7

Black-Scholes model

In this chapter we present some of the fundamental ideas of arbitrage pricing in continuous time, illustrating Black-Scholes theory from a point of view that is, as far as possible, elementary and close to the original ideas in the papers by Merton [250], Black and Scholes [49]. In Chapter 10 the topic will be treated in a more general fashion, fully exploiting martingale and PDEs theories.

In the Black-Scholes model the market consists of a non-risky asset, a bond B and of a risky asset, a stock S. The bond price verifies the equation

$$dB_t = rB_t dt$$

where r is the short-term (or locally risk-free) interest rate, assumed to be a constant. Therefore the bond follows a deterministic dynamics: if we set $B_0 = 1$, then

$$B_t = e^{rt}. \tag{7.1}$$

The price of the risky asset is a geometric Brownian motion, verifying the equation

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \tag{7.2}$$

where $\mu \in \mathbb{R}$ is the average rate of return and $\sigma \in \mathbb{R}_{>0}$ is the volatility. In (7.2), $(W_t)_{t\in[0,T]}$ is a real Brownian motion on the probability space $(\Omega, \mathcal{F}, P, (\mathcal{F}_t))$. Recall that the explicit expression of the solution of (7.2) is

$$S_t = S_0 e^{\sigma W_t + \left(\mu - \frac{\sigma^2}{2}\right)t}. \tag{7.3}$$

In what follows we study European-style derivatives in a Markovian setting and we consider payoffs of the form $F(S_T)$, where T is the maturity and F is a function defined on $\mathbb{R}_{>0}$. The most important example is the European Call option with strike K and maturity T:

$$F(S_T) = (S_T - K)^+.$$

In Section 7.6 we study Asian-style derivatives, whose payoff depends on an average of the prices of the underlying asset.

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7.1 Self-financing strategies

Let us introduce some definitions that extend in a natural way the concepts analyzed in discrete time in Chapter 2.

Definition 7.1 *A strategy (or portfolio) is a stochastic process* (α_t, β_t) *where* $\alpha \in \mathbb{L}^2_{\text{loc}}$ *and* $\beta \in \mathbb{L}^1_{\text{loc}}$. *The value of the portfolio* (α, β) *is the stochastic process defined by*

$$V_t^{(\alpha,\beta)} = \alpha_t S_t + \beta_t B_t. \tag{7.4}$$

As usual α, β are to be interpreted as the amount of S and B held by the investor in the portfolio: let us point out that short-selling is allowed, so α, β can take negative values. Where there is no risk of ambiguity, we simply write V instead of $V^{(\alpha,\beta)}$.

Intuitively the assumption that α, β have to be progressively measurable[1] describes the fact that the investment strategy depends only on the amount of information available at that moment.

Definition 7.2 *A strategy* (α_t, β_t) *is self-financing if*

$$dV_t^{(\alpha,\beta)} = \alpha_t dS_t + \beta_t dB_t \tag{7.5}$$

holds, that is

$$V_t^{(\alpha,\beta)} = V_0^{(\alpha,\beta)} + \int_0^t \alpha_s dS_s + \int_0^t \beta_s dB_s. \tag{7.6}$$

We observe that, since S is a continuous and adapted stochastic process we have that $\alpha S \in \mathbb{L}^2_{\text{loc}}$ and then the stochastic integral in (7.6) is well defined. Equation (7.5) is the continuous version[2] of the relation

$$\Delta V = \alpha \Delta S + \beta \Delta B$$

valid for discrete self-financing portfolios (cf. (2.7)): from a purely intuitive point of view, this expresses the fact that the instantaneous variation of the value of the portfolio is caused uniquely by the changes of the prices of the assets, and not by injecting or withdrawing funds from outside.

Let us now take a strategy (α, β) and define the discounted prices

$$\widetilde{S}_t = e^{-rt} S_t, \qquad \widetilde{V}_t = e^{-rt} V_t.$$

[1] In the discrete case we considered *predictable* strategies: for the sake of simplicity, in the continuous case we prefer to assume the condition (not really restrictive indeed) that α, β are progressively measurable.

[2] If α, β are Itô processes, by the two-dimensional Itô formula we have

$$dV_t^{(\alpha,\beta)} = \alpha_t dS_t + \beta_t dB_t + S_t d\alpha_t + B_t d\beta_t + d\langle \alpha, S\rangle_t,$$

and the condition that (α, β) is self-financing is equivalent to

$$S_t d\alpha_t + B_t d\beta_t + d\langle \alpha, S\rangle_t = 0.$$

The following proposition gives a remarkable characterization of the self-financing condition.

Proposition 7.3 *A strategy (α, β) is self-financing if and only if*

$$d\widetilde{V}_t^{(\alpha,\beta)} = \alpha_t d\widetilde{S}_t$$

holds, that is

$$\widetilde{V}_t^{(\alpha,\beta)} = V_0^{(\alpha,\beta)} + \int_0^t \alpha_s d\widetilde{S}_s. \tag{7.7}$$

Remark 7.4 Thanks to (7.7), *the value of a self-financing strategy (α, β) is determined uniquely by its initial value $V_0^{(\alpha,\beta)}$ and by the process α* that is the amount of risky stock held by the investor in the portfolio. The integral in (7.7) equals the difference between the final and initial discounted values and therefore represents the *gain of the strategy.*

When an initial value $V_0 \in \mathbb{R}$ and a process $\alpha \in \mathbb{L}^2_{\mathrm{loc}}$ are given, we can construct a strategy (α, β) by putting

$$\widetilde{V}_t = V_0 + \int_0^t \alpha_s d\widetilde{S}_s, \qquad \beta_t = \frac{V_t - \alpha_t S_t}{B_t}.$$

By construction (α, β) *is a self-financing strategy with initial value* $V_0^{(\alpha,\beta)} = V_0$. In other words, a self-financing strategy can be indifferently set by specifying the processes α, β or the initial value V_0 and the process α. □

Proof (of Proposition 7.3). Given a strategy (α, β), we obviously have

$$\beta_t B_t = V_t^{(\alpha,\beta)} - \alpha_t S_t. \tag{7.8}$$

Furthermore

$$d\widetilde{S}_t = -re^{-rt} S_t dt + e^{-rt} dS_t \tag{7.9}$$

$$= (\mu - r)\widetilde{S}_t dt + \sigma \widetilde{S}_t dW_t. \tag{7.10}$$

Then (α, β) is self-financing if and only if

$$\begin{aligned} d\widetilde{V}_t^{(\alpha,\beta)} &= -r\widetilde{V}_t^{(\alpha,\beta)} dt + e^{-rt} dV_t \\ &= -r\widetilde{V}_t^{(\alpha,\beta)} dt + e^{-rt} \left(\alpha_t dS_t + \beta_t dB_t\right) = \end{aligned}$$

(since $dB_t = rB_t dt$ and by (7.8))

$$\begin{aligned} &= -r\widetilde{V}_t^{(\alpha,\beta)} dt + e^{-rt} \left(\alpha_t dS_t + rV_t^{(\alpha,\beta)} dt - r\alpha_t S_t dt\right) \\ &= e^{-rt} \alpha_t \left(dS_t - rS_t dt\right) = \end{aligned}$$

(by (7.9))

$$= \alpha_t d\widetilde{S}_t,$$

and this concludes the proof. □

Remark 7.5 Thanks to (7.10), condition (7.7) takes the more explicit form

$$\widetilde{V}_t^{(\alpha,\beta)} = \widetilde{V}_0^{(\alpha,\beta)} + (\mu - r)\int_0^t \alpha_s \widetilde{S}_s ds + \sigma \int_0^t \alpha_s \widetilde{S}_s dW_s. \tag{7.11}$$

This extends the result, proved in discrete time, according to which, if the discounted prices of the assets are martingales, then also the self-financing discounted portfolios built upon those assets are martingales.

Indeed, by (7.10), the discounted price $\widetilde{S}_t$ of the underlying asset is a martingale[3] if and only if $\mu = r$ in (7.2). Under this condition $\widetilde{S}$ is a martingale and we have

$$d\widetilde{S}_t = \sigma \widetilde{S}_t dW_t; \tag{7.12}$$

moreover (7.11) becomes

$$d\widetilde{V}_t^{(\alpha,\beta)} = \sigma \widetilde{S}_t \partial_s f(t, S_t) dW_t,$$

and therefore $\widetilde{V}^{(\alpha,\beta)}$ is a (local) martingale. □

7.2 Markovian strategies and Black-Scholes equation

Definition 7.6 *A strategy* (α_t, β_t) *is Markovian if*

$$\alpha_t = \alpha(t, S_t), \qquad \beta_t = \beta(t, S_t)$$

where α, β *are functions in* $C^{1,2}([0, T[\times \mathbb{R}_{>0})$.

The value of a Markovian strategy (α, β) is a function of time and of the price of the underlying asset:

$$f(t, S_t) := V_t^{(\alpha,\beta)} = \alpha(t, S_t) S_t + \beta(t, S_t) e^{rt}, \qquad t \in [0, T[, \tag{7.13}$$

with $f \in C^{1,2}([0, T[\times \mathbb{R}_{>0})$.

We point out that the function f in (7.13) is *uniquely determined by* (α, β)*:* if

$$V_t^{(\alpha,\beta)} = f(t, S_t) = g(t, S_t) \quad \text{a.s.}$$

then $f = g$ in $[0, T[\times \mathbb{R}_{>0}$. This follows from Proposition A.59 and by the fact that S_t has a strictly positive (log-normal) density on $\mathbb{R}_{>0}$. As we are going to use Proposition A.59 often, for the reader's convenience we recall it here:

[3] In this chapter we are not going to introduce the concept of EMM: we defer the rigorous justification of the steps above to Chapter 10, where we prove the existence of a probability measure equivalent to P, under which the dynamics of S is given by (7.2) with $\mu = r$.

Proposition 7.7 *Let X be a random variable with strictly positive density on $H \in \mathscr{B}$. If $g \in m\mathscr{B}$ is such that $g(X) = 0$ a.s. ($g(X) \geq 0$ a.s.) then $g = 0$ ($g \geq 0$) almost everywhere with respect to Lebesgue measure on H. In particular if g is continuous then $g = 0$ ($g \geq 0$) on H.*

The following result characterizes the self-financing condition of a Markovian portfolio in differential terms.

Theorem 7.8 *Suppose that (α, β) is a Markovian strategy and set $f(t, S_t) = V_t^{(\alpha,\beta)}$. The following two conditions are equivalent:*

i) (α, β) *is self-financing;*
ii) f *is solution to the partial differential equation*

$$\frac{\sigma^2 s^2}{2} \partial_{ss} f(t, s) + rs\partial_s f(t, s) + \partial_t f(t, s) = rf(t, s), \tag{7.14}$$

with $(t, s) \in [0, T[\times \mathbb{R}_{>0}$, *and we have that*[4]

$$\alpha(t, s) = \partial_s f(t, s). \tag{7.15}$$

Equation (7.14) *is called Black-Scholes partial differential equation.*

We have already seen Black-Scholes partial differential equation in Section 2.3.6 as the asymptotic version of the binomial algorithm.

Theorem 7.8 relates the self-financing condition to a partial differential equation whose coefficients depend on the volatility σ of the risky asset and on the risk-free rate r, but *they do not depend on the average rate of return* μ. After examining the elementary example of Section 1.2 and the discrete case in Section 2.1, this fact should not come as a surprise: as we have already pointed out, arbitrage pricing does not depend on the subjective estimate of the future value of the risky asset.

We remark that, for a portfolio based upon formulas (7.14)-(7.15), *a inaccurate estimate of the parameters σ and r of the model might affect the self-financing property of the strategy:* for example, this means that if we change those parameters in itinere (e.g. after a re-calibration of the model), then the strategy might need more funds than the ones earmarked at the initial time. This might cause unwanted effects when we are using that strategy to hedge a derivative: if we modify the value of σ, hedging might actually cost more than expected at the beginning on the basis of the self-financing condition.

