Stochastic Properties of Riemannian Manifolds and Applications to PDE's

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Abstract The aim of this note is to describe geometric conditions under which a Riemannian manifold enjoys the Feller property and to show how the validity of the Feller property in combination with stochastic completeness provides a new viewpoint to study qualitative properties of solutions of semilinear elliptic PDE's defined outside a compact set.

Keywords Feller property · Stochastic completeness · Comparison results

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1 Introduction

The asymptotic behavior of the heat kernel of a Riemannian manifold gives rise to the classical concepts of parabolicity, stochastic completeness (or conservative property) and Feller property (or C^0-diffusion property). Both parabolicity and stochastic completeness have been subject to a systematic study which led to the discovery not only of sharp geometric conditions for their validity but also of an incredibly rich family of tools, techniques and equivalent concepts ranging from maximum principles at infinity, function theoretic tests (Khas'minskii criterion), comparison techniques and so on. The purpose of this note is twofold. First we describe geometric conditions that ensure that a manifold enjoys the Feller property, for short,

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is Feller. We will see that, although there are similarities with the case of stochastic completeness, their situation is indeed quite different.

Our second goal is to describe the consequences of the Feller property on the behavior of solutions of PDE's involving the Laplacian. It is well understood that stochastic properties of a Riemannian manifold $(M, \langle\, , \rangle)$, like parabolicity and stochastic completeness have important counterparts on the behavior of solutions of PDE's defined on the manifold. Indeed, M is parabolic, respectively stochastically complete, if every subharmonic function bounded from above is constant, respectively if every non-negative bounded solution of the differential inequality $\Delta u \geqslant \lambda u$ for $\lambda > 0$ is constant, and therefore vanishes identically. It is apparent by the very definition of these stochastic properties that global solutions must be considered. The introduction of the Feller property, that is the property that the heat semigroup maps the space of continuous functions vanishing at infinity into itself, in combination with stochastic completeness, will enable us to get important information even in the case of solutions at infinity.

In fact, using a suitable comparison theory, we are going to show that manifolds which are both stochastically complete and Feller do represent a natural environment where solutions of PDE's at infinity can be studied.

Sections 2–4 contain foundational material and the results recently obtained in [20]. In Sect. 5 we present application of the Feller property to geometry and PDE's taken from [3].

2 Stochastic Completeness vs. the Feller Property

In what follows, $(M, \langle\, , \rangle)$, often abbreviated by M, denotes a connected complete Riemannian manifold of dimension m, and $d\,\mathrm{vol}$, ∇, Δ are the Riemannian measure, the gradient and the Laplace operator of M. We denote by $B(x, r)$ and $\partial B(x, r)$ the geodesic ball of radius r centered at x and its boundary. Let g_{ij} be the components of the metric $\langle\, , \rangle$ in local coordinates x^i, g^{ij} the components of the inverse matrix, and $g = \det g_{ij}$. Recall that

$$d\,\mathrm{vol} = \sqrt{g}\,dx, \qquad \nabla f = g^{ij}\frac{\partial f}{\partial x^i}\frac{\partial}{\partial x^j}, \qquad \Delta f = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(\sqrt{g}g^{ij}\frac{\partial f}{\partial x^j}\right).$$

The heat kernel $p_t(x, y)$ of M is the minimal positive solution of the problem

$$\begin{cases} \Delta p_t = \frac{\partial p_t}{\partial t}, \\ p_{0+}(x, y) = \delta_y(x), \end{cases} \tag{1}$$

and can be obtained as limit of the Dirichlet heat kernels $p_t^{\Omega_n}(x, y)$ of any smooth, relatively compact exhaustion $\Omega_n \nearrow M$ (see details in [10]). We recall the following properties:

(i) $p_t(x, y) > 0$ is a symmetric function of x and y.
(ii) $\int_M p_t(x, z)p_s(z, y)\,d\,\mathrm{vol}(z) = p_{t+s}(x, y)$ for every $t, s > 0$ and $x, y \in M$.

(iii) $\int_M p_t(x,y)\,d\,\mathrm{vol}(y) \leqslant 1$, for every $t > 0$ and $x \in M$.
(iv) For every bounded continuous function u on M, if we set

$$P_t u(x) = \int_M p_t(x,y)u(y)\,d\,\mathrm{vol}(y),$$

then $P_t u(x)$ satisfies the heat equation on $M \times (0,+\infty)$. Moreover, by (ii) and (iii), P_t extends to a contraction semigroup on every L^p, called the heat semigroup of M.

From the probabilistic viewpoint, the heat kernel $p_t(x,y)$ represents the transition probability density of the Brownian motion $t \to X_t$ of M. In this respect, property (iii) stated above means that $t \to X_t$ is, in general, sub-Markovian.

Definition 2.1 We say that M is stochastically complete if heat is conserved, i.e., if for all $t > 0$ and some (and therefore all) $x \in M$

$$\int_M p_t(x,y)\,d\,\mathrm{vol}(y) = 1.$$

Stochastic completeness has a number of equivalent formulations. For instance,

- solutions of the heat equation with bounded initial data are unique;
- for some (and therefore all) $\lambda > 0$, bounded nonnegative solutions on M of the differential inequality $\Delta u \geqslant \lambda u$, vanish identically. See [13].

