Chapter 1
Mono-Dimensional Elastic Theory of Thin Disk

Take (Fig. I.1) a thin disk having a geometry of revolution, featuring a plane of symmetry perpendicular to its axis (plane of axial coordinate $z = 0$, in which the other two axes x, y that complete the three-dimension rectangular Cartesian reference system O(x, y, z) lie), rotating at angular velocity ω and subject to angular acceleration $\dot{\omega}$. Let it also be assumed that no surface force is applied to disk side faces 5 and 6. In addressing this type of problem, it is advantageous to use a polar reference system O(r, ϑ, z), where the position of a generic point P (x, y) $\equiv$ P (r, ϑ), in the midplane of the disk is defined by coordinate r, which establishes its distance from pole O, and by angle ϑ (angular coordinate) between axis O-r and abscissa O-x taken as reference.

The disk's mass elements will be subjected to centrifugal forces and to elementary tangential forces. These forces generate the stress and strain states that will be analysed here using the mono-dimensional theory, whose range of validity was illustrated in the introduction. Within the limits of this theory, radial stress σ_r, hoop stress σ_t and radial displacement u are functions of coordinate r alone, or in other words are constant above a cylinder of radius r whose axis is the axis of the disk.

To analyse thin disk stress and strain states, the equilibrium equations and the compatibility equations must be considered simultaneously.

1.1 Equilibrium Equations

Take an isolated mixtilinear volume element of the disk located between side surfaces 5 and 6 shown in Fig. I.1 and delimited by two diametral planes with angular coordinates ϑ and $\vartheta + d\vartheta$, which consequently form the angle $d\vartheta$ between them, and by two cylinders having radii r and $r + dr$ (Fig. 1.1). This element thus has finite dimensions in the direction of the z axis and infinitesimal dimensions along the other two polar coordinates. Consider the equilibrium of all forces acting on the element, both along line OA perpendicular to the disk axis and passing through the centre of gravity G, and along line GB perpendicular in G to OA.

V. Vullo and F. Vivio, *Rotors: Stress Analysis and Design*,
Mechanical Engineering Series, DOI 10.1007/978-88-470-2562-2_1,

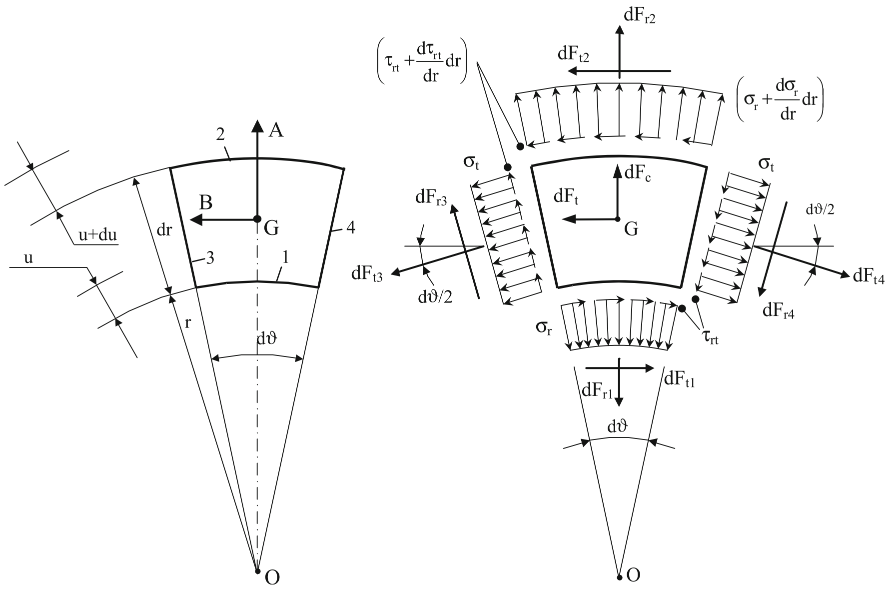

Fig. 1.1 Volume element, distribution of normal and shear stress components on its faces assuming axisymmetry, and elementary surface and body forces acting on the element

