1.

By the statistical postulate, measurement of the observable $\hat{X}$ in the state $\hat{S}$ gives the outcome x_j with the probability

$$\mu_{\hat{S}}^{\hat{E}}(x_j) = \mathrm{Tr}\hat{S}\hat{E}_j. \tag{2.8}$$

This and (2.6) imply that the mean value of $\hat{X}$ in the state $\hat{S}$ is equal to $\mathrm{Tr}\hat{S}\hat{X}$. For a pure state $\hat{S}_\psi$ this is equal to $\langle\psi|\hat{X}\psi\rangle$.

Thus the standard form of quantum mechanics is described by the statistical model $(\mathfrak{S}(\mathcal{H}), \mathfrak{M}(\mathcal{H}))$, where $\mathfrak{M}(\mathcal{H})$ is the collection of all resolutions of the identity in $\mathcal{H}$ corresponding to measurements of quantum observables. The reader have probably noticed the analogy between the relations (2.1), (2.2), (2.3) of the Kolmogorov model and the relations (2.6), (2.7), (2.8). It is also natural to consider the quantum analog of the Wald model in which the measurements are described by arbitrary (non-orthogonal) resolutions of the identity in $\mathcal{H}$, *i.e.* the families of matrices $\hat{\boldsymbol{M}} = \{\hat{M}_j\}$ satisfying

$$\sum_j \hat{M}_j = \hat{I}, \quad \hat{M}_j \geq 0.$$

While the mathematical apparatus of quantum mechanics contained all the necessary prerequisites for that, the role of non-orthogonal resolutions of the identity was recognized only in 1970-s with the advances in the statistical approach. Formally this extension of the notion of quantum observable is similar to introduction of randomized procedures in the classical picture. From this point of view the orthogonal resolutions of the identity are similar to the classical deterministic procedures. However this analogy is not complete in one crucial respect: while in the classical picture the measurement procedures that are optimal in the sense of the ultimate accuracy and maximal informativity are usually deterministic, in quantum mechanics the statistically and informationally optimal measurement procedures are often described by non-orthogonal resolutions of the identity. As it was shown in Section 2, a classical randomized measurement can be reduced to observation over a composite system including both the object and the generator of random numbers. It is intuitively clear that such a procedure cannot bring more information about the state of the classical object than direct observation of the system. However in quantum statistics, paradoxically, the use of a "quantum roulette" allows in several cases to improve the data concerning the

state of the system. The profound reason for this is the "nonseparability" of quantum-mechanical description and the specific "entanglement" between the parts of a composite system to be discussed below. It is noteworthy that the recognition of this remarkable fact was stimulated by formulation and solution of applied problems in quantum communication theory (see Chapters 3, 4 of the main text).

Since the main objective of our further discussion will be the problem of hidden variables which arose in the standard framework of quantum mechanics, we shall proceed within this framework. However the main conclusions apply also to the extended statistical model which uses arbitrary resolutions of the identity for the description of quantum measurements.

2.5. Compatibility and complementarity

Subordination of observables defines partial order in the set of measurements $\mathfrak{M}$ which has a direct statistical meaning: if M_1 is subordinated to M_2, then M_2 is a more detailed, informative measurement than M_1. If M_1 and M_2 are mutually subordinated then they are *equivalent* from the point of view of statistical information. An example is given by measurements performed with one device with differently calibrated scales U_1, U_2.

Clearly, most important are the maximal elements of the set $\mathfrak{M}$ which describe the most informative measurements admitted in the given statistical model. Mathematically, existence of such maximal measurements may be a nontrivial problem. We simply assume that every measurement in $\mathfrak{M}$ is subordinated to a maximal one. In general, there can be many inequivalent maximal measurements. Uniqueness of the maximal measurement up to equivalence turns out to be a characteristic property of the classical model.

To explain this we introduce the important definition: the measurements M_1, M_2 are called *compatible* if they are both subordinated to some measurement M. In other words, outcomes of compatible measurements can be obtained as a result of post-processings in a single-measurement experiment M. Assume now that a separable statistical model $(\mathfrak{S}, \mathfrak{M})$ is such that all the measurements are compatible. Then the maximal measurement M^* is unique up to equivalence. Let $M^* : S \to \mu_S^*$ be a representative and let Ω be the set of its outcomes. Since every measurement M is subordinated to M^*, there is a function $f_M : \Omega \to U$, where U is the set of outcomes for M such that

$$\mu_S^M(B) = \mu_S^*(f_M^{-1}(B)), \quad B \subseteq U.$$

Due to assumed separatedness of the model the map $S \to \mu_S^*$ is one-to-one affine correspondence between the set $\mathfrak{S}$ and a convex subset of the

simplex $\mathfrak{S}(\Omega)$, so that states can be identified with probability distributions on the set Ω and μ_S^M is the distribution of the observable f_M in the classical state μ_S^*.

Thus a separated statistical model in which all the measurements are compatible essentially reduces to the classical picture, the role of the phase space is played by the set of outcomes of the maximal measurement. In this connection it is useful to observe that Kolmogorov's extension theorem [78] may be interpreted as a statement about existence of the maximal measurement for an infinite projective family of mutually compatible measurements.

On the other hand, if $\mathfrak{M}$ contains incompatible measurements then the maximal measurement cannot be unique. Let us return to the model of quantum mechanics. Let $\hat{\boldsymbol{E}}$ and $\hat{\boldsymbol{F}}$ be the measurements described by orthogonal resolutions of the identity $\{\hat{E}_j\}, \{\hat{F}_j\}$ which commute in the sense that

$$\hat{E}_j \hat{F}_k = \hat{F}_k \hat{E}_j \quad \text{for all} \quad j, k. \tag{2.9}$$

Then the relation $\hat{G}_{jk} = \hat{E}_j \hat{F}_k$ defines a measurement with respect to which $\hat{\boldsymbol{E}}$ and $\hat{\boldsymbol{F}}$ are subordinated since $\hat{E}_j = \sum_k \hat{G}_{jk}, \quad \hat{F}_k = \sum_j \hat{G}_{jk}$. The condition (2.9) is not only sufficient but also necessary for the compatibility of $\hat{\boldsymbol{E}}$ and $\hat{\boldsymbol{F}}$. Observables $\hat{X}$ and $\hat{Y}$ are called *compatible* if there exist compatible measurements for them and this turns out to be equivalent to the condition $\hat{X}\hat{Y} = \hat{Y}\hat{X}$. In this way one derives the definition of compatibility from the standard formulation of quantum mechanics.

Since there are many incompatible observables described by noncommuting matrices, there are many inequivalent maximal measurements defined by different orthonormal bases in the space $\mathcal{H}$. These are the orthogonal resolutions of the identity which can not be further split into more detailed ones. In the infinite dimensional space the situation is more complicated since "continuous" maximal orthogonal resolutions of the identity appear. Notice however, that even in finite dimensional case there exist continuous maximal nonorthogonal resolutions of the identity. They arise as "overcomplete" systems of vectors obtained by projecting to $\mathcal{H}$ of the maximal orthogonal resolutions of the identity in some larger Hilbert space $\mathcal{H}'$. A typical example of an overcomplete system is the system of coherent vectors well known in quantum optics. In many cases the maximal information about quantum state is carried by the measurements described by such overcomplete systems (see Chapters 3, 4 of the main text).

There exist physical systems which are in a sense intermediate between classical and quantum such as systems with superselection rules. Let us call *center* of the statistical model $(\mathfrak{S}, \mathfrak{M})$ the collection of all measure-

ments compatible with all measurements in $\mathfrak{M}$. With the center one can associate its spectrum reflecting the classical properties of the statistical model. For a classical model the center coincides with the phase space. If the center is trivial *i.e.* consists only of the constants, as it is the case for quantum mechanics, then the model is *irreducible*. In general one can expect a decomposition of rather arbitrary statistical model into direct sum or integral of irreducible models. Such a structural theory is completely elaborated in the framework of algebraic approach and there are its generalizations to general convex sets of states [4].

Understanding the phenomenon of incompatibility was a decisive step in the creation of the "orthodox" interpretation of quantum mechanics. Incompatibility of measurements in quantum mechanics stems from the fact that physical measurements are performed with the macroscopic experimental setups, each one assuming complex specific organization of the space-time environment. Apparently two different ways of such organization can be mutually exclusive. "In quantum physics evidence about atomic objects obtained by different experimental arrangements exhibits a novel kind of peculiar complementary relationship" [155, page 4]. The classical mechanics relies upon idealization admitting theoretical compatibility of all measurement procedures which is justified in so far as it deals with macroscopic objects the interactions of which with the measurement devices can be as weak as one desires.