For the purposes of this note the most useful equivalent formulation is in terms of the weak maximum principle at infinity:

Definition 2.2 We say that the weak maximum principle at infinity holds on M if, for every $u \in C^2(M)$ with $\sup_M u = u^* < +\infty$, there exists a sequence $\{x_k\}$ along which

$$\text{(i)} \quad u(x_k) > u^\star - \frac{1}{k}, \qquad \text{(ii)} \quad \Delta u(x_k) < \frac{1}{k}.$$

It was proved in [17] (see also [18]) that

- M is stochastically complete if and only if the weak maximum principle at infinity holds on M.

The geometric conditions which imply stochastic completeness are subsumed either by a lower bound on the Ricci curvature Ricc of the underlying manifold or by an upper bound on the volume growth of geodesic balls.

Theorem 2.3 *Let M be a complete Riemannian manifold and $r(x) = \mathrm{dist}(o,x)$ denote the geodesic distance function from a reference point o. Then M is stochastically complete provided one of the following conditions hold*:

(i) *the Ricci curvature satisfies* $\mathrm{Ricc}(x) \geqslant -G^2(r(x))$, *where* G *is a positive, continuous increasing function satisfying* $\int^{+\infty} \frac{1}{G(r)}\,dr = +\infty$ [14, 22];
(ii) $\int^{+\infty} \frac{r}{\log(\mathrm{vol}\,B(o,r))}\,dr = +\infty$ [11].

These two conditions are essentially sharp, and although Ricci curvature lower bounds imply volume upper bounds, by the Bishop–Gromov volume comparison theorem, the conditions are related but independent. Note that heuristically, the obstruction to stochastic completeness is the fact that the manifold grows too fast at infinity.

We recall for comparison that M is parabolic if positive superharmonic functions are necessarily constant. This is equivalent to the non-existence of a positive minimal Green's kernel, and to the recurrence of Brownian motion. We also recall that a geodesically complete manifold is parabolic provided it has at most quadratic volume growth. Indeed, a sufficient condition for parabolicity is that

$$\int^{+\infty} \frac{1}{\mathrm{vol}\,\partial B(o,r)}\,dr = +\infty.$$

Let us now turn to the Feller condition.

Definition 2.4 We say that M satisfies the Feller condition, (for short, that M is Feller), if the heat semigroup P_t maps $C_0(M)$ into itself, that is, if

$$P_t u(x) = \int_M p_t(x,y)u(y)\,d\,\mathrm{vol}(y) \to 0, \quad \text{as } x \to +\infty \tag{2}$$

for every $u \in C_0(M) = \{u\colon M \to \mathbb{R} \text{ continuous}\colon u(x) \to 0 \text{ as } x \to \infty\}$.

Since $p_t(x,\cdot)$ is uniformly integrable, using a cut-off argument, one can easily prove the following

Lemma 2.5 *Assume that* M *is geodesically complete. Then* M *is Feller if and only if it satisfies one of the following equivalent conditions*:

(i) *the limit in* (2) *holds for every non-negative function* $u \in C_c(M)$;
(ii) *for some* (*and therefore all*) $p \in M$ *and for every* $R > 0$,

$$\int_{B(p,R)} p_t(x,y)\,d\,\mathrm{vol}(y) \to 0 \quad \text{as } x \to +\infty.$$

According to R. Azencott [1], the Feller property can be characterized in terms of asymptotic properties of solutions of exterior boundary value problems. We write

$$\Omega \Subset M \quad \text{if } \overline{\Omega} \text{ is compact and contained in } M.$$

Given a smooth open set $\Omega \Subset M$ and $\lambda > 0$, the problem

$$\begin{cases} \Delta h = \lambda h & \text{on } M \setminus \overline{\Omega}, \\ h = 1 & \text{on } \partial\Omega, \\ h > 0 & \text{on } M \setminus \Omega \end{cases} \tag{3}$$

has a (unique) minimal smooth solution $h : M \setminus \overline{\Omega} \to \mathbb{R}$. By the maximum principle $0 < h \leqslant 1$ and h is obtained as the limit $h(x) = \lim_{n\to+\infty} h_n(x)$, where Ω_n is a smooth exhaustion of M and h_n solves $\Delta h_n = \lambda h_n$ on $\Omega_n \setminus \overline{\Omega}$ and has boundary values $h_n = 1$ on $\partial\Omega$ and $h = 0$ on $\partial\Omega_n$.