The volume element is subjected to two elementary body forces applied at the centre of gravity: force dF_c acting in a radial centrifugal direction and due to the centrifugal load (this is generally the most severe load condition), and force dF_t acting in the tangential direction, i.e., perpendicular to the radius and due to angular acceleration $\dot{\omega}$. These forces are given by the following relations:

$$dF_c = r \cdot \omega^2 \cdot dm = r \cdot \omega^2 \cdot \gamma \cdot dV = \gamma \cdot \omega^2 \cdot r^2 \cdot h \cdot d\vartheta \cdot dr,$$
$$dF_t = r \cdot \dot{\omega} \cdot dm = r \cdot \dot{\omega} \cdot \gamma \cdot dV = \gamma \cdot \dot{\omega} \cdot r^2 \cdot h \cdot d\vartheta \cdot dr. \tag{1.1}$$

where dm and dV denote the elementary mass and volume of the isolated element, and γ is the mass per unit volume, or density, of the material.

The other elementary forces acting on the volume element are:

- The radial force on the element's cylindrical face 1, at radius r, directed radially and facing the interior:

$$dF_{r1} = \sigma_r \cdot h \cdot r \cdot d\vartheta; \tag{1.2}$$

- The tangential force on the element's cylindrical face 1, at radius r, directed tangentially:

$$dF_{t1} = t_{rt} \cdot h \cdot r \cdot d\vartheta, \tag{1.3}$$

where τ_{rt} is the shear stress component;

- The radial force on the element's cylindrical face 2, at radius $r + dr$, directed radially and facing the exterior:

$$dF_{r2} = \left(\sigma_r + \frac{d\sigma_r}{dr} \cdot dr\right) \cdot \left(h + \frac{dh}{dr} \cdot dr\right) \cdot (r + dr) \cdot d\vartheta = dF_{r1} + \frac{d}{dr}(dF_{r1}) \cdot dr = \\ = \left[\sigma_r \cdot h \cdot r + \frac{d}{dr}(\sigma_r \cdot h \cdot r) \cdot dr\right] \cdot d\vartheta; \quad (1.4)$$

- Tangential force on the element's cylindrical face 2, at radius $r + dr$, directed tangentially:

$$dF_{t2} = \left(\tau_{rt} + \frac{d\tau_{rt}}{dr} \cdot dr\right) \cdot \left(h + \frac{dh}{dr} \cdot dr\right) \cdot (r + dr) \cdot d\vartheta = dF_{t1} + \frac{d}{dr}(dF_{t1}) \cdot dr = \\ = \left[\tau_{rt} \cdot h \cdot r + \frac{d}{dr}(\tau_{rt} \cdot h \cdot r) \cdot dr\right] \cdot d\vartheta\ ; \quad (1.5)$$

- Two equal radial forces acting on the element's plane side faces 3 and 4 and facing in the centrifugal direction on face 3 and in the centripetal direction on face 4 (these forces are equal in modulus, neglecting higher-order infinitesimals):

$$|dF_{r3}| = |dF_{r4}| = \tau_{rt} \cdot h \cdot dr; \quad (1.6)$$

- Two equal tangential forces acting on the element's plane side faces 3 and 4, perpendicular to them and facing the exterior of the element:

$$|dF_{t3}| = |dF_{t4}| = \sigma_t \cdot h \cdot dr. \quad (1.7)$$

In formulating these relations, which express the elementary forces acting on the mixtilinear volume element, terms introducing higher-order infinitesimals in the equilibrium equations are omitted. In accordance with the initial assumptions, in any case, no forces act on the volume element's side faces 5 and 6, which are those delimiting the disk laterally (Fig. I.1).

