Pointwise Convergence of Bochner–Riesz Means in Sobolev Spaces

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Abstract The Bochner–Riesz means are defined by the Fourier multiplier operators $(S_R^\alpha * f)\hat{}(\xi) = (1 - |R^{-1}\xi|^2)_+^\alpha \hat{f}(\xi)$. Here we prove that if f has β derivatives in $L^p(\mathbf{R}^d)$, then $S_R^\alpha * f(x)$ converges pointwise to $f(x)$ as $R \to +\infty$ with a possible exception of a set of points with Hausdorff dimension at most $d - \beta p$ if one of the following conditions holds: either $\alpha > (d-1)|1/p - 1/2|$, or $\alpha > d(1/2 - 1/p) - 1/2$ and $\alpha + \beta \geqslant (d-1)/2$. If $\beta > d/p$, then pointwise convergence holds everywhere.

Keywords Bochner–Riesz means · Sobolev space · Hausdorff dimension

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1 Introduction

The Bochner–Riesz means of order α of functions in $\mathbf{R}^d$ are defined by the Fourier integrals

$$S_R^\alpha * f(x) = \int_{\{|\xi|<R\}} \left(1 - \left|R^{-1}\xi\right|^2\right)^\alpha \hat{f}(\xi) \exp(2\pi i \xi x)\, d\xi.$$

In particular, when $\alpha = 0$ one obtains the spherical partial sums, which are a natural analogue of the partial sums of one-dimensional Fourier series. The almost everywhere convergence of Bochner–Riesz means has been widely studied, however there are still many open problems. See [13] as a general reference. If $\alpha > (d-1)/2$, the critical index, then the Bochner–Riesz maximal operator $S_*^\alpha f = \sup_{R>0} |S_R^\alpha * f|$ is pointwise dominated by the Hardy-Littlewood maximal operator, hence it is of

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weak type $(1,1)$ and of strong type (∞,∞). If $p=2$ and $\alpha>0$, then S_*^α is bounded on $L^2(\mathbf{R}^d)$. Therefore, by complex interpolation, the Bochner–Riesz maximal operator is bounded on $L^p(\mathbf{R}^d)$, with $1<p\leqslant+\infty$ and $\alpha>(d-1)|1/p-1/2|$. From this the almost everywhere convergence follows: if f is in $L^p(\mathbf{R}^d)$, with $1\leqslant p\leqslant+\infty$ and $\alpha>(d-1)|1/p-1/2|$, then $\lim_{R\to+\infty} S_R^\alpha * f(x)=f(x)$ a.e.. This result is not optimal. Indeed, Carbery in [1] established the pointwise convergence of the Bochner–Riesz means when $2\leqslant p<2d/(d-1-2\alpha)$ and $d=2$. The same result for $d\geqslant 3$ has been obtained by Christ in [5], under the extra assumption that $\alpha\geqslant(d-1)/2(d+1)$. For further improvements when $p\geqslant 2$ see also [7]. Finally, in [2] Carbery, Rubio de Francia, and Vega have removed the restriction on α by showing that $S_*^\alpha f$ is bounded on the weighted space $L^2(|x|^{-\lambda}\,dx)$ if $d(1-2/p)\leqslant\lambda<1+2\alpha\leqslant d$ and observing that $L^p\subset L^2+L^2(|x|^{-\lambda}\,dx)$. See also [10]. Moreover, it has been shown by Rubio de Francia that the Bochner–Riesz means of index α are not defined in $L^p(\mathbf{R}^d)$ when $p\geqslant 2d/(d-1-2\alpha)$. Therefore the problem of the almost everywhere convergence of Bochner–Riesz means when $p\geqslant 2$ is essentially solved. On the other hand, as far as we know, sharp results when $p<2$ are not known. For related subjects, see [14].

Here we consider the pointwise convergence of Bochner–Riesz means for more regular functions, in particular functions in Sobolev classes. It has been proved by Carbery and Soria in [3] and Ma in [8] that by putting some smoothness on the function one may decrease the index of almost everywhere summability. Moreover, Carbery and Soria in [4] and Montini in [9] have considered the problem of the capacity and Hausdorff dimension of the divergence set of spherical partial sums of Fourier integrals. Then, in [6] Colzani has shown that the Bochner–Riesz means of functions with β integrable derivatives with $\alpha+\beta>(d-1)/2$ may diverge only in sets of points of Hausdorff dimension at most $d-\beta$. Here we generalise these results to functions with β fractional derivatives in $L^p(\mathbf{R}^d)$, $1<p<\infty$. In particular, we obtain conditions on α, β, p and d that ensure the pointwise convergence up to sets with Hausdorff dimension at most $d-\beta p$. The conditions are the following: either $\alpha>(d-1)|1/p-1/2|$ or $\alpha>d(1/2-1/p)-1/2$ and $\alpha+\beta\geqslant(d-1)/2$. Since the functions we are considering may be infinite precisely on sets of dimension $d-\beta p$, this estimate for the dimension of the divergence sets is the best possible. However, our analysis is not exhaustive and the ranges of the indexes are not optimal.

