Chapter 5
Disk of Uniform Strength

5.1 Profile Definition

In general, a disk of uniform strength is defined as one in which the radial and hoop stresses resulting from centrifugal loading are constant and equal to each other at all points. Consequently, the following relation applies throughout the disk:

$$\sigma_r = \sigma_t = \sigma = const, \tag{5.1}$$

where σ is the stress at which the disk material works, which is obviously equal to the permissible stress σ_{am} for that material. This configuration, where disk profile varies according to an exponential function, was first introduced by De Laval in the late 1800s.

For a disk consisting of isotropic material subjected only to centrifugal load (zero thermal gradient along the radius), the above relations satisfies compatibility equations (1.23) or (1.24). A disk with these properties is thus possible from the standpoint of compatibility of the stress field. However, as a turbine disk is also always subjected to thermal load, the condition of uniform strength given by relation (5.1) can never be ensured, as it would not respect the compatibility equation. The following treatment thus applies to a hypothetical disk subjected only to centrifugal load. Although the uniform strength profile can be considered as a starting point in designing actual turbine disks, it is better to start from the profiles that will be discussed in the following chapters, as they can ensure an optimal distribution of principal stresses even when thermal loading is involved.

Introducing relation (5.1) in the first equilibrium equation (1.10) yields the following relation, which is the solving equation for the uniform strength disk:

$$\sigma \cdot r \cdot \frac{dh}{dr} + \gamma \cdot \omega^2 \cdot r^2 \cdot h = 0. \tag{5.2}$$

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This is a first order differential equation with separable variables which can be written in the following form:

$$\frac{dh}{h} = -\frac{\gamma \cdot \omega^2}{\sigma} \cdot r \cdot dr. \tag{5.3}$$

Integrated, this relation gives:

$$\ln h = -\frac{\gamma \cdot \omega^2}{2 \cdot \sigma} \cdot r^2 + \ln C, \tag{5.4}$$

where C is the integration constant, which can be determined by establishing that the disk thickness assumes an assigned value at a given radius. Passing from the logarithm to the function, relation (5.4) gives:

$$h = C \cdot e^{-\frac{\gamma \cdot \omega^2}{2 \cdot \sigma} \cdot r^2}. \tag{5.5}$$

Independently of constant C, the profile thus defined features an inflection point at the radius

$$r = \sqrt{\frac{\sigma}{\gamma \cdot \omega^2}}, \tag{5.6}$$

which can be found by equalling the second derivative of relation (5.5) to zero. For values of radius r lesser or greater than that given by this relation, the profile's concavity will face inward or outward respectively as viewed by an observer located on the mid-plane bisecting the disk's thickness.

On the basis of relation (5.5), and depending on whether we impose the condition

$$h = h_e \quad \text{for} \quad r = r_e \tag{5.7}$$

or the condition

$$h = h_0 \quad \text{for} \quad r = 0, \tag{5.8}$$

the integration constant C will be given by:

$$C = h_e \cdot e^{\frac{\gamma \cdot \omega^2}{2 \cdot \sigma} \cdot r_e^2} \tag{5.9}$$

or

$$C = h_0. \tag{5.10}$$

Consequently, the variation function for thickness h with radius r can be expressed respectively with the following two equivalent relations:

$$h = h_e \cdot e^{\dfrac{\gamma \cdot \omega^2}{2 \cdot \sigma}\left(r_e^2 - r^2\right)}. \tag{5.11}$$

or

$$h = h_0 \cdot e^{-\dfrac{\gamma \cdot \omega^2}{2 \cdot \sigma} \cdot r^2}. \tag{5.12}$$

Finally, after collecting the term $r_e{}^2$ in the exponent in relation (5.11), multiplying and dividing the exponent appearing in relation (5.12) by $r_e{}^2$ and introducing the usual reference stress $\sigma_0 = \gamma \cdot \omega^2 \cdot r_e{}^2$ as well as the dimensionless variable ρ, relations (5.11) and (5.12) can be rewritten in the following form:

$$h = h_e \cdot e^{\dfrac{\sigma_0}{2 \cdot \sigma}\left(1 - \rho^2\right)}. \tag{5.13}$$

$$h = h_0 \cdot e^{-\dfrac{\sigma_0}{2 \cdot \sigma} \cdot \rho^2}. \tag{5.14}$$

Both relations (5.12) and (5.14) define a solid disk with thickness h_0 at the axis and thickness h decreasing rapidly from the axis outwards according to a one-parameter exponential function[1] given by the ratio $\sigma_0/2\sigma$; the first relation describes this disk in terms of r, and the second in terms of ρ. From both relations (5.12) and (5.14), we can conclude that the outer radius is not defined, and could thus be regarded as virtually infinite (Fig. 5.1a). If the disk were to be cut off at a given outer radius r_e (as is in any case necessary for design and construction reasons, as the geometrical dimensions of the interface on which the blades act are generally fixed), the condition (5.1) would not be respected on the cylindrical surface of radius $\rho = 1$, since we would necessarily have $\sigma_r = 0$ on this surface.