Accordingly, the dynamic equilibrium equations along radial direction *GA* and tangential direction *GB* are:

$$\begin{cases} -dF_{r1} + dF_{r2} + dF_{r3} \cdot cos\dfrac{d\vartheta}{2} - dF_{r4} \cdot cos\dfrac{d\vartheta}{2} - dF_{t3} \cdot \sin\dfrac{d\vartheta}{2} \\ \quad - dF_{t4} \cdot \sin\dfrac{d\vartheta}{2} + dF_c = 0 \\ -dF_{t1} + dF_{t2} + dF_{r3} \cdot \sin\dfrac{d\vartheta}{2} + dF_{r4} \cdot \sin\dfrac{d\vartheta}{2} + dF_{t3} \cdot cos\dfrac{d\vartheta}{2} \\ \quad - dF_{t4} \cdot cos\dfrac{d\vartheta}{2} + dF_t = 0. \end{cases} \quad (1.8)$$

By introducing relations (1.1, 1.2, 1.3, 1.4, 1.5, 1.6, and 1.7) derived earlier, which express the elementary forces acting on the element, in the above equilibrium equations, noting that $cos(d\vartheta/2) = 1$ and $sin(d\vartheta/2) = d\vartheta/2$ for infinitesimal angle $d\vartheta/2$, developing calculations and taking care to omit higher-order infinitesimals, we arrive at the following dynamic equilibrium equations for the volume element:

$$\begin{cases} \sigma_r \cdot h + \dfrac{d\sigma_r}{dr} \cdot h \cdot r + \dfrac{dh}{dr} \cdot \sigma_r \cdot r - \sigma_t \cdot h + \gamma \cdot \omega^2 \cdot r^2 \cdot h = 0 \\ \tau_{rt} \cdot h + \dfrac{d\tau_{rt}}{dr} \cdot h \cdot r + \dfrac{dh}{dr} \cdot \tau_{rt} \cdot r + \tau_{rt} \cdot h + \gamma \cdot \dot{\omega} \cdot r^2 \cdot h = 0. \end{cases} \tag{1.9}$$

These equations can also be rewritten in the following more compact form:

$$\begin{cases} \dfrac{d}{dr}(\sigma_r \cdot h \cdot r) - \sigma_t \cdot h + \gamma \cdot \omega^2 \cdot r^2 \cdot h = 0 \\ \dfrac{d}{dr}(\tau_{rt} \cdot h \cdot r) + \tau_{rt} \cdot h + \gamma \cdot \dot{\omega} \cdot r^2 \cdot h = 0. \end{cases} \tag{1.10}$$

Note that the shear stress components do not appear in the first of (1.10), which expresses dynamic equilibrium in the radial direction, while the normal stress components do not appear in the second of (1.10), which expresses dynamic equilibrium in the direction perpendicular to the radius. This decoupling of normal and shear stress components is because axisymmetry was assumed. Had this assumption not been made, the equilibrium equations would have been:

$$\begin{cases} \dfrac{\partial}{\partial r}(\sigma_r \cdot h \cdot r) + \dfrac{\partial}{\partial \vartheta}(\tau_{rt} \cdot h) - \sigma_t \cdot h + \gamma \cdot \omega^2 \cdot r^2 \cdot h = 0 \\ \dfrac{\partial}{\partial r}(\tau_{rt} \cdot h \cdot r) + \dfrac{\partial}{\partial \vartheta}(\sigma_t \cdot h) + \tau_{rt} \cdot h + \gamma \cdot \dot{\omega} \cdot r^2 \cdot h = 0. \end{cases} \tag{1.11}$$

These are more general than (1.10), which are a specific case for axisymmetry.