Proof (of Theorem 7.8). [$i) \Rightarrow ii)$] By the self-financing condition and expression (7.2) of S, we have that

$$dV_t^{(\alpha,\beta)} = (\alpha_t \mu S_t + \beta_t r B_t)dt + \alpha_t \sigma S_t dW_t. \tag{7.16}$$

[4] Let us recall that the expression of the process β can be obtained from α and $V_0^{(\alpha,\beta)}$, by Remark 7.4. More precisely:

$$\beta(t, s) = e^{-rt} \left(f(t, s) - s\partial_s f(t, s) \right).$$

Then, by the Itô formula and putting for brevity $f = f(t, S_t)$, we have that

$$dV_t^{(\alpha,\beta)} = \partial_t f dt + \partial_s f dS_t + \frac{1}{2}\partial_{ss} f d\langle S\rangle_t$$
$$= \left(\partial_t f + \mu S_t \partial_s f + \frac{\sigma^2 S_t^2}{2}\partial_{ss} f\right) dt + \sigma S_t \partial_s f dW_t. \tag{7.17}$$

From the uniqueness of the representation of an Itô process (cf. Proposition 5.3) we infer the equality of the terms in dt and dW_t in (7.16) and (7.17). Then, concerning the terms in dW_t, since σS_t is strictly positive, we obtain

$$\alpha_t = \partial_s f(t, S_t) \quad \text{a.s.} \tag{7.18}$$

hence, by Proposition 7.7, we get relation (7.15).

Concerning now the terms in dt, by (7.18), we get

$$\partial_t f + \frac{\sigma^2 S_t^2}{2}\partial_{ss} f - r\beta_t B_t = 0 \quad \text{a.s.} \tag{7.19}$$

Substituting the expression

$$\beta_t B_t = f - S_t \partial_s f \quad \text{a.s.}$$

in (7.19), we get

$$\partial_t f(t, S_t) + rS_t \partial_s f(t, S_t) + \frac{\sigma^2 S_t^2}{2}\partial_{ss} f(t, S_t) - rf(t, S_t) = 0, \qquad \text{a.s.} \tag{7.20}$$

therefore, by Proposition 7.7, f is a solution of the deterministic differential equation (7.14).

$[ii) \Rightarrow i)]$ By the Itô formula, we have

$$dV_t^{(\alpha,\beta)} = df(t, S_t) = \partial_s f(t, S_t) dS_t + \left(\frac{\sigma^2 S_t^2}{2}\partial_{ss} f(t, S_t) + \partial_t f(t, S_t)\right) dt =$$

(since, by assumption, f is a solution of equation (7.14))

$$= \partial_s f(t, S_t) dS_t + r(f(t, S_t) - S_t \partial_s f(t, S_t))dt = \tag{7.21}$$

(by (7.15) and since $dB_t = rB_t dt$)

$$= \alpha_t dS_t + \beta_t dB_t,$$

therefore (α, β) is self-financing. □

There is an intimate connection between the Black-Scholes equation (7.14) and the heat differential equation. To see this, let us consider the change of variables

$$t = T - \tau, \qquad s = e^{\sigma x},$$

and let us put

$$u(\tau, x) = e^{ax+b\tau} f(T-\tau, e^{\sigma x}), \qquad \tau \in [0,T],\ x \in \mathbb{R}, \tag{7.22}$$

where a, b are constants to be chosen appropriately afterwards. We obtain

$$\begin{aligned}
\partial_\tau u &= e^{ax+b\tau}\left(bf - \partial_t f\right), \\
\partial_x u &= e^{ax+b\tau}\left(af + \sigma e^{\sigma x}\partial_s f\right), \\
\partial_{xx} u &= e^{ax+b\tau}\left(a^2 f + 2a\sigma e^{\sigma x}\partial_s f + \sigma^2 e^{\sigma x}\partial_s f + \sigma^2 e^{2\sigma x}\partial_{ss} f\right),
\end{aligned} \tag{7.23}$$

hence

$$\frac{1}{2}\partial_{xx}u - \partial_\tau u = e^{ax+b\tau}\left(\frac{\sigma^2 s^2}{2}\partial_{ss}f + \left(\sigma a + \frac{\sigma^2}{2}\right)s\partial_s f + \partial_t f + \left(\frac{a^2}{2} - b\right)f\right) =$$

(if f solves (7.14))

$$= e^{ax+b\tau}\left(\left(\sigma a + \frac{\sigma^2}{2} - r\right)s\partial_s f + \left(\frac{a^2}{2} - b + r\right)f\right).$$

We have thus proved the following result.

Proposition 7.9 *Let*

$$a = \frac{r}{\sigma} - \frac{\sigma}{2}, \qquad b = r + \frac{a^2}{2}. \tag{7.24}$$

Then the function f is a solution of the Black-Scholes equation (7.14) *in* $[0,T[\times\mathbb{R}_{>0}$ *if and only if the function $u = u(\tau, x)$ defined by* (7.22) *satisfies the heat equation*

$$\frac{1}{2}\partial_{xx}u - \partial_\tau u = 0, \qquad \text{in }]0,T] \times \mathbb{R}. \tag{7.25}$$

7.3 Pricing

Let us consider a European derivative with payoff $F(S_T)$. As in the discrete case, the arbitrage price equals by definition the value of a replicating strategy. In order to guarantee the well-posedness of such a definition, we ought to prove that there exists at least one replicating strategy (problem of market completeness) and that, if there exist more than one, all the replicating strategies have the same value (problem of absence of arbitrage).

In analytic terms, completeness and absence of arbitrage in the Black-Scholes model correspond to the problem of existence and uniqueness of the solution of a Cauchy problem for the heat equation. To make use of the results on differential equations, it is necessary to impose some conditions on the payoff function F (to ensure *the existence* of a solution) and narrow the family of admissible replicating strategies to a class of uniqueness for the Cauchy problem (to guarantee *the uniqueness* of the solution).

Hypothesis 7.10 *The function F is locally integrable on $\mathbb{R}_{>0}$, lower bounded and there exist two positive constants $a < 1$ and C such that*

$$F(s) \le C e^{C|\log s|^{1+a}}, \qquad s \in \mathbb{R}_{>0}. \tag{7.26}$$

Condition (7.26) is not really restrictive: the function

$$e^{(\log s)^{1+a}} = s^{(\log s)^a}, \qquad s > 1,$$

grows, as $s \to +\infty$, less than an exponential but more rapidly than any polynomial function. This allows us to deal with the majority (if not all) of European-style derivatives actually traded on the markets.

Condition (7.26) is connected to the existence results of Appendix A.3: if we put $\varphi(x) = F(e^x)$, we obtain that φ is lower bounded and we have that

$$\varphi(x) \le C e^{C|x|^{1+a}}, \qquad x \in \mathbb{R},$$

that is a condition analogous to (A.57).

Definition 7.11 *A strategy (α, β) is admissible if it is bounded from below, i.e. there exists a constant C such that*

$$V_t^{(\alpha,\beta)} \ge C, \qquad t \in [0, T], \ a.s. \tag{7.27}$$

We denote by $\mathcal{A}$ the family of Markovian, self-financing admissible strategies.

The financial interpretation of (7.27) is that investment strategies which request unlimited debt are not allowed. This condition is indeed realistic because banks or control institutions generally impose a limit to the investor's losses. We comment further on condition (7.27) in Section 7.3.2.

If $f(t, S_t) = V_t^{(\alpha,\beta)}$ with $(\alpha, \beta) \in \mathcal{A}$, then by Proposition 7.7, f is lower bounded so it belongs to the uniqueness class for the parabolic Cauchy problem studied in Section 6.3.

Definition 7.12 *A European derivative $F(S_T)$ is replicable if there exists an admissible portfolio $(\alpha, \beta) \in \mathcal{A}$ such that*[5]

$$V_T^{(\alpha,\beta)} = F(S_T) \ in \ \mathbb{R}_{>0}. \tag{7.28}$$

We say that (α, β) is a replicating portfolio for $F(S_T)$.

[5] Let $f(t, S_t) = V_t^{(\alpha,\beta)}$. If F is a continuous function, then (7.28) simply has to be understood in the pointwise sense: the limit

$$\lim_{(t,s)\to(T,\bar{s})} f(t, s) = F(\bar{s}),$$

exists for every $\bar{s} > 0$, which is tantamount to saying that f, defined on $[0, T[\times\mathbb{R}_{>0}$ can be prolonged by continuity on $[0, T] \times \mathbb{R}_{>0}$ and, by Proposition 7.7, $f(T, \cdot) = F$. More generally, if F is locally integrable then (7.28) is to be understood in the L^1_{loc} sense, cf. Section A.3.3.

The following theorem is the central result in Black-Scholes theory and gives the definition of arbitrage price of a derivative.

Theorem 7.13 *The Black-Scholes market model is complete and arbitrage-free, this meaning that every European derivative $F(S_T)$, with F verifying Hypothesis 7.10, is replicable in a unique way. Indeed there exists a unique strategy $h = (\alpha_t, \beta_t) \in \mathcal{A}$ replicating $F(S_T)$, that is given by*

$$\alpha_t = \partial_s f(t, S_t), \qquad \beta_t = e^{-rt} \left(f(t, S_t) - S_t \partial_s f(t, S_t) \right), \tag{7.29}$$

where f is the lower bounded solution of the Cauchy problem

$$\frac{\sigma^2 s^2}{2} \partial_{ss} f + rs\partial_s f + \partial_t f = rf, \qquad \text{in } \; [0, T[\times \mathbb{R}_{>0}, \tag{7.30}$$

$$f(T, s) = F(s), \qquad s \in \mathbb{R}_{>0}. \tag{7.31}$$

By definition, $f(t, S_t) = V_t^{(\alpha,\beta)}$ is the arbitrage price of $F(S_T)$.

Proof. A strategy (α, β) replicates $F(S_T)$ if and only if:

i) (α, β) *is Markovian and admissible*, so there exists $f \in C^{1,2}([0, T[\times \mathbb{R}_{>0})$ that is lower bounded and such that $V_t^{(\alpha,\beta)} = f(t, S_t)$;
ii) (α, β) *is self-financing*, so, by Theorem 7.8, f is solution of the differential equation (7.30), the first of formulas (7.29) holds and the second one follows by Remark 7.4;
iii) (α, β) *is replicating* so, by Proposition 7.7, f verifies the final condition (7.31).

