Chapter 3
Thermal Loads and Fictitious Density Variation Along the Radius

3.1 Annular Disk, Subjected to Thermal Load

Here again, the general solution of (2.3) can be found by adding the solution of the associated homogeneous equation (2.4) to a particular solution of the complete equation. As the general solution of the associated homogeneous equation (2.4) is already known, the problem of the disk subject to a non-zero temperature gradient along the radius is reduced to determining a particular integral of (2.3). Note that the derivative dT/dr appears in this latter equation. It follows that a constant temperature resulting from slow, uniform heating or cooling of the disk does not cause stresses, but only expansion. This observation is valid in general, regardless of the disk's shape, provided the material is isotropic and its elastic and thermophysical properties are independent of the radius.

To calculate this particular integral, it is necessary to know the function $T = T(r)$ of temperature distribution along the radius. Frequently, the temperature does not depend only on radius r, but varies according to a more complex function. In many current design applications (disks for gas and steam turbines, for example), however, the assumption that temperature varies only as a function of the radius is a sufficiently close approximation from the engineering standpoint, and can thus be profitably used by the structural designer because of the simplifications that it permits.

In this context, we will consider three functions of temperature variation with the radius, viz.:

$$T = T_0 + k \cdot r^n \; ; \tag{3.1}$$

$$T = \sum_{i=0}^{n} k_i \cdot r^i \; ; \tag{3.2}$$

$$T = T(r) \; . \tag{3.3}$$

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The first two, which are taken from the literature, express $T = T(r)$ through an *n-th* degree function and a polynomial of degree n respectively, while with the third, any function $T = T(r)$ whatsoever is assumed. In the latter case, it should be noted that the function $T = T(r)$ need not necessarily be restricted to functions that can be integrated analytically; for those that cannot, it is possible to proceed with numerical integration, as the results thus obtained, though approximate, are acceptable for design purposes.

In function (3.1), k and n are constant, with n being any real exponent, whether positive or negative, an integer or a fraction, while T_0, which is also constant, is the reference temperature, and in general coincides with the ambient temperature or the assembly temperature. In function (3.2), k_i are constant and $i = 0, 1, 2, \ldots, n$ (for this function, $k_0 = T_0$ is the reference temperature, again coinciding with the ambient temperature or the assembly temperature). In even more general terms, the function $T = T(r)$ given by (3.2) can be expressed in the form $T = T_0 + \sum_{i=1}^{n} k_i \cdot r^{m_i}$, where k_i are constants and m_1, m_2, …, m_n are exponents that are not necessarily integers.

3.1.1 *Function* T *Given by an* n-th *Degree Function*

Where function $T = T(r)$ is expressed by (3.1), relation (3.4) written in the following form can be used:

$$\frac{d}{dr}\left[\frac{1}{r}\cdot\frac{d}{dr}(u\cdot r)\right] = (1+\nu)\cdot\alpha\cdot\frac{dT}{dr}\ . \tag{3.4}$$

Given that $dT/dr = k \cdot n \cdot r^{n-1}$, performing two successive integrations (here again, we have chosen for demonstration purposes to proceed with direct integration) followed by a derivation operation yields:

$$\begin{cases} u = (1+\nu)\cdot\alpha\cdot k\cdot\dfrac{r^{n+1}}{n+2} + C_1\cdot\dfrac{r}{2} + \dfrac{C_2}{r} \\ \dfrac{du}{dr} = (1+\nu)\cdot\alpha\cdot k\cdot\dfrac{n+1}{n+2}\cdot r^n + \dfrac{C_1}{2} - \dfrac{C_2}{r^2}. \end{cases} \tag{3.5}$$

It should be noted that the first term in the second member of the first of these relations is the particular integral of the non-homogeneous second order differential equation (3.4); also it can be deduced with the usual procedure that applies to this type of differential equation, i.e., by setting $u = C\cdot r^{\,n+1}$, substituting this relation together with its first and second derivatives in (3.4), where $dT/dr = k \cdot n \cdot r^{n-1}$ and then calculating constant C, once the coefficient of the power function r^{n-1} has been equated to zero.

Substituting expressions (3.5) in relations (1.27), where it is established that $\alpha{\cdot}T = \alpha \cdot k \cdot r^n$, given that, as indicated above, $\alpha{\cdot}T_0$ makes no contribution to stresses, yields the following expressions of σ_r and σ_t as a function of radius r:

$$\begin{cases} \sigma_r = \dfrac{E \cdot C_1}{2 \cdot (1 - v)} - \dfrac{E \cdot C_2}{(1 + v)} \cdot \dfrac{1}{r^2} - \dfrac{E \cdot \alpha \cdot k}{n + 2} \cdot r^n \\ \sigma_t = \dfrac{E \cdot C_1}{2 \cdot (1 - v)} + \dfrac{E \cdot C_2}{(1 + v)} \cdot \dfrac{1}{r^2} - E \cdot \alpha \cdot k \cdot \dfrac{n + 1}{n + 2} \cdot r^n. \end{cases} \tag{3.6}$$

Subsequently, by multiplying and dividing the second and third term of the second member of expressions (3.6) by ${r_e}^2$ and by ${r_e}^n$ respectively, introducing the dimensionless variable ρ, using relations (2.9) and taking

$$C' = \frac{E \cdot \alpha \cdot k}{n + 2} \cdot r_e^n \text{ e } D' = E \cdot \alpha \cdot k \cdot \frac{n + 1}{n + 2} \cdot r_e^n, \tag{3.7}$$

we arrive at the following relations, which express radial and hoop stresses as a function of ρ:

$$\begin{cases} \sigma_r = A - B/\rho^2 - C' \cdot \rho^n \\ \sigma_t = A + B/\rho^2 - D' \cdot \rho^n. \end{cases} \tag{3.8}$$

Introducing relations (3.8) in the second (1.25) and bearing in mind that, given the second relation (1.14), $u = r{\cdot}\varepsilon_t$, yields the following expression of radial displacement $u = u(\rho)$ for the configuration at reference temperature $T = T_0$, which in general is assumed to be the assembly temperature T_a:

$$\begin{aligned} u = & \frac{r_e}{E} \cdot \rho \cdot \left[A \cdot (1 - v) + \frac{B}{\rho^2} \cdot (1 + v) - \left(D' - v \cdot C' - E \cdot \alpha \cdot k \cdot r_e^n \right) \cdot \rho^n \right] = \\ & = \frac{r_e}{E} \cdot \rho \cdot \left[A \cdot (1 - v) + \frac{B}{\rho^2} \cdot (1 + v) + \frac{E \cdot \alpha \cdot k \cdot (1 + v)}{n + 2} \cdot r_e^n \cdot \rho^n \right]. \end{aligned} \tag{3.9}$$

Obviously, where the assembly temperature to differ from the reference temperature, the further term $\alpha{\cdot}\, E{\cdot}\, (T_a - T_0)$ would appear within the square brackets in expression (3.9). Relations (3.8) and (3.9) describe the distribution of radial and hoop stresses and radial displacement versus ρ in a disk subjected to a non-zero temperature gradient along the radius expressed by (3.1). In these relations, as the temperature variation along the radius is known, C' and D' represent known terms for a given material and for an assigned outer radius r_e, while A and B are the integration constants to be determined by imposing boundary conditions.

