4

Brownian integration

In this chapter we introduce the elements of stochastic integration theory that are necessary to treat some financial models in continuous time. In Paragraph 3.4 we gave grounds for the interest in the study of the limit of a Riemann-Stieltjes sum of the form

$$\sum_{k=1}^{N} u_{t_{k-1}}(W_{t_k} - W_{t_{k-1}}) \tag{4.1}$$

as the refinement parameter of the partition $\{t_0, \dots, t_N\}$ tends to zero. In (4.1) W is a real Brownian motion that represents a risky asset and u is an adapted process that represents an investment strategy: if the strategy is self-financing, the limit of the sum in (4.1) is equal to the value of the investment.

However the paths of W do not have bounded variation a.s. and this fact prevents us to define pathwise the integral

$$\int_0^T u_t dW_t$$

in the Riemann-Stieltjes sense. On the other hand W *has finite quadratic variation* and this property makes it possible to construct the stochastic integral for suitable classes of integrands u: generally speaking, we require that u is progressively measurable and satisfies some integrability conditions.

The concept of Brownian integral was introduced by Paley, Wiener and Zygmund [275] for deterministic integrand functions. The general construction is due to Itô [179]-[180] in the case of Brownian motion, and to Kunita and Watanabe [219] in $\mathscr{M}^2$. This theory lays the foundations for a rigorous study of stochastic differential equations that describe the diffusion processes introduced by Kolmogorov [213], on which the modern stochastic models for finance are based. In this chapter we confine ourselves to the Brownian case.

The aim of this chapter is to construct the Brownian integral gradually, first considering the integration of "simple" processes, i.e. processes that are

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piecewise constant with respect to the time variable, then extending the definition to a sufficiently general class of progressively measurable and square-integrable processes. Among the main consequences of the definition, we have that the stochastic integral has null expectation, it is a continuous martingale in $\mathscr{M}_c^2$ and it satisfies Itô isometry. By further extending the class of integrands, some of those properties are lost and it is necessary to introduce the more general notion of local martingale.

4.1 Stochastic integral of deterministic functions

As an introductory example, useful to see in advance some of the main results we are going to prove, we consider Paley, Wiener and Zygmund's construction [275] of the stochastic integral for deterministic functions.

Let $u \in C^1([0,1])$ be a real-valued function such that $u(0) = u(1) = 0$. Given a real Brownian motion W, we define

$$\int_0^1 u(t)dW_t = -\int_0^1 u'(t)W_t dt. \tag{4.2}$$

This integral is a random variable that verifies the following properties:

i) $E\left[\int_0^1 u(t)dW_t\right] = 0$;

ii) $E\left[\left(\int_0^1 u(t)dW_t\right)^2\right] = \int_0^1 u^2(t)dt$.

Indeed

$$E\left[\int_0^1 u'(t)W_t dt\right] = \int_0^1 u'(t)E\left[W_t\right]dt = 0.$$

Further,

$$E\left[\int_0^1 u'(t)W_t dt \int_0^1 u'(s)W_s ds\right] = \int_0^1\int_0^1 u'(t)u'(s)E\left[W_tW_s\right]dtds =$$

(since $E\left[W_tW_s\right] = t \wedge s$)

$$\begin{aligned}
&= \int_0^1 u'(t)\left(\int_0^t su'(s)ds + t\int_t^1 u'(s)ds\right)dt \\
&= \int_0^1 u'(t)\left(tu(t) - \int_0^t u(s)ds + t(u(1) - u(t))\right)dt \\
&= \int_0^1 u'(t)\left(-\int_0^t u(s)ds\right)dt = \int_0^1 u^2(t)dt.
\end{aligned}$$

More generally, if $u \in L^2(0,1)$ and (u_n) is a sequence of functions in $C_0^1(0,1)$ approximating u in the L^2 norm, by property ii) we have

$$E\left[\left(\int_0^1 u_n(t)dW_t - \int_0^1 u_m(t)dW_t\right)^2\right] = \int_0^1 (u_n(t) - u_m(t))^2 dt.$$

Therefore the sequence of integrals is a Cauchy sequence in $L^2(\Omega, P)$ and we can define

$$\int_0^1 u(t)dW_t = \lim_{n\to\infty} \int_0^1 u_n(t)dW_t.$$

We have thus constructed the stochastic integral for $u \in L^2([0,1])$ and, by passing to the limit, it is immediate to verify properties i) and ii).

Evidently this construction can be considered only an introductory step, since we are interested in defining the Brownian integral in the case u is a stochastic process. Indeed we recall that, from a financial point of view, u represents a future-investment strategy, necessarily random. On the other hand, since (4.2) seems to be a reasonable definition, in the following paragraphs we will introduce a definition of stochastic integral that agrees with the one given for the deterministic case.

4.2 Stochastic integral of simple processes

In what follows W is a real Brownian motion on the filtered probability space $(\Omega, \mathcal{F}, P, (\mathcal{F}_t))$ where the usual hypotheses hold and T is a fixed positive number.

Definition 4.1 *The stochastic process u belongs to the class $\mathbb{L}^2$ if*

i) u is progressively measurable with respect to the filtration $(\mathcal{F}_t)$;
ii) $u \in L^2([0,T] \times \Omega)$ that is

$$\int_0^T E\left[u_t^2\right] dt < \infty.$$

Condition ii) is a simple integrability condition, while i) is the property playing the crucial part in what follows. Since the definition of $\mathbb{L}^2$ depends on the given filtration $(\mathcal{F}_t)$, when it is necessary we will also write $\mathbb{L}^2(\mathcal{F}_t)$ instead of $\mathbb{L}^2$. More generally, for $p \geq 1$, we denote by $\mathbb{L}^p$ the space of progressively measurable processes in $L^p([0,T] \times \Omega)$. We note explicitly that $\mathbb{L}^p$ is a closed subspace of $L^p([0,T] \times \Omega)$.

Now we start by defining the Itô integral for a particular class of stochastic processes in $\mathbb{L}^2$.

Definition 4.2 *A process $u \in \mathbb{L}^2$ is called simple if it can be written as*

$$u_t = \sum_{k=1}^{N} e_k \mathbb{1}_{]t_{k-1},t_k]}(t), \qquad t \in [0,T], \tag{4.3}$$

where $0 \leq t_0 < t_1 < \cdots < t_N \leq T$ and e_k are random variables[1] *on $(\Omega, \mathcal{F}, P)$.*

[1] We assume also that

$$P(e_{k-1} = e_k) = 0, \qquad k = 2, \dots, N,$$

so that the representation (4.3) for u is unique a.s.

Remark 4.3 It is important to observe that, since u is progressively measurable and by hypothesis (3.27) of right-continuity of the filtration, we have that e_k in (4.3) is $\mathcal{F}_{t_{k-1}}$-measurable for every $k = 1, \dots, N$. Further, $e_k \in L^2(\Omega, P)$ and we have

$$\int_0^T E\left[u_t^2\right] dt = \sum_{k=1}^N \int_0^T E\left[e_k^2\right] \mathbb{1}_{]t_{k-1},t_k]}(t)dt = \sum_{k=1}^N E\left[e_k^2\right](t_k - t_{k-1}). \quad (4.4)$$

□

If $u \in \mathbb{L}^2$ is a simple process of the form (4.3), then we define the Itô integral in the following way:

$$\int u_t dW_t = \sum_{k=1}^N e_k (W_{t_k} - W_{t_{k-1}}) \quad (4.5)$$

and also, for every $0 \le a < b \le T$,

$$\int_a^b u_t dW_t = \int u_t \mathbb{1}_{]a,b]}(t) dW_t \quad (4.6)$$

and

$$\int_a^a u_t dW_t = 0.$$

Example 4.4 Integrating the simple process $u = \mathbb{1}_{]0,t]}$, we get

$$W_t = \int_0^t dW_s.$$

Then, going back to Example 3.8, we have

$$S_t = S_0 \left(1 + \int_0^t \mu ds\right) + \int_0^t \sigma dW_s, \qquad t > 0.$$ □

The following theorem contains some important properties of the Itô integral of simple processes.