To prove that (α, β) exists and is unique, let us transform problem (7.30)-(7.31) into a parabolic Cauchy problem in order to apply the results of existence and uniqueness of Appendices A.3 and 6.3. If we put

$$u(\tau, x) = e^{-r(T-\tau)} f(T - \tau, e^x), \qquad \tau \in [0, T], \; x \in \mathbb{R}, \tag{7.32}$$

we obtain that f is solution of (7.30)-(7.31) if and only if u is solution of the Cauchy problem

$$\begin{cases} \frac{\sigma^2}{2} \left(\partial_{xx} u - \partial_x u \right) + r\partial_x u - \partial_\tau u = 0, & (t, x) \in]0, T] \times \mathbb{R}, \\ u(0, x) = e^{-rT} F(e^x), & x \in \mathbb{R}. \end{cases}$$

By Hypothesis 7.10 and the lower boundedness of F, Theorem A.77 guarantees the existence of a lower bounded solution u. Furthermore, by Theorem 6.19, u is the only solution belonging to the class of lower bounded functions. Thus the existence of a replicating strategy and its uniqueness within the class of lower bounded functions follow immediately. □

Remark 7.14 The admissibility condition (7.27) can be replaced by the growth condition

$$|f(t, s)| \le C e^{C(\log s)^2}, \qquad s \in \mathbb{R}_{>0}, \; t \in]0, T[.$$

In this case, by the uniqueness of the solution guaranteed by Theorem 6.15, we obtain a result that is analogous to that of Theorem 7.13. □

Corollary 7.15 (Black-Scholes Formula) *Let us assume the Black-Scholes dynamics for the underlying asset*

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

and let us denote by r the short rate. Then, if K is the strike price and T is the maturity, the following formulas for the price of European Call and Put options hold:

$$\begin{aligned} c_t &= S_t\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2), \\ p_t &= Ke^{-r(T-t)}\Phi(-d_2) - S_t\Phi(-d_1), \end{aligned} \tag{7.33}$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy$$

is the standard normal distribution function and

$$d_1 = \frac{\log\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}},$$

$$d_2 = d_1 - \sigma\sqrt{T-t} = \frac{\log\left(\frac{S_t}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}.$$

Proof. The claim follows directly from the representation formula for the solution of the Cauchy problem (7.30)-(7.31) for the Black-Scholes equation (or for the heat equation, by transformation (7.22)). We are not going through the explicit computations, already carried out in Section 2.3.5. □

7.3.1 Dividends and time-dependent parameters

Black-Scholes pricing formulas can be adapted to treat the case of a dividend-paying underlying asset. The simplest case is when we suppose a continuous payment with constant return q, i.e. we suppose that in the amount of time dt the dividend paid equals $qS_t dt$. In this case, since dividends paid by a stock reduce its value, we assume the following dynamics

$$dS_t = (\mu - q)S_t dt + \sigma S_t dW_t. \tag{7.34}$$

Moreover we modify the self-financing condition (7.5) as follows:

$$dV_t^{(\alpha,\beta)} = \alpha_t (dS_t + qS_t dt) + \beta_t dB_t. \tag{7.35}$$

Then, proceeding as in the proof of Theorem 7.8, we obtain[6] the modified Black-Scholes equation

$$\frac{\sigma^2 s^2}{2} \partial_{ss} f(t,s) + (r-q)s\partial_s f(t,s) + \partial_t f(t,s) = rf(t,s).$$

[6] On one hand, inserting (7.34) in the self-financing condition (7.35), we get (cf. (7.16))

$$dV_t^{(\alpha,\beta)} = (\alpha_t \mu S_t + \beta_t r B_t)dt + \alpha_t \sigma S_t dW_t;$$

Therefore the Black-Scholes formula for the price of a dividend-paying Call option becomes

$$c_t = e^{-q(T-t)} S_t \Phi(\bar{d}_1) - K e^{-r(T-t)} \Phi(\bar{d}_1 - \sigma\sqrt{T-t}),$$

where

$$\bar{d}_1 = \frac{\log\left(\frac{S_t}{K}\right) + \left(r - q + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}.$$

We can obtain explicit pricing formulas even when the parameters r, μ, σ are time-dependent deterministic functions:

$$dB_t = r(t) B_t dt,$$
$$dS_t = \mu(t) S_t dt + \sigma(t) S_t dW_t.$$

Let us suppose, for example, that r, μ, σ are continuous functions on $[0, T]$. Then we have

$$B_t = e^{\int_0^t r(s)ds},$$
$$S_t = S_0 \exp\left(\int_0^t \sigma(s) dW_s + \int_0^t \left(\mu(s) - \frac{\sigma^2(s)}{2}\right) ds\right).$$

Following the same arguments we obtain formulas that are analogous to the ones of Corollary 7.15 where the terms $r(T-t)$ and $\sigma\sqrt{T-t}$ must be replaced by

$$\int_t^T r(s)ds \quad \text{and} \quad \left(\int_t^T \sigma^2(s)ds\right)^{\frac{1}{2}},$$

respectively.

7.3.2 Admissibility and absence of arbitrage

In this section, we comment on the concept of admissibility of a strategy and on its relation with the absence of arbitrage in the Black-Scholes model.

As in the discrete case, an arbitrage is an investment strategy that requires a null initial investment, with nearly no risk, and that has the possibility of taking a future positive value. Let us formalize the concept into the following:

---

on the other hand, by the Itô formula for $V_t^{(\alpha,\beta)} = f(t, S_t)$, we have (cf. (7.17))

$$dV_t^{(\alpha,\beta)} = \left(\partial_t f + (\mu - q) S_t \partial_s f + \frac{\sigma^2 S_t^2}{2} \partial_{ss} f\right) dt + \sigma S_t \partial_s f dW_t,$$

and the modified Black-Scholes equation follows from the uniqueness of the representation of an Itô process.

Definition 7.16 *An arbitrage is a self-financing strategy* (α, β) *whose value* $V^{(\alpha,\beta)}$ *is such that*

i) $V_0^{(\alpha,\beta)} = 0$ *a.s.;*

and there exists $t_0 \in]0, T]$ *such that*

ii) $V_{t_0}^{(\alpha,\beta)} \geq 0$ *a.s.;*
iii) $P(V_{t_0}^{(\alpha,\beta)} > 0) > 0$.

In the binomial model the absence of arbitrage strategies is guaranteed under straightforward and intuitive assumptions summed up by condition (2.39) which expresses a relation between the return of the risky asset and the return of the bond. On the contrary, in the continuous-time models, the problem of existence of arbitrage opportunities is a very delicate matter. Indeed, without imposing an admissibility condition, even in the Black-Scholes market model it is possible to construct arbitrage portfolios, i.e. one can invest in the assets (7.1) and (7.3) with a self-financing strategy of null initial cost to obtain a risk-free profit.

In very loose terms[7], the idea is to use a strategy consisting in doubling the bet in case of loss: this is well known in gambling games. To fix the ideas, let us consider a coin-tossing game in which if we bet \$1 we get \$2 if the outcome is head, and nothing if the outcome is tail. In this case the doubling strategy consists in beginning by betting \$1 and keeping on gambling, doubling the bet every time one loses and then stopping the first time one wins. Thus, if one wins for the first time at the n-th game, the amount of money gained equals the difference between what one invested and lost in the game, precisely $1 + 2 + 4 + \cdots + 2^{n-1}$, and what one won at the n-th game, i.e. 2^n: so, the total wealth is positive and equals \$1. In this way one is sure to win if the following two conditions hold:

i) one can gamble an infinite number of times;
ii) one has at his/her disposal an infinite wealth.

In a discrete market with finite horizon, these strategies are automatically ruled out by i), cf. Proposition 2.12. In a continuous-time market, even in the case of finite horizon, it is necessary to impose some restrictions in order to rule out the "doubling strategies" which constitute an arbitrage opportunity: this motivates the admissibility condition of Definition 7.11.

The choice of the family of admissible strategies must be made in a suitable way: we have to be careful not to choose a family that is too wide (this might generate arbitrage opportunities), but also not too narrow (this to guarantee a certain degree of freedom in building replicating portfolios that make the market complete). In the literature different notions of admissibility can be found, not all of them being expressed in an explicit fashion: Definition 7.11 looks a simple and intuitive choice. In order to compare our notion of ad-

[7] For further details we refer, for example, to Steele [315], Chapter 14.

missibility to other ones, let us prove now that the class $\mathcal{A}$ does not contain arbitrage opportunities.

Proposition 7.17 (No-arbitrage principle) *The family $\mathcal{A}$ does not contain arbitrage strategies.*

Proof. The claim follows directly from Corollary 6.22. By contradiction, let $(\alpha, \beta) \in \mathcal{A}$, with $V_t^{(\alpha,\beta)} = f(t, S_t)$, be an arbitrage strategy: then f is lower bounded, it is a solution of the PDE (7.30) and we have that $f(0, S_0) = 0$. Moreover there exist $t \in]0, T]$ and $\bar{s} > 0$ such that $f(t, \bar{s}) > 0$ and $f(t, s) \geq 0$ for every $s > 0$. To use Corollary 6.22, let us transform the Black-Scholes PDE into a parabolic equation by substitution(7.32)

$$u(\tau, x) = e^{-r(T-\tau)} f(T - \tau, e^x), \qquad \tau \in [0, T],\ x \in \mathbb{R}.$$

Then u is a solution of the equation

$$\frac{\sigma^2}{2} (\partial_{xx} u - \partial_x u) + r\partial_x u - \partial_\tau u = 0, \tag{7.36}$$

and Corollary 6.22 leads to the absurd inequality:

$$0 = f(0, S_0) = u(T, \log S_0) \geq \int_{\mathbb{R}} \Gamma(T, \log S_0, T - t, y) u(T - t, y) dy > 0,$$

since $u(T - t, y) = e^{-rt} f(t, e^y) \geq 0$ for every $y \in \mathbb{R}$, $u(T - t, \log \bar{s}) = e^{-rt} f(t, \bar{s}) > 0$ and $\Gamma(T, \cdot, \tau, \cdot)$, the fundamental solution of (7.36) is strictly positive when $\tau < T$. □

7.3.3 Black-Scholes analysis: heuristic approaches

We present now some alternative ways to obtain the Black-Scholes equation (7.14). The following approaches are heuristic; their good point is that they are intuitive, while their flaw is they are not completely rigorous. Furthermore they share the fact that they assume the no-arbitrage principle as a starting point, rather than a result: we will comment briefly on this at the end of the section, in Remark 7.18. What follows is informal and not rigorous.

In the first approach, we aim at pricing a derivative H with maturity T assuming that its price at a time t in the form $H_t = f(t, S_t)$ with $f \in C^{1,2}$. To this end we consider a self-financing portfolio (α, β) and impose the replication condition

$$V_T^{(\alpha,\beta)} = H_T \quad \text{a.s.}$$

By the no-arbitrage principle, it must also hold that

$$V_t^{(\alpha,\beta)} = H_t \quad \text{a.s.}$$

for $t \leq T$. Proceeding as in the proof of Theorem 7.8, we impose that the stochastic differentials $dV_t^{(\alpha,\beta)}$ and $df(t, S_t)$ are equal to get (7.14) and the hedging strategy (7.15). The result thus obtained is formally identical: nevertheless in this way one could erroneously think that the Black-Scholes equation (7.14) is a consequence of the absence of arbitrage opportunities rather than a characterization of the self-financing condition.

Concerning the second approach, let us consider the point of view of a bank that sells an option and wants to determine a hedging strategy by investing in the underlying asset. Let us consider a portfolio consisting of a certain amount of the risky asset S_t and of a short position on a derivative with payoff $F(S_T)$ whose price, at the time t, is denoted by $f(t, S_t)$:

$$V(t, S_t) = \alpha_t S_t - f(t, S_t).$$

In order to determine α_t, we want to render V neutral with respect to the variation of S_t, or, in other terms, V immune to the variation of the price of the underlying asset by imposing the condition

$$\partial_s V(t, s) = 0.$$

By the equality $V(t, s) = \alpha_t s - f(t, s)$, we get[8]

$$\alpha_t = \partial_s f(t, s), \tag{7.37}$$

and this is commonly known as the *Delta hedging*[9] strategy. By the self-financing condition we have

$$\begin{aligned} dV(t, S_t) &= \alpha_t dS_t - df(t, S_t) \\ &= \left((\alpha_t - \partial_s f)\mu S_t - \partial_t f - \frac{\sigma^2 S_t^2}{2} \partial_{ss} f \right) dt + (\alpha_t - \partial_s f)\sigma S_t dW_t. \end{aligned}$$

Therefore the choice (7.37) wipes out the riskiness of V, represented by the term in dW_t, and cancels out also the term containing the return μ of the underlying asset. Summing up we get

$$dV(t, S_t) = -\left(\partial_t f + \frac{\sigma^2 S_t^2}{2} \partial_{ss} f \right) dt. \tag{7.38}$$

Now since the dynamics of V is deterministic, *by the no-arbitrage principle* V must have the same return of the non-risky asset:

$$dV(t, S_t) = rV(t, S_t)dt = r\left(S_t \partial_s f - f\right) dt, \tag{7.39}$$

so, equating formulas (7.38) and (7.39) we obtain again the Black-Scholes equation.

The idea that an option can be used to hedge risk is very intuitive and many arbitrage pricing techniques are based upon such arguments.

[8] The attentive reader may wonder why, even though α_t is function of s, $\partial_s \alpha_t$ does not appear in the equation.

[9] In common terminology, the derivative $\partial_s f$ is usually called *Delta*.