To respect the above condition, the disk must feature a peripheral crown ring whose dimensions are such as to develop the radial stress σ at the disk interface. Usually, in fact, the outer blades are not sufficient in order to have $\sigma_r = \sigma_t = \sigma$ at radius $r = r_e$; in a disk featuring an array of blades spaced at equal angles on the

[1] Eraslan and Orçan [15] considered a profile with thickness varying according to a two-parameter exponential function, defined by the relation $h = h_0 \cdot e^{-n \cdot \rho^k}$ where, with the meaning of the other symbols remaining the same, n and k are geometric parameters controlling thickness at the outer edge relative to that at the axis and the profile shape respectively. This relation makes it possible to describe solid and annular disks with concave, convex and inflection point profiles, but not conical disks; with this relation, differential equation (1.28) can be integrated by means of the linear combination of two mutually independent hypergeometric functions. From the application standpoint, it should be noted that an optimization process for the two geometric parameters n and k leads, as would be expected, to the De Laval uniform strength disk. Consequently, this relation is of purely mathematical interest.

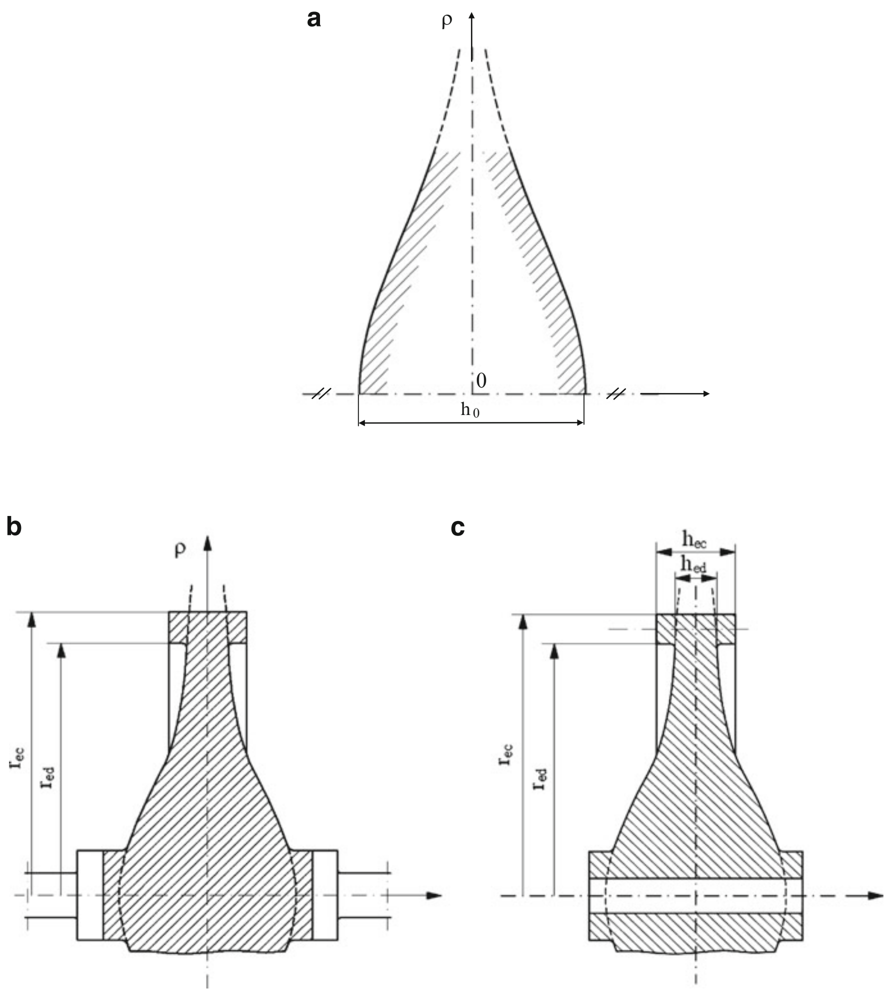


Fig. 5.1 (**a**) Uniform strength disk, unlimited at outer radius; (**b**) solid uniform strength disk with lateral flanges for connection to two flanges at the ends of a multi-pieces shaft; (**c**) uniform strength disk with annular hub to be shrunk onto a one-piece shaft

outer radius having total mass m_p and centre of gravity at a distance r_p from the rotational axis, this condition would be respected only if mass m_p were:

$$m_p = \frac{\pi \cdot \gamma \cdot r_e^3 \cdot h_0 \cdot e^{-\sigma_0/2\sigma}}{(\sigma_0/2\sigma) \cdot r_p}. \tag{5.15}$$