It should be noted that if the function of temperature variation with the radius expressed by (3.1) were to be characterized by exponent $n = -1$, we would have

$dT/dr = -k \cdot r^{-2}$ at the second member of relation (3.4). As a result, the solution of this equation would no longer be independent of that of the associated non-homogeneous equation, as two terms in r^{-2} would appear in relation (3.4). Nevertheless, a solution is still possible, but it will not be given here as it is of little design interest. The difficulty is purely mathematical and can be readily circumvented, with results that provide an excellent approximation, by using an exponent n which is close to but not equal to -1 (for example, $n = -0.999$ or $n = -1.001$).

In the case considered here of an annular disk subjected only to thermal load, the boundary conditions to be imposed are:

$$\begin{cases} \sigma_r = 0 & \text{for} \quad \rho = 1 \\ \sigma_r = 0 & \text{for} \quad \rho = \beta. \end{cases} \tag{3.10}$$

Accordingly, the system obtained from the first of relations (3.8) gives:

$$\begin{aligned} A &= \frac{E \cdot k \cdot \alpha}{n+2} \cdot r_e^n \cdot \frac{1-\beta^{n+2}}{1-\beta^2} \\ B &= \frac{E \cdot k \cdot \alpha}{n+2} \cdot r_e^n \cdot \frac{1-\beta^n}{1-\beta^2} \cdot \beta^2. \end{aligned} \tag{3.11}$$

Substituting the values of the constants thus found in relations (3.8) and (3.9) gives the following expressions for σ_r, σ_t and u which provide a univocal solution of the problem:

$$\begin{cases} \sigma_r = \dfrac{E \cdot \alpha \cdot k}{n+2} \cdot r_e^n \cdot \left[\dfrac{1-\beta^{n+2}}{1-\beta^2} - \dfrac{1-\beta^n}{1-\beta^2} \cdot \dfrac{\beta^2}{\rho^2} - \rho^n \right] \\ \sigma_t = \dfrac{E \cdot \alpha \cdot k}{n+2} \cdot r_e^n \cdot \left[\dfrac{1-\beta^{n+2}}{1-\beta^2} + \dfrac{1-\beta^n}{1-\beta^2} \cdot \dfrac{\beta^2}{\rho^2} - (n+1) \cdot \rho^n \right] \\ u = \rho \cdot \dfrac{\alpha \cdot k}{n+2} \cdot r_e^{n+1} \cdot \left[\dfrac{1-\beta^{n+2}}{1-\beta^2} \cdot (1-\nu) + \dfrac{1-\beta^n}{1-\beta^2} \cdot (1+\nu) \cdot \dfrac{\beta^2}{\rho^2} + (1+\nu) \cdot \rho^n \right]. \end{cases} \tag{3.12}$$

3.1.2 *Function* T *Given by an* n *Degree Polynomial*

By contrast, where function $T = T(r)$ is expressed by (3.2), relation (3.4) can again be used, given that $dT/dr = \sum_{i=1}^{n} i \cdot k_i \cdot r^{i-1}$, thus, performing two successive integrations followed by a derivation operation yields:

$$u = \alpha \cdot (1+\nu) \cdot \sum_{i=1}^{n} k_i \cdot \frac{r^{i+1}}{i+2} + C_1 \cdot \frac{r}{2} + \frac{C_2}{r}$$
$$\frac{du}{dr} = \alpha \cdot (1+\nu) \cdot \sum_{i=1}^{n} \frac{i+1}{i+2} \cdot k_i \cdot r^i + \frac{C_1}{2} - \frac{C_2}{r^2}. \tag{3.13}$$

It should also be noted that integrating relation (3.4) with $dT/dr = \sum_{i=1}^{n} i \cdot k_i \cdot r^{i-1}$ is also possible using the principle of superposition: in this case, the total stress state in the disk will be given by the sum of the stress states for the n terms of the summation appearing in function (3.2), each calculated by imposing the same boundary conditions. It should also be observed that if function (3.2) were characterized by a term with exponent $i = -1$, the latter would give rise to the same integration problem indicated for function (3.1), to which the reader is referred.

Substituting expressions (3.13) in relations (1.27), where it is established that $\alpha \cdot T = \alpha \cdot \sum_{i=0}^{n} k_i \cdot r^i = \alpha \cdot \sum_{i=1}^{n} k_i \cdot r^i$, given that $\alpha{\cdot}k_0$ is a constant term and thus makes no contribution to stresses, yields the following expressions of σ_r and σ_t as a function of radius r:

$$\begin{cases} \sigma_r = \dfrac{E \cdot C_1}{2 \cdot (1-\nu)} - \dfrac{E \cdot C_2}{(1+\nu)} \cdot \dfrac{1}{r^2} - \alpha \cdot E \cdot \displaystyle\sum_{i=1}^{n} k_i \cdot \dfrac{r^i}{i+2} \\ \sigma_t = \dfrac{E \cdot C_1}{2 \cdot (1-\nu)} + \dfrac{E \cdot C_2}{(1+\nu)} \cdot \dfrac{1}{r^2} - \alpha \cdot E \cdot \displaystyle\sum_{i=1}^{n} \dfrac{i+1}{i+2} \cdot k_i \cdot r^i. \end{cases} \tag{3.14}$$

Subsequently, by multiplying and dividing the second and third term of the second member of expressions (3.14) by ${r_e}^2$ and by ${r_e}^i$ respectively, introducing the dimensionless variable ρ as well as the constants A and B given by relations (2.9), we arrive at the following relations, which express radial and hoop stresses as a function of ρ:

$$\begin{cases} \sigma_r = A - \dfrac{B}{\rho^2} - \alpha \cdot E \cdot \displaystyle\sum_{i=1}^{n} k_i \cdot r_e^i \cdot \dfrac{\rho^i}{i+2} \\ \sigma_t = A + \dfrac{B}{\rho^2} - \alpha \cdot E \cdot \displaystyle\sum_{i=1}^{n} \dfrac{i+1}{i+2} \cdot k_i \cdot r_e^i \cdot \rho^i. \end{cases} \tag{3.15}$$

Relations (3.15) are more general than relations (3.8), which are a particular case thereof. It is sufficient to consider the *n-th* term of polynomial (3.2) alone, or in other words, it is sufficient to set $i = n$, $k_1 = k_2 = \ldots k_{n-1} = 0$ in the latter polynomial to reduce relations (3.15) to relations (3.8).

Introducing relations (3.15) in the second (1.25) and noting that, given the second relation (1.14), $u = r{\cdot}\varepsilon_t$, we obtain the following expression for radial displacement $u = u(\rho)$ for the configuration at reference temperature $T_0 = k_0$ (this coincides with the first term of the series expansion of relation (3.2)), which here again is in general assumed to be the assembly temperature T_a:

$$u = \frac{r_e}{E} \cdot \rho \cdot \left[A \cdot (1 - \nu) + \frac{B}{\rho^2} \cdot (1 + \nu) + \alpha \cdot E \cdot (1 + \nu) \cdot \sum_{i=1}^{n} \frac{k_i \cdot r_e^i \cdot \rho^i}{i + 2} \right]. \quad (3.16)$$

Once again, it is obvious that if the assembly temperature were to differ from the reference temperature $T_0 = k_0$, the further term $\alpha \cdot E \cdot (T_a - T_0)$ would appear within the square brackets in expression (3.16). Relation (3.16) is more general than relation (3.9), which is a particular case thereof, and it is again sufficient to consider the *n-th* term of polynomial (3.2) alone, for (3.16) to be reduced to (3.9). Relations (3.15) and (3.16) describe the distribution of radial and hoop stresses and radial displacement versus ρ in a disk subjected to a non-zero temperature gradient along the radius expressed by (3.2), while A and B are the integration constants to be determined by imposing boundary conditions.