Remark 7.18 In the approaches we have just presented, the no-arbitrage principle, under different forms, is assumed as a hypothesis in the Black-Scholes model: this certainly helps intuition, but a rigorous justification of this might be hard to find. Indeed we have seen that in the Black-Scholes model arbitrage strategies actually exist, albeit they are pathological. In our presentation, as in other more probabilistic ones based upon the notion of EMM, all the theory is built upon the self-financing condition: in this approach, the absence of arbitrage opportunities is the natural *consequence* of the self-financing property. Intuitively this corresponds to the fact that if a strategy is adapted and self-financing, then it cannot reasonably generate a risk-free profit greater than the bond: in other words it cannot be an arbitrage opportunity. □

7.3.4 Market price of risk

Let us go back to the ideas of Section 1.2.4 and analyze the pricing and hedging of a derivative whose underlying asset is not exchanged on the market, supposing though that another derivative on the same underlying asset is traded. A noteworthy case is that of a derivative on the temperature: even though it is possible to construct a probabilistic model for the value of temperature, it is not possible to build up a replicating strategy that uses the underlying asset since this cannot be bought or sold; consequently we cannot exploit the argument of Theorem 7.13. Nevertheless, if on the market there already exists an option on the temperature, we can try to price and hedge a new derivative by means of that option.

Let us assume that the underlying asset follows the geometric Brownian motion dynamics

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \tag{7.40}$$

even if the following results do not depend on the particular model considered. We suppose that a derivative on S is exchanged on the market, and that its price at time t is known. We assume also that this price can be written as $f(t, S_t)$, with $f \in C^{1,2}([0, T[\times\mathbb{R}_{>0})$. Finally we request that

$$\partial_s f \neq 0$$

and that suitable assumptions hold in order to guarantee the existence and the uniqueness of the solution of the Cauchy problem (7.49)-(7.50) below. Since we go through such conditions in Chapters 6 and 8, it seems unnecessary to recall them here.

By the Itô formula, we have

$$df(t, S_t) = Lf(t, S_t)dt + \sigma S_t \partial_s f(t, S_t)dW_t, \tag{7.41}$$

where

$$Lf(t, s) = \partial_t f(t, s) + \mu s \partial_s f(t, s) + \frac{\sigma^2 s^2}{2}\partial_{ss} f(t, s). \tag{7.42}$$

Our aim is to price a derivative with payoff $G(S_T)$. We imitate the technique of the preceding sections and build a Markovian self-financing portfolio on the bond and on the derivative f. We denote by $g(t, S_t)$ the value of such portfolio at time t,

$$g(t, S_t) = \alpha_t f(t, S_t) + \beta_t B_t, \tag{7.43}$$

and we impose the self-financing condition:

$$dg(t, S_t) = \alpha_t df(t, S_t) + \beta_t dB_t =$$

(by (7.41))

$$= (\alpha_t Lf(t, S_t) + r\beta_t B_t)\, dt + \alpha_t \sigma S_t \partial_s f(t, S_t) dW_t =$$

(since $\beta_t B_t = g(t, S_t) - \alpha_t f(t, S_t)$)

$$= (\alpha_t (Lf(t, S_t) - rf(t, S_t)) + rg(t, S_t))\, dt + \alpha_t \sigma S_t \partial_s f(t, S_t) dW_t. \tag{7.44}$$

Now we compare this expression with the stochastic differential obtained by the Itô formula

$$dg(t, S_t) = Lg(t, S_t)dt + \sigma S_t \partial_s g(t, S_t) dW_t.$$

By the uniqueness of the representation for an Itô process, we deduce the equality of the terms in dt and dW_t:

$$\alpha_t = \frac{\partial_s g(t, S_t)}{\partial_s f(t, S_t)}, \tag{7.45}$$

$$\alpha_t (Lf(t, S_t) - rf(t, S_t)) = Lg(t, S_t) - rg(t, S_t). \tag{7.46}$$

Substituting (7.45) into (7.46) and reordering the terms, we obtain

$$Lg(t, S_t) - rg(t, S_t) = \sigma S_t \lambda_f \partial_s g(t, S_t), \tag{7.47}$$

where

$$\lambda_f = \lambda_f(t, S_t) = \frac{Lf(t, S_t) - rf(t, S_t)}{\sigma S_t \partial_s f(t, S_t)}. \tag{7.48}$$

Finally, substituting expression (7.42) for L into (7.47), we have proved the following generalization of Theorems 7.8 and 7.13.

Theorem 7.19 *The portfolio given by* (7.43) *is self-financing if and only if* g *is solution of the differential equation*

$$\frac{\sigma^2 s^2}{2} \partial_{ss} g(t, s) + (\mu - \sigma \lambda_f(t, s))\, s \partial_s g(t, s) + \partial_t g(t, s) = rg(t, s), \tag{7.49}$$

with $(t, s) \in [0, T[\times \mathbb{R}_{>0}$. *Under the assumptions of Theorem 7.13, there exists a unique replicating portfolio for* $G(S_T)$*, given by the solution of the Cauchy problem for* (7.49) *with terminal condition*

$$g(T, s) = G(s), \qquad s \in \mathbb{R}_{>0}. \tag{7.50}$$

The value $(g(t, S_t))_{t \le T}$ *is the arbitrage price of* $G(S_T)$ *and the replicating strategy is given by* (7.45).

By Theorem 7.19, the replication of an option (and then the completeness of the market) is guaranteed even if the underlying asset is not exchanged, provided that on the market there exists another derivative on the same underlying asset.

If the underlying asset is traded, we can choose $f(t,s) = s$: in this case we simply denote $\lambda = \lambda_f$ and we observe that

$$\lambda = \frac{\mu - r}{\sigma}. \tag{7.51}$$

Substituting (7.51) into (7.49) we obtain exactly the Black-Scholes equation.

The coefficient λ represents the difference between the expected return μ and the riskless return r, that the investors request when buying S in order to take the risk represented by the volatility σ. For this very reason, λ is usually called *market price of risk* and it measures the investors' propensity to risk.

The market price of risk can be determined by the underlying asset (if exchanged) or by another derivative. Let us point out that (7.41) can be rewritten in a formally similar way to (7.40):

$$df(t,S_t) = \mu_f f(t,S_t)dt + \sigma_f f(t,S_t)dW_t,$$

where

$$\mu_f = \frac{Lf(t,S_t)}{f(t,S_t)}, \qquad \sigma_f = \frac{\sigma S_t \partial_s f(t,S_t)}{f(t,S_t)},$$

so, by definition (7.48), we have that

$$\lambda_f = \frac{\mu_f - r}{\sigma_f},$$

in analogy to (7.51).

We can now interpret the Black-Scholes differential equation (7.49) in a remarkable way: it is indeed equivalent to relation (7.47) that can be simply rewritten as

$$\lambda_f = \lambda_g. \tag{7.52}$$

To put this in another terms, the self-financing condition imposes that g and f share the same market price of risk. And since f and g are generic derivatives, (7.52) is actually a market consistency condition:

- *all the traded assets (or self-financing strategies) must have the same market price of risk.*

In the case of an incomplete market, where the only exchanged asset is the bond, the theoretical prices of the derivatives must verify a Black-Scholes equation similar to (7.49) but in this case the value of the market price of risk is not known, i.e. the coefficient λ_f that appears in the differential equation is unknown. Therefore the arbitrage price of an option is not unique, just as we have seen in the discrete case for the trinomial model.

7.4 Hedging

From a theoretical point of view the Delta-hedging strategy (7.37) guarantees a perfect replication of the payoff. So there would be no need to further study the hedging problem. However, in practice the Black-Scholes model poses some problems: first of all, the strategy (7.29) requires a continuous rebalancing of the portfolio, and this is not always possible or convenient, for example because of transition costs. Secondly, the Black-Scholes model is commonly considered too simple to describe the market realistically: the main issue lies in the hypothesis of constant volatility that appears to be definitely too strong if compared with actual data (see Paragraph 7.5).

The good point of the Black-Scholes model is that it yields explicit formulas for plain vanilla options. Furthermore, even though it has been severely criticized, it is still the reference model. At a first glance this might seem paradoxical but, as we are going to explain, it is not totally groundless.

The rest of the paragraph is structured as follows: in Section 7.4.1 we introduce the so-called sensitivities or Greeks: they are the derivatives of the Black-Scholes price with respect to the risk factors, i.e. the price of the underlying and the parameters of the model. In Section 7.4.2 we analyze the robustness of the Black-Scholes model, i.e. the effects its use might cause if it is not the "correct" model. In Section 7.4.3 we use the Greeks to get more effective hedging strategies than the mere Delta-hedging.

7.4.1 The Greeks

In the Black-Scholes model the value of a strategy is a function of several variables: the price of the underlying asset, the time to maturity and the parameters of the model, the volatility σ and the short-term rate r. From a practical point of view it is useful to be able to evaluate the sensitivity of the portfolio with respect to the variation of these factors: this means that we are able to estimate, for example, how the value of the portfolio behaves when we are getting closer to maturity or we are varying the risk-free rate or the volatility. The natural sensitivity indicators are the partial derivatives of the value of the portfolio with respect to the corresponding risk factors (price of the underlying asset, volatility, etc...). A Greek letter is commonly associated to every partial derivative, and for this reason these sensitivity measurements are usually called *the Greeks.*

Notation 7.20 *We denote by $f(t, s, \sigma, r)$ the value of a self-financing Markovian strategy in the Black-Scholes model, as a function of time t, of the price of the underlying s, of the volatility σ and of the short-term rate r. We put:*

$$\begin{aligned} \varDelta &= \partial_s f && \textit{(Delta)}, \\ \varGamma &= \partial_{ss} f && \textit{(Gamma)}, \end{aligned}$$

$$\mathcal{V} = \partial_\sigma f \qquad \textit{(Vega)},$$
$$\varrho = \partial_r f \qquad \textit{(Rho)},$$
$$\Theta = \partial_t f \qquad \textit{(Theta)}.$$

We say that a strategy is *neutral* with respect to one of the risk factors if the corresponding Greek is null, i.e. if the value of the portfolio is insensitive to the variation of such factor. For example, the Delta-hedging strategy is constructed in such a way that the portfolio becomes neutral to the Delta, i.e. insensitive with respect to the variation of the price of the underlying.

We can get an explicit expression for the Greeks of European Put and Call options, just by differentiating the Black-Scholes formula: some computations must be carried out, but with a little bit of shrewdness they are not particularly involved. In what follows we treat in detail only the call-option case. For the reader's convenience we recall the expression of the price at the time t of a European Call with strike K and maturity T :

$$c_t = g(d_1),$$

where g is the function defined by

$$g(d) = S_t \Phi(d) - Ke^{-r(T-t)}\Phi(d - \sigma\sqrt{T-t}), \qquad d \in \mathbb{R},$$

and

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy, \qquad d_1 = \frac{\log\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}.$$

The graph of the price of a Call option is shown in Figure 7.1. Sometimes it is convenient to use the following notation:

$$d_2 = d_1 - \sigma\sqrt{T-t} = \frac{\log\left(\frac{S_t}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}},$$

and the following lemma serves the purpose of simplifying the computations.