Before stating our result, we recall some basic definitions.

The *Bochner–Riesz kernel* S_R^α of order α, with $\alpha>0$, is defined by its Fourier transform,

$$\widehat{S_R^\alpha}(\xi)=\left(1-\left|R^{-1}\xi\right|^2\right)_+^\alpha.$$

This kernel can be written explicitly in terms of Bessel functions:

$$S_R^\alpha(x)=\pi^{-\alpha}\Gamma(\alpha+1)R^{d/2-\alpha}|x|^{-\alpha-d/2}J_{\alpha+d/2}\big(2\pi R|x|\big).$$

It follows from the asymptotic formula for Bessel functions that

$$\left|S_R^\alpha(x)\right|\leqslant cR^d\big(1+R|x|\big)^{-\alpha-(d+1)/2}.$$

The *Bessel kernel* G^β, with $\beta > 0$, is defined by its Fourier transform,

$$\widehat{G^\beta}(\xi) = \left(1 + |\xi|^2\right)^{-\beta/2}.$$

Also this kernel can be written explicitly in terms of Bessel functions, however it is more convenient to see it as a superposition of heat kernels,

$$G^\beta(x) = \Gamma(\beta/2)^{-1} \int_0^{+\infty} (4\pi t)^{-d/2} e^{-|x|^2/4t} e^{-t} t^{\beta/2-1}\, dt.$$

It follows from this representation that this kernel is positive and integrable. Moreover, if $0 < \beta < d$ then it is asymptotic to $c|x|^{\beta-d}$ when $x \to 0$ and it has an exponential decay at infinity. If $\beta = d$ then G^β has a logarithmic singularity at the origin, and if $\beta > d$ then it is bounded. See [11]. Finally, the *Riesz kernel* I^β, with $0 < \beta < d$, is given by

$$I^\beta(x) = |x|^{\beta-d}.$$

If $\beta > 0$ and $p > 1$, the *Bessel capacity* of a set $E \subset \mathbf{R}^d$ is defined by

$$B_{\beta,p}(E) = \inf\left\{\|f\|_p^p : G^\beta * f(x) \geqslant 1 \text{ on } E\right\}.$$

The *Riesz capacity* $R_{\beta,p}$ is defined in a similar way, by replacing G^β with I^β. It follows from the definitions that $R_{\beta,p}(E) \leqslant C B_{\beta,p}(E)$. Actually, it is also true that the Bessel and Riesz capacities have the same null sets. See [16, p. 67]. It can be also proved that when the $d - \beta p$ Hausdorff measure of E is finite, then $B_{\beta,p}(E) = 0$. Conversely, if $B_{\beta,p}(E) = 0$, then for every $\varepsilon > 0$ the $d - \beta p + \varepsilon$ Hausdorff measure of E is 0. See [16, Th 2.6.16].

Our main result is the following.

Theorem 1.1 *Assume that* $\alpha > 0$, $\beta > 0$, *and* $f = G^\beta * F$ *with* $F \in L^p(\mathbf{R}^d)$, $1 \leqslant p \leqslant +\infty$. *If* $0 < \beta \leqslant d/p$, *then* $S_R^\alpha * f(x)$ *converges pointwise to* $f(x)$ *as* $R \to +\infty$ *with a possible exception of a set of points with Hausdorff dimension at most* $d - \beta p$, *provided that one of the following conditions holds*:

(i) *either* $\alpha > (d-1)|1/p - 1/2|$;
(ii) *or* $\alpha > d(1/2 - 1/p) - 1/2$ *and* $\alpha + \beta \geqslant (d-1)/2$.

If $\alpha > d(1/2 - 1/p) - 1/2$ *and* $\beta > d/p$, *then the convergence is pointwise everywhere and uniform.*

As mentioned in the Introduction, almost everywhere convergence of Bochner–Riesz means has been considered in [3] and [8], and the case $\alpha = 0$ of our Theorem is contained in [4] and [9]. The cases $p = 1$ and $p = 2$ are already contained in [6], and also the case $p = \infty$ is already known. Indeed, if $p = \infty$ and $\beta > 0$, then $f = G^\beta * F$ is bounded and uniformly continuous, and if $\alpha > (d-1)/2$ then $S_R^\alpha * f$ converges to f uniformly. The assumption $\alpha > d(1/2 - 1/p) - 1/2$ in Theorem 1.1

is necessary in order to define the Bochner–Riesz means in $L^p(\mathbf{R}^d)$. Observe also that when $p < 2d/(d-1)$ this condition reduces to $\alpha + \beta \geqslant (d-1)/2$.

The main point of the proof of the Theorem is an estimate for the maximal Bochner–Riesz operator: for every f with β derivatives in $L^p(\mathbf{R}^d)$ there exists a function H in $L^p(\mathbf{R}^d)$ such that

$$\sup_{R>0} \left| S_R^\alpha * f(x) \right| \leqslant H * |x|^{\beta-d}.$$

Convergence up to a set of Riesz capacity zero follows from this.