The surrounding world which is accessible to direct human's perception is macroscopic "by the definition". Therefore is it so difficult to give an accessible image for complementarity. However Niels Bohr advocated the idea that the principle of complementarity is quite general and especially important for biology and for subtle humanitarian relations which are so difficult to model mathematically. "The integrity of living organisms and the characteristics of conscious individuals and human cultures present features of wholeness, the account of which implies a typical complementary mode of description" [155, page 7]. Nevertheless so far quantum theory remains a unique case of a mathematical theory of a segment of the reality which, in its domain, gives an exact quantitative expression of the dialectic principle of complementarity.

2.6. Classical and nonclassical models

Consider statistical model $(\mathfrak{S}, \mathfrak{M})$ which is "classical" in the sense that the states and the measurements are described in terms of some phase space as in Section 2. However we do not require that the state space $\mathfrak{S}$ contains all the probability distributions as well as the set $\mathfrak{M}$ – measurements of all observables; so we admit that $\mathfrak{S}$ and $\mathfrak{M}$ can be defined by

some apriori restrictions (like those arising from complementarity). For this reason the model $(\mathfrak{S}, \mathfrak{M})$ need not be separated: there can be different probability distributions P_1, P_2 such that $\mu^M_{P_1} = \mu^M_{P_2}$ for all $M \in \mathfrak{M}$, *i.e.* indistinguishable from the point of view of all the measurement statistics. Similarly, there can be indistinguishable measurements. By identifying such indistinguishable states and measurements into equivalence classes, we obtain the new, already separated, statistical model.

The description of states and measurements in this new model is "compressed" just to the extent sufficient to preserve the measurements statistics in the initial classical model. As we have seen in Chapter 1, "gluing" the classical states geometrically amounts to projection of the state space $\mathfrak{S}$ onto a subset $\hat{\mathfrak{S}}$ in less dimensions. Then even if the initial model admits all possible probability distributions on Ω so that $\mathfrak{S}$ is just the simplex $\mathfrak{S}(\Omega)$, its projection $\hat{\mathfrak{S}}$ in the compressed model can be rather arbitrary convex set. Its form is determined by the set $\mathfrak{M}$, *i.e.* by the restrictions upon the classical measurements.

In statistical mechanics an important role is played by reduced description related to the probabilistic notion of partial observability. Assume that in the whole variety of variables related to a classical object with the phase space Ω one can observe only the random variables $X_1, \ldots, X_n$, as well as measurable functions of them. In this case $\mathfrak{M}$ consists of the measurements of observables of the form $f(X_1, \ldots, X_n)$. Then the classical states P_1, P_2 are indistinguishable if the corresponding expectations coincide: $\mathsf{M}_{P_1} f(X_1, \ldots, X_n) = \mathsf{M}_{P_2} f(X_1, \ldots, X_n)$ for all measurable functions f (this means that the restrictions of P_1, P_2 onto $\sigma-$subalgebra $\mathcal{B}$ generated by $X_1, \ldots, X_n$ coincide). Then the equivalence classes are represented by probability distributions $P(dx_1 \ldots dx_n)$ on the space $\hat{\Omega}$ of the values of the variables $X_1, \ldots, X_n$, which can thus be taken for the phase space of the reduced description. One has $\hat{\mathfrak{S}} = \mathfrak{S}(\hat{\Omega})$ *i.e.* the simplex $\mathfrak{S}(\Omega)$ is projected onto the simplex $\mathfrak{S}(\hat{\Omega})$ and the classical nature of the description is preserved. In this case the simplex is projected "along its faces" and the geometrical picture is preserved. Thus the reduced description is an important particular case of the state space compression.

Just to present a different picture consider the following modification of partial observability: assume that one can observe only the variables of the form $f_1(X_1), \ldots, f_n(X_n)$, where $f_1, \ldots, f_n$ are arbitrary functions. Then the classical states P_1, P_2 will be indistinguishable if

$$\mathsf{M}_{P_1} f_i(X_i) = \mathsf{M}_{P_2} f_i(X_i)$$

for all measurable functions $f_i; i = 1, \ldots, n$ (*i.e.* the restrictions of P_1, P_2 onto $\sigma-$subalgebras $\mathcal{B}_1, \ldots, \mathcal{B}_n$ generated by corresponding random variables $X_1, \ldots, X_n$ coincide). Then the equivalence classes are

represented by collections $\hat{P} = (P_1(dx_1), \dots P_n(dx_n))$ of probability distributions on the spaces of values of the variables $X_1, \dots, X_n$. In that case $\hat{\mathfrak{S}}$ is a direct product of n simplexes; in particular, if $X_1, \dots, X_n$ are two-valued then $\hat{P}$ runs through $n-$dimensional hypercube. Thus introducing apriori restrictions can radically change the convex structure of the compressed state space.

As we have seen in Chapter 1, for any sufficiently regular separated statistical model $(\hat{\mathfrak{S}}, \hat{\mathfrak{M}})$ there is a classical model for which $(\hat{\mathfrak{S}}, \hat{\mathfrak{M}})$ is the compression in the sense described above. Does this mean a possibility of introducing hidden variables, in particular, for quantum mechanics? To answer this question we should analyze the requirements which has to be satisfied by a hidden variable model. Such requirements can be conventionally split into two classes. The first class is constituted by the "minimal" requirements which refer only to the general properties of statistical description for a single system. These will be considered in the first place since almost all attempts of mathematical proofs of nonexistence pretended to appeal only to such general requirements. We shall see that in fact they were based on additional assumptions lacking physical motivation. Moreover, we shall demonstrate the classical description for a single quantum system satisfying all the general requirements of statistical description.

Another class is constituted by requirements which refer to composite quantum systems and it is these requirements that present the main obstacle for a hidden variable theory.

3. The problem of hidden variables

3.1. "No-go" proofs and the minimal statistical requirements on hidden variables

A hidden variable theory pretends to give an explanation of the randomness of the experimental results through fluctuations of the values of certain variables ω describing the "real" properties of the object. Therefore attempts of the "no-go" proofs usually started with associating probability distributions $S(d\omega)$ with quantum states $\hat{S}$ and random variables $X(\omega)$ with quantum observables $\hat{X}$, where $\omega \in \Omega$ – a hypothetical phase space. Thus it was usually assumed that there are one-to-one correspondences $\hat{S} \leftrightarrow S$ and $\hat{X} \leftrightarrow X$.

However the discussion of the compressed statistical description in the previous section leads to an idea that a hidden variable theory should admit "gluing" statistically equivalent states and observables. Therefore we accept from the beginning that a classical description of a quantum system consists of the phase space Ω and a pair of mappings: $S \to \hat{S}$ from

the simplex $\mathfrak{S}(\Omega)$ onto the set of quantum states $\hat{\mathfrak{S}}$ and $X \to \hat{X}$ from the set of classical observables $\mathfrak{O}(\Omega)$ onto the set of quantum observables $\hat{\mathfrak{O}}$.

The domain of definition of the first mapping need not be the whole $\mathfrak{S}(\Omega)$ as well as that of the other – the whole $\mathfrak{O}(\Omega)$. Thus the classical description brings into correspondence some classical states $S(d\omega)$ with the density operators $\hat{S}$ and some classical observables $X(\omega)$ with Hermitian operators $\hat{X}$ in the Hilbert space $\mathcal{H}$. One and the same quantum state $\hat{S}$ can be described by several different probability distributions $S(d\omega)$ interpreted as different ways of preparation of the quantum ensemble $\hat{S}$, and one and the same quantum observable $\hat{X}$ can be described by different functions $X(\omega)$ interpreted as different ways of observation of $\hat{X}$. To denote possible non-uniqueness of the classical description Bell used the term "contextuality". The contextuality in the description of a quantum state displays in particular in the fact that one and the same density operator $\hat{S}$, depending on the context of a preparation procedure, can be obtained as completely different mixtures of pure states. Similarly, one and the same projector $\hat{P}$, depending on the context of a measurement procedure, can arise as a result of coarse-graining from different orthonormal bases.

Now let us consider requirements which where imposed on hidden variable theories. In view of the preceding discussion the following assumptions are the restrictions to be stated explicitly:

(S.0) *the mapping* $S \to \hat{S}$ *is one-to-one*;

(X.0) *the mapping* $X \to \hat{X}$ *is one-to-one.*

These assumptions which may seem "technical" from a mathematical viewpoint are in fact of crucial importance. At the same time their physical motivation may be questioned.