With the boundary conditions for the annular disk subjected only to thermal load ($\sigma_r = 0$ for both $\rho = 1$ and for $\rho = \beta$), the resulting system obtained from the first (3.15) yields:

$$\begin{aligned} A &= \frac{\alpha \cdot E}{1 - \beta^2} \cdot \sum_{i=1}^{n} \frac{k_i \cdot r_e^i}{i + 2} \cdot \left(1 - \beta^{i+2}\right) \\ B &= \frac{\alpha \cdot E}{1 - \beta^2} \cdot \beta^2 \cdot \sum_{i=1}^{n} \frac{k_i \cdot r_e^i}{i + 2} \cdot \left(1 - \beta^i\right). \end{aligned} \quad (3.17)$$

Substituting the values of the constants thus found in relations (3.15) and (3.16) gives the following expressions for σ_r, σ_t and u:

$$\begin{cases} \sigma_r = \alpha \cdot E \cdot \sum_{i=1}^{n} \frac{k_i \cdot r_e^i}{i + 2} \cdot \left[\frac{1 - \beta^{i+2}}{1 - \beta^2} - \frac{1 - \beta^i}{1 - \beta^2} \cdot \frac{\beta^2}{\rho^2} - \rho^i \right] \\ \sigma_t = \alpha \cdot E \cdot \sum_{i=1}^{n} \frac{k_i \cdot r_e^i}{i + 2} \cdot \left[\frac{1 - \beta^{i+2}}{1 - \beta^2} + \frac{1 - \beta^i}{1 - \beta^2} \cdot \frac{\beta^2}{\rho^2} - (i + 1) \cdot \rho^i \right] \\ u = \rho \cdot \alpha \cdot \sum_{i=1}^{n} \frac{k_i \cdot r_e^{i+1}}{i + 2} \cdot \left[\frac{1 - \beta^{i+2}}{1 - \beta^2} \cdot (1 - \nu) + \frac{1 - \beta^i}{1 - \beta^2} \cdot (1 + \nu) \cdot \frac{\beta^2}{\rho^2} + (1 + \nu) \cdot \rho^i \right]. \end{cases} \quad (3.18)$$

3.1.3 General Function T

Finally, in the most general possible case in which the function of temperature variation with the radius is expressed by (3.3), relation (3.4) can again be used, and performing two successive integrations followed by a derivation operation yields:

$$
\begin{aligned}
u &= \alpha \cdot (1+\nu) \cdot \frac{1}{r} \cdot \int_{r_i}^{r} T \cdot r \cdot dr + C_1 \cdot \frac{r}{2} + \frac{C_2}{r} \\
\frac{du}{dr} &= -\alpha \cdot (1+\nu) \cdot \frac{1}{r^2} \cdot \int_{r_i}^{r} T \cdot r \cdot dr + \alpha \cdot (1+\nu) \cdot T + \frac{C_1}{2} - \frac{C_2}{r^2}.
\end{aligned}
\tag{3.19}
$$

Substituting expression (3.19) in relations (1.27) yields the following expressions of σ_r and σ_t as a function of radius r:

$$
\begin{cases}
\sigma_r = \dfrac{E \cdot C_1}{2 \cdot (1-\nu)} - \dfrac{E \cdot C_2}{(1+\nu)} \cdot \dfrac{1}{r^2} - \alpha \cdot E \cdot \dfrac{1}{r^2} \cdot \displaystyle\int_{r_i}^{r} T \cdot r \cdot dr \\
\sigma_t = \dfrac{E \cdot C_1}{2 \cdot (1-\nu)} + \dfrac{E \cdot C_2}{(1+\nu)} \cdot \dfrac{1}{r^2} + \alpha \cdot E \cdot \left(\dfrac{1}{r^2} \cdot \displaystyle\int_{r_i}^{r} T \cdot r \cdot dr - T \right).
\end{cases}
\tag{3.20}
$$

Subsequently, proceeding as for the passage from relations (3.14) to (3.15), yields the following relations which express radial and hoop stresses as a function of ρ:

$$
\begin{cases}
\sigma_r = A - \dfrac{B}{\rho^2} - \alpha \cdot E \cdot \dfrac{1}{\rho^2} \cdot \displaystyle\int_{\beta}^{\rho} T \cdot \rho \cdot d\rho \\
\sigma_t = A + \dfrac{B}{\rho^2} + \alpha \cdot E \cdot \left(\dfrac{1}{\rho^2} \cdot \displaystyle\int_{\beta}^{\rho} T \cdot \rho \cdot d\rho - T \right).
\end{cases}
\tag{3.21}
$$

Introducing relations (3.21) in the second (1.25) and noting that, given the second relation (1.14), $u = r \cdot \varepsilon_t$, we obtain the following expression for radial displacement $u = u(\rho)$:

$$
u = \frac{r_e}{E} \cdot \rho \cdot \left[A \cdot (1-\nu) + \frac{B}{\rho^2} \cdot (1+\nu) + \frac{\alpha \cdot E \cdot (1+\nu)}{\rho^2} \cdot \int_{\beta}^{\rho} T \cdot \rho \cdot d\rho \right]. \tag{3.22}
$$

Relations (3.21) and (3.22) describe the distribution of radial and hoop stresses and radial displacement versus ρ in a disk subjected to a non-zero temperature gradient along the radius expressed by (3.3), while A and B are the integration constants to be determined by imposing boundary conditions.

With the usual boundary conditions for the annular disk subjected only to thermal load ($\sigma_r = 0$ for both $\rho = 1$ and for $\rho = \beta$), the resulting system obtained from the first relation (3.21) yields:

$$A = \frac{\alpha \cdot E}{1-\beta^2} \cdot \int_{\beta}^{1} T \cdot \rho \cdot d\rho$$
$$B = \frac{\alpha \cdot E \cdot \beta^2}{1-\beta^2} \cdot \int_{\beta}^{1} T \cdot \rho \cdot d\rho. \tag{3.23}$$

Substituting the values of the constants thus found in relations (3.21) and (3.22) gives the following expressions for σ_r, σ_t and u which provide a univocal solution of the problem:

$$\begin{cases} \sigma_r = \alpha \cdot E \cdot \left[\dfrac{1}{1-\beta^2} \cdot \left(1 - \dfrac{\beta^2}{\rho^2}\right) \cdot \displaystyle\int_{\beta}^{1} T \cdot \rho \cdot d\rho - \dfrac{1}{\rho^2} \cdot \displaystyle\int_{\beta}^{\rho} T \cdot \rho \cdot d\rho \right] \\ \sigma_t = \alpha \cdot E \cdot \left[\dfrac{1}{1-\beta^2} \cdot \left(1 + \dfrac{\beta^2}{\rho^2}\right) \cdot \displaystyle\int_{\beta}^{1} T \cdot \rho \cdot d\rho + \dfrac{1}{\rho^2} \cdot \displaystyle\int_{\beta}^{\rho} T \cdot \rho \cdot d\rho - T \right] \\ u = \dfrac{\alpha \cdot r_e}{\rho} \cdot \left[\dfrac{\rho^2 \cdot (1-\nu) + \beta^2 \cdot (1+\nu)}{\left(1-\beta^2\right)} \cdot \displaystyle\int_{\beta}^{1} T \cdot \rho \cdot d\rho + (1+\nu) \cdot \displaystyle\int_{\beta}^{\rho} T \cdot \rho \cdot d\rho \right]. \end{cases} \tag{3.24}$$