Theorem 4.5 *For all simple processes $u, v \in \mathbb{L}^2$, $\alpha, \beta \in \mathbb{R}$ and $0 \le a < b < c \le T$ the following properties hold:*

(1) linearity:

$$\int (\alpha u_t + \beta v_t) dW_t = \alpha \int u_t dW_t + \beta \int v_t \, dW_t;$$

(2) additivity:

$$\int_a^b u_t dW_t + \int_b^c u_t dW_t = \int_a^c u_t dW_t;$$

(3) null expectation:

$$E\left[\int_a^b u_t dW_t \mid \mathcal{F}_a\right] = 0, \tag{4.7}$$

and also

$$E\left[\int_a^b u_t dW_t \int_b^c v_t dW_t \mid \mathcal{F}_a\right] = 0; \tag{4.8}$$

(4) Itô isometry:

$$E\left[\int_a^b u_t dW_t \int_a^b v_t dW_t \mid \mathcal{F}_a\right] = E\left[\int_a^b u_t v_t dt \mid \mathcal{F}_a\right]; \tag{4.9}$$

(5) the stochastic process

$$X_t = \int_0^t u_s dW_s, \qquad t \in [0, T], \tag{4.10}$$

is a continuous $\mathcal{F}_t$-martingale, i.e. $X \in \mathscr{M}_c^2(\mathcal{F}_t)$, and we have

$$[\![X]\!]_T^2 = E\left[\sup_{t\in[0,T]} X_t^2\right] \le 4E\left[\int_0^T u_t^2 dt\right]. \tag{4.11}$$

Remark 4.6 Since

$$E[X] = E[E[X \mid \mathcal{F}_a]],$$

the non-conditional versions of (4.7), (4.8), (4.9) hold:

$$E\left[\int_a^b u_t dW_t\right] = 0,$$

$$E\left[\int_a^b u_t dW_t \int_b^c v_t dW_t\right] = 0,$$

$$E\left[\int_a^b u_t dW_t \int_a^b v_t dW_t\right] = E\left[\int_a^b u_t v_t dt\right].$$

The last identity for $u = v$ is equivalent to the L^2-norm equality

$$\left\| \int_a^b u_t dW_t \right\|_{L^2(\Omega)} = \|u\|_{L^2([a,b]\times\Omega)}$$

and this is why the fourth property is called "Itô isometry". □

Proof. Properties (1) and (2) are trivial. Concerning property (3), we have

$$E\left[\int_a^b u_t dW_t \mid \mathcal{F}_a\right] = \sum_{k=1}^N E\left[e_k(W_{t_k} - W_{t_{k-1}}) \mid \mathcal{F}_a\right] =$$

(since $t_0 \geq a$, e_k is $\mathcal{F}_{t_{k-1}}$-measurable by Remark 4.3 and so independent of $W_{t_k} - W_{t_{k-1}}$ and then we use Proposition A.107-(6))

$$= \sum_{k=1}^{N} E\left[e_k \mid \mathcal{F}_a\right] E\left[W_{t_k} - W_{t_{k-1}}\right] = 0.$$

To prove (4.8) we proceed analogously: if v is of the form

$$v = \sum_{h=1}^{M} d_h \mathbb{1}_{]t_{h-1}, t_h]},$$

then $E\left[\int_a^b u_t dW_t \int_b^c v_t dW_t \mid \mathcal{F}_a\right]$ is a sum of terms of the form

$$E\left[e_k d_h (W_{t_k} - W_{t_{k-1}})(W_{t_h} - W_{t_{h-1}}) \mid \mathcal{F}_a\right], \qquad \text{with } t_k \leq t_{h-1},$$

that are all equal to zero since $e_k d_h (W_{t_k} - W_{t_{k-1}})$ is $\mathcal{F}_{t_{h-1}}$-measurable and so independent of the increment $W_{t_h} - W_{t_{h-1}}$ whose expectation is null, since $a \leq t_{h-1}$.

Let us prove Itô isometry: if u and v are simple processes, we have

$$\begin{aligned} E\left[\int_a^b u_t dW_t \int_a^b v_t dW_t \mid \mathcal{F}_a\right] &= E\left[\sum_{k=1}^{N} \int_{t_{k-1}}^{t_k} e_k dW_t \sum_{h=1}^{N} \int_{t_{h-1}}^{t_h} d_h dW_t \mid \mathcal{F}_a\right] \\ &= \sum_{k=1}^{N} E\left[\int_{t_{k-1}}^{t_k} e_k dW_t \int_{t_{k-1}}^{t_k} d_k dW_t \mid \mathcal{F}_a\right] \\ &\quad + 2\sum_{h<k} E\left[\int_{t_{k-1}}^{t_k} e_k dW_t \int_{t_{h-1}}^{t_h} d_h dW_t \mid \mathcal{F}_a\right] = \end{aligned}$$

(by (4.8) the terms in the second summation are null)

$$= \sum_{k=1}^{N} E\left[e_k d_k (W_{t_k} - W_{t_{k-1}})^2 \mid \mathcal{F}_a\right] =$$

(by Proposition A.107-(6), since $W_{t_k} - W_{t_{k-1}}$ is independent of $e_k d_k$ and of $\mathcal{F}_a$)

$$= \sum_{k=1}^{N} E\left[e_k d_k \mid \mathcal{F}_a\right] E\left[(W_{t_k} - W_{t_{k-1}})^2\right] = E\left[\sum_{k=1}^{N} e_k d_k (t_k - t_{k-1}) \mid \mathcal{F}_a\right]$$

and the claim follows by (4.4) at least for $u = v$: the general case is analogous.

Let us now prove that the stochastic process X in (4.10) is a continuous $\mathcal{F}_t$-martingale. The continuity follows directly from the definition of stochastic integral. By definition (4.5)-(4.6) and Remark 4.3, it is obvious that X is $\mathcal{F}_t$-adapted. Further, X_t is integrable since, by Hölder's inequality, we have

$$E[|X_t|]^2 \le E\left[X_t^2\right] =$$

(by Itô isometry)

$$= E\left[\int_0^t u_s^2 ds\right] < \infty$$

since $u \in \mathbb{L}^2$. Then, for $0 \le s < t$ we have

$$E[X_t \mid \mathcal{F}_s] = E[X_s \mid \mathcal{F}_s] + E\left[\int_s^t u_\tau \, dW_\tau \mid \mathcal{F}_s\right] = X_s,$$

since X_s is $\mathcal{F}_s$-measurable and (4.7) holds: therefore X is a martingale. Finally (4.11) is consequence of Doob's inequality, Theorem 3.38, and Itô isometry: indeed we have

$$[\![X]\!]_T^2 \le 4E\left[X_T^2\right] = 4E\left[\int_0^T u_t^2 dt\right].$$

Remark 4.7 The martingale property of the stochastic integral can also be written in the following meaningful way:

$$E\left[\int_0^T u_s dW_s \mid \mathcal{F}_t\right] = \int_0^t u_s dW_s, \qquad t \le T. \qquad \square$$

4.3 Integral of $\mathbb{L}^2$-processes

We extend the definition of stochastic integral to the class $\mathbb{L}^2$ of progressively measurable and square-integrable processes. Unlike the case of simple processes, the integral will be defined only modulo indistinguishability. Apart from this, all the usual properties in Theorem 4.11 carry over to this case.

To present the general idea, we consider Itô isometry

$$\left\| \int_0^T u_t dW_t \right\|_{L^2(\Omega)} = \|u\|_{L^2([0,T]\times\Omega)}. \tag{4.12}$$

This isometry plays an essential role in the construction of the stochastic integral

$$I_T(u) := \int_0^T u_t dW_t, \tag{4.13}$$

with $u \in \mathbb{L}^2$, since it guarantees that, if (u^n) is a Cauchy sequence in $L^2([0,T]\times\Omega)$, then also $(I_T(u^n))$ is a Cauchy sequence in $L^2(\Omega)$. This fact makes it possible to define the integral in $\mathbb{L}^2$ as soon as we prove that the elements in $\mathbb{L}^2$ can be approximated by simple processes.