This value of m_p is determined by establishing that the total centrifugal force $F_c = m_p{\cdot}\omega^2{\cdot}r_p$ to which the blades are subjected is evenly distributed on the outer periphery of the disk, having a surface area of $2\pi{\cdot}r_e{\cdot}h_e$. However, as relation (5.15) is not generally satisfied in a bladed disk, it is necessary to provide a crown ring of appropriate dimensions.

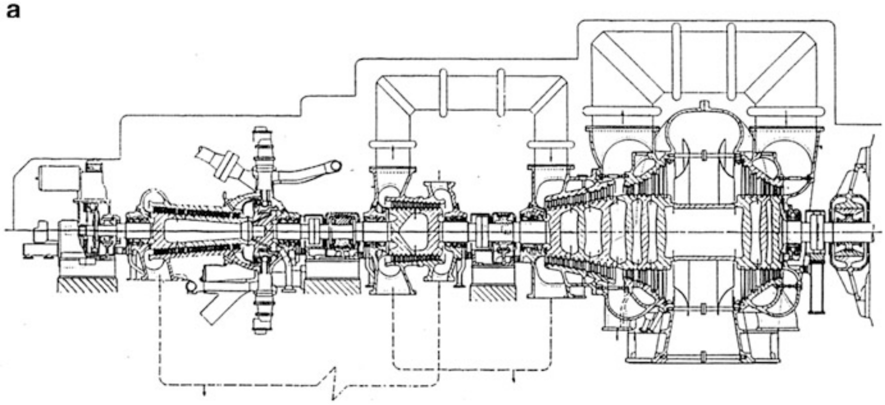
Similarly, if the disk were to feature a central hole, condition (5.1) would not be respected on the cylindrical surface of radius r_i, as here again radial stress would be zero. For this condition to be respected, the disk would have to be solid and integral with the shaft, or would have to feature a solid or annular central portion of constant thickness and blended with the uniform-strength portion, In this latter case, the central portion is a hub to be designed in such a way as to generate a tensile stress equal to σ at the interface with the variable-thickness portion. Connection to the shaft is generally accomplished by means of bolts or screws which secure the disk to flanges on the shaft.

5.2 Technical Solutions

In turbines, disks with profile as described by relation (5.14) occasionally have no central hole and are secured at both sides to two flanged piece of a multi-pieces shafts by means of threaded connections (Fig. 5.1b). More frequently, however, disks with a central hole are used, and are shrunk onto a one-piece shaft (Fig. 5.1c). In this case, the central portion of the disk is configured to form a long hub whose profile is very far from that of the uniform strength disk defined by relations (5.13) and (5.14). Figure 5.2a shows the design solution actually used in both the medium/low-pressure section and the convergent double-flow low-pressure section, which are mounted on the same shaft, of a steam turbine for a thermonuclear power plant. As can be seen, the solid disks also serve as the drive shaft, as they are torsionally and flexurally connected to each other by means of welded joints on the mating faces of the two consecutive disks' crown rings. On a larger scale, Fig. 5.2b shows a design solution similar to that represented in Fig. 5.1a. In other designs by the same manufacturer (see Fig. 5.3a, illustrating the four-flow low-pressure section of a thermonuclear power plant steam turbine), the disks are torsionally coupled, not at the frontal surfaces of the crown rings carrying seats for the blade attachments on their periphery, but at the frontal surfaces of two dummy crown rings at a smaller radius. In any case, turbine disks always feature a crown ring housing the blades on their outer periphery (Fig. 5.1b, c).

Though the hub and crown ring do not have profiles complying with the thickness variation functions characterizing the solid disk of uniform strength, if we bear the centrifugal forces involved and the deformability of the individual parts (hub, disk and crown ring) in mind, we can nevertheless determine the dimensions of these parts in such a way that the stresses σ that satisfy condition (5.1) are generated at the limit cross sections of the disk that is at outer and inner radii of the uniform strength disk. Figures 5.3a, b show two design solutions actually uses for, respectively, a steam turbine and a gas turbine.