2 Proof of Theorem 1.1

We split the proof of Theorem 1.1 into a series of lemmas.

Lemma 2.1 *If* $1 \leqslant p \leqslant +\infty$ *and* $\alpha > d(1/2 - 1/p) - 1/2$, *then for every* $F \in L^p(\mathbf{R}^d)$ *the convolution* $S_R^\alpha * G^\beta * F$ *is well defined and it is commutative and associative*:

$$S_R^\alpha * \left(G^\beta * F\right) = \left(S_R^\alpha * G^\beta\right) * F = G^\beta * \left(S_R^\alpha * F\right).$$

Proof The statement follows from Young's inequality: given $1 \leqslant p, q, r \leqslant \infty$ with $1/p + 1/q = 1 + 1/r$, if $f \in L^p(\mathbf{R}^d)$ and $g \in L^q(\mathbf{R}^d)$, then $f * g \in L^r(\mathbf{R}^d)$. It suffices to observe that the Bessel kernel G^β is an integrable function and that the Bochner–Riesz kernel S_R^α is in $L^q(\mathbf{R}^d)$, with $q > 2d/(2\alpha + d + 1)$. □

Lemma 2.2 *If* $\alpha > (d-1)/2$, *then the maximal operator* $S_*^\alpha f = \sup_{R>0} |S_R^\alpha * f|$ *is of weak type in* $L^1(\mathbf{R}^d)$. *If* $1 < p \leqslant +\infty$ *and* $\alpha > (d-1)|1/p - 1/2|$, *then this maximal operator is bounded on* $L^p(\mathbf{R}^d)$.

Proof This is a classical result of Stein. First one proves the extreme cases $p = 1$ or $p = \infty$ and $\Re(\alpha) > (d-1)/2$, then the case $p = 2$ and $\Re(\alpha) > 0$. The other cases follow by complex interpolation. See [13, Theorem 5.1]. □

Lemma 2.3 *If* $\alpha + \beta \geqslant (d-1)/2$, *then*

$$\sup_{R>0} \left| S_R^\alpha * G^\beta(x) \right| \leqslant C \begin{cases} \min\{|x|^{\beta-d}, |x|^{-\alpha-(d+1)/2}\} & \text{if } 0 < \beta < d, \\ \min\{\log(1 + 1/|x|), |x|^{-\alpha-(d+1)/2}\} & \text{if } \beta = d, \\ \min\{1, |x|^{-\alpha-(d+1)/2}\} & \text{if } \beta > d. \end{cases}$$

Proof First assume that $0 < \beta < d$. By definition

$$S_R^\alpha * G^\beta(x) = \int_{\mathbf{R}^d} \left(1 + |\xi|^2\right)^{-\beta/2} \left(1 - \left|R^{-1}\xi\right|^2\right)_+^\alpha e^{2\pi i \xi \cdot x} \, d\xi.$$

Let $\phi + \psi = 1$ be a partition of unity on $\mathbf{R}_+$, with ϕ and ψ smooth and non-negative, and

$$\phi(\rho) = 1 \quad \text{if } 0 \leqslant \rho \leqslant 1/3 \quad \text{and} \quad \phi(\rho) = 0 \quad \text{if } \rho > 2/3,$$
$$\psi(\rho) = 1 \quad \text{if } 2/3 \leqslant \rho \leqslant 1 \quad \text{and} \quad \psi(\rho) = 0 \quad \text{if } \rho < 1/3.$$

Then

$$\begin{aligned} S_R^\alpha * G^\beta(x) = &\int_{\mathbf{R}^d} \phi\left(\left|R^{-1}\xi\right|\right)\left(1+|\xi|^2\right)^{-\beta/2}\left(1-\left|R^{-1}\xi\right|^2\right)_+^\alpha e^{2\pi i\xi\cdot x}\, d\xi \\ &+ \int_{\mathbf{R}^d} \psi\left(\left|R^{-1}\xi\right|\right)\left(1+|\xi|^2\right)^{-\beta/2}\left(1-\left|R^{-1}\xi\right|^2\right)_+^\alpha e^{2\pi i\xi\cdot x}\, d\xi. \end{aligned}$$

To estimate the first integral set

$$\widehat{K}(\xi) = \phi\left(|\xi|\right)\left(1-|\xi|^2\right)_+^\alpha.$$

Then, if $K_R(x) = R^d K(Rx)$ we can rewrite the first integral as

$$\int_{\mathbf{R}^d} \phi\left(\left|R^{-1}\xi\right|\right)\left(1+|\xi|^2\right)^{-\beta/2}\left(1-\left|R^{-1}\xi\right|^2\right)_+^\alpha e^{2\pi i\xi\cdot x}\, d\xi = K_R * G^\beta(x).$$