Lemma 7.21 *We have*

$$g'(d_1) = 0, \tag{7.53}$$

and consequently

$$S_t \Phi'(d_1) = Ke^{-r(T-t)}\Phi'(d_1 - \sigma\sqrt{T-t}). \tag{7.54}$$

Proof. It is enough to observe that

$$\Phi'(x) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}.$$

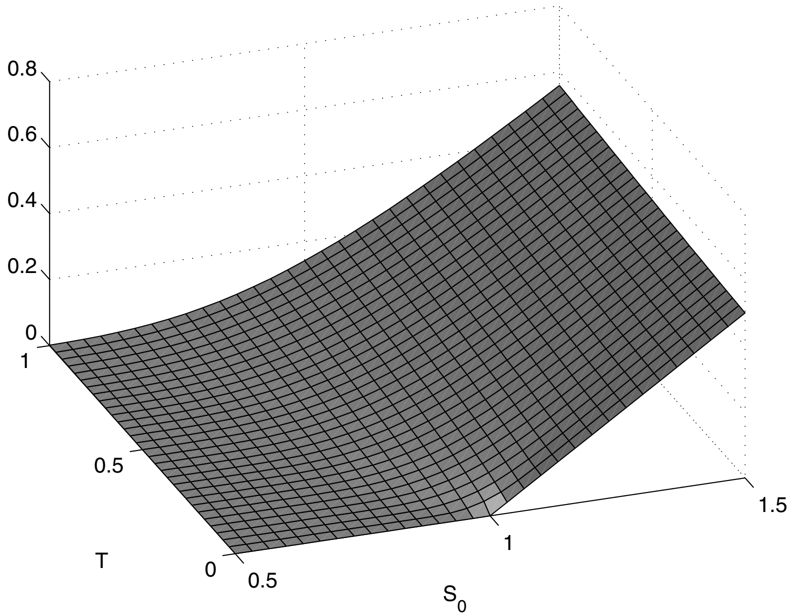

Fig. 7.1. Graph of the price of a European Call option in the Black-Scholes model, as a function of the price of the underlying asset and of time to maturity. The parameters are: strike $K = 1$, volatility $\sigma = 0.3$, risk-free rate $r = 0.05$

Then

$$\begin{aligned} g'(d) &= S_t \frac{e^{-\frac{d^2}{2}}}{\sqrt{2\pi}} - Ke^{-r(T-t)} \frac{e^{-\frac{(d-\sigma\sqrt{T-t})^2}{2}}}{\sqrt{2\pi}} \\ &= \frac{e^{-\frac{d^2}{2}}}{\sqrt{2\pi}} \left(S_t - Ke^{-\left(r+\frac{\sigma^2}{2}\right)(T-t)} e^{d\sigma\sqrt{T-t}} \right) \end{aligned}$$

and the claim follows immediately by the definition of d_1. □

Let us examine now every single Greek of a Call option.

Delta: we have

$$\Delta = \Phi(d_1). \tag{7.55}$$

Indeed

$$\Delta = \partial_s c_t = \Phi(d_1) + g'(d_1)\partial_s d_1,$$

and (7.55) follows by (7.53).

The graph of the Delta is shown in Figure 7.2. Let us point out that the Delta of the Call option is positive and less than one, because Φ is such:

$$0 < \Delta < 1.$$

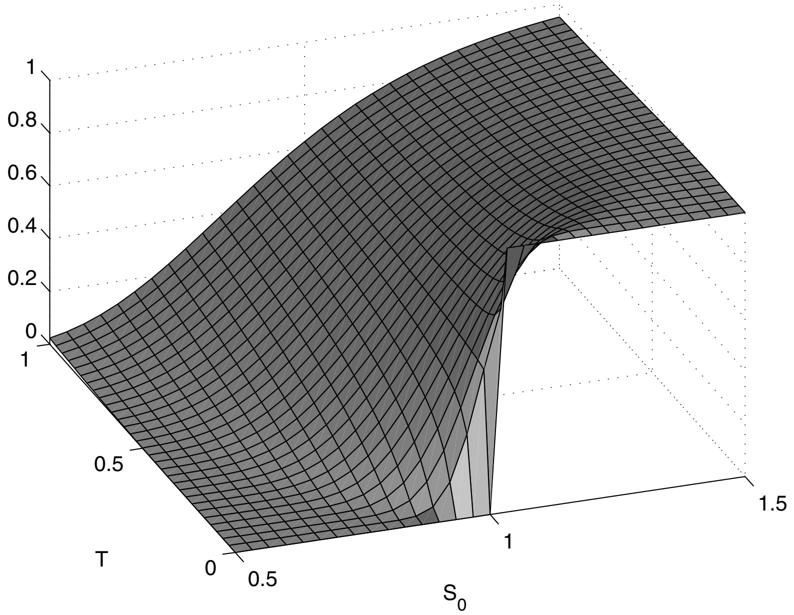

Fig. 7.2. Graph of the Delta of a European Call option in the Black-Scholes model, as a function of the price of the underlying asset and of time to maturity. The parameters are: strike $K = 1$, volatility $\sigma = 0.3$, risk-free rate $r = 0.05$

Since the Delta has to be interpreted as the amount of risky asset to be held in the Delta-hedging portfolio, this corresponds to the intuitive fact that we must buy the underlying asset in order to hedge a short position on a Call option. Let us note that

$$\lim_{s\to 0^+} d_1 = -\infty, \qquad \lim_{s\to +\infty} d_1 = +\infty,$$

so the following asymptotic expressions for price and Delta hold:

$$\lim_{s\to 0^+} c_t = 0, \qquad \lim_{s\to +\infty} c_t = +\infty,$$
$$\lim_{s\to 0^+} \Delta = 0, \qquad \lim_{s\to +\infty} \Delta = 1.$$

Gamma: we have

$$\Gamma = \frac{\Phi'(d_1)}{\sigma S_t \sqrt{T-t}}.$$

Indeed

$$\Gamma = \partial_s \Delta = \Phi'(d_1)\partial_s d_1.$$

The graph of the Gamma is shown in Figure 7.3. We note that the Gamma of a Call option is positive and therefore the price and the Delta are a

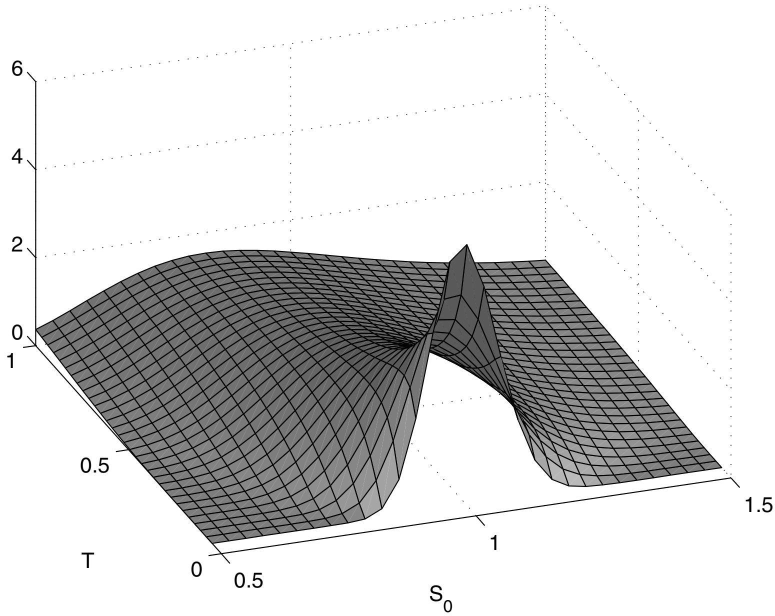

Fig. 7.3. Graph of the Gamma of a European Call option in the Black-Scholes model, as a function of the price of the underlying asset ($0.5 \leq S \leq 1.5$) and of time to maturity ($0.05 \leq T \leq 1$). The parameters are: strike $K = 1$, volatility $\sigma = 0.3$, risk-free rate $r = 0.05$

convex function and an increasing function with respect to the underlying asset, respectively. Furthermore we have that

$$\lim_{s \to 0^+} \Gamma = \lim_{s \to +\infty} \Gamma = 0.$$

Vega: we have

$$\mathcal{V} = S_t \sqrt{T-t}\, \Phi'(d_1).$$

Indeed

$$\mathcal{V} = \partial_\sigma c_t = g'(d_1)\partial_\sigma d_1 + K e^{-r(T-t)} \Phi'(d_1 - \sigma\sqrt{T-t})\sqrt{T-t} =$$

(by (7.53))

$$= K e^{-r(T-t)} \Phi'(d_1 - \sigma\sqrt{T-t})\sqrt{T-t} =$$

(by (7.54))

$$= S_t \sqrt{T-t}\, \Phi'(d_1).$$

The graph of the Vega is shown in Figure 7.4. The Vega is positive, so the price of a Call option is a strictly increasing function of the volatility (cf. Figure 7.5). Intuitively this is due to the fact that an option is a contract giving a right, not an obligation: therefore one takes advantage of

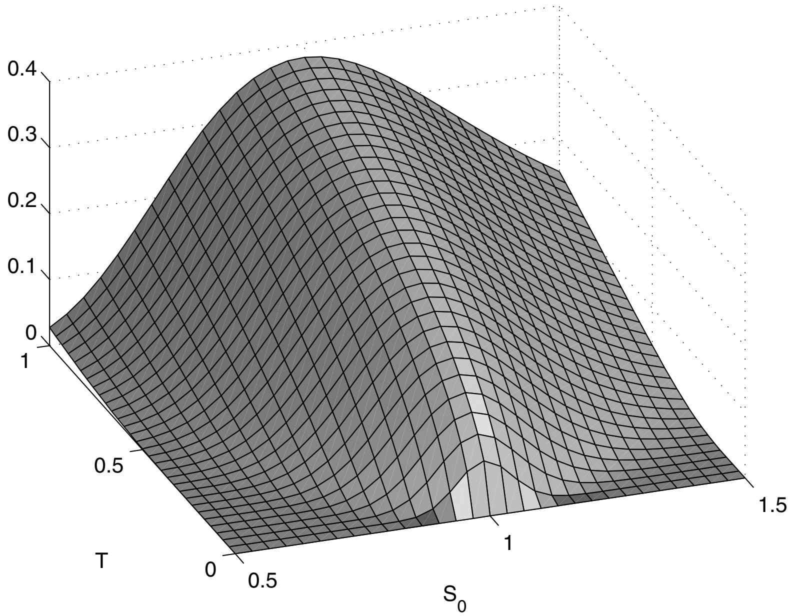

Fig. 7.4. Graph of the Vega of a European Call option in the Black-Scholes model, as a function of the price of the underlying asset and of time to maturity. The parameters are: strike $K = 1$, volatility $\sigma = 0.3$, risk-free rate $r = 0.05$

the greater riskiness of the underlying asset. It also follows that the price of the option is an *invertible* function of the volatility: in other terms, all other parameters being fixed, there is a unique value of the volatility that, plugged into the Black-Scholes formula, produces a given option price. This value is called *implied volatility.*
We show that

$$\lim_{\sigma \to 0^+} c_t = \left(S_t - K e^{-r(T-t)} \right)^+, \qquad \lim_{\sigma \to +\infty} c_t = S_t \tag{7.56}$$

and so

$$\left(S_t - K e^{-r(T-t)} \right)^+ < c_t < S_t,$$

in accordance with the estimates of Corollary 1.2, based upon arbitrage arguments. Indeed if we put

$$\lambda = \log \left(\frac{S_t}{K} \right) + r(T-t),$$

we have that $\lambda = 0$ if and only if

$$S_t = K e^{-r(T-t)},$$

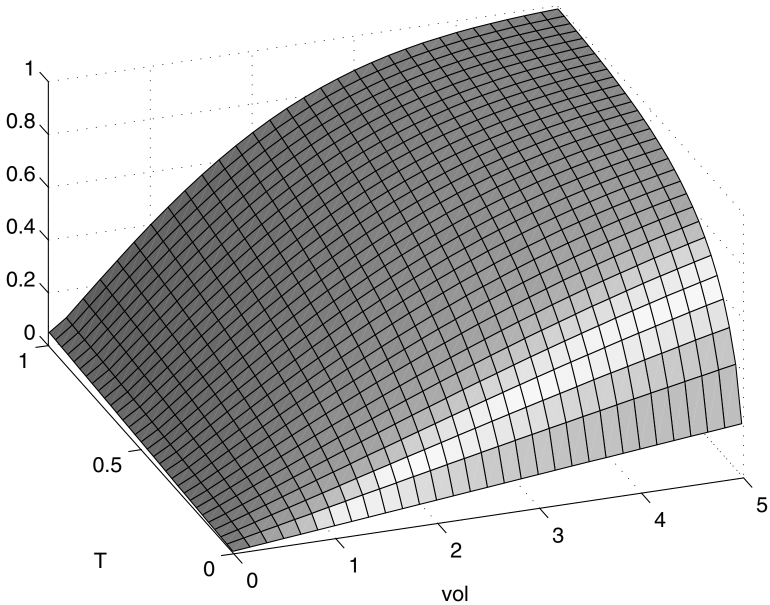

Fig. 7.5. Graph of the price of a European Call option in the Black-Scholes model, as a function of the price of the volatility ($0 \le \sigma \le 5$) and of time to maturity ($0.05 \le T \le 1$). The parameters are: $S = K = 1$, risk-free rate $r = 0.05$

and furthermore

$$\lim_{\sigma\to 0^+} d_1 = \begin{cases} +\infty, & \text{if } \lambda > 0, \\ 0, & \text{if } \lambda = 0, \\ -\infty, & \text{if } \lambda < 0. \end{cases}$$

So

$$\lim_{\sigma\to 0^+} c_t = \begin{cases} S_t - Ke^{-r(T-t)}, & \text{if } \lambda > 0, \\ 0, & \text{if } \lambda \le 0, \end{cases}$$

and this proves the first limit in (7.56). Then

$$\lim_{\sigma\to +\infty} d_1 = +\infty, \qquad \lim_{\sigma\to +\infty} d_2 = -\infty,$$

so that also the second limit in (7.56) follows easily.