Theorem 2.6 [1] *M is Feller if and only if for some* (*hence any*) *open set $\Omega \Subset M$ with smooth boundary and for some* (*hence any*) *constant $\lambda > 0$, the minimal solution $h : M \setminus \Omega \to \mathbb{R}$ of problem* (3) *satisfies*

$$h(x) \to 0 \quad \text{as } x \to \infty. \tag{4}$$

After the pioneering work of Azencott, the investigation has focused on finding optimal geometric conditions ensuring that a manifold is Feller [8, 10, 14–16, 23], and with the only exception of [8], the geometric conditions are always expressed in terms of Ricci curvature lower bounds. The methods range from estimates of solutions of parabolic equations [10, 16, 23] to estimates of the probability that the Brownian motion on M be found in certain regions before a fixed time [14]. The best known result in this direction is due to E. Hsu [14]. It uses a probabilistic approach and relies on a result by Azencott (see also [15]) according to which *M is Feller if and only if, for every compact set K and for every $t_0 > 0$, the probability that Brownian motion X_t issuing from x_0 enters K before the time t_0 tends to zero as $x_0 \to \infty$.*

Theorem 2.7 [14] *Let M be a complete, non compact Riemannian manifold of dimension* $\dim M = m$. *Assume that*

$$\mathrm{Ricc} \geqslant -(m-1)G^2\big(r(x)\big), \tag{5}$$

where $r(x) = \mathrm{dist}(x, o)$ is the distance function from a fixed reference point $o \in M$ and G is a positive, increasing function on $[0, +\infty)$ *satisfying*

$$\frac{1}{G} \notin L^1(+\infty). \tag{6}$$

Then M is Feller.

It is remarkable that, to the best to our knowledge, there is no analytic proof of this result. Note also that (6) is precisely the condition on the Ricci curvature that ensures the stochastic completeness of M. So one may be led to believe that as in the case of stochastic completeness "big volumes" are an obstruction to the Feller property. In fact this is not the case: in some sense, the obstruction is given by "small volumes". Indeed we have the following:

Theorem 2.8 [1] *If M is a Cartan–Hadamard manifold* (*complete*, *simply connected with nonpositive sectional curvature*), *then M is Feller.*

3 Model Manifolds and Comparison Results

Model manifolds also shed light on the relationship between the Feller property and the geometry of the manifold. Recall that a model manifold M^m_f is $\mathbb{R}^m = [0, \infty) \times \mathbb{S}^{m-1}$ with the metric given in polar coordinates by $\langle\, , \rangle = dr^2 + f(r)^2 d\theta^2$, where f is odd and $f'(0) = 1$. For instance if $f(r) = r$ then $M^m = \mathbb{R}^m$, if $f(r) = \sin r$ then $M^m = \mathbb{S}^m$ and if $f(r) = \sinh r$ then $M^m = \mathbb{H}^m$.

The following result holds.

Theorem 3.1 [1, 20] *An m-dimensional model manifold M^m_f with warping function f is Feller if and only if either*

$$\frac{1}{f^{m-1}(r)} \in L^1(+\infty) \tag{7}$$

or

$$\text{(i)} \quad \frac{1}{f^{m-1}(r)} \notin L^1(+\infty) \quad \textit{and} \quad \text{(ii)} \quad \frac{\int_r^{+\infty} f^{m-1}(t)\, dt}{f^{m-1}(r)} \notin L^1(+\infty). \tag{8}$$

In (8), *condition* (ii) *is considered automatically satisfied if $f^{m-1} \notin L^1(+\infty)$.*

Proof The proof is of elliptic nature. One easily observes that, on a model manifold, the minimal solution of the problem

$$\begin{cases} \Delta h = \lambda h & \text{on } M^m_f \setminus \overline{B(0,1)}, \\ h = 1 & \text{on } \partial B(0,1), \\ h > 0 & \text{on } M^m_f \setminus B(0,1) \end{cases}$$

is necessarily radial.

Next, one shows that the minimal solution $h(r)$ of the radialized 1-dimensional problem

$$\begin{cases} \left(f^{m-1} h'\right)' = \lambda f^{m-1} h & \text{on } (1, +\infty), \\ h(1) = 1 \end{cases}$$

tends to zero as $r \to +\infty$ if and only either condition (7) or condition (8) holds. □

Since $\operatorname{vol} \partial B(0, r) = c_m f^{m-1}(r)$, conditions (7) and (8) can be restated in more geometrical terms by saying that M is Feller if either

$$\frac{1}{\operatorname{vol}(\partial B_r)} \in L^1(+\infty) \tag{9}$$

or

$$\text{(i)} \quad \frac{1}{\mathrm{vol}(\partial B_r)} \notin L^1(+\infty) \quad \text{and} \quad \text{(ii)} \quad \frac{\mathrm{vol}(M_g)}{\mathrm{vol}(\partial B_r)} - \frac{\mathrm{vol}(B_r)}{\mathrm{vol}(\partial B_r)} \notin L^1(+\infty). \quad (10)$$

In particular, a model manifold with infinite volume is always Feller.

Note also that $1/f^{m-1}(r) \in L^1(+\infty)$ is the necessary and sufficient condition for a model manifold M_f to be non-parabolic. Indeed,

$$G(x,0) := \int_{r(x)}^{+\infty} \frac{dt}{f^{m-1}(t)}$$

is the Green kernel with pole at 0 of the Laplace–Beltrami operator of M_f^m. Since parabolicity implies stochastic completeness, stochastically incomplete models are always Feller.