In the literature, (3.24) are also found written in terms of variable r, i.e., in the form:

$$\begin{cases} \sigma_r = \dfrac{\alpha \cdot E}{r_e^2 - r_i^2} \cdot \left(1 - \dfrac{r_i^2}{r^2}\right) \cdot \displaystyle\int_{r_i}^{r_e} T \cdot r \cdot dr - \dfrac{\alpha \cdot E}{r^2} \cdot \displaystyle\int_{r_i}^{r} T \cdot r \cdot dr \\ \sigma_t = \dfrac{\alpha \cdot E}{r_e^2 - r_i^2} \cdot \left(1 + \dfrac{r_i^2}{r^2}\right) \cdot \displaystyle\int_{r_i}^{r_e} T \cdot r \cdot dr + \alpha \cdot E \cdot \left(\dfrac{1}{r^2} \cdot \displaystyle\int_{r_i}^{r} T \cdot r \cdot dr - T \right) \\ u = \dfrac{\alpha}{r \cdot \left(r_e^2 - r_i^2\right)} \left[r^2 \cdot (1-\nu) + r_i^2 \cdot (1+\nu)\right] \cdot \displaystyle\int_{r_i}^{r_e} T \cdot r \cdot dr + \dfrac{\alpha \cdot (1+\nu)}{r} \cdot \displaystyle\int_{r_i}^{r} T \cdot r \cdot dr. \end{cases} \tag{3.25}$$

3.1.4 Example

We will now consider an annular disk of constant thickness made of unquenched AISI 1060 steel, with $\sigma_y = 480$ MPa and having outside and inside radius $r_e = 1.0$ m and $r_i = 0.5$ m. Let the disk be subjected to centripetal heat flow characterized by a linear temperature distribution along the radius described by function $T = T_0 + kr$, with T_0 (reference temperature) and k constant, and let $T_e = 120°C$ and $T_i = T_0 = 20°C$ be the temperatures at the outer and inner radii.

We will also consider another disk, again annular and of constant thickness, having the same outer radius as the previous disk, but with $r_i = 0.1$ m and subjected to the same temperature differential $\Delta T = T_e - T_i$ across the outer and inner radii, and a linear temperature distribution along the radius.

Assuming that the material's thermophysical and mechanical properties remain unchanged up to temperature T_e, we will calculate the distribution of stresses due to thermal loading in the two disks when stationary, and we will determine the maximum values and their locations, discussing how variation in β influences the stress field.

From the problem data, we thus have: $\alpha = 12{\cdot}10^{-6}\ °C^{-1}$; $\nu = 0.3$; $E = 210$ GPa; $\beta = 0.5$ and $k = (T_e - T_i)/(r_e - r_i) = 200°C/m$ for the first disk; $\beta = 0.1$ and $k = (T_e - T_i)/(r_e - r_i) = 111.11°C/m$ for the second disk. Data for constant k appearing in linear function $T = T(r)$ are deduced from relation $T = T_i + [(T_e - T_i)/(r_e - r_i)]{\cdot}(r - r_i)$ which expresses the temperature function in explicit form.

Using the first two relations (3.12) which express the stress field due to heat flow in the annular disk, in which $n = 1$, we obtain the two pairs of dashed-line and solid-line curves shown in Fig. 3.1, which apply respectively to the disk with the larger diameter hole ($\beta = 0.5$) and the disk with the smaller diameter hole ($\beta = 0.1$). As can be seen from these curves, the radial stress for any value of β remains positive at all times, with a maximum value that, if the temperature variation function is (3.1), occurs at $\rho = \left[2\beta^2(1-\beta^n)/n(1-\beta^2)\right]^{1/(n+2)}$, and, in the case of interest to us here with $n = 1$, occurs at $\rho = \left[2\beta^2/(1+\beta)\right]^{1/3}$. Hoop stress assumes a positive absolute maximum value at the inner radius, decreases from the inner radius to the outer radius as ρ increases, becoming null for the value of ρ at which the term in square brackets appearing in the second (3.12) is zero. It then changes sign, becoming a compression stress, and again increases in absolute value until reaching its negative absolute maximum value at the outer radius.

It can thus be concluded that the material is at greatest risk at the inner radius, and that the absolute maximum value of the stress field at the inner radius is heavily influenced by ratio β and, for any given temperature difference $\Delta T = T_e - T_i$ and with all other conditions remaining equal, increases as β decreases. Conversely, the absolute value of compressive hoop stress at the outer radius drops as β decreases.

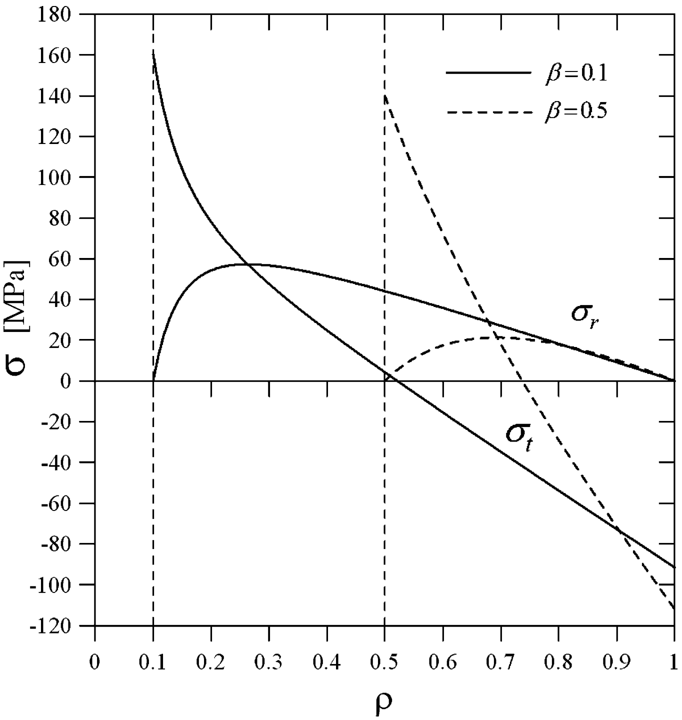


Fig. 3.1 Distribution curves of principal stresses σ_r and σ_t in two annular disks with the same outer radius and different inner radii, subjected to the same temperature differential across inner radius ($T_i = 20°C$) and outer radius ($T_e = 120°C$). Reference temperature $T_0 = 20°C$

3.2 Solid Disk, Subjected to Thermal Load

For the solid disk subjected to a temperature gradient along the radius, we will again consider the three functions $T = T(r)$ given by (3.1), (3.2) and (3.3) respectively.