Theta: we have

$$\Theta = -rKe^{-r(T-t)}\Phi(d_2) - \frac{\sigma S_t}{2\sqrt{T-t}}\Phi'(d_1). \tag{7.57}$$

Indeed

$$\Theta = \partial_t c_t = g'(d_1)\partial_t d_1 - rKe^{-r(T-t)}\Phi(d_2) - Ke^{-r(T-t)}\Phi'(d_2)\frac{\sigma}{2\sqrt{T-t}},$$

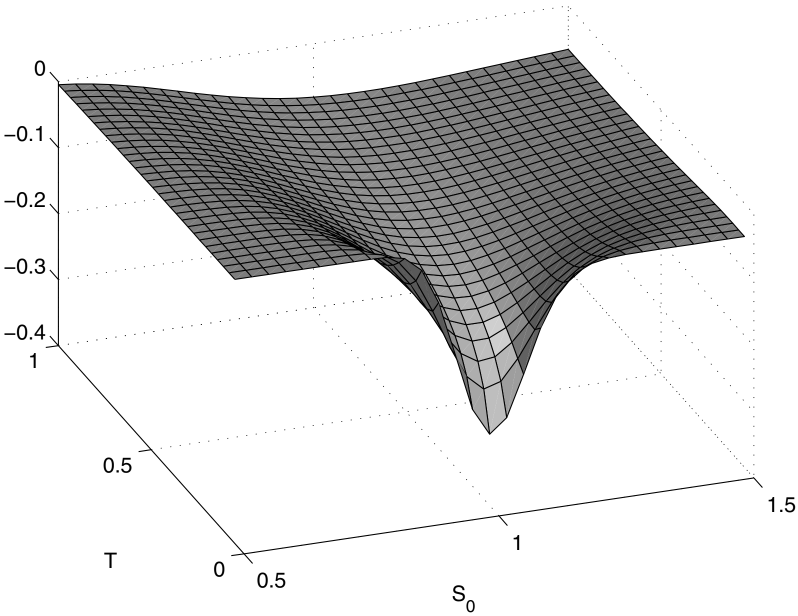

Fig. 7.6. Graph of the Theta of a European Call option in the Black-Scholes model, as a function of the price of the underlying asset ($0.5 \leq S \leq 1.5$) and of time to maturity ($0.05 \leq T \leq 1$). The parameters are: strike $K = 1$, volatility $\sigma = 0.3$, risk-free rate $r = 0.05$

and (7.57) follows from (7.54). The graph of the Theta is shown in Figure 7.6. Let us note that $\Theta < 0$ so the price of a Call option decreases when we get close to maturity: intuitively this is due to the lowering of the effect of the volatility, that is indeed multiplied in the expression for the price by a $\sqrt{T-t}$ factor.

Rho: we have

$$\varrho = K(T-t)e^{-r(T-t)}\Phi(d_2).$$

Indeed

$$\varrho = \partial_r c_t = g'(d_1)\partial_r d_1 + K(T-t)e^{-r(T-t)}\Phi(d_2),$$

and the claim follows from (7.53). The graph of the Rho is shown in Figure 7.7. Let us note that $\rho > 0$ and so the price of a Call option increases when the risk-free rate does so: this is due to the fact that if the Call is exercised, this imposes the payment of the strike K whose discounted value decreases as r increases.

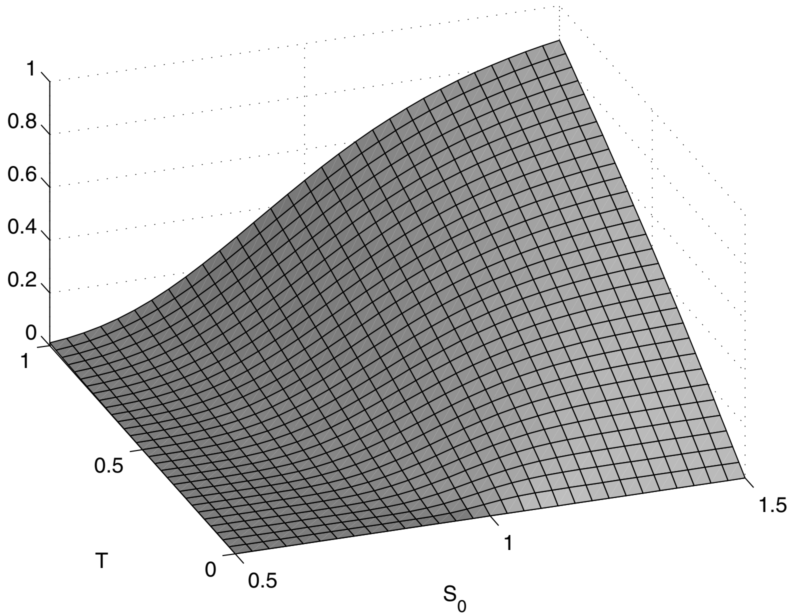

Fig. 7.7. Graph of the Rho of a European Call option in the Black-Scholes model, as a function of the price of the underlying asset and of time to maturity. The parameters are: strike $K = 1$, volatility $\sigma = 0.3$, risk-free rate $r = 0.05$

Let us mention without proof the expressions for the Greeks of a European Put option:

$$
\begin{aligned}
\Delta &= \partial_s p_t = \Phi(d_1) - 1, \\
\Gamma &= \partial_{ss} p_t = \frac{\Phi'(d_1)}{\sigma S_t \sqrt{T-t}}, \\
\mathcal{V} &= \partial_\sigma p_t = S_t \sqrt{T-t}\,\Phi'(d_1), \\
\Theta &= \partial_t p_t = rKe^{-r(T-t)}\left(1 - \Phi(d_2)\right) - \frac{\sigma S_t}{2\sqrt{T-t}}\Phi'(d_1), \\
\rho &= \partial_r p_t = K(T-t)e^{-r(T-t)}\left(\Phi(d_2) - 1\right).
\end{aligned}
$$

We point out that the Delta of a Put option is negative. Gamma and Vega have the same expression for both Put and Call options: in particular, the Vega is positive and so also the price of the Put option increases when the volatility does so. The Theta of a Put option may assume positive and negative values. The Rho of the Put is negative.

7.4.2 Robustness of the model

We assume the Black-Scholes dynamics for the underlying asset

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{7.58}$$

where μ, σ are constant parameters and we denote by r the short-term rate. Then the price $f(t, S_t)$ of an option with payoff $F(S_T)$ is given by the solution of the Cauchy problem

$$\frac{\sigma^2 s^2}{2}\partial_{ss} f + rs\partial_s f + \partial_t f = rf, \qquad \text{in } \; [0, T[\times\mathbb{R}_{>0}, \tag{7.59}$$

$$f(T, s) = F(s), \qquad s \in \mathbb{R}_{>0}. \tag{7.60}$$

Moreover

$$f(t, S_t) = \alpha_t S_t + \beta_t B_t$$

is the value of the Delta-hedging strategy given by $\alpha_t = \partial_s f(t, S_t)$ and $\beta_t = f(t, S_t) - S_t \partial_s f(t, S_t)$.

Let us suppose now that the actual dynamics of the underlying asset is different from (7.58) and are described by an Itô process of the form

$$d\bar{S}_t = \mu_t \bar{S}_t dt + \sigma_t \bar{S}_t dW_t, \tag{7.61}$$

with $\mu_t \in \mathbb{L}^1_{\text{loc}}$ and $\sigma_t \in \mathbb{L}^2_{\text{loc}}$. On the basis of the final condition (7.60), the Delta-hedging strategy *replicates the payoff $F(\bar{S}_T)$ on any trajectory of the underlying asset.* However the fact that the actual dynamics (7.61) is different from the Black-Scholes' ones causes the *loss of the self-financing property:* in practice, this means that hedging has a different cost (possibly greater) with respect to the Black-Scholes price $f(0, \bar{S}_0)$. Indeed we have

$$df(t, \bar{S}_t) = \partial_s f d\bar{S}_t + \left(\partial_t f + \frac{\sigma_t^2 \bar{S}_t^2}{2}\partial_{ss} f\right) dt =$$

(by (7.59))

$$= \partial_s f d\bar{S}_t + \left(rf - r\bar{S}_t \partial_s f - \frac{(\sigma^2 - \sigma_t^2)\bar{S}_t^2}{2}\partial_{ss} f\right) dt$$

$$= \partial_s f d\bar{S}_t + \left(f - \bar{S}_t \partial_s f\right) dB_t - \frac{(\sigma^2 - \sigma_t^2)\bar{S}_t^2}{2}\partial_{ss} f dt. \tag{7.62}$$

More explicitly we have the following integral expression of the payoff

$$F(\bar{S}_T) = f(T, \bar{S}_T) = I_1 + I_2 + I_3$$

where

$$I_1 = f(0, \bar{S}_0)$$

is the Black-Scholes price,

$$I_2 = \int_0^T \partial_s f(t, \bar{S}_t) d\bar{S}_t + \int_0^T \left(f(t, \bar{S}_t) - \bar{S}_t \partial_s f(t, \bar{S}_t)\right) dB_t$$

is the gain of the Delta-hedging strategy,

$$I_3 = -\frac{1}{2} \int_0^T (\sigma^2 - \sigma_t^2) \bar{S}_t^2 \partial_{ss} f(t, \bar{S}_t) dt \tag{7.63}$$

is a correction term due to the erroneous specification of the model for the underlying asset. Clearly $I_3 = 0$ if $\sigma = \sigma_t$ and only in that case the strategy is self-financing.

We remark that I_3 depends only on the misspecification of the volatility term and *not on the drift.* More precisely I_3, which also represents *the replication error of the Delta-hedging strategy*, depends on the Vega which measures the convexity of the Black-Scholes price as a function of the price of the underlying asset. In particular the error is small if $\partial_{ss} f$ is small. Furthermore, if the price is convex, $\partial_{ss} f \geq 0$, as in the case of Call and Put options, then the Black-Scholes strategy (whose final value is $I_1 + I_2$) super-replicates the derivative *for any dynamics of the underlying asset* as long as we choose the volatility sufficiently large, $\sigma \geq \sigma_t$, since in this case $I_3 \leq 0$.

In this sense the Black-Scholes model is robust and, if used with all due precautions, can be effectively employed to hedge derivatives. Let us note finally that there exist options whose price is not a convex function of the underlying asset and so the Vega is not necessarily positive: this is the case of the digital option, corresponding to the Delta of a Call (see Figure 7.2), and also of some barrier options. Consequently in some cases in order to super-replicate it may be necessary to decrease the volatility.

7.4.3 Gamma and Vega-hedging

The Greeks can be used to determine more efficient hedging strategies than Delta-hedging. Here we consider the replication problem from a practical point of view: it is clear that theoretically the Delta-hedging approach offers perfect replication; nevertheless we have already mentioned some substantial problems we might have to face:

- the strategies are discrete and there are transition costs;
- the volatility is not constant.