The requirement that all the statistical predictions of the hidden variable theory coincide in all respects with the quantum mechanical ones can be expressed as equality between the expectations:

(E.1) $\mathrm{Tr}\hat{S}\hat{X} = \int_\Omega S(d\omega)X(\omega)$ for all X, S.

If this requirement does not hold, then there is a disagreement between statistical predictions of the theories which in principle can be detected experimentally. This would lead to physical questions which are beyond the scope of the present mathematical consideration.

The next group of requirements concerns the properties of the mapping $X \to \hat{X}$. The relation (2.7) shows that the outcomes of observable $f(\hat{X})$ can be obtained from the outcomes of $\hat{X}$ by mere computation $x_i \to f(x_i)$. But in a hidden variable theory a measurement of

$\hat{X}$ is reduced to observation of some $X(\omega)$. The subsequent computation $x \to f(x)$ is equivalent to a direct observation of $f(X(\omega))$. Thus in the hidden variable theory observation of $f(X)$ represents a way of measuring the quantum observable $f(\hat{X})$. This motivates the following functional condition:

(X.1) *if* $X \to \hat{X}$ *then* $f(X) \to f(\hat{X})$.

This is closely related to the following two conditions:

(X.2) *if* $X \to \hat{X}$ *then any value* $X(\omega)$ *belongs to the spectrum* $\{x_i\}$ *of quantum observable* $\hat{X}$.

The meaning of this *spectral condition* is clear: a hidden variable description should preserve "objective values" of observables.

(X.3) *for any pair of compatible quantum observables* $\hat{X}, \hat{Y}$ *there exist the corresponding classical observables* X, Y *such that* $X \to \hat{X}, Y \to \hat{Y}$ *and* $X + Y \to \hat{X} + \hat{Y}$.

Compatibility of quantum observables means that there is a measuring device which produces the outcomes of measurement of both $\hat{X}$ and $\hat{Y}$. Complementing this with a summator we obtain a device for measuring $\hat{X} + \hat{Y}$. The *finite sum rule* (X.3) reflects this possibility in the hidden variable theory. It can be replaced with the following *finite product rule*:

(X.4) *for any pair of compatible quantum observables* $\hat{X}, \hat{Y}$ *there exist the corresponding classical observables* X, Y *such that* $X \to \hat{X}, Y \to \hat{Y}$ *and* $XY \to \hat{X}\hat{Y}$.

Lemma 3.1. *The functional condition* (X.1) *implies the finite sum rule* (X.3) *and the finite product rule* (X.4). *Under* (X.0)*, the condition* (X.1) *implies also the spectral condition* (X.2).

Proof. If $\hat{X}, \hat{Y}$ are compatible then there exists $\hat{Z}$ such that $\hat{X} = f(\hat{Z}), \hat{Y} = g(\hat{Z})$. Let $Z \to \hat{Z}$ be the corresponding classical observable, then by (X.1) $X = f(Z) \to \hat{X}$ and $Y = g(Z) \to \hat{Y}$. Therefore $X + Y = (f + g)(Z) \to (f + g)(\hat{Z}) = \hat{X} + \hat{Y}$. This proves (X.3), and (X.4) is proved similarly.

Let the correspondence $X \to \hat{X}$ be one-to-one so that we can write $X \leftrightarrow \hat{X}$. Consider the function $f_0(x) \equiv 0$. Applying (X.1) to this function, we have $0 \leftrightarrow \hat{0}$, where $\hat{0}$ is the zero operator and 0 is the classical observable identically equal to zero. Let $P(x)$ be the characteristic polynomial of an Hermitian operator $\hat{X}$, so that $P(\hat{X}) = \hat{0}$. If $X \leftrightarrow \hat{X}$ then by (X.1) $P(X) \leftrightarrow P(\hat{X})$ hence $P(X(\omega)) \equiv 0$ *i.e.* any value $X(\omega)$ belongs to the spectrum of quantum observable $\hat{X}$. □

The conditions (E.1) and (X.1) imply the following amplification of the property of statistical correspondence:

(E.2) $\mathrm{Tr}\hat{S}f(\hat{X}) = \int_\Omega S(d\omega)f(X(\omega))$ for all X, S, f.

Notice that while (E.2) trivially implies (E.1), the condition (X.1) follows only under some additional assumption. A subset $\mathfrak{S}_0 \subseteq \mathfrak{S}(\Omega)$ will be called *separating* if the equality

$$\int_\Omega S(d\omega)X_1(\omega) = \int_\Omega S(d\omega)X_2(\omega) \quad \text{for all } S \in \mathfrak{S}_0$$

implies $X_1 = X_2$. If the set of probability distributions which correspond to all possible quantum states in the given classical description (let us denote it $\mathfrak{S}_0$) is separating then the condition (E.1) implies (X.0) and (E.2) implies (X.1). Indeed, let $Y \to f(\hat{X})$ then by (E.2)

$$\int_\Omega S(d\omega)Y(\omega) = \mathrm{Tr}\hat{S}f(\hat{X}) = \int_\Omega S(d\omega)f(X(\omega)) \quad \text{for all } S \in \mathfrak{S}_0$$

whence $f(X) = Y \to f(\hat{X})$.

Finally let us discuss the linearity condition:

(X.5) $\widehat{(\lambda X + \mu Y)} = \lambda\hat{X} + \mu\hat{Y}$ for all X, Y and real λ, μ .

This mathematically innocent condition is not justified physically as it was mentioned already in the von Neumann's book [138]. If $\hat{X}$ and $\hat{Y}$ are incompatible then measurements of observables $\hat{X}$,$\hat{Y}$ and $\hat{X}+\hat{Y}$ may have nothing in common apart from equality of the mean values. This last relation was used by von Neumann to motivate the condition (X.5). Let us analyze the corresponding argument as given in[2]: "In quantum mechanics the mean values satisfy the relation $\langle\hat{X}+\hat{Y}\rangle = \langle\hat{X}\rangle + \langle\hat{Y}\rangle$. Therefore in a hidden variable model it should hold

$$\int_\Omega S(d\omega)(X+Y)(\omega) = \int_\Omega S(d\omega)X(\omega) + \int_\Omega S(d\omega)Y(\omega). \qquad (3.10)$$

Since to different density matrices $\hat{S}$ correspond different probability distributions $S(d\omega)$, it is natural to require that

$$(X+Y)(\omega) = X(\omega) + Y(\omega) \qquad (3.11)$$

[2] A.I. Akhiezer, R.V. Polovin, Why it is impossible to introduce hidden variables in quantum mechanics? UFN 107(3) (1972) 463-487.

for arbitrary classical observables X, Y which may correspond to commuting as well as noncommuting operators $\hat{X}, \hat{Y}$."

However (3.11) follows from (3.10) only under additional assumption that the set of classical states $\mathfrak{S}_0$ in the hidden variable theory is separating. Is such an assumption "natural"? A very simple (and generic) example of nonseparating set is the subset of probability distributions $S(d\omega)$ on a product $\Omega' \times \Omega''$ which have the form $S(d\omega) = S'(d\omega')P(d\omega'')$, where $S'(d\omega')$ is arbitrary distribution on Ω' and $P(d\omega'')$ is a fixed distribution on Ω''. This is just the case in the explicit hidden variable models. The distribution $P(d\omega'')$ plays a role of the equilibrium state of the "hidden" subsystem which provides stochasticity to the measurement outcomes [10].

Under the condition of one-to-one correspondence (X.0) the conditions (X.1)-(X.5) turn into the requirements which were used in the "no-go" proofs for hidden variables. Let us give here the most significant results in this direction in the formulation adapted to the proposed general classification.

The first statement is close to what is called the von Neumann theorem.

Proposition 3.2. *There is no classical description satisfying the conditions* (X.0), (X.2), (X.5).

From the discussion above we see that this can be also formulated as follows: *there is no classical description with separating set of classical states, satisfying the condition* (E.2).