Lemma 4.8 *For every $u \in \mathbb{L}^2$ there exists a sequence (u^n) of simple processes in $\mathbb{L}^2$ such that*

$$\lim_{n\to+\infty} E\left[\int_0^T (u_t - u_t^n)^2 dt\right] = \lim_{n\to+\infty} \|u - u^n\|^2_{L^2([0,T]\times\Omega)} = 0.$$

In particular an approximating sequence is defined by

$$u^n = \sum_{k=1}^{2^n-1} \left(\frac{1}{t_k - t_{k-1}} \int_{t_{k-1}}^{t_k} u_s ds \right) \mathbb{1}_{]t_k,t_{k+1}]}, \tag{4.14}$$

where $t_k := \frac{kT}{2^n}$ for $0 \le k \le 2^n$: for this sequence we also have

$$\|u^n\|_{L^2([0,T]\times\Omega)} \le \|u\|_{L^2([0,T]\times\Omega)}.$$

We shall soon prove the lemma in a meaningful particular case (cf. Proposition 4.20 and Remark 4.21): for the general case we refer, for instance, to Steele [315], Theorem 6.5.

Thus we consider a sequence (u^n) of simple processes approximating $u \in \mathbb{L}^2$: since it converges, (u^n) is a Cauchy sequence in $L^2([0,T] \times \Omega)$, that is

$$\lim_{m,n\to\infty} \|u^n - u^m\|_{L^2([0,T]\times\Omega)} = 0.$$

Then, by Itô isometry, the sequence of stochastic integrals $(I_T(u^n))$ is a Cauchy sequence in $L^2(\Omega)$ and therefore it is convergent. It seems natural to define

$$\int_0^T u_t dW_t = \lim_{n\to+\infty} I_T(u^n) \quad \text{in } L^2(\Omega). \tag{4.15}$$

Note that (4.15) defines the stochastic integral only except for a negligible event $N_T \in \mathcal{N}$. This causes problems in defining the integral as a stochastic process, i.e. as T varies. Indeed T belongs to an uncountable set and therefore the previous definition is questionable since the set $\bigcup_{T\ge0} N_T$ might not be measurable, or if it is measurable, it might not have null probability.

On the other hand, this problem can be solved by using Doob's inequality, Theorem 3.38. Indeed, let us consider a sequence (u^n) of *simple* stochastic processes in $\mathbb{L}^2$ approximating u in $L^2([0,T] \times \Omega)$: we put

$$I_t(u^n) = \int_0^t u_s^n dW_s, \qquad t \in [0,T]. \tag{4.16}$$

By (4.11) we obtain

$$[\![I(u^n) - I(u^m)]\!]_T \le 2\|u^n - v^n\|_{L^2([0,T]\times\Omega)},$$

and so $(I(u^n))$ is a Cauchy sequence in $\left(\mathscr{M}_c^2, [\![\cdot]\!]_T\right)$ that is a complete space by Lemma 3.43. So there exists $I(u) \in \mathscr{M}_c^2$, unique up to indistinguishability, such that

$$\lim_{n\to\infty} [\![I(u) - I(u^n)]\!]_T = 0. \tag{4.17}$$

We observe that $I(u)$ does not depend on the approximating sequence, i.e. if v^n is another sequence of simple processes in $\mathbb{L}^2$ approximating u, we have

$$\begin{aligned}[\![I(u^n) - I(v^n)]\!]_T &\le 2\|u^n - v^n\|_{L^2([0,T]\times\Omega)} \\ &\le 2\|u^n - u\|_{L^2([0,T]\times\Omega)} + 2\|u - v^n\|_{L^2([0,T]\times\Omega)} \longrightarrow 0\end{aligned}$$

as $n \to \infty$.

Definition 4.9 *The stochastic integral of $u \in \mathbb{L}^2$ is defined (up to indistinguishability) by* (4.17)*, that is*

$$\int_0^t u_s dW_s := \lim_{n\to\infty} \int_0^t u_s^n dW_s \quad \text{in } \mathscr{M}_c^2,$$

where (u^n) is a sequence of simple processes, approximating u in $\mathbb{L}^2$.

Remark 4.10 Just as in classical functional analysis it is common practice to identify functions that are equal almost everywhere (cf., for example, Brezis [62] Chapter 4) in what follows *we will identify indistinguishable stochastic processes.* □

The following result is the natural extension of Theorem 4.5.

Theorem 4.11 *For every $u, v \in \mathbb{L}^2$, $\alpha \in \mathbb{R}$ and $0 \le a < b < c$, we have:*

(1) linearity:

$$\int_0^a (\alpha u_t + \beta v_t) dW_t = \alpha \int_0^a u_t dW_t + \beta \int_0^a v_t dW_t;$$

(2) additivity:

$$\int_a^c u_t dW_t = \int_a^b u_t dW_t + \int_b^c u_t dW_t;$$

(3) null expectation:

$$E\left[\int_a^b u_t dW_t \mid \mathcal{F}_a\right] = 0,$$

and also

$$E\left[\int_a^b u_t dW_t \int_b^c v_t dW_t \mid \mathcal{F}_a\right] = 0;$$

(4) Itô isometry:

$$E\left[\int_a^b u_t dW_t \int_a^b v_t dW_t \mid \mathcal{F}_a\right] = E\left[\int_a^b u_t v_t dt \mid \mathcal{F}_a\right];$$

(5) the process

$$X_t = \int_0^t u_s dW_s, \qquad t \in [0, T], \tag{4.18}$$

belongs to the space $\mathscr{M}_c^2$ *and we have*

$$[\![X]\!]_T^2 \leq 4E\left[\int_0^T u_t^2 dt\right]. \tag{4.19}$$

As in Remark 4.6 the "non-conditional versions" of the identities in (3) and (4) hold.

Proof. The theorem can be proved by taking the limit in the analogous relations that hold for the integral of simple stochastic processes: the details are left as an exercise. □

Remark 4.12 An immediate but important consequence of the estimate (4.19) is that *if* $u, v \in \mathbb{L}^2$ *are* $(m \otimes P)$*-equivalent (or, in particular, if they are modifications) then their stochastic integrals coincide.* This is a fundamental consistency property of the integral (recall Example 3.27). The converse is true as well, by Corollary 4.13 below. □

Corollary 4.13 *If* $u \in \mathbb{L}^2$ *and for a fixed positive* T *we have*

$$\int_0^T u_t dW_t = 0,$$

then u *is* $(m \otimes P)$*-equivalent to the null process on* $[0, T] \times \Omega$*, that is*

$$\{(t, \omega) \in [0, T] \times \Omega \mid u_t(\omega) \neq 0\}$$

has null $(m \otimes P)$*-measure.*

Proof. The thesis follows from Itô isometry, since we have

$$0 = E\left[\left(\int_0^T u_t dW_t\right)^2\right] = E\left[\int_0^T u_t^2 dt\right]. \qquad \square$$

We wish to point out that the stochastic integral is not defined pathwise and the value of the integral in $\omega \in \Omega$ does not only depend on the paths $u(\omega)$ and $W(\omega)$ but on the entire processes u and W. For this reason the following "identity principle" for the stochastic integral will be useful later on:

Corollary 4.14 *Let* $F \in \mathcal{F}$ *and let* $u, v \in \mathbb{L}^2$ *be modifications on* F*, i.e.* $u_t(\omega) = v_t(\omega)$ *for almost all* $\omega \in F$ *and for every* $t \in [0, T]$*. If*

$$X_t = \int_0^t u_s dW_s, \qquad Y_t = \int_0^t v_s dW_s,$$

then X *and* Y *are indistinguishable on* F*.*

Proof. Let us consider the approximation by simple processes u^n, v^n in $\mathbb{L}^2$ defined in (4.14). By construction u^n and v^n are modifications on F for every n. Hence it follows directly that, if

$$X_t^n = \int_0^t u_s^n dW_s, \qquad Y_t^n = \int_0^t v_s^n dW_s,$$

then X^n and Y^n are modifications on F for every n.