With reference to relations (5.13) and (5.14), Fig. 5.4 shows ratio h/h_e (Fig. 5.4a) and ratio h/h_0 (Fig. 5.4b) versus ρ, for different values of parameter $\sigma_0/2\sigma$, as can be seen, bell shapes are heavily influenced by ratio $\sigma_0/2\sigma$, with maximum thickness at the axis. Pronounced thickness gradients do not exist only where the values of ratio a



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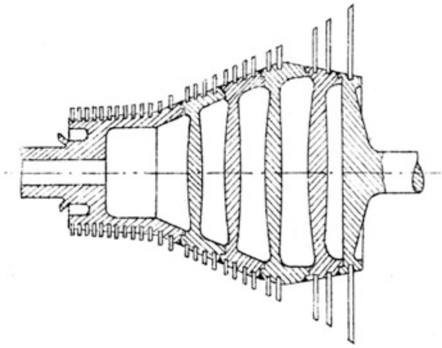

Fig. 5.2 (**a**) Uniform strength disks in the medium/low-pressure and double-flow low-pressure sections of a Brown Boveri steam turbine for a thermonuclear power plant, with disks connected at the crown rings; (**b**) detail of a similar steam turbine design

$\sigma_0/2\sigma$ are not particularly high. If ratio h_0/h_e exceeds 3, and $\sigma_0/2\sigma$ is thus higher than 1.1, the disk shape may be impossible to actually produce, or may be extremely expensive both from the manufacturing standpoint and because of its axial dimensions. As a result, the use of uniform strength disks is now limited to values of $\sigma_0/2\sigma$ below 1.1 (though certain design applications involve values of $\sigma_0/2\sigma$ up to (1.6–1.8), but always below 2) and, consequently, to relatively low peripheral velocities $\omega \cdot r_e$ or high strength materials, i.e., materials with high permissible stress σ.

In view of the foregoing considerations, the limit often given in the literature (Giovannozzi, [29]) for the ratio $\sigma_0/2\sigma = 3.22$, which applies to disks having a thickness at the outer radius equal to 1/25 of the thickness at the axis, must be regarded as quite far from that imposed by current manufacturing and functional requirements. It should also be specified that, when designing the disk, thickness at the axis must not exceed $(0.25–0.30) \cdot r_e$.

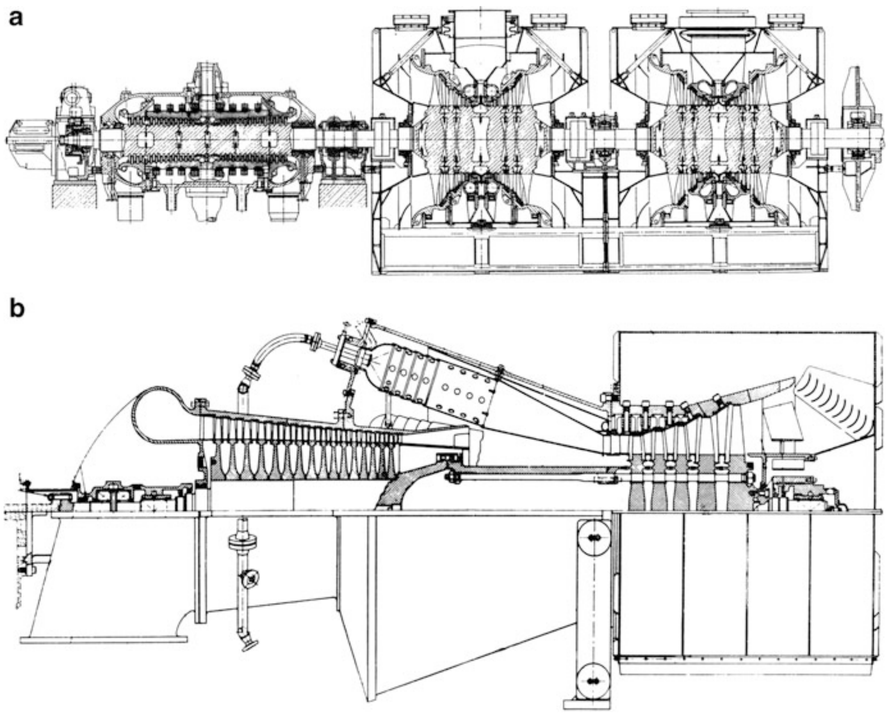

Fig. 5.3 (**a**) Uniform strength disks in the four flow low-pressure section of a Brown Boveri steam turbine for a thermonuclear power plant, with disks connected at dummy crown rings at a smaller radius than that of the bladed crown rings; (**b**) longitudinal section of a Westinghouse-Fiat gas turbine

