On the damped geometric telegrapher's process

Antonio Di Crescenzo, Barbara Martinucci, and Shelemyahu Zacks

Abstract. The geometric telegrapher's process has been proposed in 2002 as a model to describe the dynamics of the price of risky assets. In this contribution we consider a related stochastic process, whose trajectories have two alternating slopes, for which the random times between consecutive slope changes have exponential distribution with linearly increasing parameters. This leads to a process characterized by a damped behavior. We study the main features of the transient probability law of the process, and of its stationary limit.

Key words: Geometric telegrapher's process, damped processes, exponential times, linear rates, log-logistic stationary distribution, moment generating function

1 Introduction

Motivated by the need of describing the price of a risky asset by means of a process with bounded variations, which seems quite realistic in true markets, [4] introduced the geometric telegrapher's process expressed via an exponential transformation of the telegrapher's process. Paper [8] proposed a similar financial market model that is free of arbitrage under suitable conditions, and is based on a continuous time random motion with alternating constant velocities and jumps occurring when the

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velocities are switching. Other contributions on stochastic processes characterized by alternating finite velocities are given in [2, 7, 9] and [10]. Moreover, the problem of estimating the parameters of the geometric telegrapher's process has been faced in [1].

In this contribution we study a modified version of the geometric telegrapher's process under the assumption that the random times between consecutive slope changes are exponentially distributed with linearly increasing parameters. This is suggested in the recent paper [3], where a damped telegrapher's process is studied. In this framework the trajectories of damped processes are continuous curves composed by stochastically smaller and smaller paths. Some examples of damped diffusion processes can be found in the literature of financial modeling, such as [6].

The damped geometric telegrapher's process is introduced in Section 2, where we obtain its probability law and study the asymptotic behavior. The moment generating function-approach is then used to evaluate the m-th moment of the process.

We remark that our contribution can be seen as an initial attempt to modify the geometric telegrapher's process. Specific problems of mathematical finance, such as the problem of existence of arbitrage opportunities, will be the object of future investigations.

2 The stochastic model and probability laws

Let us assume that the price of risky assets is described by the following stochastic process, named damped geometric telegrapher's process:

$$S_t = s_0 \exp\left[a\, t + X_t\right], \qquad \text{with} \qquad X_t = c \int_0^t (-1)^{N_\tau} \mathrm{d}\tau, \quad t \geq 0, \tag{1}$$

where $s_0 > 0$, $a \in \mathbb{R}$, $c > 0$, and where N_t is an alternating counting process characterized by independent random times U_k, D_k, $k \geq 1$. Hence,

$$N_0 = 0, \qquad N_t = \sum_{n=1}^{\infty} \mathbf{1}_{\{T_n \leq t\}}, \qquad t > 0,$$

where $T_{2k} = U^{(k)} + D^{(k)}$ and $T_{2k+1} = T_{2k} + U_{k+1}$ for $k = 0, 1, \ldots$, with $U^{(0)} = D^{(0)} = 0$ and

$$U^{(k)} = U_1 + U_2 + \cdots + U_k, \qquad D^{(k)} = D_1 + D_2 + \cdots + D_k, \qquad k = 1, 2, \ldots. \tag{2}$$

We assume that $\{U_k\}$ and $\{D_k\}$ are mutually independent sequences of independent random variables characterized by exponential distribution with parameters

$$\lambda_k = \lambda\, k, \qquad \mu_k = \mu\, k, \qquad (\lambda, \mu > 0;\ k = 1, 2, \ldots), \tag{3}$$

respectively. We remark that process S_t has bounded variations and its sample-paths are constituted by connected lines having exponential behavior, characterized alter-