1.2 Compatibility Equations

In the general case in which there is no axisymmetry, normal strain components (radial strain ε_r and tangential strain ε_t) and shear strain component γ_{rt} are linked to components of the displacement in the radial and tangential directions u and v by the following geometric relations:

$$\begin{cases} \varepsilon_r = \dfrac{\partial u}{\partial r} \\ \varepsilon_t = \dfrac{u}{r} + \dfrac{1}{r} \cdot \dfrac{\partial v}{\partial \vartheta} \\ \gamma_{rt} = \dfrac{\partial v}{\partial r} - \dfrac{v}{r} + \dfrac{1}{r} \cdot \dfrac{\partial u}{\partial \vartheta}. \end{cases} \tag{1.12}$$

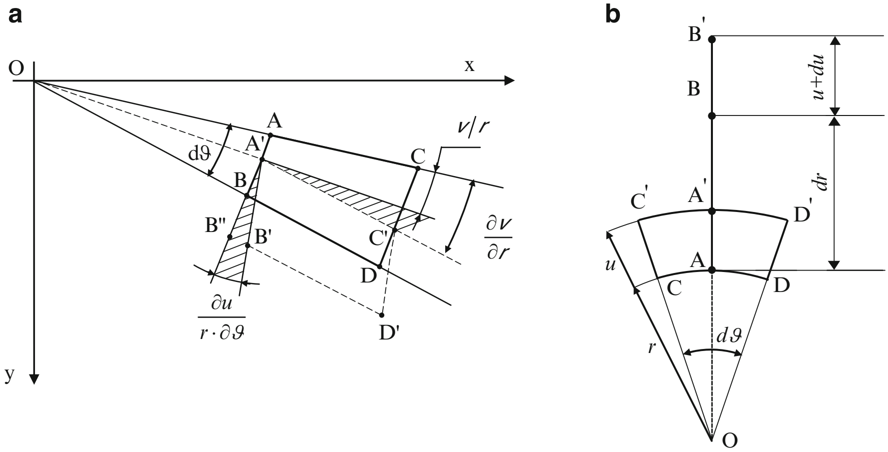

Fig. 1.2 (**a**) Displacements and rotations of a mixtilinear element, without axisymmetry; (**b**) elementary radial and tangential fibers and displacements of their ends, with axisymmetry

The first of relations (1.12) is obtained by considering (Fig. 1.2a) the undeformed mixtilinear element $ABCD$ located between two cylinders of radius r and $r + dr$ and between two diametral planes with angular coordinates ϑ and $\vartheta + d\vartheta$ (given that $d\vartheta$ is an elementary angle, arcs AB and CD, which are not shown in the figure in order to avoid over-complication, can be replaced by their chords) and bearing in mind that the radial displacements of sides $\overline{AB}$ and $\overline{CD}$ are given by u and by $u + (\partial u/\partial r) \cdot dr$ respectively. It follows that the unit elongation of this element in the radial direction is given by relation $\varepsilon_r = \partial u/\partial r$, or in other words by the first of the two expressions (1.12).

Tangential strain component ε_t, on the other hand, depends on both radial displacement u and tangential displacement v. For side $\overline{AB}$ of the element in question, the contribution of radial displacement u is evaluated by considering that this displacement causes initial arc $AB = r \cdot d\vartheta$ to acquire a length $(r + u) \cdot d\vartheta$. Consequently (Fig. 1.2b), the tangential strain related to displacement u is given by relation:

$$\frac{(r + u) \cdot d\vartheta - r \cdot d\vartheta}{r \cdot d\vartheta} = \frac{u}{r}. \tag{1.13}$$

The contribution of tangential displacement v is determined by considering that it moves point A to A', with a tangential displacement v, and point B moves to point B'', with a tangential displacement $v + (\partial v/\partial\vartheta) \cdot d\vartheta$. It follows that there is a difference of tangential displacement equal to $(\partial v/\partial\vartheta) \cdot d\vartheta$, and the related tangential strain will be $(1/r) \cdot \partial v/\partial\vartheta$. Adding the two contributions gives the total tangential strain expressed by the second of relations (1.12).

Shear strain component γ_{rt} represents the total variation of angle BAC, initially a right angle. It is determined by comparing the element's position $A'B'C'D'$ after strain-induced deformation, and its initial position $ABCD$. The angle between directions AB and $A'B'$ is due to radial displacement u and is $(1/r) \cdot \partial u/\partial\vartheta$.