Now, for fixed $t \in]0, T]$, we have that X_t^n, Y_t^n converge in $L^2(\Omega, P)$-norm (and pointwise a.s. after taking a subsequence) to X_t and Y_t, respectively. Therefore $X_t = Y_t$ a.s. in F and this proves that they are modifications in F. The claim follows from Proposition 3.25, since X and Y are continuous processes. □

Example 4.15 Let us consider a process of the form

$$S_t = S_0 + \int_0^t \mu(s)ds + \int_0^t \sigma(s)\, dW_s$$

where $S_0 \in \mathbb{R}$ and $\mu, \sigma \in L^2([0, T])$ are *deterministic functions*. By the previous theorem, we have

$$E[S_t] = S_0 + \int_0^t \mu(s)ds$$

and

$$\operatorname{var}(S_t) = E\left[\left(S_t - S_0 - \int_0^t \mu(s)ds\right)^2\right] =$$

(by Itô isometry)

$$= \int_0^t \sigma(s)^2 ds.$$

We will see later on that S_t *has normal distribution*: we shall prove this stronger result only after proving the Itô formula (cf. Proposition 5.13). □

Exercise 4.16 Under the hypotheses of Theorem 4.11, prove that, for every σ-algebra $\mathcal{G} \subseteq \mathcal{F}_a$, we have

$$E\left[\int_a^b u_t dW_t \mid \mathcal{G}\right] = \int_a^b E[u_t \mid \mathcal{G}]\, dW_t.$$

4.3.1 Itô and Riemann-Stieltjes integral

In this section we show that, in the case of continuous processes, the stochastic integral is the limit of Riemann sums and so it is the natural extension of the Riemann-Stieltjes integral.

Definition 4.17 *A process X is called L^2-continuous at t_0 if*

$$\lim_{t\to t_0} E\left[(X_t - X_{t_0})^2\right] = 0.$$

Example 4.18 Given $u \in \mathbb{L}^2$, the process

$$X_t = \int_0^t u_s dW_s, \qquad t \geq 0,$$

is L^2-continuous at every point. Indeed, if $t > t_0$,

$$E\left[(X_t - X_{t_0})^2\right] = E\left[\left(\int_{t_0}^t u_s dW_s\right)^2\right] =$$

(by Itô isometry)

$$= \int_{t_0}^t E\left[u_s^2\right] ds \longrightarrow 0, \qquad \text{as } t \to t_0,$$

by Lebesgue's dominated convergence theorem and the case $t < t_0$ is analogous. In particular every Brownian motion is L^2-continuous. □

Example 4.19 Let X be a continuous process such that $|X_t| \leq Y$ a.s. with $Y \in L^2(\Omega)$. Then, as an immediate consequence of the dominated convergence theorem, the process X is L^2-continuous at any point. In particular, if X is continuous and f is a bounded continuous function, then $f(X)$ is L^2-continuous. □

Proposition 4.20 *Let $u \in \mathbb{L}^2$ be an L^2-continuous process on $[0,T]$. If we put*

$$u^{(\varsigma)} = \sum_{k=1}^N u_{t_{k-1}} \mathbb{1}_{]t_{k-1},t_k]},$$

where $\varsigma = \{t_0, t_1, \dots, t_N\}$ is a partition of $[0,T]$, then $u^{(\varsigma)}$ is a simple process in $\mathbb{L}^2$ and we have

$$\lim_{|\varsigma|\to 0^+} u^{(\varsigma)} = u, \qquad \text{in } L^2([0,T]\times\Omega). \tag{4.20}$$

Proof. For every $\varepsilon > 0$, there exists[2] $\delta_\varepsilon > 0$ such that, if $|\varsigma| < \delta_\varepsilon$, then we have

$$\int_0^T E\left[\left(u_t - u_t^{(\varsigma)}\right)^2\right] dt = \sum_{k=1}^N \int_{t_{k-1}}^{t_k} E\left[(u_t - u_{t_{k-1}})^2\right] dt \leq \varepsilon T. \qquad \square$$

[2] By the Heine-Cantor theorem, if X is L^2-continuous on the compact set $[0,T]$, then it is also uniformly L^2-continuous.

Remark 4.21 Proposition 4.20 states that $u^{(\varsigma)}$ is a simple stochastic process in $\mathbb{L}^2$ approximating u in $L^2([0,T]\times\Omega)$ for $|\varsigma|\to 0^+$. Then by definition we have

$$\lim_{|\varsigma|\to 0^+}\int_0^T u_t^{(\varsigma)}dW_t = \int_0^T u_t dW_t, \qquad \text{in } \mathscr{M}_c^2,$$

or equivalently

$$\lim_{|\varsigma|\to 0^+}\sum_{k=1}^N u_{t_{k-1}}(W_{t_k}-W_{t_{k-1}}) = \int_0^T u_t dW_t, \qquad \text{in } \mathscr{M}_c^2. \tag{4.21}$$

In this sense the Itô integral, being the limit of Riemann-Stieltjes sums as in (4.1), generalizes the Riemann-Stieltjes integral. □

4.3.2 Itô integral and stopping times

Some properties of the stochastic integral are similar to those of the Lebesgue integral, even though in general it is necessary to be careful: for example, let us consider the following (false) equality

$$X\int_0^T u_t dW_t = \int_0^T X u_t dW_t,$$

where $u\in\mathbb{L}^2$ and X is a $\mathcal{F}_{t_0}$-measurable random variable for some $t_0>0$. Although X is *constant with respect to the variable* t, the member on the right-hand side of the equality does not make sense since the integrand $Xu\notin\mathbb{L}^2$ and is not in general adapted. However, the equality

$$X\int_{t_0}^T u_t dW_t = \int_{t_0}^T X u_t dW_t \tag{4.22}$$

holds true, since (4.22) is true for every simple process u in $\mathbb{L}^2$ and can be proved in general by approximation.

The following result contains the definition of stochastic integral with *a random time as upper integration limit:* the statement might seem tautological but, in the light of the previous remark, it requires a rigorous proof.

Proposition 4.22 *Given $u\in\mathbb{L}^2(\mathcal{F}_t)$, we set*

$$X_t = \int_0^t u_s dW_s, \qquad t\in[0,T]. \tag{4.23}$$

If τ is an $(\mathcal{F}_t)$-stopping time such that $0\le\tau\le T$ a.s. then $\left(u_t\mathbb{1}_{\{t\le\tau\}}\right)\in\mathbb{L}^2$ and

$$X_\tau = \int_0^\tau u_s dW_s = \int_0^T u_s\mathbb{1}_{\{s\le\tau\}}dW_s \quad a.s. \tag{4.24}$$

Proof. It is clear that, by definition of stopping time, the process $\left(u_t \mathbb{1}_{\{t\le\tau\}}\right)$ belongs to $\mathbb{L}^2$ and in particular is adapted. We put

$$Y = \int_0^T u_s \mathbb{1}_{\{s\le\tau\}} dW_s,$$

and we prove that

$$X_\tau = Y \quad \text{a.s.}$$

First of all, we consider the case

$$\tau = \sum_{k=1}^n t_k \mathbb{1}_{F_k} \tag{4.25}$$

with $0 < t_1 < \cdots < t_n = T$ and $F_k \in \mathcal{F}_{t_k}$ disjoint events such that

$$F := \bigcup_{k=1}^n F_k \ \in \mathcal{F}_0.$$

It is apparent that τ is a stopping time. Given X in (4.23), we have $X_\tau = 0$ on $\Omega \setminus F$ and

$$X_\tau = \int_0^T u_s dW_s - \int_{t_k}^T u_s dW_s, \qquad \text{on } F_k,$$

or, in other terms,

$$X_\tau = \mathbb{1}_F \int_0^T u_s dW_s - \sum_{k=1}^n \mathbb{1}_{F_k} \int_{t_k}^T u_s dW_s.$$

On the other hand, we have

$$Y = \int_0^T u_s \left(1 - \mathbb{1}_{\{s>\tau\}}\right) dW_s =$$

(by linearity)

$$\begin{aligned} &= \int_0^T u_s dW_s - \int_0^T u_s \left(\mathbb{1}_{\Omega\setminus F} + \sum_{k=1}^n \mathbb{1}_{F_k} \mathbb{1}_{\{s>t_k\}}\right) dW_s \\ &= \mathbb{1}_F \int_0^T u_s dW_s - \sum_{k=1}^n \int_{t_k}^T u_s \mathbb{1}_{F_k} dW_s, \end{aligned}$$

and we conclude that $X_\tau = Y$ by (4.22). To use (4.22) we have written the integral from 0 to t as the difference of the integral from 0 to T and the integral from t to T.