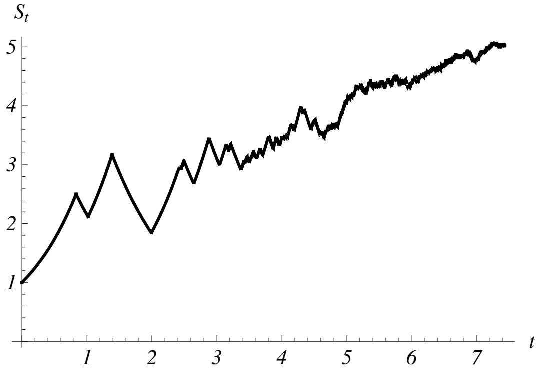

Fig. 1. A simulated sample path of S_t

nately by growth rates $a + c$ and $a - c$, where a is the growth rate of risky assets' price in the absence of randomness, and c is the intensity of the random factor of alternating type. Assumption (3) implies that the reversal rates λ_k and μ_k linearly increase with the number of reversals, so that the sample paths of S_t are subject to an increasing number of slope changes when t increases, this giving a damped behavior. An example is shown in Fig. 1.

Denoting by $F^{(k)}(u)$ the distribution function of the k-fold convolution of random variables U_j (see (2)), hereafter we show a suitable method to disclose it.

Proposition 1. *For $k = 1, 2, \dots$ we have*

$$F^{(k)}(u) := P(U^{(k)} \le u) = (1 - \mathrm{e}^{-\lambda u})^k, \qquad u \ge 0. \tag{4}$$

Proof. We proceed by induction on k. For $k = 1$, the result is obvious. Let us now assume (4) holding for all $m = 1, \dots, k - 1$. Hence, due to independence,

$$\begin{aligned}
F^{(k)}(u) &= \lambda k \int_0^u \mathrm{e}^{-\lambda k y}(1 - \mathrm{e}^{-\lambda(u-y)})^{k-1}\mathrm{d}y \\
&= \lambda k \sum_{j=0}^{k-1}(-1)^j \binom{k-1}{j} \mathrm{e}^{-\lambda j u} \int_0^u \mathrm{e}^{-\lambda(k-j)y}\mathrm{d}y \\
&= \frac{k}{k-j} \sum_{j=0}^{k-1}(-1)^j \binom{k-1}{j} \mathrm{e}^{-\lambda j u}[1 - \mathrm{e}^{-\lambda(k-j)u}] \\
&= \sum_{j=0}^{k-1}(-1)^j \binom{k}{j}[\mathrm{e}^{-\lambda j u} - \mathrm{e}^{-\lambda k u}] = \sum_{j=0}^{k-1}(-1)^j \binom{k}{j}\mathrm{e}^{-\lambda j u} + (-1)^k \mathrm{e}^{-\lambda k u} \\
&= (1 - \mathrm{e}^{-\lambda u})^k,
\end{aligned}$$

this giving (4). □

Making use of a similar reasoning, for $k = 1, 2, \dots$ we also have

$$G^{(k)}(u) := P(D^{(k)} \le u) = (1 - \mathrm{e}^{-\mu u})^k, \qquad u \ge 0. \tag{5}$$

Note that (4) and (5) identify with the distribution functions of the maximum of k independent and exponentially distributed random variables with parameters λ and μ, respectively. Moreover, denoting by $\widetilde{U}_j$ ($\widetilde{D}_j$), $j \ge 1$, independent and exponentially distributed random variables with parameters λ (μ), recalling (2) and (3) we remark that

$$U^{(k)} \stackrel{\mathrm{d}}{=} \sum_{j=1}^{k} \frac{\widetilde{U}_j}{j}, \qquad \left(D^{(k)} \stackrel{\mathrm{d}}{=} \sum_{j=1}^{k} \frac{\widetilde{D}_j}{j} \right), \qquad k = 1, 2, \dots.$$

In order to obtain the distribution function of process X_t, let us now introduce the compound process

$$Y_t = \sum_{n=0}^{M_t} D_n, \qquad \text{where} \quad M_t := \max\{n \ge 0 : \sum_{j=1}^{n} U_j \le t\}, \qquad t > 0.$$

Hereafter we obtain the distribution function of Y_t.