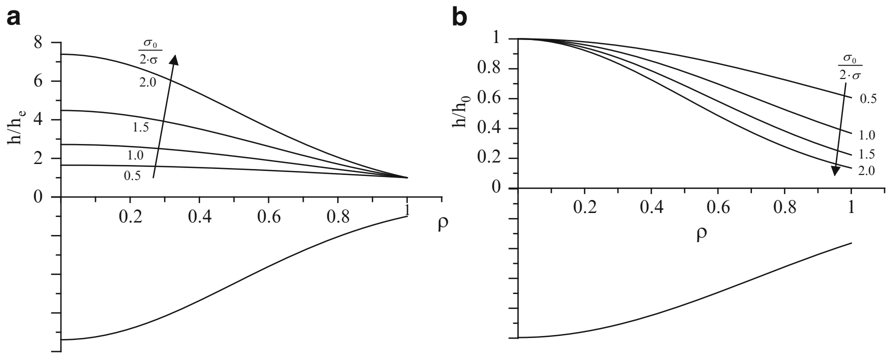

Fig. 5.4 (**a**) Distributions of ratio h/h_e; (**b**) ratio h/h_0 versus radius ρ for different values of parameter $\sigma_0/2\sigma$ in uniform strength disks

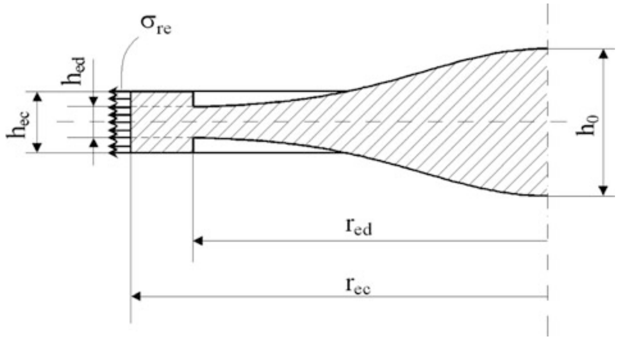

Fig. 5.5 Geometry of a uniform strength disk with a crown ring of uniform thickness

5.3 Crown Ring Design

To determine (Fig. 5.1b, c) the radial thickness ($r_{ec} - r_{ed}$) of the crown ring which, together with the blades at its outer radius, can guarantee condition (5.1) at the interface with the disk, we will consider a uniform strength disk with no central hole and with a crown ring of constant thickness, featuring the geometry shown in Fig. 5.5. Let $\beta = r_{ed}/r_{ec}$ and $\delta = h_{ec}/h_{ed}$; the crown ring's outer radius r_{ec} and its axial thickness h_{ec} are generally known, as they are determined beforehand when designing the blades and the associated seats in the crown ring.

For a rough calculation, we can consider the crown ring to be a disk of constant thickness stressed at the outer radius r_{ec} by a radial stress σ_{re} due to the blades, and at the inner radius r_{ed} by a radial stress $\sigma_{ri} = \sigma/\delta$ due to the disk. Stress σ_{ri} is justified by the fact that, for there to be equilibrium, the product ($\sigma_r \cdot h$) must remain constant at the disk/ring interface. At this interface, in any case, there is a discontinuity in function $h = h(r)$ if $\delta \neq 1$ or in its first derivative if $\delta = 1$.

Assuming a plane stress state in this area is thus a very rough approximation. With this interface regarded as part of the crown ring, there would be a redistribution of stress σ_{ri} for $\delta \neq 1$ in areas that are not in fact loaded. Nevertheless, this approximation is indispensable if we wish to use the relations of the mono-dimensional thin disk theory, and in any case leads to results that are acceptable from the design standpoint, given that, according to Saint Venant's principle, if a system of forces acting on a small portion of the surface of an elastic body is replaced by another statically equivalent system of forces acting on the same portion of the surface, the redistribution of loading produces substantial changes in the stresses only at the local level, but has negligible effects on the stress state at distances which are large by comparison with the linear dimensions of the surface to which the equivalent system of loads was applied.

For the structure to satisfy compatibility conditions, the radial displacement of the outer edge of the disk must be equal to the radial displacement of the inner edge of the crown ring. Bearing in mind the second geometric relation (1.14) and the second (1.25) from which the temperature term is omitted, and considering condition (5.1), the disk's radial displacement at radius r_{ed} will be:

$$u = r_{ed} \cdot \varepsilon_t = \frac{r_{ed}}{E} \cdot (\sigma_t - v \cdot \sigma_r) = \frac{r_{ed}}{E} \cdot \sigma \cdot (1 - v). \tag{5.16}$$

By applying the principle of superposition, the crown ring's radial displacement at the same radius r_{ed} can be calculated as the sum of the displacements resulting from centrifugal load, stress at the inner radius, and stress at the outer radius. Consequently, the third relations of (2.34), (2.16) and (2.19) written for $\rho = \beta$ yield:

$$u = \frac{r_{ed}}{E} \cdot \beta \cdot \left\{ \frac{\sigma_0}{4} \cdot \left[(3 + v) + \beta^2 \cdot (1 - v) \right] - \frac{\sigma}{\delta} \cdot \left[\frac{\beta^2 \cdot (1 - v) + (1 + v)}{1 - \beta^2} \right] + \frac{2 \cdot \sigma_{re}}{1 - \beta^2} \right\}, \tag{5.17}$$

where $\sigma_0 = \gamma \cdot \omega^2 \cdot {r_{ec}}^2$.