Proof. Assume that a classical description with the properties (X.0), (X.2), (X.5) exists. By (X.0) there is unique X such that $X \to \hat{X}$. Fix a point ω_0 of the phase space and consider the functional on operators given by $F(\hat{X}) = X(\omega_0)$. By (X.5) this functional is linear. In the finite dimensional case we are considering, it is almost obvious that every such functional has the form $F(\hat{X}) = \mathrm{Tr}\hat{P}\hat{X}$, where $\hat{P} \in \hat{\mathfrak{O}}$. Let $\hat{X}$ run over all possible projections in $\mathcal{H}$ so that its eigenvalues are $x_i = 0, 1$. Then by (X.2) the quantity $X(\omega_0) = \mathrm{Tr}\hat{P}\hat{X}$ assumes only the values 0, 1 which is apparently impossible for any choice of $\hat{P}$. □

Of historical interest is the following remark of Wigner to his article on hidden variables [198]: "The discussion of Von Neumann, most commonly quoted, is that contained in his book, Sections 4.1 and 4.2. As an old friend of Von Neumann, and in order to preserve historical accuracy, the present writer may be permitted the observation that the proof contained in this book was not the one which was principally responsible for Von Neumann's conviction of the inadequacy of hidden variable

theories. . . "[3] However independently of the meaning given to these arguments by von Neumann, published in his monographs they were considered as decisive argument against hidden variables for the years until 1950-s with the appearance of papers of Bohm (1952) and Wiener and Siegel (1953), which contained explicit (although not completely transparent) constructions of hidden variable models. On the other hand, several improvements of the von Neumann "no-go theorem" appeared. The situation was radically clarified with the works of Bell (1966) and also Kochen and Specker (see [11, 77]).

In the year 1957 Gleason [168] proved the following quite nontrivial theorem which gave an answer to a question of Mackey:

Let $F(\hat{E})$ be a probability measure on projections in the Hilbert space $\mathcal{H}$ of dimensionality ≥ 3, i.e. the real function satisfying the conditions: 1) $F(\hat{E}) \geq 0$; 2) *for any orthogonal resolution of the identity $\{\hat{E}_i\}$ it holds $\sum_i F(\hat{E}_i) = 1$. Then $F(\hat{E}) = \mathrm{Tr}\hat{S}\hat{E}$, where $\hat{S}$ is a density operator in $\mathcal{H}$.*

This implies

Proposition 3.3. *If* $\dim \mathcal{H} \geq 3$ *then there is no classical description satisfying the conditions* (X.0), (X.2), (X.3).

Indeed, if such a description exists then the function $F(\hat{E}) = E(\omega_0)$, where $E \leftrightarrow \hat{E}$, satisfies the conditions of Gleason's theorem. Then $F(\hat{E}) = \mathrm{Tr}\hat{S}\hat{E}$ and we come to the contradiction as in the proof of Proposition 1.

The proof of Gleason's theorem remains difficult even after a number of subsequent simplifications. Gleason himself did not apply his theorem to the hidden variable problem. This possibility was noticed by Bell. Moreover he extracted the geometrical idea behind Gleason's proof which is essential from the viewpoint of hidden variables issue and gave a short direct proof of the following statement which in our presentation follows from Proposition 3.3 and the Lemma 3.1:

Proposition 3.4. *If* $\dim \mathcal{H} \geq 3$ *then there is no classical description satisfying the conditions* (X.0) *and* (X.1).

Similar result was independently and in a quite different way obtained by Kochen and Specker [77] who gave an explicit construction of 117 (later more "economical" constructions were designed) unit vectors in three-dimensional space on which one cannot define a measure with the

[3] This unpublished argument of von Neumann will be considered in the next section.

properties 1), 2) from Gleason's theorem and assuming only the values 0 or 1. In other words, the geometry of this system is such that the conditions defining such a measure become self-contradictory. Later it was observed that the proof of Bell in fact amounts to construction of a certain system of 13 vectors [10].

The case $\dim \mathcal{H} = 2$ corresponding to spin-1/2 particle is special. In the papers of Bell and of Kochen-Specker explicit hidden variable constructions were given for this case. One more construction follows from the general model which will be given in the next section. These constructions clearly demonstrate insufficiency of the "no-go" proofs of the type of Proposition 3.2.

Notice that the Propositions 3.3 and 3.4 do not use at all the condition of statistical correspondence (E.1) which in fact should be central for a hidden variable theory. These results do not concern statistics and speak only of impossibility of the one-to-one correspondence between the "quantum logic" of projections and the Boolean algebra of classical events preserving the algebraic relations between the compatible variables. The key condition here is (X.0) as it was understood in the work of Bell.

These mathematical results do not contradict explicit hidden variable models such as Bohm's model and Wiener-Siegel model: the assumption which is not fulfilled in these models is the condition (X.0) of one-to-one correspondence. We already noticed that a reasonable assumption would be to admit that one and the same quantum observable can be measured in many different ways (which in particular is reflected in possible nonuniqueness of the resolution of the identity entering the representation (2.6)). Propositions 3.3, 3.4 imply that in a hidden variable theory satisfying functional conditions of the type (X.1)-(X.4) such a nonuniqueness is unavoidable.

A different approach to "no-go" proofs is related to the ideas of Wigner and Blokhintsev on the impossibility of defining a joint distribution for incompatible quantum observables and is based on the analysis of the properties of the mapping $S \to \hat{S}$.

Consider the classical state S and the corresponding quantum state $\hat{S}$. Then the classical ensemble defined by the probability distribution $S(d\omega)$ replaces the quantum ensemble representing $\hat{S}$. The mixture $pS_1 + (1 - p)S_2$ of the classical ensembles represents the quantum ensemble $p\hat{S}_1 + (1 - p)\hat{S}_2$. This leads to the following affinity condition:

(S.1) *for* $S_1 \to \hat{S}_1$, $S_2 \to \hat{S}_2$ *and a real* p, *satisfying* $0 < p < 1$ *it holds* $pS_1 + (1 - p)S_2 \to p\hat{S}_1 + (1 - p)\hat{S}_2$.

Proposition 3.5. *There is no classical description satisfying the conditions* (E.2), (S.0), (S.1).

Proof. Assume that such a description exists and consider two arbitrary quantum observables $\hat{X}, \hat{Y}$. Let X, Y be some corresponding classical observables. Due to the condition (S.0), there is one-to-one affine correspondence $S \leftrightarrow \hat{S}$ and the formula

$$\mu_{\hat{S}}(B) = S(\omega : (X(\omega), Y(\omega)) \in B)$$

defines a probability distribution on the plane $\mathbb{R}^2$. Due to (S.1) the mapping $\hat{S} \to \mu_{\hat{S}}$ is affine and hence defines a generalized quantum measurement with outcomes (x, y). By using the condition (E.2) we obtain that distributions of observables $\hat{X}, \hat{Y}$ are marginals of the distribution $\mu_{\hat{S}}(dx, dy)$. Hence we come to an absurd conclusion that any two quantum observables $\hat{X}, \hat{Y}$ are compatible. □

This argument follows essentially the paper of Srinivas [195], where it is stressed that the key condition here is (S.0). Any quantum state $\hat{S}$ can be represented in many different ways as the mixture of pure states. If a hidden variable theory satisfying the conditions (E.2), (S.1) is possible then it should necessarily admit such a nonuniqueness. In this respect the condition (S.0) is similar to (X.0).

Thus among the requirements which were proposed for hidden variable theories one should distinguish (E.1), (X.1), (S.1) (as well as the related conditions (E.2), (X.2)-(X.4)) which have sound statistical motivation. Essentially, these conditions require that the classical description should preserve the main structural properties of statistical model expressed by the axioms (A.1)-(A.3). The "no-go" theorems in fact do not forbid such theories. A mathematically and physically motivated conclusion is that a classical description satisfying these requirements should have necessarily non-unique nature admitting possibility of compression in transition to quantum theory.

3.2. A hidden variable model for a single quantum system

In the work of Kochen and Specker [77] the authors gave a "trivial" hidden variable model satisfying the statistical condition (E.1) but not preserving the structure of functional dependences in quantum mechanics. The idea was straightforward and amounts to introducing a personal "hidden variable" for each observable $\hat{X}$ producing stochasticity in outcomes of measurement of $\hat{X}$. The totality of all such hidden variables is then the phase space variable ω of the model. The purpose of this construction was to demonstrate insufficiency of the statistical condition (E.1) and

the power of the functional condition (X.1). However we will show that basing on a similar idea one can construct a hidden variable model satisfying all the minimal requirements (E.1), (X.1), (S.1). The key feature is of course the non-uniqueness of the classical description.

Denote by Ω' the set of all pure quantum states, so that for any $\omega' \in \Omega'$ there is a unit vector $\psi_{\omega'} \in \mathcal{H}$ such that $\mathrm{Tr}\omega'\hat{X} = \langle\psi_{\omega'}|\hat{X}\psi_{\omega'}\rangle$. Every quantum state can be written as a mixture of pure states

$$\hat{S} = \int_{\Omega'} \omega' S'(d\omega'), \tag{3.12}$$

where $S'(d\omega')$ is a probability distribution on Ω', and this representation is of course not unique. The relation (3.12) defines affine mapping $S' \to \hat{S}$ of the simplex $\mathfrak{S}(\Omega')$ onto the convex set $\hat{\mathfrak{S}}$.