Proposition 2. *For any fixed $t > 0$ and $y \in [0, +\infty)$, we have*

$$H(y, t) := P(Y_t \le y) = \frac{\mathrm{e}^{-\lambda t}}{\mathrm{e}^{-\lambda t} + \mathrm{e}^{-\mu y} (1 - \mathrm{e}^{-\lambda t})}. \tag{6}$$

Proof. For $t > 0$ the distribution function of Y_t can be expressed as

$$H(y, t) = \sum_{n=0}^{+\infty} P(M_t = n)\, G^{(n)}(y),$$

where, due to (4),

$$P(M_t = n) = F^{(n)}(t) - F^{(n+1)}(t) = \mathrm{e}^{-\lambda t}(1 - \mathrm{e}^{-\lambda t})^n, \quad n = 0, 1, \dots.$$

Hence, recalling (5), we obtain

$$H(y, t) = \mathrm{e}^{-\lambda t} \sum_{n=0}^{+\infty} (1 - \mathrm{e}^{-\lambda t})^n (1 - \mathrm{e}^{-\mu y})^n,$$

so that (6) immediately follows. □

Notice that $P(Y_t = 0) = \mathrm{e}^{-\lambda t}$.

Let us now define the stochastic process identifying the total time spent by S_t going upward:

$$W_t = \int_0^t \mathbf{1}_{\{N_s \text{ even}\}} \mathrm{d}s, \qquad t > 0,$$

so that

$$X_t = c(2W_t - t), \qquad t > 0. \tag{7}$$

Proposition 3. *For all* $0 < \tau < t$*, the distribution function of* W_t *is:*

$$P(W_t \le \tau) = \frac{\mathrm{e}^{-\mu(t-\tau)}(1-\mathrm{e}^{-\lambda\tau})}{\mathrm{e}^{-\lambda\tau} + \mathrm{e}^{-\mu(t-\tau)}(1-\mathrm{e}^{-\lambda\tau})}. \tag{8}$$

Moreover,

$$P(W_t < t) = 1 - \mathrm{e}^{-\lambda t}, \qquad P(W_t \le t) = 1.$$

Proof. Note that, for a fixed value $t_0 > 0$,

$$W_{t_0} = \inf\{t > 0 : Y(t) \ge t_0 - t\}. \tag{9}$$

Moreover, if $W_{t_0} = \tau$, $\tau \le t_0$, and $Y_\tau = t_0 - \tau$ ($Y_\tau > t_0 - \tau$), then the motion is going upward (downward) at time t_0. Finally, since Y_t is an increasing process, due to (9), the survival function $P(W_t > \tau)$ is equal to $H(t-\tau, \tau)$ for $0 < \tau \le t$. Hence, (8) immediately follows from (6). □

Due to (7) and Proposition 3, the probability law of X_t can be easily obtained.

Proposition 4. *Let* $\tau_* = \tau_*(x,t) = (x+ct)/(2c)$*. For all* $t > 0$ *and* $x < ct$ *we have*

$$P(X_t \le x) = \frac{\mathrm{e}^{-\mu(t-\tau_*)}(1-\mathrm{e}^{-\lambda\tau_*})}{\mathrm{e}^{-\lambda\tau_*} + \mathrm{e}^{-\mu(t-\tau_*)}(1-\mathrm{e}^{-\lambda\tau_*})}.$$

Moreover, $P(X_t < ct) = 1 - \mathrm{e}^{-\lambda t}$ *and* $P(X_t \le ct) = 1$.

In the following proposition we finally obtain the distribution function of S_t.