3.2.1 Function T *Given by an* n-th *Degree Function*

If the function of temperature variation with the radius is expressed by (3.1), we must impose the condition that radial displacement at the centre is zero, as was done for the rotating solid disk. From the first relation (3.5), for $r = 0$, we obtain $C_2 = 0$ and thus, given the second relation (2.9), $B = 0$. As a result, expressions (3.8) become:

$$\begin{aligned}\sigma_r &= A - C' \cdot \rho^n \\ \sigma_t &= A - D' \cdot \rho^n.\end{aligned} \tag{3.26}$$

Here again, we will have $\sigma_r = \sigma_t = A$ at the disk axis, where $\rho = 0$. But as stress σ_r is zero at the outer radius, where $\rho = 1$, we can conclude from the first relation (3.26) that $A = C'$. Accordingly, at the centre of the disk we have:

$$\sigma_r = \sigma_t = A = C' = \frac{E \cdot \alpha \cdot k}{n+2} \cdot r_e^n. \tag{3.27}$$

For the solid disk, (3.12) become:

$$\begin{cases} \sigma_r = \dfrac{E \cdot \alpha \cdot k}{n+2} \cdot r_e^n \cdot (1 - \rho^n) \\ \sigma_t = \dfrac{E \cdot \alpha \cdot k}{n+2} \cdot r_e^n \cdot [1 - (n+1) \cdot \rho^n] \\ u = \dfrac{\alpha \cdot k \cdot r_e^{n+1}}{n+2} \cdot \rho \cdot [(1 - \nu) + (1 + \nu) \cdot \rho^n]. \end{cases} \tag{3.28}$$

These equations express the radial and hoop stresses and radial displacement as a function of ρ. Equation (3.28) can also be derived from (3.12) by setting $\beta = 0$ in the latter.

3.2.2 *Function* T *Given by an* n *Degree Polynomial*

If function $T = T(r)$ is expressed by (3.2), and again imposing the condition that radial displacement at the centre is zero, the first relation (3.13) yields $C_2 = 0$ for $r = 0$, and thus, given the second relation (2.9), $B = 0$. Accordingly, relations (3.15) become:

$$\begin{cases} \sigma_r = A - \alpha \cdot E \cdot \displaystyle\sum_{i=1}^{n} k_i \cdot r_e^i \cdot \frac{\rho^i}{i+2} \\ \sigma_t = A - \alpha \cdot E \cdot \displaystyle\sum_{i=1}^{n} \frac{i+1}{i+2} \cdot k_i \cdot r_e^i \cdot \rho^i. \end{cases} \tag{3.29}$$

Obviously, at the disk axis, where $\rho = 0$, we will again have $\sigma_r = \sigma_t = A$. But as stress σ_r is zero at the outer radius, where $\rho = 1$, we can conclude from the first relation (3.29) that:

$$A = \alpha \cdot E \cdot \sum_{i=1}^{n} \frac{k_i \cdot r_e^i}{i+2}. \tag{3.30}$$

Accordingly, at the centre of the disk we have:

$$\sigma_r = \sigma_t = A = \alpha \cdot E \cdot \sum_{i=1}^{n} \frac{k_i \cdot r_e^i}{i+2}. \tag{3.31}$$

For the solid disk, relations (3.18) become:

$$\begin{cases} \sigma_r = \alpha \cdot E \cdot \sum_{i=1}^{n} \frac{k_i \cdot r_e^i}{i+2} \cdot \left(1 - \rho^i\right) \\ \sigma_t = \alpha \cdot E \cdot \sum_{i=1}^{n} \frac{k_i \cdot r_e^i}{i+2} \cdot \left[1 - (i+1) \cdot \rho^i\right] \\ u = \rho \cdot \alpha \cdot \sum_{i=1}^{n} \frac{k_i \cdot r_e^{i+1}}{i+2} \cdot \left[(1-\nu) + (1+\nu) \cdot \rho^i\right]. \end{cases} \tag{3.32}$$

These equations can be derived from (3.18) by setting $\beta = 0$ in the latter. Here again, relations (3.32) are more general than relations (3.28), which are a particular case thereof. It is sufficient to consider the *n-th* term of polynomial (3.2) alone, or in other words, it is sufficient to set $i = n$, $k_1 = k_2 = \ldots k_{n-1} = 0$ in the latter polynomial to reduce relations (3.32) to relations (3.28).

3.2.3 General Function T

Finally, in the most general possible case in which the function of temperature variation with the radius is expressed by (3.3), we again impose the condition that radial displacement at the centre is zero; thus, for $r = r_i = 0$, we obtain $C_2 = 0$ from the first relation (3.19), and consequently, given the second relation (2.9), $B = 0$.

However, rather than calculating stresses σ_r and σ_t and radial displacement u by specializing expressions (3.21) to the case considered here ($B = 0$), calculating the integration constant A from the first of these expressions (3.21) thus specialized, with the condition that $\sigma_r = 0$ at the outer radius, i.e., for $\rho = 1$ and then substituting the constant found in this way in expressions (3.21) and in relation (3.22), we prefer here to start directly from (3.24) and (3.25), establishing that ratio β tends to zero in (3.24) and inner radius r_i tends to zero in (3.25). In this way, the following relations are obtained from (3.24) and (3.25) respectively:

$$\begin{cases} \sigma_r = \alpha \cdot E \cdot \left[\int_0^1 T \cdot \rho \cdot d\rho - \frac{1}{\rho^2} \cdot \int_0^\rho T \cdot \rho \cdot d\rho \right] \\ \sigma_t = \alpha \cdot E \cdot \left[\int_0^1 T \cdot \rho \cdot d\rho + \frac{1}{\rho^2} \cdot \int_0^\rho T \cdot \rho \cdot d\rho - T \right] \\ u = \frac{\alpha \cdot r_e}{\rho} \cdot \left[\rho^2 \cdot (1 - \nu) \cdot \int_0^1 T \cdot \rho \cdot d\rho + (1 + \nu) \cdot \int_0^\rho T \cdot \rho \cdot d\rho \right]. \end{cases} \tag{3.33}$$

$$\begin{cases} \sigma_r = \alpha \cdot E \cdot \left[\frac{1}{r_e^2} \cdot \int_0^{r_e} T \cdot r \cdot dr - \frac{1}{r^2} \cdot \int_0^r T \cdot r \cdot dr \right] \\ \sigma_t = \alpha \cdot E \cdot \left[\frac{1}{r_e^2} \cdot \int_0^{r_e} T \cdot r \cdot dr + \frac{1}{r^2} \cdot \int_0^r T \cdot r \cdot dr - T \right] \\ u = \frac{\alpha \cdot (1 - \nu) \cdot r}{r_e^2} \cdot \int_0^{r_e} T \cdot r \cdot dr + \frac{\alpha \cdot (1 + \nu)}{r} \cdot \int_0^r T \cdot r \cdot dr. \end{cases} \tag{3.34}$$

Relations (3.34) can be used to calculate stresses and displacement at the centre of the disk. In fact, with a Taylor series expansion of $T = T(r)$, such that

$$T(r) = T(0) + T'(0) \cdot r + \frac{1}{2}\ T''(0) \cdot r^2 + \ldots \tag{3.35}$$

and noting that

$$\begin{aligned} &\lim_{r \to 0} \frac{1}{r^2} \int_0^r T \cdot r \cdot dr = \frac{1}{2} \cdot T(0) \\ &\lim_{r \to 0} \frac{1}{r} \int_0^r T \cdot r \cdot dr = 0, \end{aligned} \tag{3.36}$$

we arrive at the conclusion, which is in any case fairly intuitive and has already been demonstrated using other methods, that at the centre of the disk we have:

$$(\sigma_r)_{r=0} = (\sigma_t)_{r=0};\ (u)_{r=0} = 0. \tag{3.37}$$

3.3 Summary of Results for Constant Thickness Disk Subjected to Thermal Load

In summary form, Table 3.1 shows all results obtained for annular and solid disks of constant thickness subjected to a temperature gradient along the radius described by three different functions $T = T(r)$. For each geometry and each of the three functions describing thermal load, the table indicates boundary conditions and the general relations for σ_r, σ_t and u as a function of dimensionless variable ρ.