Recall that the Hardy-Littlewood maximal function of a locally integrable function is defined by

$$\mathscr{M} f(x) = \sup_{r>0} \frac{1}{|B_r|}\int_{B_r} \left|f(x-y)\right| dy,$$

where B_r denotes the ball of radius r centred at the origin and $|B_r|$ is its volume. Since the multiplier $\widehat{K}$ is smooth with compact support, the kernel $K(x)$ is bounded and rapidly decreasing at infinity. This implies that

$$\sup_{R>0}\left|K_R * G^\beta(x)\right| \leqslant C\mathscr{M} G^\beta(x).$$

Since $G^\beta(x) \leqslant C|x|^{\beta-d}$ and since the Hardy-Littlewood maximal function of a radial homogeneous function is radial homogeneous, it also follows that

$$\mathscr{M} G^\beta(x) \leqslant C\mathscr{M}|x|^{\beta-d} = C|x|^{\beta-d}.$$

We now estimate the second integral. When $|Rx| \leqslant 3$ a crude estimate gives

$$\begin{aligned} &\left|\int_{\mathbf{R}^d} \psi\left(\left|R^{-1}\xi\right|\right)\left(1+|\xi|^2\right)^{-\beta/2}\left(1-\left|R^{-1}\xi\right|^2\right)_+^\alpha e^{2\pi i\xi\cdot x}\, d\xi\right| \\ &\leqslant \int_{\mathbf{R}^d} \psi\left(\left|R^{-1}\xi\right|\right)\left(1+|\xi|^2\right)^{-\beta/2}\left(1-\left|R^{-1}\xi\right|^2\right)_+^\alpha d\xi \\ &= CR^{d-\beta}\int_0^1 \psi(\rho)\left(R^{-2}+\rho^2\right)^{-\beta/2}\left(1-\rho^2\right)^\alpha \rho^{d-1}\, d\rho \\ &\leqslant CR^{d-\beta} \\ &\leqslant C|x|^{\beta-d}. \end{aligned}$$

To estimate the integral when $|Rx| > 3$, we introduce another smooth cut-off function $0 \leqslant \chi \leqslant 1$ such that $\chi(\rho) = 1$ if $0 \leqslant \rho \leqslant 1 - 2/|Rx|$ and $\chi(\rho) = 0$ if $\rho \geqslant 1 - 1/|Rx|$. Moreover we require that for all $j = 0, 1, 2, \ldots$,

$$\left|\frac{d^j}{d\rho^j}\chi(\rho)\right| \leqslant C(j)|Rx|^j.$$

Then

$$\begin{aligned}&\int_{\mathbf{R}^d}\psi\big(|R^{-1}\xi|\big)\big(1+|\xi|^2\big)^{-\beta/2}\big(1-|R^{-1}\xi|^2\big)_+^{\alpha}e^{2\pi i\xi\cdot x}\,d\xi\\ &\quad=\int_{\mathbf{R}^d}\big(1-\chi\big(|R^{-1}\xi|\big)\big)\psi\big(|R^{-1}\xi|\big)\big(1+|\xi|^2\big)^{-\beta/2}\big(1-|R^{-1}\xi|^2\big)_+^{\alpha}e^{2\pi i\xi\cdot x}\,d\xi\\ &\qquad+\int_{\mathbf{R}^d}\chi\big(|R^{-1}\xi|\big)\psi\big(|R^{-1}\xi|\big)\big(1+|\xi|^2\big)^{-\beta/2}\big(1-|R^{-1}\xi|^2\big)_+^{\alpha}e^{2\pi i\xi\cdot x}\,d\xi.\end{aligned}$$

In polar coordinates the first integral becomes

$$\begin{aligned}&\int_{\mathbf{R}^d}\big(1-\chi\big(|R^{-1}\xi|\big)\big)\psi\big(|R^{-1}\xi|\big)\big(1+|\xi|^2\big)^{-\beta/2}\big(1-|R^{-1}\xi|^2\big)_+^{\alpha}e^{2\pi i\xi\cdot x}\,d\xi\\ &=R^{d-\beta}\int_0^1\rho^{d-1}\big(1-\chi(\rho)\big)\psi(\rho)\big(R^{-2}+\rho^2\big)^{-\beta/2}\big(1-\rho^2\big)^{\alpha}\int_{|\theta|=1}e^{2\pi iR\rho x\cdot\theta}\,d\theta\,d\rho.\end{aligned}$$

It is well known that

$$\left|\int_{|\theta|=1}e^{2\pi ix\cdot\theta}\,d\theta\right| \leqslant C|x|^{-(d-1)/2}.$$

This is a standard estimate for oscillatory integrals with non-degenerate critical points, which also follows from the decay of Bessel functions and the explicit formula

$$\int_{|\theta|=1}e^{2\pi ix\cdot\theta}\,d\theta = 2\pi|x|^{(2-d)/2}J_{(d-2)/2}\big(2\pi|x|\big).$$