Proposition 5. *For all* $t > 0$ *and* $x < s_0\,\mathrm{e}^{(a+c)t}$*, we have*

$$P(S_t \le x) = \frac{A_\mu(t)\left[x/s_0\right]^{(\lambda+\mu)/(2c)} - A_\mu(t)A_\lambda(t)\left[x/s_0\right]^{\mu/(2c)}}{A_\lambda(t) + A_\mu(t)\left[x/s_0\right]^{(\lambda+\mu)/(2c)} - A_\mu(t)A_\lambda(t)\left[x/s_0\right]^{\mu/(2c)}},$$

where $A_\lambda(t) = \exp\left\{-\frac{\lambda}{2}\left(1-\frac{a}{c}\right)t\right\}$ *and* $A_\mu(t) = \exp\left\{-\frac{\mu}{2}\left(1+\frac{a}{c}\right)t\right\}$*. Moreover,*

$$P(S_t < s_0\,\mathrm{e}^{(a+c)t}) = 1 - \mathrm{e}^{-\lambda t}, \qquad P(S_t \le s_0\,\mathrm{e}^{(a+c)t}) = 1.$$

Proof. It immediately follows from (1) and recalling Proposition 4. □

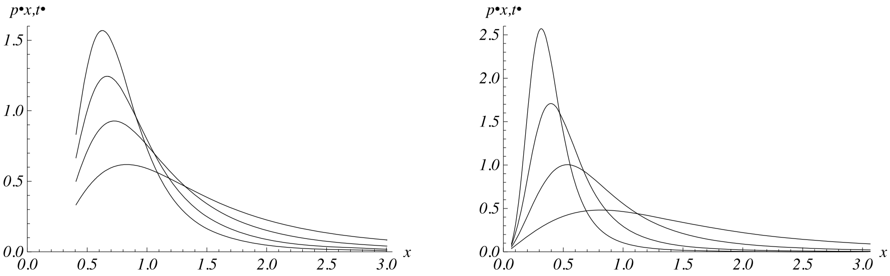

Fig. 2. Plot of $p(x,t)$ for $s_0 = 1$, $a = 0.1$, $c = 1$, $\mu = 2$, and $\lambda = 2, 3, 4, 5$, from bottom to top near the origin, with $t = 1$ (left-hand-side) and $t = 3$ (right-hand-side)

By straightforward use of Proposition 5, hereafter we come to the probability law of S_t, which is characterized by a discrete component on $s_0\,\mathrm{e}^{(a+c)t}$ having probability $\mathrm{e}^{-\lambda t}$, and by an absolutely continuous component on $(s_0\,\mathrm{e}^{(a-c)t}, s_0\,\mathrm{e}^{(a+c)t})$.

Proposition 6. *The absolutely continuous component of the probability law of S_t for $t > 0$ and $x \in (s_0\,\mathrm{e}^{(a-c)t}, s_0\,\mathrm{e}^{(a+c)t})$ is given by:*

$$p(x,t) := \frac{\mathrm{d}}{\mathrm{d}x} P(S_t \le x) = \frac{\lambda + \mu - \mu \left(\frac{x}{s_0}\right)^{-\frac{\lambda}{2c}} \exp\{-\frac{\lambda t(c-a)}{2c}\}}{2cx}$$
$$\times \frac{1}{\{2\cosh\{\frac{\lambda+\mu}{4c}\log\left(\frac{x}{s_0}\right) + \frac{\lambda t(c-a)-\mu t(c+a)}{4c}\} - \left(\frac{x}{s_0}\right)^{-\frac{\lambda-\mu}{4c}} \exp\{-\frac{\lambda t(c-a)+\mu t(c+a)}{4c}\}\}^2}.$$

Some plots of density $p(x,t)$ are shown in Fig. 2 for various choices of λ and t. Let us now analyze the behavior of $p(x,t)$ in the limit as t tends to $+\infty$.

Corollary 1. *If $\lambda(c-a) = \mu(c+a)$ then*

$$\lim_{t\to+\infty} p(x,t) = \frac{\beta}{s_0}\,\frac{(x/s_0)^{\beta-1}}{[1+(x/s_0)^{\beta}]^2}, \qquad x \in (0,+\infty),$$

where $\beta = \lambda/(c+a)$; whereas, if $\lambda(c-a) \neq \mu(c+a)$ then

$$\lim_{t\to+\infty} p(x,t) = 0.$$

Hence, under condition $\lambda(c-a) = \mu(c+a)$, process S_t has a stationary density which is of log-logistic type with shape parameter β and scale parameter s_0. We remark that a similar result also holds under the suitable scaling conditions given hereafter.