3.4 Constant Thickness Disk Subjected to Centrifugal and Thermal Loads

The stress state in a constant thickness annular disk subjected simultaneously to centrifugal and thermal loads is given by the following three pairs of general relations:

$$\begin{cases} \sigma_r = A - B/\rho^2 - C \cdot \rho^2 - C' \cdot \rho^n \\ \sigma_t = A + B/\rho^2 - D \cdot \rho^2 - D' \cdot \rho^n. \end{cases} \tag{3.38}$$

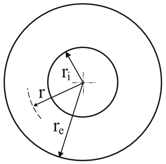
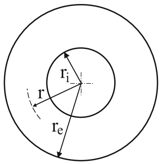
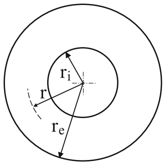
$$\begin{cases} \sigma_r = A - \dfrac{B}{\rho^2} - C \cdot \rho^2 - \alpha \cdot E \cdot \displaystyle\sum_{i=1}^{n} k_i \cdot r_e^i \cdot \dfrac{\rho^i}{i+2} \\ \sigma_t = A + \dfrac{B}{\rho^2} - D \cdot \rho^2 - \alpha \cdot E \cdot \displaystyle\sum_{i=1}^{n} \dfrac{i+1}{i+2} \cdot k_i \cdot r_e^i \cdot \rho^i. \end{cases} \tag{3.39}$$

$$\begin{cases} \sigma_r = A - \dfrac{B}{\rho^2} - C \cdot \rho^2 - \alpha \cdot E \cdot \dfrac{1}{\rho^2} \cdot \displaystyle\int_{\beta}^{\rho} T \cdot \rho \cdot d\rho \\ \sigma_t = A + \dfrac{B}{\rho^2} - D \cdot \rho^2 + \alpha \cdot E \cdot \left(\dfrac{1}{\rho^2} \cdot \displaystyle\int_{\beta}^{\rho} T \cdot \rho \cdot d\rho - T \right). \end{cases} \tag{3.40}$$

These relations apply in cases where the function $T = T(r)$ is expressed by relations (3.1), (3.2) and (3.3).

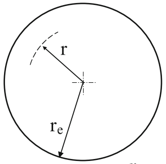
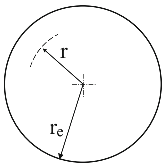
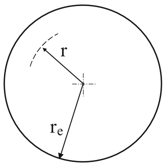
In these relations, the coefficients C, D, C' and D', as well as the terms in the summations and integrals, are zero in load conditions resulting from surface forces acting on the inner edge and/or the outer edge. If the disk is only rotating, the coefficients C' and D' and the terms in the summations and integrals are zero, while if the disk is stationary but subject to a non-zero temperature gradient along the radius, the coefficients C and D are zero.

Table 3.1 Constant thickness disks; geometry (annular disk or solid disk), thermal load represented by three different functions $T = T(r)$, boundary conditions and general relations for calculating stresses σ_r and σ_t and radial displacement u

Thermal load	Boundary conditions	Stresses σ_r and σ_t and radial displacement u
 $T = T_0 + k \cdot r^n$	$\sigma_r = 0$ for $\rho = \beta$ $\sigma_r = 0$ for $\rho = 1$	$\begin{cases} \sigma_r = \dfrac{E \cdot \alpha \cdot k}{n+2} \cdot r_e^n \cdot \left[\dfrac{1-\beta^{n+2}}{1-\beta^2} - \dfrac{1-\beta^n}{1-\beta^2} \cdot \dfrac{\beta^2}{\rho^2} - \rho^n\right] \\ \sigma_t = \dfrac{E \cdot \alpha \cdot k}{n+2} \cdot r_e^n \cdot \left[\dfrac{1-\beta^{n+2}}{1-\beta^2} + \dfrac{1-\beta^n}{1-\beta^2} \cdot \dfrac{\beta^2}{\rho^2} - (n+1) \cdot \rho^n\right] \\ u = \rho \cdot \dfrac{\alpha \cdot k}{n+2} \cdot r_e^{n+1} \cdot \left[\dfrac{1-\beta^{n+2}}{1-\beta^2} \cdot (1-v) + \dfrac{1-\beta^n}{1-\beta^2} \cdot (1+v) \cdot \dfrac{\beta^2}{\rho^2} + (1+v) \cdot \rho^n\right] \end{cases}$
 $T = \sum_{i=0}^{n} k_i \cdot r^i$	$\sigma_r = 0$ for $\rho = \beta$ $\sigma_r = 0$ for $\rho = 1$	$\begin{cases} \sigma_r = \alpha \cdot E \cdot \sum\limits_{i=1}^{n} \dfrac{k_i \cdot r_e^i}{i+2} \cdot \left[\dfrac{1-\beta^{i+2}}{1-\beta^2} - \dfrac{1-\beta^i}{1-\beta^2} \cdot \dfrac{\beta^2}{\rho^2} - \rho^i\right] \\ \sigma_t = \alpha \cdot E \cdot \sum\limits_{i=1}^{n} \dfrac{k_i \cdot r_e^i}{i+2} \cdot \left[\dfrac{1-\beta^{i+2}}{1-\beta^2} + \dfrac{1-\beta^i}{1-\beta^2} \cdot \dfrac{\beta^2}{\rho^2} - (i+1) \cdot \rho^i\right] \\ u = \rho \cdot \alpha \cdot \sum\limits_{i=1}^{n} \dfrac{k_i \cdot r_e^{i+1}}{i+2} \cdot \left[\dfrac{1-\beta^{i+2}}{1-\beta^2} \cdot (1-v) + \dfrac{1-\beta^i}{1-\beta^2} \cdot (1+v) \cdot \dfrac{\beta^2}{\rho^2} + (1+v) \cdot \rho^i\right] \end{cases}$
 $T = T(r)$	$\sigma_r = 0$ for $\rho = \beta$ $\sigma_r = 0$ for $\rho = 1$	$\begin{cases} \sigma_r = \alpha \cdot E \cdot \left[\dfrac{1}{1-\beta^2} \cdot \left(1 - \dfrac{\beta^2}{\rho^2}\right) \cdot \int_\beta^1 T \cdot \rho \cdot d\rho - \dfrac{1}{\rho^2} \cdot \int_\beta^\rho T \cdot \rho \cdot d\rho\right] \\ \sigma_t = \alpha \cdot E \cdot \left[\dfrac{1}{1-\beta^2} \cdot \left(1 + \dfrac{\beta^2}{\rho^2}\right) \cdot \int_\beta^1 T \cdot \rho \cdot d\rho + \dfrac{1}{\rho^2} \cdot \int_\beta^\rho T \cdot \rho \cdot d\rho - T\right] \\ u = \frac{\alpha \cdot r_e}{\rho} \cdot \left[\dfrac{\rho^2 \cdot (1-v) + \beta^2 \cdot (1+v)}{(1-\beta^2)} \cdot \int_\beta^1 T \cdot \rho \cdot d\rho + (1+v) \cdot \int_\beta^\rho T \cdot \rho \cdot d\rho\right]. \end{cases}$