Let $\hat{\boldsymbol{E}} = \{\hat{E}_i\}$ be an orthogonal resolution of the identity in $\mathcal{H}$ describing a quantum measurement. It suffices to restrict to the maximal measurements for which $\hat{E}_i$ are projections onto the vectors e_i of an orthonormal basis. Put

$$M_i(\omega') = \mathrm{Tr}\omega'\hat{E}_i = |\langle\psi_{\omega'}|e_i\rangle|^2, \tag{3.13}$$

then $\hat{\boldsymbol{M}} = \{M_i(\omega')\}$ will be a classical randomized measurement on Ω'. The relations (3.12), (3.13) imply

$$\mathrm{Tr}\hat{S}\hat{E}_i = \int_{\Omega'} S'(d\omega')M_i(\omega'). \tag{3.14}$$

Thus we succeeded to build a classical model in which states are described by probability distributions on the "phase space" Ω' and the (maximal) quantum measurements – by randomized classical measurements M, with the statistical correspondence condition (3.14) fulfilled. The quantum theory is the compressed description of this model in the sense of Section 2.6 (*cf.* also Theorem 1.7.1).

The next step is to realize every classical randomized measurement (3.13) with the help of randomizing probability space $\left(\Lambda_{\hat{\boldsymbol{E}}}, d\lambda_{\hat{\boldsymbol{E}}}\right)$ for example as it is described at the end of Section 2.2. One can say that to every maximal quantum measurement corresponds a "roulette" $\left(\Lambda_{\hat{\boldsymbol{E}}}, d\lambda_{\hat{\boldsymbol{E}}}\right)$ which models stochasticity of the measurement outcomes in any pure state ω'. If the state $\hat{S}$ is mixed, then the probability distribution of the measurement outcomes is given by the corresponding mixture

$$\mathrm{Tr}\hat{S}\hat{E}_i = \int_{\Omega'}\int_{\Lambda_{\hat{\boldsymbol{E}}}} S'(d\omega')d\lambda_{\hat{\boldsymbol{E}}}E_i\left(\omega', \lambda_{\hat{\boldsymbol{E}}}\right), \tag{3.15}$$

as it is seen from (3.14) and (2.5). To embrace the totality of the maximal quantum measurements, let us introduce the product of probability spaces

$\left(\Omega'', P''\right) = \prod_{\hat{E}} \left(\Lambda_{\hat{E}}, d\lambda_{\hat{E}}\right)$. Thus every $\omega'' \in \Omega''$ is a collection $\omega'' = \prod_{\hat{E}} \lambda_{\hat{E}}$ of independent "roulettes" corresponding to all possible maximal quantum measurements so that $P(d\omega'') = \prod_{\hat{E}} d\lambda_{\hat{E}}$.

Now define the phase space of the sought classical description as $\Omega = \Omega' \times \Omega''$, so that $\omega = \left(\omega', \omega''\right)$. The classical states will be given by the probability distributions on Ω having the form $S(d\omega) = S'(d\omega')P(d\omega'')$, where S' is an arbitrary probability distribution on Ω'. The mapping $S \to \hat{S}$ is defined as

$$\hat{S} = \int_{\Omega} \omega' S'(d\omega) = \int_{\Omega'} \omega' S'(d\omega'). \tag{3.16}$$

To every maximal quantum measurement $\hat{\boldsymbol{E}}$ corresponds uniquely the deterministic classical measurement $\boldsymbol{E} = \{E_i(\omega)\}$, where $E_i(\omega) = E_i\left(\omega', \lambda_{\hat{E}}\right)$, and $\lambda_{\hat{E}} = \pi_{\hat{E}}(\omega'')$ is the coordinate projection of the point ω''. From (3.15) it follows that

$$\mathrm{Tr}\hat{S}\hat{E}_i = \int_{\Omega} S(d\omega)E_i(\omega). \tag{3.17}$$

It remains to establish correspondence between quantum observables and random variables on Ω. Let $\hat{X}$ be a quantum observable and $\hat{\boldsymbol{E}} = \{\hat{E}_i\}$ - one of the corresponding *maximal* measurements, so that $\hat{X} = \sum_i x_i \hat{E}_i$. Let $\boldsymbol{E} = \{E_i(\omega)\}$ be the corresponding classical deterministic measurement. Consider the random variable $X(\omega) = \sum_i x_i E_i(\omega)$. Since $\hat{\boldsymbol{E}}$ is in general nonunique, we obtain a collection of random variables $X(\omega) \to \hat{X}$ corresponding to different $\hat{\boldsymbol{E}}$. It is important that $X(\omega)$ depends on ω'' only via the coordinate projection $\lambda_{\hat{E}} = \pi_{\hat{E}}(\omega'')$. If $X(\omega) \neq \text{const}$ this allows to reconstruct $\boldsymbol{E}$ uniquely given $X(\omega)$. The values x_i are also reconstructed by X. Then from $\boldsymbol{E}$ one recovers $\hat{\boldsymbol{E}}$ and hence $\hat{X} = \sum_i x_i \hat{E}_i$. Therefore the mapping $X \to \hat{X}$ is well defined for $X(\omega) \neq \text{const}$. In case $X(\omega) \equiv \lambda$ we have $x_i \equiv \lambda$, hence $\hat{X} = \lambda \hat{I}$ so the mapping is unambiguously defined also in this case.

From (3.17) it follows that the statistical correspondence (E.1) holds. The mapping $S \to \hat{S}$ is affine, so that (S.1) also holds. Let us check the functional condition (X.1). If X is a random variable, $X \to \hat{X}$, and f a function, then $f(X(\omega)) = \sum_i f(x_i)E_i(\omega)$. Without loss of generality we can assume both X and f nonconstant. Then $\hat{\boldsymbol{E}}$ and hence $\hat{X}$ is recovered from X uniquely, so that $\widehat{f(X)} = \sum_i f(x_i)\hat{E}_i = f(\hat{X})$.

Notice that in the case $\dim \mathcal{H} = 2$ every nontrivial resolution of the identity in $\mathcal{H}$ is maximal, therefore our construction in this case satisfies even the uniqueness condition (X.0). This explains the restriction $\dim \mathcal{H} \geq 3$ in Propositions 3.3, 3.4.

In brief, the proposed classical description is constructed so as to make distinguishable all the mixtures giving the same quantum state as well as all the measurements giving the same quantum observable. The correspondence $S \to \hat{S}$ is one-to-one only for the pure states $\hat{S}$ and the correspondence $X \to \hat{X}$ – only for the maximal observables with simple spectrum.

Certainly this model does not pretend to replace the mathematical apparatus of quantum mechanics. It is also extremely wasteful from the viewpoint of representing the measurement statistics as it contains a lot of irrelevant "detail". However it is relevant in making explicit the structural properties of quantum theory which can be preserved in classical description and in showing that "no-go" proof for hidden variables cannot be based solely on the minimal statistical requirements (E.1), (S.1), (X.1). Complementarity does not prevent from a classical description as soon as the classical-quantum correspondence is not required to be one-to-one. Similar conclusions hold for rather arbitrary separated statistical model since the minimal statistical requirements concern only the common properties of quantum mechanical description. Thus consideration of the hidden variable hypothesis should involve more specific properties of the quantum mechanical description.

3.3. Hidden variables and evolutions of quantum system

Here we discuss the question: can a classical description reproduce temporal quantum evolutions including a) the reversible dynamics as determined by the Schrödinger equation; b) state changes (reductions) due to repeated quantum measurements.

The quantum dynamics of a single system is translated into the classical description proposed in the previous section without great difficulties. The complex unit state vector ψ defines the coordinates $\left[\psi_j, \bar{\psi}_j\right]$ on the variety of pure states Ω', and the Schrödinger equation $\frac{d\psi}{dt} = -iH\psi$ with the Hamiltonian H generates a flow $\left\{T'_t\right\}$ on Ω' which can be written in the coordinates $\left[\psi_j, \bar{\psi}_j\right]$ as a classical Hamiltonian system

$$\frac{d\psi_j}{dt} = -i\frac{\partial}{\partial\bar{\psi}_j}\Gamma\left(\psi, \bar{\psi}\right),$$
$$\frac{d\bar{\psi}_j}{dt} = i\frac{\partial}{\partial\psi_j}\Gamma\left(\psi, \bar{\psi}\right); \quad j = 1, \ldots, n,$$

with the Hamiltonian function $\Gamma\left(\psi, \bar{\psi}\right) = (\psi|H\psi)$. Putting $T''_t\omega'' = \omega''$, we obtain a flow $T_t = T'_t \times T''_t$ defining the dynamics of the corresponding classical model.