For comparison, it may be interesting to notice that a model manifold M_f^m is stochastically complete if

$$\int^{+\infty} \frac{\int_0^r f^{m-1}(s)\,ds}{f^{m-1}(r)}\,dr = +\infty\,,$$

that is if and only if

$$r \to \frac{\mathrm{vol}\,B(0,r)}{\mathrm{vol}\,\partial B(0,r)} \notin L^1(+\infty).$$

Indeed, the function

$$u(x) = \int_0^{r(x)} \frac{\int_0^r f^{m-1}(s)\,ds}{f^{m-1}(r)}\,dr$$

satisfies $\Delta u = 1$. Therefore, if it is bounded, it violates the weak maximum principle at infinity and M_f^m is not stochastically complete. The other implication follows from a comparison argument.

However, neither parabolicity nor, a fortiori, stochastic completeness imply the Feller property. Indeed, fix $\beta > 2$ and $\alpha > 0$, and let $f(t) : \mathbb{R} \to \mathbb{R}$ be any smooth, positive, odd function satisfying $f'(0) = 1$ and $f(r) = \exp(-\alpha r^\beta)$ for $r \geqslant 10$. Then,

$$\frac{1}{f(r)} = \exp\big(\alpha r^\beta\big) \notin L^1(+\infty) \quad \text{and} \quad \frac{\int_r^{+\infty} f(t)\,dt}{f(r)} \asymp r^{1-\beta} \in L^1(+\infty)$$

and the 2-dimensional model M_f^2 is parabolic and therefore stochastically complete, but it is not Feller. Actually we shall see in Sect. 4 below that, using a gluing technique, one can construct Feller manifolds which are neither parabolic nor stochastically complete.

Parabolicity and stochastic completeness of a general manifold can be deduced from those of a model manifold via curvature comparisons (see, e.g., [13]). We are going to describe how this technique may be extended to the Feller property.

To this goal, recall that the minimal solution h of the exterior problem (3) is the limit of solutions h_n which vanish on the boundary $\partial\Omega_n$ of an exhaustion of M. Therefore standard comparison results show that, if u is a supersolution of (3), that is

$$\begin{cases} \Delta u \leqslant \lambda u & \text{on } M \setminus \overline{\Omega}, \\ u \geqslant 1 & \text{on } \partial\Omega, \end{cases}$$

then

$$h \leqslant u, \quad \text{on } M \setminus \Omega.$$

In particular, if $u(x) \to 0$ as $x \to \infty$, then M is Feller.

In the case where the manifold M is stochastically complete we obtain a somewhat complementary result.

Theorem 3.2 *Let M be stochastically complete, and let u be a bounded solution $u > 0$ of*

$$\Delta u \geqslant \lambda u$$

outside a smooth domain $\Omega \Subset M$. If $h > 0$ is the minimal solution of

$$\begin{cases} \Delta h = \lambda h & \text{on } M \setminus \overline{\Omega}, \\ h = 1 & \text{on } \partial\Omega, \end{cases}$$

then there is a constant $c > 0$ such that

$$u(x) \leqslant \mathrm{ch}(x) \quad \text{on } M \setminus \Omega.$$

Proof Let $c = \sup_{\partial\Omega} u$. Then, for every $\varepsilon > 0$, $\Delta(u - ch - \varepsilon) \geqslant \lambda(u - ch) \geqslant \lambda(u - ch - \varepsilon)$ on $M \setminus \overline{\Omega}$ and $u - ch - \varepsilon \leqslant -\varepsilon$ on $\partial\Omega$. Therefore the function $v_\varepsilon = \max\{0, u - ch - \varepsilon\}$ is bounded, non-negative and satisfies $\Delta v_\varepsilon \geqslant \lambda v_\varepsilon$. Since M is stochastically complete $v_\varepsilon \equiv 0$, that is, $u \leqslant ch + \varepsilon$. The conclusion follows letting $\varepsilon \to 0$. □

In particular, if $h(x) \to 0$ as $x \to \infty$, we can deduce that the same holds for the original function u. This leads to the following

Corollary 3.3 [20] *Let M be stochastically complete. If M is Feller, then every bounded solution $v > 0$ of*

$$\Delta v \geqslant \lambda v \quad \text{on } M \setminus \overline{\Omega}$$

satisfies

$$v(x) \to 0 \quad \text{as } x \to \infty.$$

To state the announced result of comparison with models, given a smooth even function $G : \mathbb{R} \to \mathbb{R}$, we let $f : [0, +\infty) \to [0 + \infty)$ be the unique solution of the Cauchy problem

$$\begin{cases} f'' + Gf = 0, \\ f(0) = 0, \quad f'(0) = 1. \end{cases} \tag{11}$$

Then we have:

Theorem 3.4 [20] *Let M be a complete Riemannian m-manifold.*

(i) *Assume that M has a pole at o and that the radial sectional curvature with respect to o satisfies*

$$^M\operatorname{Sec}_{\mathrm{rad}} \leqslant G\bigl(r(x)\bigr) \quad \textit{on } M \tag{12}$$

for some smooth even function $G : \mathbb{R} \to \mathbb{R}$. If the m-dimensional model M_f^m is Feller then M is Feller.