Likewise, the angle between directions AC and $A'C'$ is due to tangential displacement v and is $\partial v/\partial r$. However, only a part of the latter contributes to shear strain component (the part represented with a dashed area), as the remaining part, v/r represents the angular displacement due to rotation of the element $ABCD$ as a rigid body about the axis through O. It can be concluded from these considerations that shear strain component γ_{rt} is given by the third relation (1.12).

Where the assumption of axisymmetry applies, relations (1.12) are reduced to the simpler form:

$$\varepsilon_r = \frac{du}{dr}; \quad \varepsilon_t = \frac{u}{r}; \quad \gamma_{rt} = \frac{dv}{dr} - \frac{v}{r}. \tag{1.14}$$

In the case of axisymmetry, as can thus be seen, normal strain components ε_r and ε_t are linked only to radial displacement u, while shear strain component γ_{rt} is linked only to tangential displacement v. Assuming axisymmetry, expressions ε_r and ε_t can be derived from even simpler geometrical considerations. Take, for instance, a radial fiber, considering (Fig. 1.2b) a portion thereof of infinitesimal length dr prior to deformation, located between points A and B at distances r and $r + dr$ from the centre respectively: after deformation, end A moves to A', displaced by u, while end B moves to B', displaced by $u + du$. Similarly, if we take a tangential fiber, considering (again with reference to Fig. 1.2b) an infinitesimal arc thereof of length $r{\cdot}d\vartheta$ prior to displacement, located between points C and D both at distance r from the centre, after displacement end C moves to C', displaced by u, while end D moves to D', also displaced by u. We thus have:

$$\varepsilon_r = \frac{\overline{A'B'} - \overline{AB}}{\overline{AB}} = \frac{du}{dr}, \tag{1.15}$$

$$\varepsilon_t = \frac{(r+u)\cdot d\vartheta - r\cdot d\vartheta}{r\cdot d\vartheta} = \frac{u}{r}. \tag{1.16}$$

It is also clear that, where loading is entirely centrifugal ($\omega \neq 0$ and $\dot{\omega} = 0$) and for an axisymmetric stress field, tangential displacement v will be zero, and $\gamma_{rt} = 0$. Essentially, then, if axisymmetry is assumed, when the mixtilinear element shown in Fig. 1.1 is subjected to axisymmetric centrifugal load, its movement will be entirely radial, with no tangential displacement.

The first two geometric relations (1.14), which apply when axisymmetry is assumed, give the following compatibility equation in terms of strain:

$$\varepsilon_r = \frac{d}{dr}(r\cdot\varepsilon_t). \tag{1.17}$$

Were the assumption of axisymmetry not to apply, the compatibility equation would be expressed as follows:

$$r\cdot\frac{\partial^2\gamma_{rt}}{\partial r\cdot\partial\vartheta} + \frac{\partial\gamma_{rt}}{\partial\vartheta} - r^2\cdot\frac{\partial^2\varepsilon_t}{\partial r^2} - 2r\cdot\frac{\partial\varepsilon_t}{\partial r} + r\cdot\frac{\partial\varepsilon_r}{\partial r} - \frac{\partial^2\varepsilon_r}{\partial\vartheta^2} = 0, \tag{1.18}$$

which is more general than (1.17) and subsumes it.

For an axisymmetric orthotropic material, the constitutive equations which relate the stress components to strain components (these equations, in the linear elastic range, represent the generalised Hooke's law) can be written in the following form:

$$\begin{Bmatrix} \varepsilon_r \\ \varepsilon_t \\ \gamma_{rt} \end{Bmatrix} = [\mathbf{S}] \cdot \begin{Bmatrix} \sigma_r \\ \sigma_t \\ \tau_{rt} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_r} & \frac{-v_{tr}}{E_t} & 0 \\ \frac{-v_{rt}}{E_r} & \frac{1}{E_t} & 0 \\ 0 & 0 & \frac{1}{G_{rt}} \end{bmatrix} \cdot \begin{Bmatrix} \sigma_r \\ \sigma_t \\ \tau_{rt} \end{Bmatrix}. \quad (1.19)$$