The question of describing repeated measurements and reductions is more complicated. A genuine hidden variable model should completely reduce measurements to observations and in this way resolve the enigmatic "measurement problem". A lot of attention was paid to this problem in physical and philosophical literature (see, *e.g.* [71, 138, 171, 186, 198]) while our remarks will be rather sketchy.

First we describe repeated measurements in quantum mechanics. Let a measurement $\hat{\boldsymbol{E}} = \{\hat{E}_i\}$ and then a measurement $\hat{\boldsymbol{F}} = \{\hat{F}_j\}$ be performed over a quantum system in the initial state $\hat{S}$. The formula (2.8) is insufficient in this case and an additional assumption is required. The "repeatability hypothesis" of von Neumann leads to the following expression for the joint probability distribution of the repeated quantum measurements

$$\mu_{\hat{S}}^{\hat{\boldsymbol{E}},\hat{\boldsymbol{F}}}(i,\, j) = \mathrm{Tr}\hat{S}\hat{E}_i\hat{F}_j\hat{E}_i. \tag{3.18}$$

If $\hat{\boldsymbol{E}}$ and $\hat{\boldsymbol{F}}$ are compatible, then these probabilities do not depend on the order of measurements and are equal to the probabilities for the joint measurement $\mathrm{Tr}\hat{S}\hat{E}_i\hat{F}_j$. If $\hat{\boldsymbol{E}} = \hat{\boldsymbol{F}}$ then $\mu_{\hat{S}}^{\hat{\boldsymbol{E}},\hat{\boldsymbol{F}}}(i,\, j) = 0$ for $i \neq j$, which explains the term "repeatability hypothesis" . In the general case of incompatible measurements $\mu_{\hat{S}}^{\hat{\boldsymbol{E}},\hat{F}} \neq \mu_{\hat{S}}^{\hat{\boldsymbol{F}},\hat{\boldsymbol{E}}}$, reflecting impossibility to give an objective meaning to the joint probability distribution. Notice also that for a given observable $\hat{X}$ the distribution (3.18) in general depends on the choice of its measurement $\hat{\boldsymbol{E}}$.

The relation (3.18) can be directly generalized to the case of arbitrary number of repeated measurements. For example, for three measurements

$$\mu_{\hat{S}}^{\hat{\boldsymbol{E}},\hat{\boldsymbol{F}},\hat{\boldsymbol{G}}}(i,\, j,\, k) = \mathrm{Tr}\hat{S}\hat{E}_i\hat{F}_j\hat{G}_k\hat{F}_j\hat{E}_i. \tag{3.19}$$

The unpublished von Neumann's argument mentioned in the remark of Wigner was just about the possibility of reproducing the statistics of repeated quantum measurements. That remark concerns the case of spin-1/2 particle which is described by two-dimensional Hilbert space $\mathcal{H}$ and the spin components are $2 \times 2-$matrices.

"... Rather, Von Neumann often discussed the measurement of the spin component of a spin-J particle in various directions. Clearly, the probabilities for the two possible outcomes of a single such measurement can be easily accounted for by hidden variables (see, *e.g.*, the rest of the present section or the more specific discussion on page 448 of Bell's article, Reference 2). However, Von Neumann felt that this is not the case for many consecutive measurements of the spin component in various different directions. The outcome of the first such measurement restricts

the range of values which the hidden parameters must have had before that first measurement was undertaken. The restriction will be present also after the measurement so that the probability distribution of the hidden variables characterizing the spin will be different for particles for which the measurement gave a positive result from that of the particles for which the measurement gave a negative result. The range of the hidden variables will be further restricted in the particles for which a second measurement of the spin component, in a different direction, also gave a positive result. A great number of consecutive measurements will select particles the hidden variables of which are all so closely alike that the spin component has, with a high probability, a definite sign in all directions. However, according to quantum mechanical theory, no such state is possible. Schrödinger raised the objection against this argument that the measurement of a spin component in one direction, while possibly specifying some hidden variables, may restore a random distribution of some other hidden variables. It is this writer's impression that Von Neumann did not accept Schrödinger's objection. His point was that the objection presupposed hidden variables in the apparatus used for the measurement. Von Neumann's argument needs to assume only two apparata, with perpendicular magnetic fields, and a succession of measurements alternating between the two apparata. Eventually, even the hidden variables of both apparata will be fixed by the outcomes of many subsequent measurements of the spin component in their respective directions so that the whole system's hidden variables will be fixed. Von Neumann did not publish this apparent refutation of Schrödinger's objection."

Let us see how one should change the state in a hidden variable model of Section 2.2 to reproduce the outcomes of the repeated measurements. For simplicity we restrict to pure states and maximal measurements. If in the state ω' one performs the measurement $\hat{\boldsymbol{E}}$ and then the measurement $\hat{\boldsymbol{F}}$ then according to (3.18) the probability of the outcome (i, j) is equal to

$$\mu_{\omega'}^{\hat{\boldsymbol{E}},\hat{\boldsymbol{F}}}(i,\, j) = |\langle \psi_{\omega'} | e_i \rangle|^2 \left| \langle e_i | f_j \rangle \right|^2 ,$$

where $\{e_i\}$, $\left\{f_j\right\}$ are the bases determining the measurements $\hat{\boldsymbol{E}}$, $\hat{\boldsymbol{F}}$. One can see that such values of the probabilities will be ensured if after the first measurement the initial point $\omega = \left(\omega', \omega''\right)$ of the phase space will go into the point $(\hat{E}_i, \omega'')$ under the condition that the outcome of the first measurement was i, *i.e.* if $\lambda_{\hat{\boldsymbol{E}}} = \pi_{\hat{\boldsymbol{E}}}(\omega'')$ satisfies the inequality

$$\sum_{k=1}^{i-1} |\langle \psi_{\omega'} | e_i \rangle|^2 < \lambda_{\hat{\boldsymbol{E}}} \leq \sum_{k=1}^{i} |\langle \psi_{\omega'} | e_i \rangle|^2 .$$

The change of the component ω' is thus a controlled Markov process. We also have to describe the change $\omega'' \to \tilde{\omega}''$. If the second measurement is the same as the first one, then (3.18) implies that its outcome should coincide with the outcome of the first measurement. To ensure this it should be $\pi_{\hat{E}}(\tilde{\omega}'') = \pi_{\hat{E}}(\omega'')$ *i.e.* the "roulette" corresponding to the measurement $\hat{E}$ should preserve its state after the first measurement. It is not important what then happens with the other "roulettes" unless they preserve the uniform distribution.

Consider, however, three repeated measurements $\hat{E}, \hat{F}, \hat{E}$. According to (3.19) the probability of the outcome (i, j, k) is equal to

$$\mu_{\omega'}^{\hat{E},\hat{F},\hat{E}}(i, j, k) = |\langle \psi_{\omega'}|e_i\rangle|^2 \left|\langle e_i|f_j\rangle\right|^2 \left|\langle f_j|e_k\rangle\right|^2 .$$

In order to obtain such an expression in the classical model, it is necessary that after the second measurement the probability distribution of the "roulette" corresponding to the measurement $\hat{E}$ should be completely renewed, *i.e.* $\pi_{\hat{E}}(\omega'')$ should become a random variable independent of the previous values. Since $\hat{E}$ is arbitrary, this leads to the following rule of state change for the roulettes: after a measurement $\hat{F}$ the states of all the roulettes $\lambda_{\hat{E}} = \pi_{\hat{E}}(\omega'')$ with $\hat{E} \neq \hat{F}$ are completely renewed while $\lambda_{\hat{F}} = \pi_{\hat{F}}(\omega'')$ preserves its value. This completely corresponds to the Schrödinger's remark. One can check that such a prescription allows to reproduce probabilities for all possible repeated maximal measurements. To include not necessarily maximal measurements one has to extend further the collection of "roulettes" in the classical descriptions.

The hidden variable ω have the two components, the first of which ω' can be considered as the characteristic of the system itself. The question – to what corresponds ω'' – properly to the system or to the measuring devices is very interesting although it does not give direct arguments contradicting to Schrödinger's prescription. On one hand it appears natural to consider $\lambda_{\hat{E}} = \pi_{\hat{E}}(\omega'')$ as parameters of the measuring device $\hat{E}$. Interaction of $\hat{E}$ with the system fixes the parameters of the device and $\lambda_{\hat{E}}$ does not change after the repeated measurement $\hat{E}$. On the other hand, the totality of all possible measurements $\hat{E}$ which can be performed over the system and the corresponding collection of the parameters $\omega'' = \prod_{\hat{E}} \lambda_{\hat{E}}$ can be regarded as the characteristic of the whole system, taking into account that measurement $\hat{F}$ affects all the parameters $\lambda_{\hat{E}}, \hat{E} \neq \hat{F}$. Here it is appropriate to remind of Bohr's saying that in quantum physics "the interaction between the measuring instruments and the objects forms an integral part of the phenomena" [155, page 4]. Anyhow we have seen that there exists a way to define the stochastic rule of state change of a classi-

cal system which allows to reproduce probabilities of repeated quantum measurements.