See for example [12, p. 347]. Therefore, if $\alpha+\beta \geqslant (d-1)/2$ and $|Rx| > 3$,

$$\begin{aligned}&\left|\int_{\mathbf{R}^d}\big(1-\chi\big(|R^{-1}\xi|\big)\big)\psi\big(|R^{-1}\xi|\big)\big(1+|\xi|^2\big)^{-\beta/2}\big(1-|R^{-1}\xi|^2\big)_+^{\alpha}e^{2\pi i\xi\cdot x}\,d\xi\right|\\ &\quad\leqslant CR^{(d+1)/2-\beta}|x|^{-(d-1)/2}\int_0^1\big(1-\chi(\rho)\big)(1-\rho)^{\alpha}\,d\rho\\ &\quad\leqslant CR^{(d+1)/2-\beta}|x|^{-(d-1)/2}|Rx|^{-\alpha-1}\\ &\quad= C|Rx|^{(d-1)/2-\alpha-\beta}|x|^{\beta-d}\\ &\quad\leqslant C|x|^{\beta-d}.\end{aligned}$$

To estimate the second integral, we introduce the Laplacian $\Delta_\xi = -\sum_{j=1}^{d} \partial^2/\partial\xi_j^2$. Since this operator is self-adjoint and $\Delta_\xi^k(e^{2\pi i\xi\cdot x}) = |2\pi x|^{2k} e^{2\pi i\xi\cdot x}$, we obtain

$$\begin{aligned}
&\int_{\mathbf{R}^d} \chi\big(\big|R^{-1}\xi\big|\big)\psi\big(\big|R^{-1}\xi\big|\big)\big(1+|\xi|^2\big)^{-\beta/2}\big(1-\big|R^{-1}\xi\big|^2\big)_+^{\alpha} e^{2\pi i\xi\cdot x}\,d\xi \\
&\quad= \int_{\mathbf{R}^d} \chi\big(\big|R^{-1}\xi\big|\big)\psi\big(\big|R^{-1}\xi\big|\big)\big(1+|\xi|^2\big)^{-\beta/2}\big(1-\big|R^{-1}\xi\big|^2\big)_+^{\alpha} \Delta_\xi^k\left(\frac{e^{2\pi i\xi\cdot x}}{|2\pi x|^{2k}}\right) d\xi \\
&\quad= |2\pi x|^{-2k} R^{-\beta} \int_{\mathbf{R}^d} \Delta_\xi^k\big(g\big(R^{-1}\xi\big)\big) e^{2\pi i\xi\cdot x}\,d\xi,
\end{aligned}$$

where we have set

$$g(\xi) = \chi\big(|\xi|\big)\psi\big(|\xi|\big)\big(R^{-2}+|\xi|^2\big)^{-\beta/2}\big(1-|\xi|^2\big)_+^{\alpha}.$$

Now, denoting by Δ_ρ^k the radial part of the Laplacian and setting $g_0(|\xi|) = g(\xi)$, we get

$$\begin{aligned}
&\int_{\mathbf{R}^d} \chi\big(\big|R^{-1}\xi\big|\big)\psi\big(\big|R^{-1}\xi\big|\big)\big(1+|\xi|^2\big)^{-\beta/2}\big(1-\big|R^{-1}\xi\big|^2\big)_+^{\alpha} e^{2\pi i\xi\cdot x}\,d\xi \\
&\quad= |2\pi x|^{-2k} R^{-\beta-2k} \int_{\mathbf{R}^d} \big(\Delta_\xi^k g\big)\big(R^{-1}\xi\big) e^{2\pi i\xi\cdot x}\,d\xi \\
&\quad= |2\pi x|^{-2k} R^{d-\beta-2k} \int_0^{+\infty} \Delta_\rho^k g_0(\rho)\rho^{d-1} \int_{|\theta|=1} e^{2\pi iR\rho x\cdot\theta}\,d\theta\,d\rho.
\end{aligned}$$

Then, recalling the properties of the cut-off functions χ and ψ, if $k > (\alpha-1)/2$ we finally get

$$\begin{aligned}
&\left|\int_{\mathbf{R}^d} \chi\big(\big|R^{-1}\xi\big|\big)\psi\big(\big|R^{-1}\xi\big|\big)\big(1+|\xi|^2\big)^{-\beta/2}\big(1-\big|R^{-1}\xi\big|^2\big)_+^{\alpha} e^{2\pi i\xi\cdot x}\,d\xi\right| \\
&\quad\leqslant C|x|^{-2k-(d-1)/2} R^{(d+1)/2-\beta-2k} \int_0^{+\infty} \big|\Delta_\rho^k g_0(\rho)\big|\rho^{(d-1)/2}\,d\rho \\
&\quad\leqslant C|x|^{-2k-(d-1)/2} R^{(d+1)/2-\beta-2k} \int_{1/3}^{1-1/R|x|} (1-\rho)^{\alpha-2k}\,d\rho \\
&\quad\leqslant C|x|^{-2k-(d-1)/2} R^{(d+1)/2-\beta-2k} |Rx|^{-\alpha+2k-1} \\
&\quad= C|Rx|^{(d-1)/2-\alpha-\beta}|x|^{\beta-d} \\
&\quad\leqslant C|x|^{\beta-d}.
\end{aligned}$$