In the case of a general stopping time τ, we adapt the approximation result of Remark 3.55 and we consider the following decreasing sequence (τ_n) of stopping times of the form (4.25):

$$\tau_n = \sum_{k=0}^{2^n} \frac{T(k+1)}{2^n} \mathbb{1}_{\left\{\frac{Tk}{2^n} < \tau \le \frac{T(k+1)}{2^n}\right\}}.$$

We have that (τ_n) converges to τ a.s. and, by continuity, X_{τ_n} converges to X_τ a.s. Further, if we put

$$Y^n = \int_0^t u_s \mathbb{1}_{\{s \le \tau_n\}} dW_s,$$

by the dominated convergence theorem, we have that Y^n converges to Y in $L^2(\Omega, P)$ and this is enough to conclude. □

The following proposition extends the usual properties of the Itô integral when the integration limit is a stopping time.

Corollary 4.23 *Let $t_0 \in [0, T[$ and $\tau \in [t_0, T]$ be a stopping time. If $u, v \in \mathbb{L}^2$ then we have*

$$E\left[\int_{t_0}^{\tau} u_t dW_t \mid \mathcal{F}_{t_0}\right] = 0,$$
$$E\left[\int_{t_0}^{\tau} u_t dW_t \int_{\tau}^{T} v_t dW_t \mid \mathcal{F}_{t_0}\right] = 0,$$
$$E\left[\int_{t_0}^{\tau} u_t dW_t \int_{t_0}^{\tau} v_t dW_t \mid \mathcal{F}_{t_0}\right] = E\left[\int_{t_0}^{\tau} u_t v_t dt \mid \mathcal{F}_{t_0}\right].$$

Proof. By (4.24) we have

$$\int_{t_0}^{\tau} u_t dW_t = \int_{t_0}^{T} u_t \mathbb{1}_{\{t \le \tau\}} dW_t$$

with $u_t \mathbb{1}_{\{t \le \tau\}} \in \mathbb{L}^2$ and so the claim follows from Theorem 4.11. □

4.3.3 Quadratic variation process

In Theorem 3.74 we computed the quadratic variation of a Brownian motion W, showing that

$$\langle W \rangle_t = t, \quad t \ge 0.$$

On the other hand, in Proposition 3.37 we proved that

$$M_t = W_t^2 - \langle W \rangle_t \tag{4.26}$$

is a martingale. Since W_t^2 is a sub-martingale (cf. Remark 3.36), this result is in line with the Doob's decomposition Theorem A.119 that states that, in discrete time, any sub-martingale can be decomposed as the sum of a martingale M and an increasing predictable process A with null initial value. Thus, in the Brownian framework, (4.26) can be interpreted as a Doob-type decomposition where the role of the process A is played by the quadratic variation $\langle W \rangle$.

In this section we aim at getting similar results for the stochastic integral process

$$X_t = \int_0^t u_s dW_s, \tag{4.27}$$

with $u \in \mathbb{L}^2$. We already proved that $X \in \mathscr{M}_c^2$. Now we introduce the quadratic variation process $\langle X \rangle$ and show that $X^2 - \langle X \rangle$ is a martingale.

Proposition 4.24 *Let X be as in* (4.27) *with $u \in \mathbb{L}^2$. Then for any $t > 0$, there exists the limit*

$$\lim_{\substack{|\varsigma| \to 0 \\ \varsigma \in \mathcal{P}_{[0,t]}}} \sum_{k=1}^{N} \left| X_{t_k} - X_{t_{k-1}} \right|^2 = \int_0^t u_s^2 ds \qquad in\ L^2(\Omega, P). \tag{4.28}$$

We set

$$\langle X \rangle_t = \int_0^t u_s^2 ds, \qquad t \in [0, T], \tag{4.29}$$

and we say that $\langle X \rangle$ is the quadratic variation process of X. We have that $X^2 - \langle X \rangle$ is a martingale.

Proof. If u is a simple $\mathbb{L}^2$-process, (4.28) can be proved by proceeding as in Theorem 3.74. In general the claim follows approximating X by integrals of simple processes.

Next we verify that $X^2 - \langle X \rangle$ is a martingale. For every $0 \le s < t$ we have

$$E\left[X_t^2 - \langle X \rangle_t \mid \mathcal{F}_s\right] = E\left[(X_t - X_s)^2 + 2X_s(X_t - X_s) + X_s^2 - \langle X \rangle_t \mid \mathcal{F}_s\right] =$$

(by (3) in Theorem 4.11)

$$= E\left[(X_t - X_s)^2 - \langle X \rangle_t \mid \mathcal{F}_s\right] + X_s^2 =$$

(by Itô isometry)

$$= E\left[\int_s^t u_\tau^2 d\tau - \langle X \rangle_t \mid \mathcal{F}_s\right] + M_s^2 = M_s^2 - \langle X \rangle_s. \qquad \square$$

Remark 4.25 Since the L^2-convergence implies convergence in probability (cf. Theorem A.136), the limit in (4.28) converges in probability as well. Moreover, by Theorem A.136, we also have that, for any sequence of partitions (ς_n) of $[0,t]$, with mesh converging to zero, there exists a subsequence (ς_{k_n}) such that

$$\lim_{n\to\infty} V_t^{(2)}(X,\varsigma_{k_n}) = \int_0^t u_s^2 ds \qquad \text{a.s.}$$

where $V_t^{(2)}$ is the quadratic variation of Definition 3.72. □

Proposition 4.24 is a particular case of the classical Doob-Meyer decomposition theorem which we state below: the interested reader can find an organic presentation of the topic, for example, in Chapter 1.4 of Karatzas-Shreve [201].

In what follows, $(\Omega, \mathcal{F}, P, (\mathcal{F}_t))$ is a filtered probability space verifying the usual hypotheses. We recall that a process A is *increasing* if almost all the paths of A are increasing functions. Moreover if $M \in \mathscr{M}^2$ then, by Jensen's inequality, $|M|^2$ is a sub-martingale.

Theorem 4.26 (Doob-Meyer decomposition theorem) *For every $M = (M_t)_{t\in[0,T]} \in \mathscr{M}_c^2(\mathcal{F}_t)$ there exists a unique (up to indistinguishability) increasing continuous process A such that $A_0 = 0$ a.s. and $|M|^2 - A$ is a $\mathcal{F}_t$-martingale. We call A the quadratic-variation process of M and we write $A_t = \langle M\rangle_t$. Moreover, for any $t \le T$ we have*

$$A_t = \lim_{\substack{|\varsigma|\to 0\\ \varsigma\in\mathcal{P}_{[0,t]}}} V_t^{(2)}(M,\varsigma) \tag{4.30}$$

in probability.

We explicitly remark that the general definition of quadratic variation agrees with that given in (4.29): indeed, for X as in (4.27), $\langle X\rangle$ in (4.29) is an increasing continuous process such that $\langle X\rangle_0 = 0$ a.s. and $|X|^2 - \langle X\rangle$ is a martingale (cf. Proposition 4.24).

It is remarkable that $\langle M\rangle$ in (4.30) does not depend on the filtration that we consider: in the case $(\mathcal{F}_t)$ is the Brownian filtration, the martingale representation Theorem 10.11 states that *any square-integrable $(\mathcal{F}_t)$-martingale can be represented as a stochastic integral of the form* (4.29)*;* as a consequence, in this particular case Theorem 4.26 follows by Proposition 4.24.

The proof of Theorem 4.26 is based on a discrete approximation procedure: we observe that, if (M_n) is a real discrete martingale, then the process (A_n) defined by $A_0 = 0$ and

$$A_n = \sum_{k=1}^n (M_k - M_{k-1})^2, \qquad n \ge 1,$$

is increasing and such that $M^2 - A$ is a martingale. Indeed

$$E\left[M_{n+1}^2 - A_{n+1} \mid \mathcal{F}_n\right] = M_n^2 - A_n$$

if and only if

$$E\left[M_{n+1}^2 - (M_{n+1} - M_n)^2 \mid \mathcal{F}_n\right] = M_n^2,$$

hence the claim.