In the case of spin-1/2 system this was observed by Clauser [159] in a discussion which followed the publication of Wigner's paper. Having considered the Bell's model, Clauser proposed a simple rule for state change due to measurement which in fact uses Schrödinger's prescription and reproduces probabilities of all repeated measurements. In his answer Wigner raised the following objection which is applicable also to our model: a hidden variable model should give explanation for state changes in "mechanistic" *i.e.* deterministic but not in stochastic terms. A permanent flow of randomness which is necessary for partial renewal of the distribution of ω'' after each measurement is in the conflict with Liouville type theorem concerning conservation of the phase volume. To describe such a permanent renewal, a radical increase of dimensionality is required. Let us explain this on the simplest example of a sequence of independent random variables $\{X_i\}$ with values in $\mathbb{R}$. To describe it as a dynamical system with an invariant measure one has to pass to the trajectory space $\mathbb{R}^{\mathbb{Z}}$. The stochastic renewal of the state of one dimensional system can be represented as a mechanical evolution only in the infinite dimensional space of sequences.

The comment of Wigner is concluded with explanation that "all schemes of hidden parameters which either Von Neumann himself, or anyone else whom he knew, could think of and which reproduced the probabilities of outcomes of several successive measurements of the spin directions, had some feature which made it unattractive, in fact unreasonable" and "this was the true reason for his (von Neumann's) conviction of the inadequacy of the theories of hidden variables".

Thus the requirement of reproducing the results of repeated measurements in a hidden variable theory seems to lead to unappealing constructions which evoke negative emotions both from physicists and mathematicians. However no definite results which would close search in this direction were obtained. After publication of Bell's works the edge of investigations shifted to a different aspect of the hidden variable problem related to the description of composite systems[4].

3.4. Composite systems, EPR paradox and the Bell inequality

The most important physical applications of the mathematical apparatus of quantum mechanics concern the special features of *interactions* of

[4] Recently there was a revival of interest to noncontextual hidden variables due to discovery of tests for noncontextuality, including experimental work (see [169, 183] and the references therein).

microobjects which have no place in the classical physics. Nevertheless the first step in the quantum mechanical description interaction follows the classical pattern. Namely, one starts from a collection of noninteracting "single" components and then the interaction is described in terms of this composite system. Thus speaking about interaction one deals not so much with a model for single system, but rather with a category of these models equipped with the operation of product defining the rule for making a composite system. In classical mechanics one considers (symplectic) phase spaces with the operation of Cartesian product, while in quantum mechanics – the Hilbert spaces with the operation of tensor product.

In our discussion of this operation, for simplicity, we will not take into account additional complications related to possible indistinguishability of particles. One calls by tensor product of the vector $\psi_1 \in \mathcal{H}_1$ with the components $[\psi_1^i]$ and the vector $\psi_2 \in \mathcal{H}_2$ with the components $[\psi_2^j]$ the vector $\psi_1 \otimes \psi_2$ with the components $[\psi_1^i \psi_2^j]$ which is conveniently represented by the matrix. The space $\mathcal{H}_1 \otimes \mathcal{H}_2$ consists of all possible linear combinations (superpositions) of vectors of the form $\psi_1 \otimes \psi_2$, *i.e.* of all complex matrices $[\psi^{ij}]$. Consider a pure state of the composite system defined by a unit vector $\psi \in \mathcal{H}_1 \otimes \mathcal{H}_2$. There is a lot of vectors which cannot be written in the form $\psi_1 \otimes \psi_2$ corresponding to the case where the first and the second components are in the uniquely defined pure states. Most of the vectors ψ are superpositions of such vectors. For a superposition it is not possible to separate uniquely the first and the second components of the composite system. These unfactorizable *entangled* states represent a holistic entity in which the components exists, as one is accustomed to say, virtually. This reflects the property of quantum *nonseparability*.

At a first glance it looks unclear how such a merging of the components could happen before the interaction of the subsystems. The explanation is that preparation of an entangled pure state of a composite system assumes preliminary interaction between the components. Indeed, any vector ψ can be obtained from factorizable one $\psi_1 \otimes \psi_2$ by action of a unitary operator U in $\mathcal{H}_1 \otimes \mathcal{H}_2$ so that $\psi = U(\psi_1 \otimes \psi_2)$. Then the preparation consists of the interaction described by the operator U. As distinct from the classical mechanics where state preparation can be described in purely kinematical terms, preparation of many quantum mechanical states requires a dynamical interaction.

Consider from this point of view the hidden variable model of Section 2.2. Since the set of pure states of the composite system Ω' is larger than Cartesian product $\Omega'_1 \times \Omega'_2$, where Ω'_j is the set of pure states of the $j-$th component, the phase space of the classical description of the

composite system will be larger than the product of phase spaces for the components: $\Omega_1 \times \Omega_2 \subsetneq \Omega$. Therefore this classical description is not a correspondence between the categories of classical and quantum system preserving the operation of forming the composite systems.

Moreover, it appears that there is no way to establish such a correspondence. In any classical description of a composite quantum system the variables corresponding to observables of the components are necessarily entangled in the way unusual for classical subsystems. To give exact formulation, notice that observables corresponding to the first and the second components have the form $\hat{X} = \hat{X}_1 \otimes \hat{I}_2$, $\hat{Y} = \hat{I}_1 \otimes \hat{Y}_2$, where $\hat{X}_1$ is operator in $\mathcal{H}_1$, $\hat{Y}_2$ operator in $\mathcal{H}_2$ and $\hat{I}_j$ are the unit operators in $\mathcal{H}_j$. Certainly $\hat{X}\hat{Y} = \hat{Y}\hat{X}$ so that $\hat{X}, \hat{Y}$ are compatible, but they have even stronger property of algebraic independence: if $f(\hat{X}) = g(\hat{Y})$ then $f(\hat{X}) = g(\hat{Y}) = c\hat{I}$ (such a property is not satisfied *e.g.*by the commuting variables $\hat{X}$ and $\hat{X}^2$ with $\hat{X}^2 \neq c\hat{I}$).

Proposition 3.6. *There is no classical description for a composite quantum system in the space $\mathcal{H}_1 \otimes \mathcal{H}_2$ satisfying* (E.1), (X.2) *and the following separability condition:*

(X.6) *for any observables $\hat{X}_1, \dots, \hat{X}_n$ of the first subsystem and $\hat{Y}_1, \dots, \hat{Y}_m$ of the second subsystem there are random variables $X_1, \dots, X_n$; $Y_1, \dots, Y_m$ such that $X_i \to \hat{X}_i$, $Y_j \to \hat{Y}_j$ and $X_i Y_j \to \hat{X}_i \hat{Y}_j$.*

It will be shown that there is no such a description already for $n = m = 2$. Notice that the conditions (X.0) or (S.0) of one-to-one correspondence are not required here. The proposition means that while for any pair $\hat{X}, \hat{Y}$, where $\hat{X}$ (respectively $\hat{Y}$) refers to the first (respectively to the second) system, it is always possible to find X, Y such that $X \to \hat{X}$, $Y \to \hat{Y}$ and $XY \to \hat{X}\hat{Y}$ (this follows from the possibility to satisfy the product rule (X.4)), it is impossible to do it in the way that Y would be the same for all choices of $\hat{X}$ and X – the same for all $\hat{Y}$. The expression $\hat{X}\hat{Y}$ enters into the correlation

$$\left\langle \hat{X}\hat{Y} \right\rangle = \mathrm{Tr}\hat{S}\hat{X}\hat{Y} \tag{3.20}$$

of the outcomes of joint measurements of $\hat{X}$ and $\hat{Y}$. Thus to reproduce the quantum mechanical correlations between the subsystems, a hidden variable theory should possess the following strange property: the observation method for an observable of the second subsystem $\hat{Y}$ should necessarily depend on which $\hat{X}$ is observed over the first subsystem.