We have proved that, if $0 < \beta < d$, then for every x

$$\sup_{R>0}\big|S_R^\alpha * G^\beta(x)\big| \leqslant C|x|^{\beta-d}.$$

This estimate is the best possible if $\alpha+\beta=(d-1)/2$, or if $\alpha+\beta>(d-1)/2$ and $|x|\leqslant 1$. If $(d-1)/2-\beta<\alpha<(d-1)/2$ and $|x|\geqslant 1$, then

$$\begin{aligned}\sup_{R>0}\left|S_R^\alpha * G^\beta(x)\right| &\leqslant G^{\alpha+\beta-(d-1)/2} * \sup_{R>0}\left|S_R^\alpha * G^{(d-1)/2-\alpha}\right|(x)\\ &\leqslant C G^{\alpha+\beta-(d-1)/2} * |x|^{-\alpha-(d+1)/2}\\ &\leqslant C|x|^{-\alpha-(d+1)/2}.\end{aligned}$$

The first inequality follows from the fact that Bessel kernels are positive, and the last inequality follows from the fact that these kernels are integrable with an exponential decay at infinity. If $\alpha\geqslant(d-1)/2$, then

$$\left|S_R^\alpha(x)\right|\leqslant CR^d\big(1+|Rx|\big)^{-\alpha-(d+1)/2}.$$

Again by this estimate it follows that, if $|x|\geqslant 1$,

$$\sup_{R>0}\left|S_R^\alpha * G^\beta(x)\right|\leqslant C|x|^{-\alpha-(d+1)/2}.$$

Hence we have proved that, if $0<\beta<d$,

$$\sup_{R>0}\left|S_R^\alpha * G^\beta(x)\right|\leqslant C\min\left\{|x|^{\beta-d},|x|^{-\alpha-(d+1)/2}\right\}.$$

The proof of the cases $\beta\geqslant d$ is similar. □

Lemma 2.4 *Let* $1<p<+\infty$, $\alpha>0$, $0<\beta<d$, *and assume that one of the following properties holds*:

(i) *either* $\alpha>(d-1)|1/p-1/2|$;
(ii) *or* $\alpha>d(1/2-1/p)-1/2$ *and* $\alpha+\beta\geqslant(d-1)/2$.

Then for every function $F\in L^p(\mathbf{R}^d)$ *there exists a function* $H\in L^p(\mathbf{R}^d)$ *with* $\|H\|_p\leqslant C\|F\|_p$ *and such that*

$$S_*^\alpha\big(G^\beta * F\big)(x)=\sup_{R>0}\left|S_R^\alpha * G^\beta * F(x)\right|\leqslant I^\beta * H(x),$$

where $I^\beta(x)=|x|^{\beta-d}$ *is the Riesz kernel.*

Proof First assume that $\alpha>(d-1)|1/p-1/2|$. This assumption is stronger than the one in Lemma 2.1, therefore $S_*^\alpha(G^\beta * F)$ is well-defined. Since the Bessel kernel G^β is positive and it is dominated by I^β, we can estimate the maximal function as

$$\sup_{R>0}\left|S_R^\alpha * G^\beta * F(x)\right|\leqslant CI^\beta * \sup_{R>0}\left|S_R^\alpha * F\right|(x).$$

As stated in Lemma 2.2, if $\alpha>(d-1)|1/p-1/2|$ and $F\in L^p(\mathbf{R}^d)$, then also $\sup_{R>0}|S_R^\alpha * F|\in L^p(\mathbf{R}^d)$. Hence, (i) follows with $H=\sup_{R>0}|S_R^\alpha * F|$.

Now assume that $\alpha > d(1/2 - 1/p) - 1/2$ and $\alpha + \beta \geqslant (d-1)/2$. Then, by Lemma 2.3,

$$\sup_{R>0} \left| S_R^\alpha * G^\beta * F(x) \right| \leqslant |F| * \sup_{R>0} \left| S_R^\alpha * G^\beta \right|(x) \leqslant C I^\beta * |F|(x).$$