As will be recalled, an orthotropic material is a non-isotropic material which has three mutually orthogonal planes of symmetry at every point. From the standpoint of its elastic behaviour, complete characterization requires nine parameters, viz., the three Young's moduli E, the three moduli of rigidity G and the three Poisson's ratios v in the directions normal to the planes of symmetry. For complete elastic characterization of an isotropic material, on the other hand, only two of the three parameters E, G, v, are required, as they are linked by a linear dependence. If the structure of the orthotropic material is such that it has three mutually orthogonal planes of symmetry at every point, but also an overall axis of symmetry, the material is *axisymmetric orthotropic*. A composite material produced by winding a fiber embedded in a matrix around a cylinder fulfils these conditions if the fiber is wound tangentially: in this case, the radial, tangential and axial directions are normal to the planes of symmetry at every point.

In formulating the constitutive equations given above, the tangential and radial directions of the axisymmetric orthotropic material were considered to be the principal directions. If this were not true, no component of the matrix of the material compliances **[S]** would be zero. In this case, normal strain components ε_r and ε_t and shear strain component γ_{rt} would be linked simultaneously to normal stresses σ_r and σ_t and to shear stress τ_{rt}, and a number of the simplifications that will be discussed below would not be possible. Note that in writing matrix **[S]** appearing in (1.19), the first term in the second line, i.e., $-v_{rt}/E_r$, is often substituted by $-v_{tr}/E_t$, as this matrix is symmetrical with respect to the principal diagonal.

Though we will be dealing chiefly with homogeneous isotropic materials in this text, generalizing for axisymmetric orthotropic materials presents no difficulties. The constitutive equations for homogeneous isotropic materials take the following form:

$$\begin{Bmatrix} \varepsilon_r \\ \varepsilon_t \\ \gamma_{rt} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E} & \frac{-v}{E} & 0 \\ \frac{-v}{E} & \frac{1}{E} & 0 \\ 0 & 0 & \frac{1}{G} \end{bmatrix} \cdot \begin{Bmatrix} \sigma_r \\ \sigma_t \\ \tau_{rt} \end{Bmatrix}. \quad (1.20)$$

If the effect of a generic temperature variation deriving from a thermal load is to be taken into account, the strain components due to this thermal load must obviously be added to strains ε_r and ε_t caused by stresses $\{\sigma\}$. These additional strain components are given by $\alpha \cdot T$, where α and T are the material's coefficient of linear thermal expansion and the temperature respectively. If the material is homogeneous and isotropic, α does not depend on direction and, consequently, (1.20) becomes:

$$\begin{Bmatrix} \varepsilon_r \\ \varepsilon_t \\ \gamma_{rt} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E} & \frac{-v}{E} & 0 \\ \frac{-v}{E} & \frac{1}{E} & 0 \\ 0 & 0 & \frac{1}{G} \end{bmatrix} \cdot \begin{Bmatrix} \sigma_r \\ \sigma_t \\ \tau_{rt} \end{Bmatrix} + \alpha \cdot T \cdot \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix}. \tag{1.21}$$

For an axisymmetric orthotropic material, (1.21) becomes:

$$\begin{Bmatrix} \varepsilon_r \\ \varepsilon_t \\ \gamma_{rt} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_r} & \frac{-v_{tr}}{E_t} & 0 \\ \frac{-v_{rt}}{E_r} & \frac{1}{E_t} & 0 \\ 0 & 0 & \frac{1}{G_{rt}} \end{bmatrix} \cdot \begin{Bmatrix} \sigma_r \\ \sigma_t \\ \tau_{rt} \end{Bmatrix} + T \cdot \begin{Bmatrix} \alpha_r \\ \alpha_t \\ 0 \end{Bmatrix}. \tag{1.22}$$