Proof. The proof is based on *Clauser-Horne-Shimony-Holt inequality* which is a version of the famous Bell inequality [160]. Assume that one

can satisfy the condition (X.6) with $n = m = 2$ and let us prove that for all observables $\hat{X}_1$, $\hat{X}_2$ of the first component and all observables $\hat{Y}_1$, $\hat{Y}_2$ of the second, with the outcomes in $[-1, 1]$, the correlations (3.20) satisfy

$$\left|\left\langle \hat{X}_1\hat{Y}_1\right\rangle + \left\langle \hat{X}_1\hat{Y}_2\right\rangle + \left\langle \hat{X}_2\hat{Y}_1\right\rangle - \left\langle \hat{X}_2\hat{Y}_2\right\rangle\right| \leq 2. \tag{3.21}$$

By the assumptions (E.1), (X.6) it is sufficient to prove this for correlations of classical random variables $X_1(\omega)$, $X_2(\omega)$, $Y_1(\omega)$, $Y_2(\omega)$ given by

$$\langle XY\rangle = \int_\Omega S(d\omega)\, X(\omega)\, Y(\omega).$$

But taking into account the spectral condition (X.2) one has $|X_j| \leq 1$, $|Y_k| \leq 1$; $j, k = 1, 2$, which implies

$$|X_1Y_1+X_1Y_2+X_2Y_1-X_2Y_2| \leq |Y_1+Y_2|+|Y_1-Y_2| \leq 2\max\{|Y_1|,|Y_2|\} \leq 2,$$

hence $-2 \leq X_1Y_1 + X_1Y_2 + X_2Y_1 - X_2Y_2 \leq 2$, whence, taking the expectation, one obtains the required inequality.

It remains to show that in any composite quantum system one can find observables $\hat{X}_1$, $\hat{X}_2$; $\hat{Y}_1$, $\hat{Y}_2$ and the state $\hat{S}$ violating the inequality (3.21). For this consider first the system of two distinguishable spin-1/2 particles so that $\mathcal{H}_1$ and $\mathcal{H}_2$ are two-dimensional. Denote by $\hat{X}(\vec{a})$ the spin observable in the direction $\vec{a} = (a_x, a_y, a_z)$ for the first particle and by $\hat{Y}(\vec{b})$ – the spin observable in the direction $\vec{b}$ for the second particle. In the basis

$$|\uparrow\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, |\downarrow\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

one has

$$\hat{X}(\vec{a}) = \begin{bmatrix} a_z & a_x - ia_y \\ a_x + ia_y & -a_z \end{bmatrix},$$

and similarly for $\hat{Y}(\vec{b})$. Consider the entangled pure state $\hat{S}_\psi$ of the composite system given by the vector

$$|\psi\rangle = \frac{1}{\sqrt{2}}\big[|\uparrow\rangle \otimes |\downarrow\rangle - |\downarrow\rangle \otimes |\uparrow\rangle\big]. \tag{3.22}$$

A simple calculation shows that

$$\langle\psi|\sigma(\vec{a}) \otimes \sigma(\vec{b})|\psi\rangle = -\vec{a}\cdot\vec{b}. \tag{3.23}$$

If one chooses the four vectors $\vec{a}_j, \vec{b}_k$, $(j, k = 1, 2)$ as shown on Fig. 6.3, then the correlations between $\hat{X}(\vec{a}_j)$ and $\hat{Y}(\vec{b}_k)$ give the value $2\sqrt{2}$ for the left side of (3.21), which breaks the inequality.

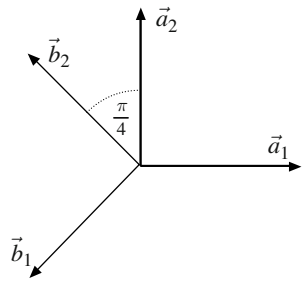

Figure 6.3. Choice of $\vec{a}_j$ and $\vec{b}_k$.

For an arbitrary composite quantum system $\mathcal{H}_1 \otimes \mathcal{H}_2$ one can always take the two-dimensional subspaces of $\mathcal{H}_1$ and $\mathcal{H}_2$ to build this construction which proves the Proposition in the general case. □

The underlying case of two spatially separated particles has quite important physical consequences. This was realized already in 1935 after publication of the paper of Einstein, Podolsky and Rosen (EPR) [162] in the subsequent discussion on completeness of quantum mechanics (considering entangled state of two spin-1/2 particles was proposed later by Bohm). EPR believed that their argument shows "incompleteness" of quantum mechanical description. The argument of Bell shifts the accent substantially: if the quantum mechanical description is correct then any attempt to "complete" it with hidden variables leads necessarily to contradiction with the physical principle of *locality*. Assume that $\mathcal{H}_1$ and $\mathcal{H}_2$ describe spin degrees of freedom of two spin-1/2 particles located in spatially separated domains. Quantum mechanics implies existence of states for such a system in which the spins are entangled as in (3.22). Moreover, such a state in principle can be realized experimentally as a product of decay of a spin-zero system.

Now assume that a joint measurement is performed for the spin of the first particle in the direction $\vec{a}$ and the spin of the second particle in the direction $\vec{b}$. After a long series of independent repetitions the experimenter computes statistical estimates for the correlation between $\hat{X}(\vec{a})$ and $\hat{Y}(\vec{b})$. Consider the following three statements:

I the correct values for the correlations are given by the quantum mechanical formula (3.20);

II there is a classical description of the composite quantum system satisfying the spectral condition *i.e.* preserving the "objective values" of observables;

III "the real factual situation of the system S_1 is independent of what is done with the system S_2, which is spatially separated from the former".

The last property is the *Einstein locality* [162] or *separability* [163] related to the principle "no instantaneous action at a distance". Notice that the property (III) has an unambiguous meaning only in the classical picture *i.e.* under the condition (II). The classical description satisfying the requirement (III) is called a local hidden variable theory or "local realism". The Bell argument shows that the properties (I)-(III) are incompatible *i.e. a local hidden variable theory reproducing the statistical predictions of quantum mechanics does not exist*. This can be considered as modern interpretation of the "EPR paradox" .

Moreover, the Bell type inequalities in principle open a possibility for experimental test: quantum mechanics versus local hidden variable theory. A practical implementation of such experiments is however aggravated by several possible loopholes and requires great efforts. The famous Aspect's experiment performed in 1981-1982 showed the agreement with quantum mechanical formula for correlations. After that there was a continuing series of experiments with similar conclusions while a couple of them claimed possible agreement with the Bell inequality [160, 194]. However the question is not closed because of the high price of rejecting the "realistic" description of the Nature [163].

We are not speaking of defenders of the naive realism who would be happy with a pictorial image of the microworld as something similar to the world directly accessible to the human perception where mechanistic idealizations like a material point still have sense. They continue attempts to find a gap in the argument leading to the conclusion that any local hidden variable theory cannot reproduce the statistical predictions of quantum theory concerning correlations between the parts of a composite quantum system. It should be noticed that while the Bell inequality as such is elementary, the logic of its application in the hidden variable problem is far from trivial; any critique of Bell's argument with its subsequent refinements, however it might seem sophisticated, sooner or later was found based on a misunderstanding. A critical survey of recent discussions and different opinions is given in the article [164] where in particular a proof is given of a Bell inequality taking into account finiteness of the sample and possibility of arbitrary local correlations between the subsequent experiments.

What is really disturbing is the apparent impossibility of a peaceful coexistence of quantum mechanics and "local realistic" theory such as general relativity (which was in fact the motivation of Einstein's concern).

Admitting a local hidden variable theory would mean that quantum mechanics is limited at least in some of its most basic predictions which would be quite an extraordinary development. Some compromise has to be found if one believes in the possibility of unified quantum theory of interactions including gravitation.

On the other hand, macroscopic or at least mesoscopic manifestations of the entanglement are critically important for the future technological implementations of the new quantum information processing protocols including quantum computation.

Therefore much attention is paid to close possible logical loopholes in the actual Bell-test experiments. The most significant one is the *detection loophole* which still exists after almost 40 years of experimental research. Soon after appearance of Bell's work it was recognized that the inequality can be violated in local hidden variable models with post-selection with a positive probability of failure for the particle detection if the "no detection" outcomes are just neglected. A simple explicit local hidden variable model for such correlation experiment with two spin-1/2 particles was demonstrated by N. Gisin and B. Gisin [167] implying in particular that experiments with the detector efficiency below 75% cannot be considered as decisive. Therefore researchers continue to improve experimental techniques and search for the more sensitive inequalities [187]. Nowadays theoretical study of the Bell-type inequalities providing quantitative boarders between "classical" and "quantum" is one of the topics in the modern quantum information theory with applications to quantum cryptography, entanglement detection, multipartite interactive proof systems, communication complexity etc., see [181] and the references therein. However these exciting subjects are already beyond the scope of the present essay.