(ii) *Assume that the radial Ricci curvature of M satisfies*

$$^M\operatorname{Ricc}(\nabla r, \nabla r) \geqslant (m-1)G\bigl(r(x)\bigr),$$

where $r(x) = \operatorname{dist}(x, o)$.

If the m-dimensional model M_f^m is not Feller (thus, it has finite volume) then also M is not Feller.

Proof We give only an outline of the proof. In case (i), one shows that the minimal radial solution α of the exterior problem on $M_f \setminus B^{M_f}(0,1)$ is decreasing and since M_f^m is Feller it tends to zero at infinity.

Let $u(x) = \alpha(r(x))$. Then the curvature condition and the Laplacian Comparison Theorem imply that $\Delta r \geqslant (m-1)f'/f$, hence

$$\Delta u = \alpha''\bigl(r(x)\bigr) + \alpha'\bigl(r(x)\bigr)\Delta r \leqslant \alpha''\bigl(r(x)\bigr) + (m-1)\frac{f'}{f}\alpha'\bigl(r(x)\bigr) = \lambda u.$$

By the comparison result, the minimal solution of the exterior Dirichlet problem h satisfies $h \leqslant u$. Since $u(r(x)) \to 0$ as $r(x) \to \infty$, M is Feller.

To prove (ii), let us note that since M_f is not Feller, by Theorem 3.1, $f^{m-1} \in L^1(+\infty)$, $1/f^{m-1} \notin L^1(+\infty)$ and

$$\frac{\int_r^{+\infty} f^{m-1}(t)\,dt}{f^{m-1}(r)} \in L^1(+\infty).$$

Define

$$\alpha(r) = \int_r^{+\infty} \frac{\int_s^{+\infty} f^{m-1}(t)\,dt}{f^{m-1}(s)}\,ds.$$

A direct computation shows that

$$^{M_f}\Delta\alpha = 1.$$

Now consider

$$v(x) = \alpha\bigl(r(x)\bigr) + 1 \quad \text{on } M \setminus B_1.$$

Clearly, v is a positive bounded function, and since $\alpha' \leqslant 0$, by Laplacian comparison we have

$$\Delta v \geqslant 1 \geqslant \lambda v,$$

where $\lambda = 1/\sup v$. Since $v(x) \to 1$ as $x \to \infty$, by Corollary 3.3 M is not Feller. □

Note that in (i) above the (sectional) curvature is bounded from *above*. This is the opposite of the inequality assumed in Hsu's result, and it shows that, in contrast with what happens for stochastic completeness, Hsu's result is a genuine estimation result, and does not follow from a comparison argument.

4 Ends and Further Geometric Conditions for the Feller Property

It is clear that the Feller property is affected only by the properties of M outside a compact set Ω. The set $M \setminus \Omega$ has a finite number of unbounded connected components E_i, called the ends of M with respect to Ω. Thus, the minimal solution h of

$$\begin{cases} \Delta h = \lambda h & \text{on } M \setminus \overline{\Omega}, \\ h = 1 & \text{on } \partial \Omega, \\ h > 0 & \text{on } M \setminus \Omega, \end{cases}$$

restricts to the minimal solution h_j of the same Dirichlet problem on E_j with respect to the compact boundary ∂E_j. Furthermore, h tends to zero at infinity in M if and only if each function $h_j(x)$ tends to 0 as $E_j \ni x \to \infty$.

This suggests that the property of being Feller may be localized at the ends of M.

We say that an end E is Feller if, for some $\lambda > 0$, the minimal solution $g : E \to (0, 1]$ of the Dirichlet problem

$$\begin{cases} \Delta g = \lambda g & \text{on int}(E), \\ g = 1 & \text{on } \partial E \end{cases}$$

satisfies $g(x) \to 0$ as $x \to \infty$. The usual exhausting procedure shows that g actually exists. The following statement holds:

Proposition 4.1 *Let $(M, \langle\, , \rangle)$ be a complete Riemannian manifold and let $E_1, \ldots, E_k$ be the ends of M with respect to the smooth compact domain Ω. Then, the following are equivalent:*

(i) *M is Feller;*
(ii) *each end E_j has the Feller property;*
(iii) *the double $\mathcal{D}(E_j)$ of each end has the Feller property.*

Using this observation, one can easily construct new Feller or non-Feller manifolds from old ones by adding suitable ends. For instance, consider complete Riemannian manifolds M and N of the same dimension m and form their connected sum $M\#N$. This latter is Feller if and only if both M and N has the Feller property.

Since the same property also holds for parabolicity and stochastic completeness (for the latter see [2]), one may then construct examples of manifolds which show that are no obvious implications between stochastic completeness and the Feller property. By way of example, consider the warped product $M = \mathbb{R} \times_f \mathbb{S}^{m-1}$ with warping function $f(t)$ such that

$$f(t) = \begin{cases} e^{t^4} & \text{if } t \geqslant 1, \\ e^{-t^4} & \text{if } t \leqslant -1. \end{cases}$$

Then the "positive" end of M is Feller and stochastically incomplete, while the "negative" end of M is parabolic and non-Feller, so that M is both non-Feller and stochastically incomplete.