As an example, in this section we consider the *Delta-Gamma* and *Delta-Vega-hedging* strategies whose purpose is to reduce the replication error due to the fact that rebalancing is not continuous in the first case and to the variation of the volatility in the second.

The reason why it is necessary to rebalance the Black-Scholes hedging portfolio is that the Delta changes as the underlying price varies. So, to minimize the number of times we have to rebalance (and the relative costs, of course), it seems natural to create a strategy that is neutral not only to the Delta but also to the Gamma. With all due adjustments, the procedure is similar to the Delta-hedging one in Section 7.3.3. Nevertheless in order to impose two neutrality conditions, one unknown is no longer sufficient, so it is necessary to build a portfolio with three assets. The situation is analogous to that of an incomplete market (cf. Section 2.4.1): indeed if continuous rebalancing is not allowed, not all derivatives are replicable and the Black-Scholes model loses its completeness property.

Let us suppose that we have sold a derivative $f(t, S_t)$ and we try to hedge the short position by investing on the underlying asset and on another derivative $g(t, S_t)$: the typical situation is when f is an exotic derivative and g is a plain vanilla option and we suppose it is exchanged on the market. We consider

$$V(t, S_t) = -f(t, S_t) + \alpha_t S_t + \beta_t g(t, S_t), \tag{7.64}$$

and we determine α, β by imposing the neutrality conditions

$$\partial_s V = 0, \qquad \partial_{ss} V = 0.$$

We get the system of equations

$$\begin{cases} -\partial_s f + \alpha_t + \beta_t \partial_s g = 0, \\ -\partial_{ss} f + \beta_t \partial_{ss} g = 0, \end{cases}$$

hence we deduce the Delta-Gamma-hedging strategy

$$\beta_t = \frac{\partial_{ss} f(t, S_t)}{\partial_{ss} g(t, S_t)}, \qquad \alpha_t = \partial_s f(t, S_t) - \frac{\partial_{ss} f(t, S_t)}{\partial_{ss} g(t, S_t)} \partial_s g(t, S_t).$$

We use a similar argument to reduce the uncertainty risk of the volatility parameter. The main assumption of the Black-Scholes model is that the volatility is constant, therefore the Delta-Vega-hedging strategy that we present in what follows is, in a certain sense, “beyond” the model. In this case also, the underlying asset is not sufficient and so we suppose there exists a second derivative which is exchanged on the market. Let us consider the portfolio (7.64) and let us impose the neutrality conditions

$$\partial_s V = 0, \qquad \partial_\sigma V = 0.$$

We get the system of equations

$$\begin{cases} -\partial_s f + \alpha_t + \beta_t \partial_s g = 0, \\ -\partial_\sigma f + \alpha_t \partial_\sigma S_t + \beta_t \partial_\sigma g = 0, \end{cases}$$

and then we can obtain easily the hedging strategy by observing that $\partial_\sigma S_t = S_t(W_t - \sigma t)$.

7.5 Implied volatility

In the Black-Scholes model the price of a European Call option is a function of the form

$$C_{\mathrm{BS}} = C_{\mathrm{BS}}(\sigma, S, K, T, r)$$

where σ is the volatility, S is the current price of the underlying asset, K is the strike, T is the maturity and r is the short-term rate. Actually the price can also be expressed in the form

$$C_{\mathrm{BS}} := S\varphi\left(\sigma, \frac{S}{K}, T, r\right),$$

where φ is a function whose expression can be easily deduced from the Black-Scholes formula (7.33). The number $m = \frac{S}{K}$ is usually called "moneyness" of the option: if $\frac{S}{K} > 1$, we say that that the Call option is "in the money", since we are in a situation of potential profit; if $\frac{S}{K} < 1$, the Call option is "out of the money" and has null intrinsic value; finally, if $\frac{S}{K} = 1$ i.e. $S = K$, we say that the option is "at the money".

Of all the parameters determining the Black-Scholes price, the volatility σ is the only one that is not directly observable. We recall that

$$\sigma \mapsto C_{\mathrm{BS}}(\sigma, S, K, T, r)$$

is a strictly increasing function and therefore invertible: having fixed all the other parameters, a Black-Scholes price of the option corresponds to every value of σ; conversely, a unique value of the volatility σ^* is associated to every value C^* on the interval $]0, S[$ (the interval to which the price must belong by arbitrage arguments). We set

$$\sigma^* = \mathrm{VI}(C^*, S, K, T, r),$$

where σ^* is the unique value of the volatility parameter such that

$$C^* = C_{\mathrm{BS}}(\sigma^*, S, K, T, r).$$

The function

$$C^* \mapsto \mathrm{VI}(C^*, S, K, T, r)$$

is called *implied volatility function.*

The first problem when we price an option in the Black-Scholes model is the choice of the parameter σ that, as we have already said, is not directly observable. The first idea could be to use a value of σ obtained from an estimate on the historical data on the underlying asset, i.e. the so-called *historical volatility.* Actually, the most widespread and simple approach is that of using directly, where it is available, the implied volatility of the market: we see, however, that this approach is not free from problems.

The concept of implied volatility is so important and widespread that, in financial markets, the plain vanilla options are commonly quoted in terms of implied volatility, rather than explicitly by giving their price. As a matter of fact, using the implied volatility is convenient for various reasons. First of all, since the Put and Call prices are increasing functions of the volatility, the quotation in terms of the implied volatility immediately gives the idea of the "cost" of the option. Analogously, using the implied volatility makes it easy to compare the prices of options on the same asset, but with different strikes and maturities.

For fixed S and r, and given a family of prices

$$\{C_i^* \mid i = 1, \dots M\} \tag{7.65}$$

where C_i^* denotes the price of the Call with strike K^i and maturity T^i, the *implied volatility surface* relative to (7.65) is the graph of the function

$$\left(K^i, T^i\right) \mapsto \mathrm{VI}\left(C_i^*, S, K^i, T^i, r\right).$$

If we assume the Black-Scholes dynamics for the underlying asset

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

and $\left(C_{\mathrm{BS}}^i\right)_{i\in I}$ is a family of Black-Scholes prices relative to the strikes K^i and maturities T^i, then the corresponding implied volatilities must obviously coincide:

$$\mathrm{VI}\left(C_{\mathrm{BS}}^i, S, K^i, T^i, r\right) = \sigma \qquad \text{for any } i \in I.$$

In other terms, the *implied volatility surface* relative to the prices obtained by the Black-Scholes model is flat and coincides with the graph of the function that is constant and equal to σ.

On the contrary, for an *empirical* implied volatility surface, inferred from quoted prices in real markets, the result is generally quite different: it is well known that the market prices of European options on the same underlying asset have implied volatilities that vary with strike and maturity. By way of example, in Figure 7.8 we depict the implied volatility surface of options on the London FTSE index on March 31st 2006.

Typically every section, with T fixed, of the implied volatility surface takes a particular form that is usually called "smile" (in the case of Figure 7.9) or "skew" (in the case of Figure 7.8). Generally we can say that market quotation tends to give more value (greater implied volatility) to the extreme cases "in" or "out of the money". This reflects that some situations in the market are perceived as more risky, in particular the case of extreme falls or rises of the quotations of the underlying asset.

Also the dependence on T, the time to maturity, is significant in the analysis of the implied volatility: this is called the *term-structure of the implied volatility*. Typically when we get close to maturity ($T \to 0^+$), we see that the smile or the skew become more marked.

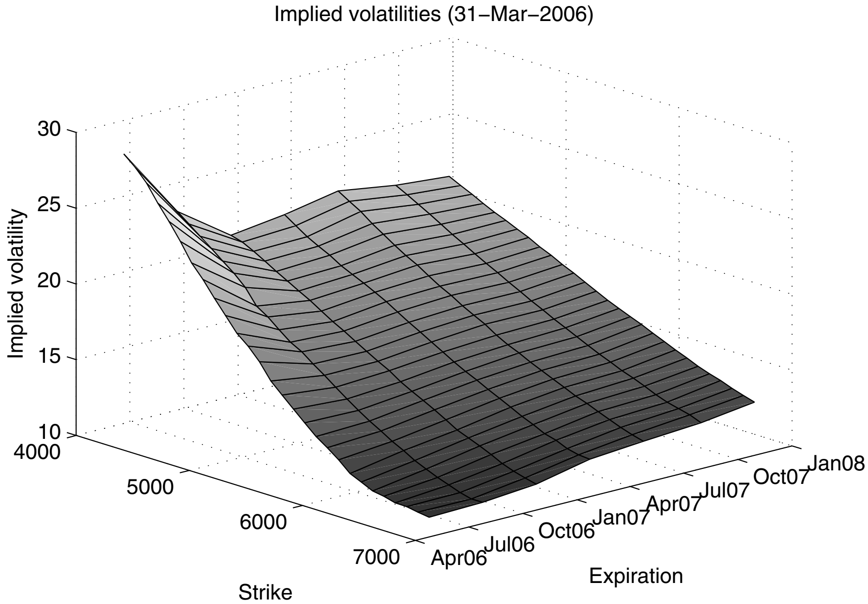

Fig. 7.8. Implied-volatility surface of options on the FTSE index on March 31st 2006

Other characteristic features make definitely different the implied volatility surface of the market from the constant Black-Scholes volatility: for example, in Figure 7.9 we show the dependence of the implied volatility of options on the S&P500 index, with respect to the so-called "deviation from trend" of the underlying asset, defined as the difference between the current price and a weighted mean of historical prices. Intuitively this parameter indicates if there have been sudden large movements of the quotation of the underlying asset.

Finally we note that the implied volatility depends also on time *in absolute terms*: indeed, it is well known that the shape of the implied volatility surface on the S&P500 index has significantly changed from the beginning of the eighties until today. The market crash of 19 October 1987 may be taken as the date marking the end of flat volatility surfaces.

This also reflects the fact that, though based on the same mathematical and probabilistic tools, the modeling of financial and, for instance, physical phenomena are essentially different: indeed, the financial dynamics strictly depends on the behaviour and beliefs of investors and therefore, differently from the general laws in physics, may vary drastically over time.

The analysis of the implied volatility surface makes it evident that the Black-Scholes model is not realistic: more precisely, we could say that nowadays Black-Scholes is the *language* of the market (since prices are quoted in

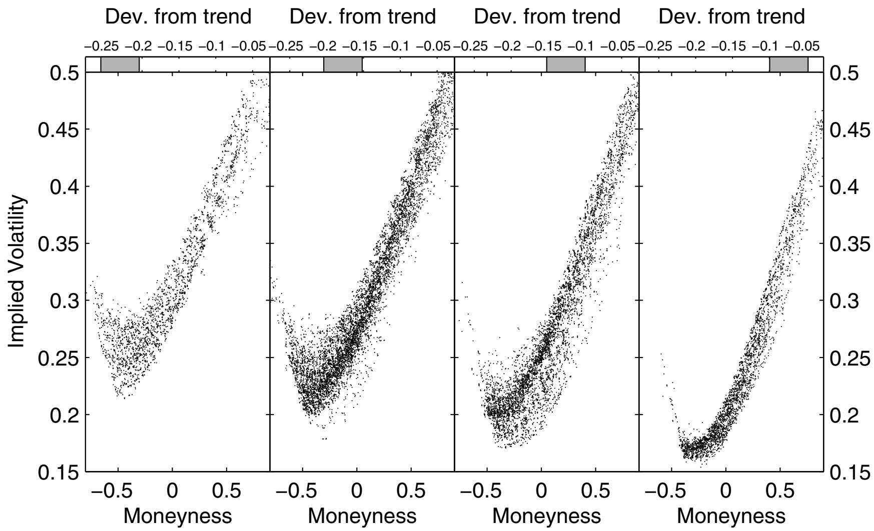

Fig. 7.9. Effect of the deviation from trend on the implied volatility. The volatility smiles for options on the S&P500 index are grouped for different values of the deviation, as indicated on top of each box

terms of implied volatility), but usually it is not the model really used by investors to price and hedge derivatives. Indeed the use of the Black-Scholes model poses some not merely theoretical problem: for instance, let us suppose that, despite all the evidence against the Black-Scholes model, we wish to use it anyway. Then we have seen that we have to face the problem of the choice of the volatility parameter for the model. If we use the historical volatility, we might get quotations that are "out of the market", especially when compared with those obtained from the market-volatility surface in the extreme "in" and "out of money" regions. On the other hand, if we want to use the implied volatility, we have to face the problem of choosing one value among all the values given by the market, since the volatility surface is not "flat". Evidently, if our goal is to price and hedge a plain vanilla option, with strike, say, K and maturity, say, T, the most natural idea is to use the implied volatility corresponding to (K, T). But the problem does not seem to be easily solvable if we are interested in the pricing and hedging of an exotic derivative: for example, if the derivative does not have a unique maturity (e.g. a Bermudan option) or if a fixed strike does not appear in the payoff (e.g., an Asian option with floating strike).