Equating (5.16) and (5.17) and solving for β, we obtain:

$$\beta = \sqrt{\frac{2 \cdot \sigma}{\sigma_0 \cdot \delta} \cdot \left[(\delta - 1) + 2 \cdot \sqrt{\frac{(\sigma_0 \cdot \delta^2 / 2\sigma) \cdot [(\sigma_0 / 2\sigma) - 1 + v]}{(1 - v^2)} + \frac{(\delta - 1)^2}{4} + \frac{\sigma_0 \cdot \delta^2 \cdot \sigma_{re}}{2\sigma^2 \cdot (1 - v)}} \right] - \frac{1 + v}{1 - v}} \tag{5.18}$$

Equation (5.18) can be used to determine the crown ring's radial thickness once all other parameters have been determined. The ratio of crown ring thickness h_{ec} to thickness h_0 at the centre of the disk can be readily determined by means of relation (5.14), thus leading to the following relation:

$$h_{ec} / h_0 = \delta \cdot e^{-\sigma_0 / 2\sigma}. \tag{5.19}$$

In design calculations, crown ring outer radius r_{ec} and axial length h_{ec} are imposed by manufacturing reasons, and rotor angular velocity ω is assigned. Once ratio δ and permissible stress σ are established as design choices (the latter on the basis of the material to be used), (5.18) can be used to obtain ratio β and, consequently, outer radius r_{ed} of the uniform strength disk. Disk profile is then determined by means of (5.13).

The disk profile can also be constructed via a graphic procedure, using the Rowe diagram [16] shown in Fig. 5.6 and taken from several technical manuals. To this end, once r_{ed} has been determined, we introduce the peripheral load per unit of tangential length $P = h_{ed} \cdot \sigma$ and use the diagram to find, for a selected number of radii ρ, the ratio $(h \cdot \sigma)/P$ for an assigned σ_0/σ. The thicknesses corresponding to these radii are obtained with the relation:

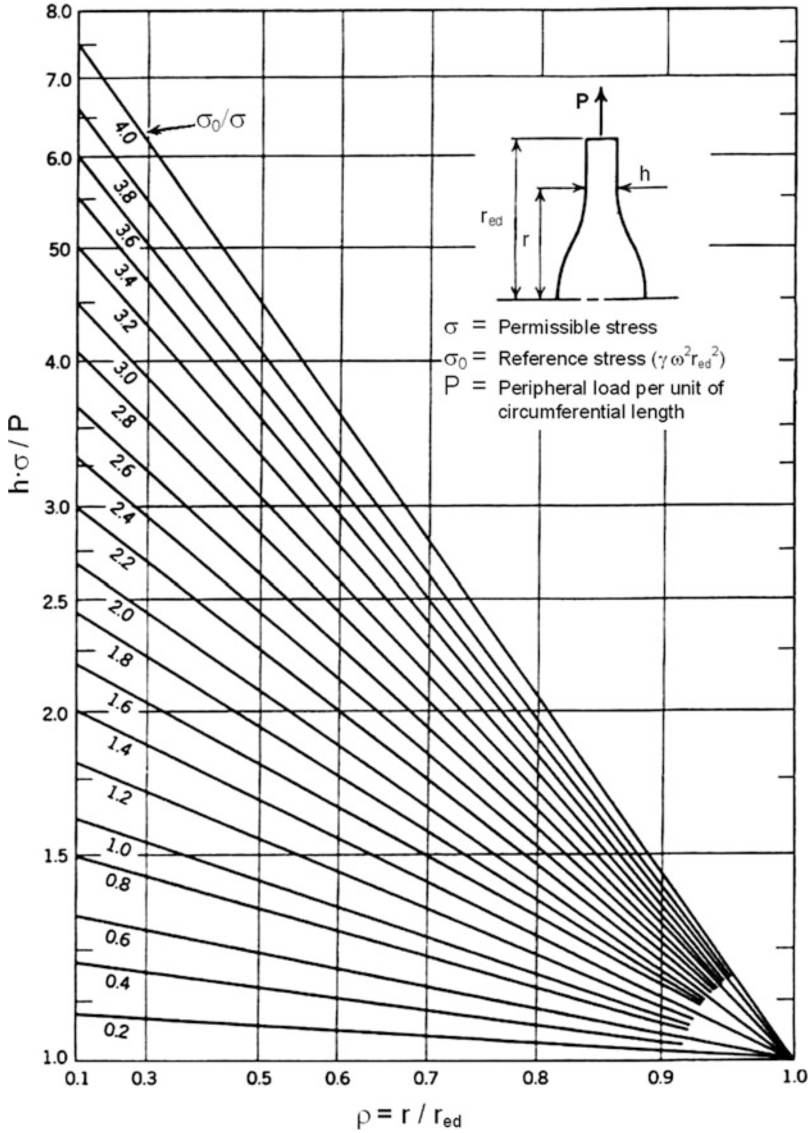

Fig. 5.6 Rowe diagram for constructing the profile of a uniform strength disk via a graphic procedure

$$h = \left(\frac{h \cdot \sigma}{P}\right) \cdot \left(\frac{P}{\sigma}\right). \tag{5.20}$$

Though modern computing power might appear to have made this diagram obsolete, it has been included here because it clarifies the influence of the various parameters on disk profile at a glance.

To conclude our discussion of the uniform strength disk, it should be borne in mind that, from the manufacturing standpoint, the large number of design constraints, some of which are dictated by the process cycle, can at times make it impracticable to employ disks which are of uniform strength in the strict sense. As a result, it may be preferable to forego the undeniable advantages they offer in terms of stress and strain states, and opt for hyperbolic disks (or, as we will see in Chaps. 6 and 7, conical disks and non-linearly variable thickness disks) which approximate them within certain limits (for instance, a uniform strength disk is characterized by an inflection point, which does not exist in hyperbolic, conical or non-linearly variable thickness disks).

5.4 Example

We will now consider a uniform strength steel disk of the type shown in Fig. 5.1c, whose geometry at the centre is defined by magnitudes $r_e = r_{ed} = 0.250$ m, $r_i = 0.050$ m, $h_e = h_{ed} = 0.015$ m and $h_0 = 0.075$ m, and rotating at angular velocity $\omega = 750$ rad/s ($n \approx 7{,}160$ rpm). We will determine the stress in the disk in question and its profile. We will then consider an annular hyperbolic disk, also consisting of steel, and also having at the central part $r_e = 0.250$ m, $r_i = 0.050$ m, $h_e = 0.015$ m and h_i corresponding to those of the uniform strength disk at radius $r_i = 0.050$ m, comparing the stress states at radius r_i of the two disks, as well as the radial displacements at the same radius.

Equating relations (5.13) and (5.14), we obtain:

$$\frac{h_e}{h_0} = \frac{e^{-\frac{\sigma_0}{2\cdot\sigma}\cdot\rho^2}}{e^{\frac{\sigma_0}{2\cdot\sigma}\cdot\left(1-\rho^2\right)}} = e^{-\frac{\sigma_0}{2\cdot\sigma}}.$$

whereby

$$\frac{0.015}{0.075} = \frac{1}{5} = e^{-\frac{\sigma_0}{2\cdot\sigma}\cdot}, \text{ then } \ln\frac{1}{5} = -\frac{\sigma_0}{2\cdot\sigma}$$

and hence

$$\sigma = \frac{\sigma_0}{3.219} = 0.310\cdot\sigma_0 = 85.19 \text{ MPa}$$

given that $\sigma_0 = \gamma\cdot\omega^2\cdot r_e{}^2 = 7.8\cdot 10^3\cdot 750^2\cdot 0.25^2 = 274.22$ MPa.

Using relation (5.14), we can then conclude that the profile of the uniform strength disk in question is defined by the following relation

$$h = h_0\cdot e^{-\frac{\sigma_0}{2\cdot\sigma}\cdot\rho^2} = 0.075\cdot e^{-1.609\cdot\rho^2}$$