Taking the expressions of ε_r and ε_t from (1.21) and introducing them in the compatibility equation (1.17) gives the following compatibility equation in terms of stress:

$$(\sigma_t - \sigma_r) \cdot (1 + v) + r \cdot \frac{d\sigma_t}{dr} - v \cdot r \cdot \frac{d\sigma_r}{dr} + \alpha \cdot E \cdot r \frac{dT}{dr} = 0. \tag{1.23}$$

This equation applies assuming that quantities E, v and α are constant throughout the disk. Were these quantities variable, (1.23) would become:

$$(\sigma_t - \sigma_r) \cdot (1 + v) + r \cdot \frac{d\sigma_t}{dr} - v \cdot r \cdot \frac{d\sigma_r}{dr} + \cdot E \cdot r \cdot \sigma_t \frac{d}{dr}\left(\frac{1}{E}\right) - E \cdot r \cdot \sigma_r \cdot \frac{d}{dr}\left(\frac{v}{E}\right) + E \cdot r \cdot \frac{d}{dr}(\alpha \cdot T) = 0. \tag{1.24}$$

1.3 General Differential Equation for Rotating Disk Subjected to Thermal Load

Here, analysis will be limited to the stress and strain state in an axisymmetric disk of arbitrary profile only rotating and subjected to a non-zero temperature gradient along the radius.

If the disk is only rotating ($\omega = const$), the only body force acting on it is that due to the centrifugal load. Accordingly, the tangential equilibrium equation given by the second relation (1.10) reduces to the identity $\tau_{rt} = 0$, which clearly satisfies the compatibility equation. Thus, if functions $h = h(r), \gamma = \gamma(r), E = E(r)$, $v = v(r), T = T(r)$ and $\alpha = \alpha(r)$ are known, the radial equilibrium equation given by the first relation (1.10) and compatibility equation (1.23) or (1.24) are sufficient to solve the problem completely, i.e., to determine how the two unknowns σ_r and σ_t vary with radius. From the two equilibrium and compatibility equations, which are first order differential equations in two unknowns σ_r and σ_t, we can obtain only one second order differential equation in one unknown.

Within certain arbitrary limits, two alternatives are usually considered in selecting this unknown: the first is to express stress components as a function of radial displacement u by means of the compatibility equation and then introduce these expressions in the equilibrium equation, while the second alternative is to first derive the expression of stress σ_t as a function of σ_r from the equilibrium equation and then introduce this expression in the compatibility equation. The two approaches are equivalent and lead to very similar solving equations. The first approach will be followed below, assuming that quantities γ, E, v and α are constant and thus independent of r. In dealing with the disk whose profile varies non-linearly according to a power of a linear function and which is loaded beyond yielding, the second approach will be used (see Chap. 12, Sect. 12.2).

Equation 1.21 give:

$$\begin{cases} \varepsilon_r = \dfrac{1}{E}(\sigma_r - v \cdot \sigma_t) + \alpha \cdot T \\ \varepsilon_t = \dfrac{1}{E}(\sigma_t - v \cdot \sigma_r) + \alpha \cdot T, \end{cases} \tag{1.25}$$

which, solved for σ_r and σ_t, yield the relationships:

$$\begin{cases} \sigma_r = \dfrac{E}{1 - v^2} \cdot [(\varepsilon_r - \alpha \cdot T) + v \cdot (\varepsilon_t - \alpha \cdot T)] \\ \sigma_t = \dfrac{E}{1 - v^2} \cdot [(\varepsilon_t - \alpha \cdot T) + v \cdot (\varepsilon_r - \alpha \cdot T)]. \end{cases} \tag{1.26}$$

Bearing the first two geometric relations (1.12) in mind, relations (1.26) can be written in the form:

$$\begin{cases} \sigma_r = \dfrac{E}{1 - v^2} \cdot \left[\left(\dfrac{du}{dr} - \alpha \cdot T\right) + v \cdot \left(\dfrac{u}{r} - \alpha \cdot T\right)\right] \\ \sigma_t = \dfrac{E}{1 - v^2} \cdot \left[\left(\dfrac{u}{r} - \alpha \cdot T\right) + v \cdot \left(\dfrac{du}{dr} - \alpha \cdot T\right)\right]. \end{cases} \tag{1.27}$$