We conclude this summary of the geometric properties leading to the Feller property with two last results.

Using heat kernel estimates in the presence of an isoperimetric inequality of A. Grigor'yan [12] and a result of G. Carron [7] we obtain the following result.

Theorem 4.2 [20] *Assume that M supports an L^2-Sobolev inequality of the form*

$$\|\nabla u\|_{L^2} \geqslant S_{2,p} \|u\|_{L^{\frac{2p}{p-2}}}, \quad \textit{for every } u \in C^1_c(M).$$

Then M is Feller.

Note that according to a result of Carron [6], if the L^2-isoperimetric inequality holds off a compact set then it holds everywhere and M is Feller.

Corollary 4.3 [20] *Let M be isometrically immersed into a Cartan–Hadamard manifold. If its mean curvature vector field H satisfies*

$$\|H\|_{L^m(M)} < +\infty,$$

then M is Feller. In particular,

(i) *every Cartan–Hadamard manifold is Feller*;
(ii) *every complete, minimal submanifold in a Cartan–Hadamard manifold is Feller.*

The above result has been completed in the very recent paper [5], where it is shown that bounded mean curvature hypersurfaces properly immersed in Cartan–Hadamard manifold are Feller. The proof relies upon the comparison principle (Theorem 3.2), by means of a suitable test function $u(x)$.

Finally, we address the following

Problem 4.4 Suppose we are given a Riemannian covering

$$\pi : \left(\widehat{M}, \widehat{\langle\, , \rangle}\right) \to \left(M, \langle\, , \rangle\right).$$

What are the relationships between the validity of the Feller property on the covering space $\widehat{M}$ and on the base manifold M?

By comparison, recall that M is stochastically complete if and only if so is $\widehat{M}$ (see, e.g., [9]). As for parabolicity, the situation is quite different. Using subharmonic functions it is easy to see that if $\widehat{M}$ is parabolic then the base manifold M is also parabolic. In general, the converse is not true, as shown e.g. by the twice punctured complex plane, which is a parabolic manifold, as can be seen by using the well know Khas'minskii test [13], and which is universally covered by the (non-parabolic) Poincaré disk.

Let us now consider the Feller property. To begin with, consider the easiest case of coverings with a finite number of sheets.

Observe that in a finite covering one can pass from functions on M to functions on $\widehat{M}$, and vice-versa, and that a sequence of points in $\widehat{M}$ goes to infinity if and only if their projections tend to infinity in M. So one has

Proposition 4.5 [20] *Let $\pi : (\widehat{M}, \widehat{\langle\, , \rangle}) \to (M, \langle\, , \rangle)$ be a k-fold Riemannian covering, with $k < +\infty$. Then $\widehat{M}$ is Feller if and only if M is Feller.*

In general

$$\widehat{M} \text{ Feller} \not\Longleftarrow M \text{ Feller}.$$

Consider the 2-dimensional warped product $M = \mathbb{R} \times_f \mathbb{S}^1$ where $f(t) = e^{t^3}$. Combining the necessary and sufficient condition for a model to be Feller and the results on the Feller property for manifolds with ends, we see that M is not Feller. But since the Gaussian curvature of M is given by

$$K(t, \theta) = -\frac{f''(t)}{f(t)} \leqslant 0,$$

the universal covering $\widehat{M}$ is Cartan–Hadamard, and hence Feller by Azencott's result.

However the reverse implication

$$M \text{ is Feller} \Longrightarrow \widehat{M} \text{ is Feller}$$

always holds. Indeed, by means of results by M. Bordoni [4] on the relationship between the heat kernel of M and that of its covering $\widehat{M}$, we obtain

Theorem 4.6 [20] *If M is Feller then so is $\widehat{M}$.*

5 Applications to Geometry and PDE's

The weak maximum principle at infinity is a powerful tool to deduce qualitative information on the solutions of differential inequalities of the form

$$\Delta u \geqslant \Lambda(u). \tag{13}$$

Indeed, it implies that every solution u of (13) on the whole manifold M such that $u^* = \sup_M u < +\infty$ satisfies

$$\Lambda\big(u^*\big) \leqslant 0.$$

This fact has many applications in geometric analysis. Our aim is to apply the Feller property to investigate qualitative properties of solutions of (13) which are defined only in a neighborhood of infinity. This section is based on [3].