Hence, (ii) follows with $H = |F|$. □

Proof of Theorem 1.1 As observed before, the case $p = \infty$ is already known. Indeed, if $p = \infty$ and $\beta > 0$, then $f = G^\beta * F$ is bounded and uniformly continuous. Hence, if $\alpha > (d-1)/2$ then $S_R^\alpha * f$ converges to f uniformly. The case $p = 1$ is already contained in [6], however it is also a consequence of the case $p > 1$. Indeed, write $f = G^\beta * F = G^{\beta-\varepsilon} * G^\varepsilon * F$ with $0 < \varepsilon < \beta$. If F is in $L^1(\mathbf{R}^d)$ then, by the Hardy–Littlewood–Sobolev theorem of fractional integration [11, Theorem 5.1], $G^\varepsilon * F$ is in $L^p(\mathbf{R}^d)$ for every $p < d/(d-\varepsilon)$. Hence, assuming that the Theorem holds with $p > 1$, then $S_R^\alpha f$ converges up to a set with Hausdorff dimension at most $d - (\beta - \varepsilon)p$. Finally, letting $\varepsilon \to 0$ and $p \to 1$, one obtains convergence up to a set of dimension at most $d - \beta$. The case $1 < p < \infty$ and $0 < \beta \leqslant d/p$ follows from Lemma 2.4 and the notion of capacity. The proof is standard, anyhow for completeness we include some details. Let $\{F_n\}$ be a sequence of functions in the Schwartz class which converges to F in the metric of $L^p(\mathbf{R}^d)$ and let $f_n = G^\beta * F_n$. Since also f_n is in the Schwartz class, $\lim_{R\to+\infty} S_R^\alpha * f_n = f_n$ pointwise everywhere. Then, for every $t > 0$,

$$\begin{aligned}
&\left\{x : \limsup_{R\to+\infty} \left| S_R^\alpha * f(x) - f(x) \right| > t \right\} \\
&\quad \subseteq \left\{x : \sup_{R>0} \left| S_R^\alpha * f(x) - S_R^\alpha * f_n(x) \right| > t/2 \right\} \cup \left\{x : \left| f_n(x) - f(x) \right| > t/2 \right\}.
\end{aligned}$$

The Bessel capacity of the second term can be estimated by

$$\begin{aligned}
&B_{\beta,p}\left(\left\{x : \left| f_n(x) - f(x) \right| > t/2 \right\}\right) \\
&\quad \leqslant B_{\beta,p}\left(\left\{x : G^\beta * |F_n - F|(x) > t/2 \right\}\right) \\
&\quad \leqslant \left(\frac{t}{2}\right)^{-p} \|F_n - F\|_p^p.
\end{aligned}$$

Hence, this capacity tends to zero as $n \to +\infty$. In case (i) when $\alpha > (d-1)|1/p - 1/2|$, the Bessel capacity of the first term can be estimated by

$$\begin{aligned}
&B_{\beta,p}\left(\left\{x : \limsup_{R\to+\infty} \left| S_R^\alpha * f(x) - S_R^\alpha * f_n(x) \right| > t/2 \right\}\right) \\
&\quad \leqslant B_{\beta,p}\left(\left\{x : G^\beta * \sup_{R>0} \left| S_R^\alpha * (F - F_n) \right|(x) > t/2 \right\}\right)
\end{aligned}$$

$$\leqslant \left(\frac{t}{2}\right)^{-p} \left\| \sup_{R>0} \left| S_R^\alpha * (F - F_n) \right| \right\|_p^p$$

$$\leqslant C\left(\frac{t}{2}\right)^{-p} \|F_n - F\|_p^p.$$

Since $\|F_n - F\|_p \to 0$ as $n \to +\infty$, we obtain

$$B_{\beta,p}\left(\left\{x : \limsup_{R\to+\infty} \left|S_R^\alpha * f(x) - f(x)\right| > t\right\}\right) = 0.$$

In case (ii), when $\alpha \geqslant d(1/2 - 1/p) - 1/2$ and $\alpha + \beta \geqslant (d-1)/2$, the Riesz capacity of the second term can be estimated by

$$R_{\beta,p}\left(\left\{x : \limsup_{R\to+\infty} \left|S_R^\alpha * (f - f_n)\right|(x) > t/2\right\}\right)$$

$$\leqslant R_{\beta,p}\left(\left\{x : I^\beta * \left|(F - F_n)\right|(x) > Ct/2\right\}\right)$$

$$\leqslant \left(\frac{Ct}{2}\right)^{-p} \|F_n - F\|_p^p.$$

The last term tends to zero as $n \to +\infty$. Since the Riesz and Bessel capacities have the same null sets, see [16, p. 67], in both cases we get

$$B_{\beta,p}\left(\left\{x : \limsup_{R\to+\infty} \left|S_R^\alpha * f(x) - f(x)\right| > 0\right\}\right)$$

$$\leqslant \sum_{k=1}^{\infty} B_{\beta,p}\left(\left\{x : \limsup_{R\to+\infty} \left|S_R^\alpha * f(x) - f(x)\right| > 1/k\right\}\right)$$

$$= 0.$$

Therefore, applying [16, Theorem 2.6.16]), we obtain that the Hausdorff dimension of the set $\{x : \limsup_{R\to+\infty} |S_R^\alpha * f(x) - f(x)| > 0\}$ is at most $d - \beta p$.