Corollary 2. *Let $\alpha_t = s_0 \exp\{at\}$. If $\lambda = \mu \to +\infty$, $c \to +\infty$, with $\lambda/c \to \theta$, then*

$$p(x,t) \to \frac{\theta}{\alpha_t}\,\frac{(x/\alpha_t)^{\theta-1}}{[1+(x/\alpha_t)^{\theta}]^2}, \qquad x \in (0,+\infty).$$

Let us now analyse the behavior of $p(x,t)$ when x approaches the endpoints of its support, i.e. the interval $[s_1, s_2] := [s_0\, e^{(a-c)t}, s_0\, e^{(a+c)t}]$.

Corollary 3. *For any fixed $t > 0$, we have*

$$\lim_{x\downarrow s_1} p(x,t) = \frac{\lambda}{2cs_0} e^{(c-a-\mu)t}, \qquad \lim_{x\uparrow s_2} p(x,t) = \frac{[\lambda + \mu(1-e^{-\lambda t})]\, e^{-(c+a+\lambda)t}}{2cs_0}.$$

Hereafter we express the m-th moment of S_t in terms of the Gauss hypergeometric function ${}_2F_1$.

Proposition 7. *Let m be a positive integer. Then, for $t > 0$,*

$$E[S_t^m] = s_0^m\, e^{m\,(a-c)t}\left\{1 + \frac{2\,m\,c}{\lambda}\sum_{k=0}^{+\infty}\frac{(1-e^{-\lambda t})^{k+1}}{k+1}\right.$$
$$\left.\times\sum_{r=0}^{k}\binom{k}{r}(-e^{-\mu t})^r\,{}_2F_1\left(\frac{2\,m\,c}{\lambda}+\frac{\mu}{\lambda}r, k+1; k+2; 1-e^{-\lambda t}\right)\right\}. \quad (10)$$

Proof. Due to Proposition 4, by setting $y = (ct + x)/2c$ we have

$$M_{X_t}(s) := E\left[e^{sX(t)}\right] = e^{-sct}\left\{1 + 2sc\int_0^t \frac{e^{-(\lambda-2cs)y}}{e^{-\lambda y} + e^{-\mu(t-y)}(1-e^{-\lambda y})}\mathrm{d}y\right\}. \quad (11)$$

After some calculations (11) gives

$$M_{X_t}(s) = e^{-sct}\left\{1 + \frac{2sc}{\lambda}\sum_{k=0}^{+\infty}\sum_{r=0}^{k}\binom{k}{r}(-e^{-\mu t})^{k-r}\int_{\mathcal{I}} x^k(1-x)^{-[2cs+\mu(k-r)]/\lambda}\,\mathrm{d}x\right\},$$

where $\mathcal{I} = (0, 1-e^{-\lambda t})$. Hence, recalling the equation (3.194.1) of [5], and noting that $E[S_t^m] = s_0^m\, e^{m\,at} M_{X_t}(m)$, the right-hand-side of (10) immediately follows. □

Figures 3 and 4 show some plots of mean and variance of S_t, respectively, evaluated by using (10). The right-hand-sides of both figures show cases when condition $\lambda(c-a) = \mu(c+a)$ is fulfilled.

Remark 1. If $\lambda = \mu$, then the moment (10) can be expressed as:

$$E[S_t^m] = s_0^m\, e^{m\,(a-c)t}\left\{1 + \frac{2\,m\,c}{\lambda}\sum_{k=0}^{+\infty}\frac{(k\,!)^2(1-e^{-\lambda t})^{2k+1}}{(2k+1)\,!}\right.$$
$$\left.\times\; {}_2F_1\left(\frac{2\,m\,c}{\lambda}+k, k+1; 2k+2; 1-e^{-\lambda t}\right)\right\}.$$