Recall that, according to Corollary 3.3, if M is stochastically complete and Feller, then every bounded solution $v > 0$ of $\Delta v \geqslant \lambda v$ on $M \setminus \overline{\Omega}$ satisfies $v(x) \to 0$ as $x \to \infty$. On the basis of these remarks, we prove the following:

Theorem 5.1 *Let M be a stochastically complete and Feller manifold. Consider the differential inequality*

$$\Delta u \geqslant \Lambda(u) \quad \text{on } M \setminus \Omega, \tag{14}$$

where $\Omega \Subset M$ and $\Lambda : [0, +\infty) \to [0, +\infty)$ is either continuous or it is a non-decreasing function which satisfies the following conditions:

(i) $\Lambda(0) = 0$; (ii) $\Lambda(t) > 0$ *for every* $t > 0$; (iii) $\liminf_{t \to 0+} \dfrac{\Lambda(t)}{t^\xi} > 0$

for some $0 \leqslant \xi \leqslant 1$. *Then every bounded solution $u > 0$ of (14) must satisfy*

$$\lim_{x \to \infty} u(x) = 0.$$

Proof Suppose Λ is non-decreasing. By assumption, there exists $0 < \varepsilon < 1/2$ and $c > 0$ such that

$$\Lambda(t) \geqslant c t^\xi \quad \text{on } (0, 2\varepsilon).$$

As $t^\xi \geqslant t$ on $(0, 1]$, and Λ is non-decreasing, then

$$\Lambda\big(u(x)\big) \geqslant \Lambda_\varepsilon\big(u(x)\big) = \begin{cases} cu & \text{if } u(x) < \varepsilon, \\ c\varepsilon & \text{if } u(x) \geqslant \varepsilon. \end{cases}$$

Since $u > 0$ is bounded, if we set $u^\star = \sup_{M \setminus \Omega} u$, then

$$c\varepsilon \geqslant \frac{c\varepsilon}{u^\star} u^\star \geqslant \frac{c\varepsilon}{u^\star} u.$$

It follows that

$$\Delta u \geqslant \Lambda_\varepsilon(u) \geqslant \lambda u,$$

where

$$\lambda = c \min\left\{1, \frac{\varepsilon}{u^\star}\right\} > 0.$$

Using the Feller property we now conclude that $u(x) \to 0$ as $x \to \infty$. □

As shown in Theorem 5.1, using the Feller property on a stochastically complete manifold enables one to extend the investigation of qualitative properties of solution of PDEs to the case where these are defined only in a neighborhood of infinity.

We are going to exemplify the use of this viewpoint in various geometric and analytic settings. We stress that the needed stochastic assumptions are enjoyed by a very rich family of examples. For instance, as seen in Sect. 2, we have the class of complete manifolds such that Ricc $\geqslant -G^2(r)$, where $G(r) > 0$ is an increasing function satisfying $1/G \notin L^1(+\infty)$. Another admissible category is given by Cartan–Hadamard manifolds, or minimal submanifolds of Cartan–Hadamard manifolds (which are Feller by Corollary 4.3) with at most quadratic exponential volume growth (to guarantee stochastic completeness).

5.1 Isometric Immersions

An application of the weak maximum principle shows that if a Riemannian manifold $(M, \langle\,,\rangle)$ is stochastically complete, then the mean curvature $\mathbf{H}$ of a bounded isometric immersion $f : M \to \mathbb{B}(O, R) \subset \mathbb{R}^n$ must satisfy

$$\sup_M |\mathbf{H}| R \geqslant 1.$$

In particular, a stochastically complete minimal submanifold in Euclidean space is necessarily unbounded.

The next result extends this to the case where the complement of a compact domain in M admits a bounded isometric immersion into $\mathbb{R}^n$.

Theorem 5.2 *Let the Riemannian manifold M be stochastically complete and Feller. Assume that, outside a compact set $\Omega \subset M$, there exists a bounded isometric immersion $f : M \setminus \Omega \to \mathbb{B}(O, R) \subset \mathbb{R}^n$. Then*

$$\sup_{M\setminus\Omega} |\mathbf{H}| R \geqslant 1.$$

Proof Assume by contradiction that

$$\sup_{M\setminus\Omega} |\mathbf{H}| R < 1, \tag{15}$$

and let $u(x) = |f(x) - O|^2 \geqslant 0$. Then

$$\Delta u \geqslant c \quad \text{on } M \setminus \overline{\Omega},$$

where we have set $c = 2m(1 - \sup_{M\setminus\Omega} |\mathbf{H}|R) > 0$. By Theorem 5.1, $u \to 0$ and therefore $f(x) \to O$ as $x \to \infty$.

Now, as strict inequality holds in (15), for $R' > R$ sufficiently close to R we have $\sup_{M\setminus\Omega} |\mathbf{H}|R' < 1$, and clearly $f(M \setminus \Omega) \subset \mathbb{B}(O', R')$ provided $|O' - O| < R' - R$. Thus we can repeat the argument with $u'(x) = |f(x) - O'|^2$ for which again we have

$$\Delta u' \geqslant c$$

with the same value c, and then $u'(x) \to 0$, i.e., $f(x) \to O' \neq O$, as $x \to \infty$. This yields the required contradiction and proves the theorem. □

We note that a modification of the above argument allows to consider just one of the ends of M with respect to Ω is isometrically immersed in a ball.