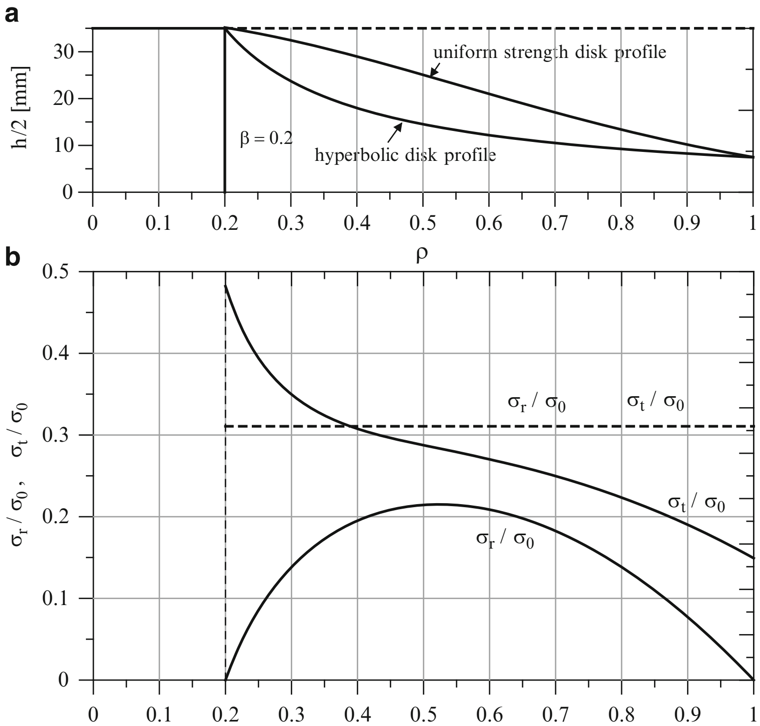

Fig. 5.7 (**a**) Profiles of the uniform strength disk and of the hyperbolic disk with the same h_i; (**b**) dimensionless principal stresses σ_r/σ_0 and σ_t/σ_0 versus dimensionless radius ρ in the uniform strength disk and in the hyperbolic disk with the same h_i, both subjected only to centrifugal load

as a function of dimensionless variable ρ, or as a function of variable r by the relation

$$h = h_0 \cdot e^{-\dfrac{\gamma \cdot \omega^2 \cdot r^2}{2 \cdot \sigma}} = 0.075 \cdot e^{-25.751 \cdot r^2}.$$

From both of these relations, it can be concluded that at radius $r_i = 0.050$ m, i.e., for $\beta = r_i/r_e = 0.2$, thickness h_i of the uniform strength disk is 0.070 m.

The hyperbolic disk to be compared with that considered here is thus defined by the following magnitudes: $r_e = 0.250$ m, $r_i = 0.050$ m, $h_e = 0.015$ m and $h_i = 0.070$ m.

As $\beta = 0.2$, (4.3) written for $\rho = 1$ gives $a = -0.957$. For the steel disk ($\nu = 0.3$), relations (4.14) yield the following values of roots p and q: $p = 1.710$, $q = -0.753$. As $\beta^2 = 0.04$, $\beta^{p-1} = \beta^{0.710} = 0.32$ and $\beta^{q-1} = \beta^{-1.753} = 16.80$, the first two relations (4.31) give:

$$\sigma_r = 0.68 \cdot \frac{\left(-16.76 \cdot \rho^{0.71} + 0.28 \cdot \rho^{-1.75} + 16.48 \cdot \rho^2\right)}{(-16.48)} \cdot \sigma_0$$
$$\sigma_t = \frac{\left(-41.65 \cdot \rho^{0.71} - 1.58 \cdot \rho^{-1.75} + 31.31 \cdot \rho^2\right)}{(-79.79)} \cdot \sigma_0.$$

In particular, we obtain:

$$(\sigma_r)_{\rho=\beta} = (\sigma_r)_{\rho=1} = 0; \quad (\sigma_t)_{\rho=\beta} = 0.482 \cdot \sigma_0; \quad (\sigma_t)_{\rho=1} = 0.149 \cdot \sigma_0.$$

In comparing the stress states of the two disks, we thus see that the hoop stress at the inner radius of the hyperbolic disk is over 55% higher than that in the uniform strength disk (Fig. 5.7).

The radial displacement at the inner radius of the uniform strength disk is:

$$(u)_{\rho=\beta} = \frac{r_e \cdot \beta}{E} \cdot \left[(\sigma_t)_{\rho=\beta} - \nu \cdot (\sigma_r)_{\rho=\beta}\right] = \frac{r_e \cdot \beta}{E} \cdot \sigma \cdot (1 - \nu) = \frac{r_e \cdot \beta}{E} \cdot 0.217 \cdot \sigma_0,$$

while that at the inner radius of the corresponding hyperbolic disk is:

$$(u)_{\rho=\beta} = \frac{r_e \cdot \beta}{E} \cdot \left[(\sigma_t)_{\rho=\beta} - \nu \cdot (\sigma_r)_{\rho=\beta}\right] = \frac{r_e \cdot \beta}{E} \cdot 0.482 \cdot \sigma_0.$$

From the standpoint of displacements as well as that of stresses, then, there can be no doubt that using a uniform strength disk is more advantageous, all other conditions remaining equal, than employing a comparable hyperbolic disk.