These problems make it necessary to introduce more sophisticated models than the Black-Scholes one, that can be calibrated in such a way that it is possible to price plain vanilla options in accordance with the implied volatility surface of the market. In this way such models can give prices to exotic derivatives that are consistent with the market Call and Put prices. This result is not particularly difficult and can be obtained by various models with non-constant volatility such as those in Chapter 10.5. A second goal that poses many more delicate questions and is still a research topic consists in finding a model that gives the "best" solution to the hedging problem and that is stable with respect to perturbations of the value of the parameters involved (see for instance Schoutens, Simons, and Tistaert [302] and Cont [75]).

7.6 Asian options

An Asian option is a derivative whose payoff depends on an average of the prices of the underlying asset. This kind of derivative is quite often used, for example in the currencies or commodities markets: one of the reasons to introduce this derivative is to limit speculation on plain vanilla options. Indeed it is known that the European Call and Put option prices close to maturity can be influenced by the investors through manipulations on the underlying asset.

Asian options can be classified by the payoff function and by the particular average that is used. As usual we assume that the underlying asset follows a geometric Brownian motion S verifying equation (7.2) and we denote by M_t the value of the average at time t: for an Asian option with *arithmetic average* we have

$$M_t = \frac{A_t}{t} \quad \text{with} \quad A_t = \int_0^t S_\tau d\tau; \tag{7.66}$$

for an Asian option with *geometric average* we have

$$M_t = e^{\frac{G_t}{t}} \quad \text{with} \quad G_t = \int_0^t \log(S_\tau)\, d\tau. \tag{7.67}$$

Even though arithmetic Asian options are more commonly traded in real markets, in the literature geometric Asian options have been widely studied because they are more tractable from a theoretical point of view and, under suitable conditions, they can be used to approximate the corresponding arithmetic version.

Concerning the payoff, the most common versions are the Asian Call *with fixed strike K*

$$F(S_T, M_T) = (M_T - K)^+,$$

the Asian Call *with floating strike*

$$F(S_T, M_T) = (S_T - M_T)^+,$$

and the corresponding Asian Puts.

Formally, the pricing and hedging problems for Asian options have a lot in common with their standard European counterparts: the main difference is that an Asian option depends not only on the spot price of the underlying asset but also on its entire trajectory. Nevertheless, as already mentioned in the discrete case in Section 2.3.3, it is possible to preserve the Markovian property of the model by using a technique now standard: this consists in augmenting the space by introducing an additional state variable related to the average process A_t in (7.66) or G_t in (7.67).

7.6.1 Arithmetic average

In order to make the previous ideas precise, let us examine first the arithmetic average case. We say that $(\alpha_t, \beta_t)_{t\in[0,T]}$ is a Markovian portfolio if

$$\alpha_t = \alpha(t, S_t, A_t), \qquad \beta_t = \beta(t, S_t, A_t), \qquad t \in [0, T],$$

where α, β are functions in $C^{1,2}([0,T[\times\mathbb{R}_{>0}\times\mathbb{R}_{>0}) \cap C([0,T]\times\mathbb{R}_{>0}\times\mathbb{R}_{>0})$, and we denote by

$$f(t, S_t, A_t) = \alpha_t S_t + \beta_t B_t, \qquad t \in [0, T],$$

the corresponding value. The following result extends Theorems 7.8 and 7.13:

Theorem 7.22 *The following conditions are equivalent:*

i) $(\alpha_t, \beta_t)_{t\in[0,T]}$ *is self-financing, i.e. we have*

$$df(t, S_t, A_t) = \alpha_t dS_t + \beta_t dB_t;$$

ii) f *is a solution of the partial differential equation*

$$\frac{\sigma^2 s^2}{2}\partial_{ss} f(t,s,a) + rs\partial_s f(t,s,a) + s\partial_a f(t,s,a) + \partial_t f(t,s,a) = rf(t,s,a), \tag{7.68}$$

for $(t,s,a) \in [0,T[\times\mathbb{R}_{>0}\times\mathbb{R}_{>0}$, *and we have that*

$$\alpha(t,s,a) = \partial_s f(t,s,a).$$

The arbitrage price $f = f(t, S_t, A_t)$ *of an Asian arithmetic option with payoff function* F *is the solution of the Cauchy problem for equation* (7.68) *with final datum*

$$f(T,s,a) = F\left(s, \frac{a}{T}\right), \qquad s, a \in \mathbb{R}_{>0}.$$

For example, in the case of a fixed strike Asian Call, the final condition for equation (7.68) is

$$f(T,s,a) = \left(\frac{a}{T} - K\right)^+, \qquad s, a \in \mathbb{R}_{>0}. \tag{7.69}$$

For the floating strike Asian Call, the final condition becomes

$$f(T,s,a) = \left(s - \frac{a}{T}\right)^+, \qquad s, a \in \mathbb{R}_{>0}. \tag{7.70}$$

The proof of Theorem 7.22 is formally analogous to the ones of Theorems 7.8 and 7.13. Let us observe that *equation* (7.68) *cannot be transformed into a parabolic equation by a change of variables* as in the European case. In particular the results of existence and uniqueness for the Cauchy problem of Appendix A.3 and Section 6.2 are not sufficient to prove the completeness of the market and the existence and uniqueness of the arbitrage price: these results have been recently proved, for a generic payoff function, by Barucci, Polidoro and Vespri [33].

Equation (7.68) is *degenerate parabolic*, because the matrix of the second-order part of the equation is singular and only positive semi-definite: indeed, in the standard notation (A.45) of Appendix A.3, the matrix $\mathcal{C}$ corresponding to (7.68) is

$$\mathcal{C} = \begin{pmatrix} \sigma^2 s^2 & 0 \\ 0 & 0 \end{pmatrix}$$

and has rank one for every $(s,a) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}$. This does not have to come as a surprise: equation (7.68) was deduced by using the Itô formula and the second-order derivative appearing in it is "produced" by the Brownian motion of the process S. The average A brings an additional state variable in, thus augmenting the dimension of the problem, setting it in $\mathbb{R}^3$, but it does not bring a new Brownian motion in (nor second-order derivative with respect to the variable a).

In some particular cases there exists a suitable transformation to take back the problem to two dimensions. In the floating strike case, Ingersoll [178] suggests the change of variable $x = \frac{a}{s}$: if we put

$$f(t,s,a) = su\left(t, \frac{a}{s}\right) \tag{7.71}$$

we have

$$\partial_t f = s\partial_t u, \qquad \partial_s f = u - \frac{a}{s}\partial_x u, \qquad \partial_{ss} f = \frac{a^2}{s^3}\partial_{xx} u, \qquad \partial_a f = \partial_x u.$$

So f solves the Cauchy problem (7.68)-(7.70) if and only if the function $u = u(t,x)$ defined in (7.71) is a solution of the Cauchy problem in $\mathbb{R}^2$

$$\begin{cases} \frac{\sigma^2 x^2}{2}\partial_{xx} u + (1 - rx)\partial_x u + \partial_t u = 0, & t \in [0,T[,\ x > 0, \\ u(T,x) = \left(1 - \frac{x}{T}\right)^+, & x > 0. \end{cases}$$

More generally, transformation (7.71) allows to reduce the dimension of the problem in case the payoff is a homogeneous function of degree one, that is

$$F(s,a) = sF\left(1, \frac{a}{s}\right), \qquad s, a > 0.$$

For the fixed strike Asian option, Rogers and Shi [291] suggest the change of variable

$$x = \frac{\frac{a}{T} - K}{s}.$$

If we put

$$f(t,s,a) = su\left(t, \frac{\frac{a}{T} - K}{s}\right) \tag{7.72}$$

we have

$$\partial_s f = u - \frac{\frac{a}{T} - K}{s}\partial_x u, \qquad \partial_{ss} f = \frac{\left(\frac{a}{T} - K\right)^2}{s^3}\partial_{xx} u, \qquad \partial_a f = \frac{\partial_x u}{T}.$$

So f solves the Cauchy problem (7.68)-(7.69) if and only if the function $u = u(t,x)$ defined in (7.72) is a solution of the Cauchy problem in $\mathbb{R}^2$

$$\begin{cases} \frac{\sigma^2 x^2}{2}\partial_{xx} u + \left(\frac{1}{T} - rx\right)\partial_x u + \partial_t u = 0, & t \in [0,T[, \ x \in \mathbb{R}, \\ u(T,x) = x^+, & x \in \mathbb{R}. \end{cases}$$

Note that the reduction of the dimension of the problem is possible only in particular cases and assuming the Black-Scholes dynamics for the underlying asset.

7.6.2 Geometric average

We consider a geometric average Asian option: in this case the value $f = f(t,s,g)$ of the replicating portfolio is function of t, S_t and G_t in (7.67). Furthermore a result analogous to Theorem 7.22 holds, where (7.68) is replaced by the differential equation

$$\frac{\sigma^2 s^2}{2}\partial_{ss} f(t,s,g) + rs\partial_s f(t,s,g) + (\log s)\partial_g f(t,s,g) + \partial_t f(t,s,g) = rf(t,s,g), \tag{7.73}$$

with $(t,s,g) \in [0,T[\times \mathbb{R}_{>0} \times \mathbb{R}$.

Similarly to Proposition 7.9, we change the variables by putting

$$t = T - \tau, \qquad s = e^{\sigma x}, \qquad g = \sigma y,$$

and

$$u(\tau,x,y) = e^{ax+b\tau} f(T-\tau, e^{\sigma x}, \sigma y), \qquad \tau \in [0,T], \ x,y \in \mathbb{R}, \tag{7.74}$$

where a, b are constants to be determined appropriately later. Let us recall formulas (7.23) and also that

$$\partial_y u = e^{ax+b\tau}\sigma\partial_g f;$$

it follows that

$$\frac{1}{2}\partial_{xx} u + x\partial_y u - \partial_\tau u =$$

$$e^{ax+b\tau}\left(\frac{\sigma^2 s^2}{2}\partial_{ss} f + \left(\sigma a + \frac{\sigma^2}{2}\right)s\partial_s f + (\log s)\partial_g f + \partial_t f + \left(\frac{a^2}{2} - b\right)f\right) =$$

(if f solves (7.73))

$$= \left(\sigma a + \frac{\sigma^2}{2} - r\right) s\partial_s f + \left(\frac{a^2}{2} - b + r\right) f.$$

This proves the following result.

Proposition 7.23 *By choosing the constants a and b as in* (7.24)*, the function f is a solution of the equation* (7.73) *in* $[0,T[\times\mathbb{R}_{>0}\times\mathbb{R}$ *if and only if the function $u = u(\tau,x,y)$ defined in* (7.74) *satisfies the equation*

$$\frac{1}{2}\partial_{xx}u + x\partial_y u - \partial_\tau u = 0, \qquad \text{in } \;]0,T]\times\mathbb{R}^2. \tag{7.75}$$

(7.75) is a degenerate parabolic equation, called Kolmogorov equation which will be studied in Section 9.5 and whose fundamental solution will be constructed explicitly in Example 9.53.