(continued)

Table 3.1 (continued)

Thermal load	Boundary conditions	Stresses σ_r and σ_t and radial displacement u
 $T = T_0 + k \cdot r^n$	$\sigma_r = 0$ for $\rho = 1$ $u = 0$ for $\rho = 0$	$\begin{cases} \sigma_r = \dfrac{E \cdot \alpha \cdot k}{n+2} \cdot r_e^n \cdot (1 - \rho^n) \\ \sigma_t = \dfrac{E \cdot \alpha \cdot k}{n+2} \cdot r_e^n \cdot [1 - (n+1) \cdot \rho^n] \\ u = \dfrac{\alpha \cdot k \cdot r_e^{n+1}}{n+2} \cdot \rho \cdot [(1 - v) + (1 + v) \cdot \rho^n] \end{cases}$
 $T = \sum_{i=0}^{n} k_i \cdot r^i$	$\sigma_r = 0$ for $\rho = 1$ $u = 0$ for $\rho = 0$	$\begin{cases} \sigma_r = \alpha \cdot E \cdot \sum\limits_{i=1}^{n} \dfrac{k_i \cdot r_e^i}{i+2} \cdot \left(1 - \rho^i\right) \\ \sigma_t = \alpha \cdot E \cdot \sum\limits_{i=1}^{n} \dfrac{k_i \cdot r_e^i}{i+2} \cdot \left[1 - (i+1) \cdot \rho^i\right] \\ u = \rho \cdot \alpha \cdot \sum\limits_{i=1}^{n} \dfrac{k_i \cdot r_e^{i+1}}{i+2} \cdot \left[(1 - v) + (1 + v) \cdot \rho^i\right] \end{cases}$
 $T = T(r)$	$\sigma_r = 0$ for $\rho = 1$ $u = 0$ for $\rho = 0$	$\begin{cases} \sigma_r = \alpha \cdot E \cdot \left[\int_0^1 T \cdot \rho \cdot d\rho - \frac{1}{\rho^2} \cdot \int_0^\rho T \cdot \rho \cdot d\rho\right] \\ \sigma_t = \alpha \cdot E \cdot \left[\int_0^1 T \cdot \rho \cdot d\rho + \frac{1}{\rho^2} \cdot \int_0^\rho T \cdot \rho \cdot d\rho - T\right] \\ u = \frac{\alpha \cdot r_e}{\rho} \cdot \left[\rho^2 \cdot (1 - v) \cdot \int_0^1 T \cdot \rho \cdot d\rho + (1 + v) \cdot \int_0^\rho T \cdot \rho \cdot d\rho\right] \end{cases}$

If C, D, C', D' and the terms making up the summations and the integrals are all zero, strain ε_z in the axial direction will in accordance with the generalized form of Hooke's law be:

$$\varepsilon_z = -\frac{v}{E} \cdot (\sigma_r + \sigma_t) = -2 \cdot \frac{v}{E} \cdot A = const. \tag{3.41}$$

For the disk subjected to surface forces acting on the inner and/or outer edges, axial expansion will be constant, and the strain state can be considered as a generalized plane strain state. In this case, the assumption of a plane stress state, and that of an uniform axial translation of one generic cross section, will all lead to the same result. It was mentioned earlier that the stress state due to surface forces in annular disks coincides with that in thick-walled tubes; however, this is no longer true when C, D, C', D' and the terms making up the summations and the integrals are not zero (for example, in disks subjected to thermal load or to centrifugal load).

The stress state in a constant thickness solid disk subjected simultaneously to centrifugal and thermal loads is also given by the three pairs of relations (3.38), (3.39) and (3.40), where we set $B = 0$.

3.5 Stresses in Rotating Disks Having a Fictitious Density Variation Along the Radius

As was indicated in the introduction, the peripheral surface of a disk featuring blades spaced at equal angles, the slots serving as seats for the blades and the material between each slot and the next can be simulated by means of a fictitious, discrete increase in the density of the material variously distributed therein. In other design applications (impellers for centrifugal compressors, impellers for centrifugal pumps and the like), the problem is that of evaluating the stress and strain states arising in the disk as a result of blades evenly distributed on the two side faces (5 and 6 in Fig. I.1). These blades produce a significant increase in centrifugal load stresses, without contributing appreciably to the disk's strength.

The body forces due to these blades are simulated by considering the disk without blades and introducing a fictitious variation in the disk's mass per unit volume (or density) along the radius. This fictitious variation will be discrete for peripheral blades and continuous for lateral blades. A continuous density variation function expressed in the following form is normally used:

$$\gamma(r) = \gamma_0 \cdot (1 + dm'/dm), \tag{3.42}$$

where γ_0 is the density of the basic disk material and dm' and dm are respectively the elementary masses of the blades and of the portion of the disk between two coaxial cylinders of radius r and $r + dr$.

Among the relations expressing density variation with radius that have been introduced in the literature, mention should be made of the exponential function, often uses in literature (see, as an example Güven 1992):

$$\gamma(r) = \gamma_e \cdot r^m, \tag{3.43}$$

where γ_e is the disk's density at its outer radius and m is any exponent, and of the most general relation, used by Giovannozzi [28], which expresses function $\gamma = \gamma(r)$ as follows:

$$\gamma(r) = \sum_{i=0}^{n} \gamma_i \cdot r^i, \tag{3.44}$$

or in other words through an *n-th* degree polynomial in r; as indicated earlier, in this latter function γ_0 represents the density of the rotor material, while $\gamma_1, \gamma_2, \ldots$ are constants and $i = 0, 1, 2, \ldots, n$. Here again, in even more general terms, the function $\gamma = \gamma(r)$ can be expressed in the form $\gamma = \gamma_0 + \sum_{i=1}^{n} \gamma_i \cdot r^{m_i}$, where γ_i are constants and $m_1, m_2, \ldots, m_n$ are exponents that are not necessarily integers. This function was introduced by Botto [6].

Using the exponential function (3.43) and considering a constant thickness disk, (2.2) becomes:

$$\frac{d}{dr}\left[\frac{1}{r} \cdot \frac{d}{dr}(u \cdot r)\right] = -\frac{(1 - \nu^2)}{E} \cdot \gamma_e \cdot \omega^2 \cdot r^{m+1}. \tag{3.45}$$

Integrating the latter equation directly in successive passages yields:

$$\begin{aligned} u &= -\frac{(1 - \nu^2) \cdot \gamma_e \cdot \omega^2 \cdot r^{m+3}}{E \cdot (m+2) \cdot (m+4)} + C_1 \cdot \frac{r}{2} + \frac{C_2}{r} \\ \frac{du}{dr} &= -\frac{(1 - \nu^2) \cdot \gamma_e \cdot \omega^2 \cdot (m+3) \cdot r^{m+2}}{E \cdot (m+2) \cdot (m+4)} + \frac{C_1}{2} - \frac{C_2}{r^2}. \end{aligned} \tag{3.46}$$