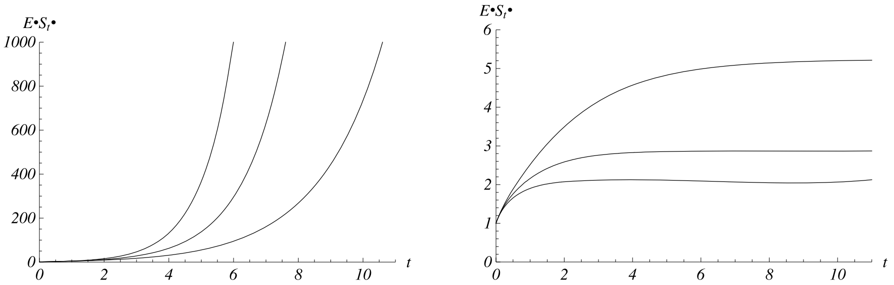

Fig. 3. Plot of $E(S_t)$ for $(\lambda, \mu) = (1.5, 0.9), (1.75, 1.05), (2, 1.2)$ (left-hand-side) and for $(\lambda, \mu) = (3, 1.8), (3.5, 2.1), (4, 2.4)$ (right-hand-side) from top to bottom, with $s_0 = 1$, $a = 0.5$, $c = 2$

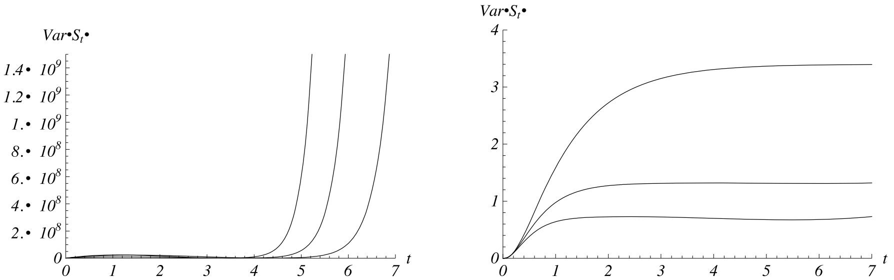

Fig. 4. Plot of $\mathrm{Var}(S_t)$ for $(\lambda, \mu) = (1, 0.6), (1.5, 0.9), (2, 1.2)$ (left-hand-side) and for $(\lambda, \mu) = (6, 3.6), (7, 4.2), (8, 4.8)$ (right-hand-side) from top to bottom, with $s_0 = 1$, $a = 0.5$, $c = 2$.

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References

1. De Gregorio, A., Iacus, S.M.: Parametric estimation for the standard and geometric telegraph process observed at discrete times, Stat. Infer. Stoch. Process. **11**, 249–263 (2008)
2. Di Crescenzo, A.: On random motions with velocities alternating at Erlang-distributed random times, Adv. Appl. Prob. **33**, 690–701 (2001)
3. Di Crescenzo, A., Martinucci, B.: A damped telegraph random process with logistic stationary distribution, J. Appl. Prob. **47**, 84–96 (2010)
4. Di Crescenzo, A., Pellerey, F.: On prices' evolutions based on geometric telegrapher's process, Appl. Stoch. Models Bus. Ind. **18**, 171–184 (2002)
5. Gradshteyn, I.S., Ryzhik, I.M.: Tables of Integrals, Series and Products. 7th Ed. Academic Press, Amsterdam (2007)
6. Li, M.: A damped diffusion framework for financial modeling and closed-form maximum likelihood estimation, J. Econ. Dyn. Control **34**, 132–157 (2010)
7. Orsingher, E.: Probability law, flow function, maximum distribution of wave-governed random motions and their connections with Kirchoff's laws, Stoch. Proc. Appl. **34**, 49–66 (1990)
8. Ratanov, N.: A jump telegraph model for option pricing, Quant. Finance **7**, 575–583 (2007)
9. Stadje, W., Zacks, S.: Telegraph processes with random velocities, J. Appl. Prob. **41**, 665–678 (2004)
10. Zacks, S.: Generalized integrated telegraph processes and the distribution of related stopping times, J. Appl. Prob. **41**, 497–507 (2004)