The proof of (4.30) is similar to that of Theorem 3.74 and it is based on the fact that the mean of the product of increments of a martingale M over non-overlapping intervals is equal to zero[3]. More precisely, in the scalar case, for $0 \le s < t \le u < v$ we have

$$E\left[(M_v - M_u)(M_t - M_s)\right] = E\left[E\left[(M_v - M_u) \mid \mathcal{F}_u\right](M_t - M_s)\right] = 0. \quad (4.31)$$

Formula (4.31) is very simple yet useful and meaningful: for instance, (4.31) is one of the key ingredients in the construction of the stochastic integral for a general martingale.

Given $M \in \mathscr{M}_c^2$, as a consequence of Theorem 4.26, we also have that

$$E\left[|M_t|^2 - |M_s|^2 \mid \mathcal{F}_s\right] = E\left[\langle M\rangle_t - \langle M\rangle_s \mid \mathcal{F}_s\right], \qquad s \le t, \quad (4.32)$$

that follows from the fact that $|M|^2 - \langle M\rangle$ is a martingale.

4.3.4 Martingales with bounded variation

As a consequence of the Doob-Meyer Theorem 4.26 we have that if a martingale $M \in \mathscr{M}_c^2$ has bounded variation, then it is indistinguishable from the null process: this means that almost all the paths of a non-trivial martingale M are irregular in the sense that they do not have bounded variation. More precisely, we have:

Proposition 4.27 *Let $M \in \mathscr{M}_c^2$. For almost any ω such that $\langle M\rangle_T(\omega) > 0$, the function $t \mapsto M_t(\omega)$ does not have bounded variation over $[0,T]$. Moreover, for almost any ω such that $\langle M\rangle_T(\omega) = 0$ the function $t \mapsto M_t(\omega)$ is null.*

Proof. By Theorem 4.26 there exists a sequence of partitions (ς_n) in $\mathcal{P}_{[0,T]}$, with mesh converging to zero, such that

$$\langle M\rangle_T = \lim_{n\to\infty} V_T^{(2)}(M, \varsigma_n) \qquad \text{a.s.}$$

Thus, by Proposition 3.73, the condition $\langle M\rangle_T(\omega) > 0$ is a.s. incompatible with the fact that $M(\omega)$ has bounded variation.

Concerning the second part of the claim, we set

$$\tau = \inf\{t \mid \langle M\rangle_t > 0\} \wedge T.$$

By Theorem 3.52, τ is a stopping time and since $M^2 - \langle M\rangle$ is a martingale, then, by Theorem 3.58, also[4]

$$M_{t\wedge\tau}^2 - \langle M\rangle_{t\wedge\tau} = M_{t\wedge\tau}^2$$

---

[3] For further details see, for example, Karatzas-Shreve [201], Chapter 1.5.

[4] The equality follows from the fact that $\langle M\rangle_t = 0$ for $t \le \tau$.

is a martingale. Therefore

$$E\left[M^2_{T\wedge\tau}\right] = E\left[M^2_0\right] = 0.$$

Consequently, by Doob's inequality, $(M^2_{t\wedge\tau})$ has a.s. null paths and the claim follows from the fact that $M = (M^2_{t\wedge\tau})_{t\in[0,T]}$ over $\{\langle M\rangle_T = 0\}$. □

4.3.5 Co-variation process

For the sake of simplicity, in this section we consider only real-valued processes. We remark that, by Theorem 4.26, for any $X, Y \in \mathscr{M}^2_c$ the processes

$$(X+Y)^2 - \langle X+Y\rangle, \qquad (X-Y)^2 - \langle X-Y\rangle$$

are martingales and therefore so is the following process, obtained as their difference,

$$4XY - (\langle X+Y\rangle - \langle X-Y\rangle).$$

This motivates the following:

Definition 4.28 *For any $X, Y \in \mathscr{M}^2_c$, the process*

$$\langle X, Y\rangle := \frac{1}{4}\left(\langle X+Y\rangle - \langle X-Y\rangle\right)$$

is called co-variation process of X and Y.

By Theorem 4.26, $\langle X, Y\rangle$ is the unique (up to indistinguishability) continuous adapted process with bounded variation[5] such that $\langle X, Y\rangle_0 = 0$ a.s. and $XY - \langle X, Y\rangle$ is a continuous martingale. Moreover, for any $t \le T$ we have

$$\langle X, Y\rangle_t = \lim_{\substack{|\varsigma|\to 0\\ \varsigma\in\mathcal{P}_{[0,t]}}} \sum_{k=1}^{N} (X_{t_k} - X_{t_{k-1}})(Y_{t_k} - Y_{t_{k-1}})$$

in probability. Note that $\langle X, X\rangle = \langle X\rangle$ and the following identity (that extends (4.32)) holds:

$$\begin{aligned} E\left[(X_t - X_s)(Y_t - Y_s) \mid \mathcal{F}_s\right] &= E\left[X_tY_t - X_sY_s \mid \mathcal{F}_s\right] \\ &= E\left[\langle X, Y\rangle_t - \langle X, Y\rangle_s \mid \mathcal{F}_s\right], \end{aligned}$$

for every $X, Y \in \mathscr{M}^2_c$ and $0 \le s < t$. In the following proposition we collect other straightforward properties of the co-variation process.

[5] A process has bounded variation if almost all its paths are functions with bounded variation.

Proposition 4.29 *The co-variation $\langle\cdot,\cdot\rangle$ is a bi-linear form in $\mathscr{M}_c^2$: for every $X, Y, Z \in \mathscr{M}_c^2$, $\lambda, \mu \in \mathbb{R}$ we have*

i) $\langle X, Y\rangle = \langle Y, X\rangle$;
ii) $\langle \lambda X + \mu Y, Z\rangle = \lambda\langle X, Z\rangle + \mu\langle Y, Z\rangle$;
iii) $|\langle X, Y\rangle|^2 \le \langle X\rangle\langle Y\rangle$.

Example 4.30 A particularly important case is when

$$X_t = \int_0^t u_s dW_s, \qquad Y_t = \int_0^t v_s dW_s,$$

with $u, v \in \mathbb{L}^2$. Then, proceeding as in Proposition 4.24, we can show that

$$X_t Y_t - \int_0^t u_s v_s dW_s$$

is a martingale and therefore[6]

$$\langle X, Y\rangle_t = \int_0^t u_s v_s dW_s, \qquad t \in [0, T], \tag{4.33}$$

is the quadratic variation process of X, Y. Proceeding as in Theorem 3.74, we can also directly prove that

$$\int_0^t u_s v_s dW_s = \lim_{\substack{|\varsigma|\to 0\\ \varsigma\in\mathcal{P}_{[0,t]}}} \sum_{k=1}^N (X_{t_k} - X_{t_{k-1}})(Y_{t_k} - Y_{t_{k-1}})$$

where the limit is in $L^2(\Omega, P)$-norm and therefore also in probability. □

Next we recall that, by Proposition 3.79, if X is a continuous process and Y is a process with bounded variation then

$$\lim_{\substack{|\varsigma|\to 0\\ \varsigma\in\mathcal{P}_{[0,t]}}} \sum_{k=1}^N \left(X_{t_k}(\omega) - X_{t_{k-1}}(\omega)\right)\left(Y_{t_k}(\omega) - Y_{t_{k-1}}(\omega)\right) = 0$$

for any $t \le T$ and $\omega \in \Omega$. Hence, if Z and V are continuous processes with bounded variation and $X, Y \in \mathscr{M}_c^2$, we formally have

$$\langle X + Z, Y + V\rangle = \langle X, Y\rangle + \underbrace{\langle Z, Y + V\rangle + \langle X + Z, V\rangle}_{=0}.$$

Therefore it seems natural to extend Definition 4.28 as follows:

[6] Note also that the process

$$I_t = \int_0^t u_s v_s dW_s, \qquad t \in [0, T],$$

has bounded variation in view of Example 3.60-iii) and $I_0 = 0$.