Substituting relations (3.46) in (1.27) from which temperature terms are omitted gives the following expressions of σ_r and σ_t as a function of radius r:

$$\begin{cases} \sigma_r = \dfrac{E \cdot C_1}{2 \cdot (1 - \nu)} - \dfrac{E \cdot C_2}{(1 + \nu)} \cdot \dfrac{1}{r^2} - \gamma_e \cdot \omega^2 \cdot \dfrac{m + 3 + \nu}{(m+2) \cdot (m+4)} \cdot r^{m+2} \\ \sigma_t = \dfrac{E \cdot C_1}{2 \cdot (1 - \nu)} + \dfrac{E \cdot C_2}{(1 + \nu)} \cdot \dfrac{1}{r^2} - \gamma_e \cdot \omega^2 \cdot \dfrac{1 + \nu \cdot (m+3)}{(m+2) \cdot (m+4)} \cdot r^{m+2}. \end{cases} \tag{3.47}$$

Continuing to consider radius r as an independent variable, setting $B = E \cdot C_2/(1 + \nu)$ and bearing the first relation (2.9) in mind, we arrive at the following relations expressing radial and hoop stresses as a function of r:

$$\begin{cases} \sigma_r = A - \dfrac{B}{r^2} - \gamma_e \cdot \omega^2 \cdot \dfrac{m+3+\nu}{(m+2)\cdot(m+4)} \cdot r^{m+2} \\ \sigma_t = A + \dfrac{B}{r^2} - \gamma_e \cdot \omega^2 \cdot \dfrac{1+\nu\cdot(m+3)}{(m+2)\cdot(m+4)} \cdot r^{m+2}. \end{cases} \quad (3.48)$$

Finally, introducing relations (3.48) in the expression for ε_t from which the temperature term has been omitted and noting that $u = r \cdot \varepsilon_t$ yields the following relation for displacement $u(r)$ at the generic radius r:

$$u = \frac{r}{E} \cdot \left[A \cdot (1-\nu) + \frac{B}{r^2} \cdot (1+\nu) - \frac{(1-\nu^2)\cdot\gamma_e\cdot\omega^2}{(m+2)\cdot(m+4)} \cdot r^{m+2} \right]. \quad (3.49)$$

Introducing the dimensionless variable ρ and noting that in this case the reference stress is $\sigma_0 = \gamma_e \cdot \omega^2 \cdot r_e^2$, relations (3.48) and (3.49) become:

$$\begin{cases} \sigma_r = A - \dfrac{B}{\rho^2} - \dfrac{m+3+\nu}{(m+2)\cdot(m+4)} \cdot \sigma_0 \cdot r_e^m \cdot \rho^{m+2} \\ \sigma_t = A + \dfrac{B}{\rho^2} - \dfrac{1+\nu\cdot(m+3)}{(m+2)\cdot(m+4)} \cdot \sigma_0 \cdot r_e^m \cdot \rho^{m+2} \\ u = \dfrac{r_e \cdot \rho}{E} \cdot \left[A \cdot (1-\nu) + \dfrac{B}{\rho^2} \cdot (1+\nu) - \dfrac{(1-\nu^2)}{(m+2)\cdot(m+4)} \cdot \sigma_0 \cdot r_e^m \cdot \rho^{m+2} \right], \end{cases} \quad (3.50)$$

where, however, B is given by the second relation (2.9).

From this point onwards, the procedure involves steps that are entirely similar to those described above, in the first of which integration constants A and B are calculated by imposing boundary conditions, which will obviously differ according to whether the disk is annular or solid. These steps will not be further illustrated here.

If the most general polynomial function (3.44) is used and a constant thickness disk is again considered, (2.2) becomes

$$\frac{d}{dr}\left[\frac{1}{r} \cdot \frac{d}{dr}(u \cdot r)\right] = -\left(1-\nu^2\right) \cdot \frac{\omega^2 \cdot r}{E} \cdot \sum_{i=0}^{n} \gamma_i \cdot r^i; \quad (3.51)$$

Integrating this equation directly in successive passages yields:

$$\begin{aligned} u &= -\left(1-\nu^2\right) \cdot \frac{\omega^2}{E} \cdot \sum_{i=0}^{n} \gamma_i \cdot \frac{r^{i+3}}{(i+2)\cdot(i+4)} + C_1 \cdot \frac{r}{2} + \frac{C_2}{r} \\ \frac{du}{dr} &= -\left(1-\nu^2\right) \cdot \frac{\omega^2}{E} \cdot \sum_{i=0}^{n} \gamma_i \cdot \frac{(i+3)}{(i+2)\cdot(i+4)} \cdot r^{i+2} + \frac{C_1}{2} - \frac{C_2}{r^2}. \end{aligned} \quad (3.52)$$

Substituting relations (3.52) in (1.27) from which temperature terms are omitted gives the following expressions of σ_r and σ_t as a function of radius r:

$$\begin{cases} \sigma_r = \dfrac{E \cdot C_1}{2 \cdot (1-\nu)} - \dfrac{E \cdot C_2}{(1+\nu)} \cdot \dfrac{1}{r^2} - \omega^2 \cdot \sum_{i=0}^{n} \gamma_i \cdot \dfrac{(i+3+\nu)}{(i+2)\cdot(i+4)} \cdot r^{i+2} \\ \sigma_t = \dfrac{E \cdot C_1}{2 \cdot (1-\nu)} + \dfrac{E \cdot C_2}{(1+\nu)} \cdot \dfrac{1}{r^2} - \omega^2 \cdot \sum_{i=0}^{n} \gamma_i \cdot \dfrac{1+(i+3)\cdot\nu}{(i+2)\cdot(i+4)} \cdot r^{i+2}. \end{cases} \tag{3.53}$$

Then, continuing to consider radius r as an independent variable, setting $B = E{\cdot}C_2/(1 + \nu)$, bearing in mind the first relation (2.9) and (2.27), which express the constant A and known terms σ_0, C and D respectively, and isolating the γ_0 term from the summations ($\sigma_0 = \gamma_0{\cdot}\omega^2{\cdot}r_e{}^2$), we arrive at the following relations expressing radial and hoop stresses as a function of r:

$$\begin{cases} \sigma_r = A - \dfrac{B}{r^2} - C \cdot r^2 + \omega^2 \cdot \sum_{i=1}^{n} \gamma_i \cdot \dfrac{(i+3+\nu)}{1-(i+3)^2} \cdot r^{i+2} \\ \sigma_t = A + \dfrac{B}{r^2} - D \cdot r^2 + \omega^2 \cdot \sum_{i=1}^{n} \gamma_i \cdot \dfrac{1+(i+3)\cdot\nu}{1-(i+3)^2} \cdot r^{i+2}. \end{cases} \tag{3.54}$$

Finally, introducing relations (3.54) in the expression for ε_t from which the temperature term has been omitted and noting that $u = r{\cdot}\varepsilon_t$ yields the following relation for displacement $u(r)$:

$$u = \frac{r}{E} \cdot \left[A \cdot (1-\nu) + \frac{B}{r^2} \cdot (1+\nu) - (D - \nu \cdot C) \cdot r^2 + \omega^2 \cdot \left(1-\nu^2\right) \cdot \sum_{i=1}^{n} \gamma_i \cdot \frac{r^{i+2}}{1-(i+3)^2} \right]. \tag{3.55}$$

Here again, from this point onwards, the procedure involves steps that are entirely similar to those described above, in the first of which integration constants A and B are calculated by imposing boundary conditions, which will obviously differ according to whether the disk is annular or solid. These steps will also not be further illustrated here.