5.2 *Conformal Deformations*

Given a Riemannian manifold $(M, \langle\, ,\rangle)$ of dimension $m \geqslant 3$ consider the conformally related metric $\overline{\langle\, ,\rangle} = v^{\frac{4}{m-2}} \langle\, ,\rangle$ where $v > 0$ is a smooth function. Thus, the conformality factor v obeys the Yamabe equation

$$c_m^{-1} \Delta v - Sv = -\overline{S} v^{\frac{m+2}{m-2}},$$

where S and $\overline{S}$ denote the scalar curvatures of $\langle\, ,\rangle$ and $\overline{\langle\, ,\rangle}$, respectively. Assume that M is stochastically complete and that

$$\sup_M S(x) \leqslant S^\star, \qquad \inf_M \overline{S}(x) \geqslant \overline{S}_\star,$$

for some constants $S^\star \geqslant 0$ and $\overline{S}_\star > 0$. An application of the weak minimum principle at infinity to the Yamabe equation shows that

$$\left(\frac{S^\star}{\overline{S}_\star}\right)^{\frac{m-2}{4}} \geqslant v_\star = \inf_M v.$$

In particular, if $S(x) \leqslant 0$ on M, then $v_\star = 0$. Actually, since the infimum of v cannot be attained, for every $\Omega \Subset M$

$$\inf_{M\setminus\Omega} v = 0.$$

Clearly, to reach these conclusions the scalar curvature bound must hold on M. As a consequence of Theorem 5.1, we obtain the following non-existence result.

Note that this applies e.g. to an expanding, gradient Ricci soliton M. Indeed, in this case, the scalar curvature assumption is compatible with the restriction $\inf_M S \leqslant 0$ imposed by the soliton structure.

Theorem 5.3 *Let $(M, \langle\, ,\rangle)$ be a stochastically complete and Feller manifold of dimension $m \geqslant 6$ such that, for some relatively compact domain Ω*

$$\sup_{M \setminus \Omega} S(x) \leqslant 0.$$

On M, one cannot perform a conformal change $\overline{\langle\, ,\rangle} = v^{\frac{4}{m-2}} \langle\, ,\rangle$ in such a way that

$$0 < v_\star \leqslant v(x) \leqslant v^\star < +\infty$$

and

$$\liminf_{x\to\infty} \overline{S}(x) = \overline{S}_\star > 0.$$

Proof Simply observe that the positive, bounded function $u(x) = v(x)^{-1}$ satisfies

$$c_m^{-1} \Delta u \geqslant -Su + \overline{S} u^{\frac{m-6}{m-2}} \geqslant \overline{S} u^{\frac{m-6}{m-2}}.$$

Since

$$0 \leqslant \frac{m-6}{m-2} < 1,$$

Theorem 5.1 yields

$$u(x) \to 0 \quad \text{as } x \to \infty. \qquad \square$$

One may wonder if the assumption that S be nonnegative at infinity implies that it can be made nonnegative everywhere on M with a conformal change of metric. However this in general would require a somewhat implicit control on the positive part of S in the set Ω (see, e.g., Proposition 1.2 in [21]).

5.3 Compact Support Property of Bounded Solutions of PDEs

We say that a certain PDE satisfies the compact support principle if all solutions in the exterior of a compact set which are non-negative and decay at infinity must have compact support. We are going to analyze some situations where the decay assumption can be relaxed. This has applications to the Yamabe problem.

Theorem 5.4 *Let M be a geodesically complete and stochastically complete, Cartan–Hadamard manifold. Let $u \geqslant 0$ be a bounded solution of*

$$\Delta u \geqslant \Lambda(u) \quad \text{on } M \setminus \Omega \tag{16}$$

for some domain $\Omega \Subset M$ *and for some non-decreasing function* $\Lambda : [0, +\infty) \to [0, +\infty)$ *satisfying the following conditions*:

$$\text{(i)} \quad \Lambda(0) = 0; \qquad \text{(ii)} \quad \Lambda(t) > 0 \quad \textit{for every } t > 0; \qquad \text{(iii)} \quad \liminf_{t \to 0+} \frac{\Lambda(t)}{t^{\xi}} > 0 \tag{17}$$

for some $0 \leqslant \xi < 1$. *Then* u *has compact support.*

Proof Recall that a Cartan–Hadamard manifold is Feller. By Theorem 5.1 we know that $u(x) \to 0$ as $x \to \infty$. The conclusion now follows from the compact support principle, that is valid under the stated assumptions on M and Λ [19, Theorem 1.1]. □

Of course for the conclusion of Theorem 5.4 to hold it suffices that M be stochastically complete, Feller and that the compact support principle hold for solutions of (16).

The above theorems can be applied to obtain nonexistence results. For instance, combining Theorems 5.4 and 5.3 we obtain

Corollary 5.5 *Let* $(M, \langle\,,\rangle)$ *be a stochastically complete Cartan–Hadamard manifold of dimension* $m \geqslant 6$. *Then the metric of* M *cannot be conformally deformed to a new metric* $\overline{\langle\,,\rangle} = v^2 \langle\,,\rangle$ *with* $v_* > 0$ *and scalar curvature* $\overline{S}$ *satisfying* $\liminf_{x \to \infty} \overline{S} > 0$.

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