Definition 4.31 *Let Z and V be continuous processes with bounded variation and $X, Y \in \mathscr{M}^2_{\rm c}$. We call*

$$\langle X + Z, Y + V\rangle := \langle X, Y\rangle$$

the co-variation process of $X + Z$ and $Y + V$.

Example 4.32 We go back to Example 4.15 and consider

$$S_t = S_0 + \int_0^t \mu(s)ds + \int_0^t \sigma(s)\, dW_s$$

with $\mu, \sigma \in L^2([0,T])$ *deterministic functions*. We proved that

$$\mathrm{var}(S_t) = \int_0^t \sigma(s)^2 ds.$$

Now we observe that the process $S_0 + \int_0^t \mu(s)ds$ is continuous and has bounded variation by Example 3.60-iii). Therefore, according to Definition 4.31 and formula (4.29), we have

$$\langle S\rangle_t = \mathrm{var}(S_t), \qquad t \in [0,T],$$

i.e. the quadratic variation process is deterministic and equal to the variance function. □

4.4 Integral of $\mathbb{L}^2_{\rm loc}$-processes

In this paragraph we further extend the class of processes for which the stochastic integral is defined. This generalization is necessary because simple processes like $f(W_t)$, where f is a continuous function, do not generally belong to $\mathbb{L}^2$: indeed we have

$$E\left[\int_0^T f(W_t)dt\right] = \frac{1}{\sqrt{2\pi t}}\int_0^T \int_{\mathbb{R}} \exp\left(-\frac{x^2}{2t}\right) f(x)dxdt.$$

Then, for example, $f(W_t) \notin \mathbb{L}^2$ if $f(x) = e^{x^4}$. Luckily it is not difficult to extend the construction of the Itô integral to a class of progressively measurable processes that verify an integrability condition that is weaker than in Definition 4.1-ii) and that is sufficiently general to handle most applications. However, when this generalization is made, some important properties are lost: in particular the stochastic integral is not in general a martingale.

Definition 4.33 *We denote by $\mathbb{L}^2_{\rm loc}$ the family of processes $(u_t)_{t\in[0,T]}$ that are progressively measurable with respect to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ and such that*

$$\int_0^T u_t^2\, dt < \infty \quad a.s. \tag{4.34}$$

Example 4.34 Every stochastic process that is progressively measurable and has a.s. continuous paths belongs to $\mathbb{L}^2_{\mathrm{loc}}$. In particular $\exp(W_t^4)$, where W is a Brownian motion, belongs to $\mathbb{L}^2_{\mathrm{loc}}$. □

It is interesting to note that the space $\mathbb{L}^2_{\mathrm{loc}}$ is invariant with respect to changes of equivalent probability measures: if (4.34) holds and $Q \sim P$ then we have of course

$$\int_0^T u_t^2\, dt < \infty, \qquad Q\text{-a.s.}$$

On the contrary, the space $\mathbb{L}^2$ depends on the fixed probability measure.

Now we define the stochastic integral $u \in \mathbb{L}^2_{\mathrm{loc}}$ step by step: the rest of the paragraph can be skipped on first reading.

I) Given $u \in \mathbb{L}^2_{\mathrm{loc}}$, the process[7]

$$A_t = \int_0^t u_s^2 ds, \qquad t \in [0,T],$$

is continuous and adapted to the filtration. Indeed it is enough to observe that u can be approximated pointwise by a sequence of simple and adapted processes.

II) For every $n \in \mathbb{N}$ we put

$$\tau_n = \inf\{t \in [0,T] \mid A_t \geq n\} \wedge T.$$

By Theorem 3.52, τ_n is a stopping time and

$$\tau_n \nearrow T \qquad \text{a.s. as } n \to \infty.$$

We have

$$F_n := \{\tau_n = T\} = \{A_T \leq n\}, \tag{4.35}$$

and so, since $u \in \mathbb{L}^2_{\mathrm{loc}}$,

$$\bigcup_{n \in \mathbb{N}} F_n = \Omega \setminus N, \qquad N \in \mathcal{N}. \tag{4.36}$$

III) We put

$$u_t^n = u_t \mathbb{1}_{\{t \leq \tau_n\}}, \qquad t \in [0,T],$$

and note that $u^n \in \mathbb{L}^2$ since

$$E\left[\int_0^T (u_t^n)^2\, dt\right] = E\left[\int_0^{\tau_n} u_t^2 dt\right] \leq n.$$

Therefore the process

$$X_t^n = \int_0^t u_t^n dW_t, \qquad t \in [0,T] \tag{4.37}$$

is well-defined and $X^n \in \mathscr{M}_c^2$.

---

[7] We put $A(\omega) = 0$ if $u(\omega) \notin L^2(0,T)$.

IV) For every $n, h \in \mathbb{N}$, we have $u^n = u^{n+h} = u$ on F_n in (4.35). So, by Corollary 4.14, the processes X^n and X^{n+h} are indistinguishable on F_n. Recalling that (F_n) is an increasing sequence for which (4.36) holds, the following definition is well-posed.

Definition 4.35 *Given $u \in \mathbb{L}^2_{\mathrm{loc}}$, let F_n and X^n be defined as in* (4.35) *and* (4.37)*, respectively. Then the stochastic integral of u is the continuous and $\mathcal{F}_t$-adapted stochastic process X that is indistinguishable from X^n on F_n, for every $n \in \mathbb{N}$. We write*

$$X_t = \int_0^t u_s dW_s, \qquad t \in [0, T].$$

Note that, by construction, we have

$$X_t = \lim_{n\to\infty} \int_0^t u_t^n dW_t, \qquad t \in [0, T], \text{ a.s.} \tag{4.38}$$

Remark 4.36 Given $p \geq 1$, we denote by $\mathbb{L}^p_{\mathrm{loc}}$ the family of progressively measurable processes $(u_t)_{t\in[0,T]}$ such that

$$\int_0^T |u_t|^p dt < \infty \quad \text{a.s.} \tag{4.39}$$

By Hölder's inequality we have

$$\mathbb{L}^p_{\mathrm{loc}} \subseteq \mathbb{L}^q_{\mathrm{loc}}, \qquad p \geq q \geq 1,$$

and in particular $\mathbb{L}^2_{\mathrm{loc}} \subseteq \mathbb{L}^1_{\mathrm{loc}}$. Since $\mathbb{L}^p_{\mathrm{loc}}$ depends on the filtration $(\mathcal{F}_t)$, whenever it is necessary we write more explicitly $\mathbb{L}^p_{\mathrm{loc}}(\mathcal{F}_t)$. The space $\mathbb{L}^2_{\mathrm{loc}}$ is the natural setting for the definition of stochastic integral: we refer to Steele [315], Paragraph 7.3, for an interesting discussion about the impossibility of defining the Itô integral of $u \in \mathbb{L}^p_{\mathrm{loc}}$ for $1 \leq p < 2$. □

4.4.1 Local martingales

In general, the stochastic integral of a process $u \in \mathbb{L}^2_{\mathrm{loc}}$ *is not a martingale:* however, in the sense that we are going to explain, it is not "far off to be a martingale".

Definition 4.37 *A process $M = (M_t)_{t\in[0,T]}$ is a $\mathcal{F}_t$-local martingale if there exists an increasing sequence (τ_n) of $\mathcal{F}_t$-stopping times, called localizing sequence for M, such that*

$$\lim_{n\to\infty} \tau_n = T \quad a.s. \tag{4.40}$$

and, for every $n \in \mathbb{N}$, the stochastic process $M_{t\wedge\tau_n}$ is a $\mathcal{F}_t$-martingale. We denote by $\mathscr{M}_{\mathrm{c,loc}}$ the space of continuous local martingales such that $M_0 = 0$ a.s.

To put it simply, a local martingale is a stochastic process that can be approximated by a sequence of true martingales. Sometimes, when we want to emphasize the fact that a process M is a true martingale and not simply a local martingale, we say that M is a *strict martingale*. An interesting example of a local martingale that is not a strict martingale is given in Example 9.34.