If $1 < p \leqslant 2$ and $\beta > d/p$ then, by the Hausdorff-Young inequality, the Fourier transform of f is absolutely integrable. Indeed, if $1/p + 1/q = 1$,

$$\int_{\mathbf{R}^d} \left|\hat{f}(\xi)\right| d\xi = \int_{\mathbf{R}^d} \left(1 + |\xi|^2\right)^{-\beta/2} \left|\widehat{F}(\xi)\right| d\xi$$

$$\leqslant \left(\int_{\mathbf{R}^d} \left(1 + |\xi|^2\right)^{-\beta p/2} d\xi\right)^{1/p} \left(\int_{\mathbf{R}^d} \left|\widehat{F}(\xi)\right|^q d\xi\right)^{1/q}$$

$$\leqslant \left(\int_{\mathbf{R}^d} \left(1 + |\xi|^2\right)^{-\beta p/2} d\xi\right)^{1/p} \left(\int_{\mathbf{R}^d} \left|F(x)\right|^p dx\right)^{1/p}.$$

Hence, by the integrability of the Fourier transform, the inversion formula for the Fourier transform holds everywhere and the convergence of the Bochner–Riesz means is uniform,

$$\begin{aligned} \left|S_R^\alpha * f(x) - f(x)\right| &= \left|\int_{\mathbf{R}^d} \left(\left(1-\left|R^{-1}\xi\right|^2\right)_+^\alpha - 1\right)\hat{f}(\xi)\exp(2\pi i\xi x)\,d\xi\right| \\ &\leqslant \int_{\mathbf{R}^d} \left|\left(1-\left|R^{-1}\xi\right|^2\right)_+^\alpha - 1\right|\left|\hat{f}(\xi)\right|d\xi. \end{aligned}$$

It remains to consider the case $p > 2$, $\alpha > d(1/2-1/p)-1/2$ and $\beta > d/p$. In this case $\alpha+\beta > (d-1)/2$ and Lemma 2.3 applies. Hence, if $1/p+1/q=1$, then

$$\begin{aligned} \sup_{R>0}\left|S_R^\alpha * G^\beta * F(x)\right| &\leqslant \left(\sup_{R>0}\left|S_R^\alpha * G^\beta\right|\right) * |F|(x) \\ &\leqslant \left(\int_{\mathbf{R}^d}\left|\sup_{R>0}\left|S_R^\alpha * G^\beta(x)\right|\right|^q dx\right)^{1/q}\left(\int_{\mathbf{R}^d}\left|F(x)\right|^p dx\right)^{1/p}. \end{aligned}$$

Uniform convergence everywhere now follows from the uniform boundedness of this maximal function, via a standard density argument. □

Since functions with β derivatives in $L^p(\mathbf{R}^d)$ may be infinite on sets with Hausdorff dimension $d-\beta p$, the dimension of the divergence set in the statement of the Theorem cannot be decreased. When $p \geqslant 2$, part (i) of Theorem 1.1 can be easily improved by using the bounds for the maximal Bochner–Riesz operator in [1, 5, 7], and [10] for even dimensions. Finally, we want to remark again that our analysis is not exhaustive and the ranges of the indexes are not optimal. However, at least for radial functions, we can prove some definitive results:

Let $\alpha \geqslant 0$, $\beta \geqslant 0$, $1 \leqslant p \leqslant +\infty$, and $2d/(d+1+2\alpha+2\beta) < p < 2d/(d-1-2\alpha)$. Then the Bochner–Riesz means with index α of radial functions with β derivatives in $L^p(\mathbf{R}^d)$ converge pointwise, with the possible exception of a set of points Ω with the following properties:

(i) *if $\beta p \leqslant 1$, then the Hausdorff dimension of Ω is at most $d-\beta p$;*
(ii) *if $1 < \beta p \leqslant d$, then Ω either is empty or it reduces to the origin;*
(iii) *if $\beta p > d$, then Ω is empty.*

For these results, see [15]. The range of the indexes for non-radial functions cannot be larger than for radial function. Figure 1 shows the largest possible region of convergence for our problem. The dark shaded area corresponds to condition (i) and (ii) of Theorem 1.1, whereas we have proved convergence in the light shaded areas only for radial functions. In particular, observe the asymmetry between $p < 2$ and $p > 2$. When $p < 2$ the regularity β lowers the summability index α. On the other hand, when $p > 2$ the indexes of summability α and of regularity β are unrelated.

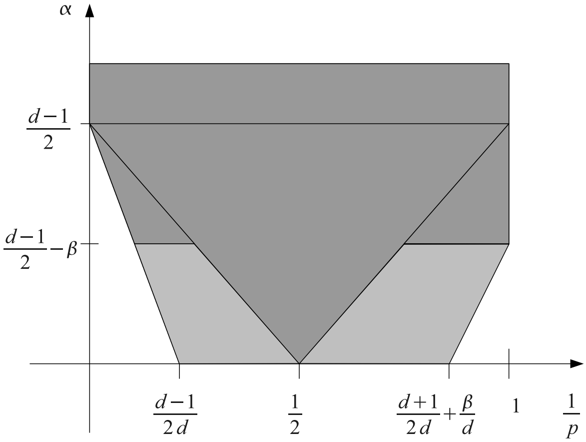

Fig. 1 The region of convergence

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