Deriving the first of relations (1.27) by respect to r and introducing this derivative, together with relations (1.27) in the equilibrium equation, which in this case is the first (1.10), and carrying out a few other passages yields the following second order differential equation in u, which is the equation that solves the disk of arbitrary profile only rotating and subject to a non-zero temperature gradient along the radius:

$$\frac{d^2u}{dr^2} + \left(\frac{1}{h}\cdot\frac{dh}{dr} + \frac{1}{r}\right)\cdot\frac{du}{dr} + \left(\frac{\nu}{h\cdot r}\cdot\frac{dh}{dr} - \frac{1}{r^2}\right)\cdot u - (1+\nu)\cdot\alpha\cdot\left(\frac{dT}{dr} + \frac{T}{h}\cdot\frac{dh}{dr}\right) + \left(1-\nu^2\right)\cdot\frac{\gamma\cdot\omega^2\cdot r}{E} = 0. \tag{1.28}$$

Whether this equation can be integrated analytically depends on the function $h = h\,(r)$. Closed form integration is possible, and relatively simple, in three special cases: constant thickness disk, uniform strength disk, and hyperbolic disk.[1]

In other cases, such as those of the linear tapered disk (the so-called conical profile disk) and the non-linearly variable thickness disk, a hypergeometric series solution is possible. Series solutions are also possible for the following other profiles (these will also be hypergeometric in all cases below except for the two-parameter exponential profile, whose solution involves the use of confluent hypergeometric series):

- Profile whose thickness varies according to an exponential function of the type characterizing the uniform strength disk, but with two parameters, and thus defined by relation $h = h_0{\cdot}e^{-n\cdot\rho^k}$, where h_0 is the thickness at the axis, $\rho = r/r_e$ is the radius made dimensionless relative to the outside radius r_e, and n and k are the two geometric parameters controlling thickness at the outer edge relative to that at the axis and the profile shape respectively. This function makes it possible to describe solid and annular disks with concave, convex and inflection point profiles, but not conical disks;
- Profile whose thickness varies according to a generalization of Stodola's hyperbolic function, defined by the relation $h = h_0{\cdot}(1+\rho)^a$, where a is a parameter controlling disk shape. This function does not give rise to singularity at the axis and thus, unlike Stodola's relationship, can be used to describe also the hyperbolic profile of a solid disk;
- Profile whose thickness varies according to an elliptical function defined by the relation $h = h_0{\cdot}\,(1-n\cdot\rho^2)^{1/2}$, with one parameter n, whereby convex converging and concave diverging profiles can be described;
- Profile whose thickness varies according to the following two parabolic functions, both with two parameters, $h = h_0{\cdot}\,\left(1-n\cdot\rho^k\right)$ and $h = h_0{\cdot}\left[1-\left(\frac{r_e\cdot\rho}{r_e+n}\right)^k\right]$, whereby solid and annular convex and linear tapered disk profiles can be described.

[1] For a broad overview of these importacnt topics, see, in particular, Stodola [70], Love [44], Giovannozzi [29], Timoshenko and Goodier [74], Saada [62], Burr [7], Ugural and Fenster [76].

In all other cases, it is necessary to use numerical solutions which, however, pose no difficulties and yield results that, though approximate, are acceptable from the design standpoint.

In the following pages, the general (1.28) will be specialized and integrated for the three cases that can be solved in closed form, as well as for the two families consisting of the conical disk and the non-linearly variable thickness disk, with the disk subject to simple load conditions. In the linear elastic field (where stress is proportional to strain), the stress and strain states in a disk subjected to a complex load condition can obviously be determined by using the method of the superposition, that is as the superposition of the individual load conditions into which actual loading can be broken down, which are assumed to operate separately. The treatment used for the other disk profiles indicated above will not be discussed, as the analytical developments are formally similar to those for the two families we have just mentioned. These developments will thus be left to the reader.