By definition, we have that

$$M_{s\wedge\tau_n} = E\left[M_{t\wedge\tau_n} \mid \mathcal{F}_s\right], \qquad 0 \le s \le t \le T, \tag{4.41}$$

and if M is continuous, since $\tau_n \to T$ a.s., we have

$$\lim_{n\to\infty} M_{t\wedge\tau_n} = M_t \quad \text{a.s.}$$

Consequently, whenever we can take the limit inside the conditional expectation in (4.41), we have that M is a strict martingale: as particular cases, see Propositions 4.39 and 4.40 below.

Clearly every martingale is also a local martingale: it is enough to choose $\tau_n = T$ for every n. Further, we remark that every local martingale admits a right-continuous modification: indeed it is enough to note that, by Theorem 3.41, this holds true for the stopped processes $M_{t\wedge\tau_n}$. In what follows we shall always consider the right-continuous version of every local martingale.

Remark 4.38 Every continuous local martingale M admits an approximating sequence of continuous and *bounded* martingales. Indeed let (τ_n) be a localizing sequence for M and let us put

$$\sigma_n = \inf\{t \in [0,T] \mid |M_t| \ge n\} \wedge T, \qquad n \in \mathbb{N}.$$

Since M is continuous we have that σ_n satisfies (4.40) and also $(\tau_n \wedge \sigma_n)$ is a localizing sequence for M: indeed

$$M_{t\wedge(\tau_n\wedge\sigma_n)} = M_{(t\wedge\tau_n)\wedge\sigma_n}$$

and so, by Doob's Theorem 3.58, $M_t^n := M_{t\wedge(\tau_n\wedge\sigma_n)}$ is a bounded martingale such that

$$|M_t^n| \le n, \qquad t \in [0,T].$$

□

We present now some simple properties of continuous local martingales.

Proposition 4.39 *If* $M \in \mathscr{M}_{\mathrm{c,loc}}$ *and*

$$\sup_{t\in[0,T]} |M_t| \in L^1(\Omega, P),$$

then M *is a martingale. In particular every bounded*[8] $M \in \mathscr{M}_{\mathrm{c,loc}}$ *is a martingale.*

[8] There exists a constant c such that $|M_t| \le c$ a.s. for every $t \in [0,T]$.

Proof. The claim follows directly from (4.41), applying the dominated convergence theorem for conditional expectation. □

Proposition 4.40 *Every continuous non-negative local martingale M is also a super-martingale. Further, if*

$$E[M_T] = E[M_0] \tag{4.42}$$

then $(M_t)_{0\le t\le T}$ is a martingale.

Proof. Applying Fatou's lemma for conditional expectation to (4.41), we get

$$M_s \ge E[M_t \mid \mathcal{F}_s], \qquad 0 \le s \le t \le T, \tag{4.43}$$

and this proves the first part of the claim.

By taking the expectation in the previous relation we get

$$E[M_0] \ge E[M_t] \ge E[M_T], \qquad 0 \le t \le T.$$

By assumption (4.42), we infer that $E[M_t] = E[M_0]$ for every $t \in [0,T]$. Eventually, by (4.43), if we had $M_s > E[M_t \mid \mathcal{F}_s]$ on an event of strictly positive probability, then we would get a contradiction. □

Proposition 4.41 *If $M \in \mathscr{M}_{\rm c,loc}$ and τ is a stopping time, then also $M_{t\wedge\tau} \in \mathscr{M}_{\rm c,loc}$.*

Proof. If (τ_n) is a localizing sequence for M and $X_t = M_{t\wedge\tau}$, we have

$$X_{t\wedge\tau_n} = M_{(t\wedge\tau)\wedge\tau_n} = M_{(t\wedge\tau_n)\wedge\tau}.$$

Consequently, by Theorem 3.58 and since by assumption $M_{t\wedge\tau_n}$ is a continuous martingale, we have that (τ_n) is a localizing sequence for X. □

4.4.2 Localization and quadratic variation

The following theorem states that the stochastic integral of a process $u \in \mathbb{L}^2_{\rm loc}$ is a continuous local martingale. In the whole section we use the notations

$$X_t = \int_0^t u_s dW_s, \qquad A_t = \int_0^t u_s^2 ds, \qquad t \in [0,T]. \tag{4.44}$$

Theorem 4.42 *We have:*

i) if $u \in \mathbb{L}^2$, then $X \in \mathscr{M}^2_{\rm c}$;
ii) if $u \in \mathbb{L}^2_{\rm loc}$, then $X \in \mathscr{M}_{\rm c,loc}$ and a localizing sequence for X is given by

$$\tau_n = \inf\{t \in [0,T] \mid A_t \ge n\} \wedge T, \qquad n \in \mathbb{N}. \tag{4.45}$$

Proof. We only prove *ii*). We saw at the beginning of Paragraph 4.4 that (τ_n) in (4.45) is an increasing sequence of stopping times such that $\tau_n \to T$ a.s. for $n \to \infty$.

By Definition 4.35, on $F_k = \{A_T \le k\}$ with $k \ge n$, we have

$$X_{t\wedge\tau_n} = \int_0^{t\wedge\tau_n} u_s \mathbb{1}_{\{s\le\tau_k\}} dW_s =$$

(by Proposition 4.22, since $u_s \mathbb{1}_{\{s\le\tau_k\}} \in \mathbb{L}^2$)

$$= \int_0^t u_s \mathbb{1}_{\{s\le\tau_k\}} \mathbb{1}_{\{s\le\tau_n\}} dW_s =$$

(since $n \le k$)

$$= \int_0^t u_s \mathbb{1}_{\{s\le\tau_n\}} dW_s, \qquad \text{on } F_k.$$

By the arbitrariness of k and by (4.36), we get

$$X_{t\wedge\tau_n} = \int_0^t u_s \mathbb{1}_{\{s\le\tau_n\}} dW_s, \qquad t \in [0,T], \text{ a.s.} \tag{4.46}$$

The claim follows from the fact that $u_s \mathbb{1}_{\{s\le\tau_n\}} \in \mathbb{L}^2$ and so $X_{t\wedge\tau_n} \in \mathscr{M}_c^2$ and τ_n is a localizing sequence for X. □

Next we extend Proposition 4.24.

Proposition 4.43 *Given $u \in \mathbb{L}^2_{\text{loc}}$, let X and A be the processes in* (4.44). *Then $X^2 - A$ is a continuous local martingale: A is called quadratic variation process of X and we write $A = \langle X \rangle$.*

Proof. Let us consider the localizing sequence (τ_n) for X defined in Theorem 4.42. We proved that (cf. (4.46))

$$X_{t\wedge\tau_n} = \int_0^t u_s \mathbb{1}_{\{s\le\tau_n\}} dW_s$$

with $u_s \mathbb{1}_{\{s\le\tau_n\}} \in \mathbb{L}^2$. Therefore, by Proposition 4.24, we have that the following process is a martingale:

$$X^2_{t\wedge\tau_n} - \int_0^t u_s^2 \mathbb{1}_{\{s\le\tau_n\}} ds = X^2_{t\wedge\tau_n} - A_{t\wedge\tau_n} = \left(X^2 - A\right)_{t\wedge\tau_n}.$$

Hence $X^2 - A$ is a local martingale and τ_n is a localizing sequence for X. □

Proposition 4.43 has the following extension: for every $X, Y \in \mathscr{M}_{\text{c,loc}}$ there exists a unique (up to indistinguishability) continuous process $\langle X, Y \rangle$ with bounded variation, such that $\langle X, Y \rangle_0 = 0$ a.s. and

$$XY - \langle X, Y \rangle \in \mathscr{M}_{\text{c,loc}}.$$

We call $\langle X, Y \rangle$ the *co-variation process of* X, Y. Note that $\langle X \rangle = \langle X, X \rangle$.

Remark 4.44 If

$$X_t = \int_0^t u_s dW_s, \qquad Y_t = \int_0^t v_s dW_s,$$

with $u, v \in \mathbb{L}^2_{\rm loc}$, then

$$\langle X, Y\rangle_t = \int_0^t u_s v_s ds. \qquad \square$$

More generally, by analogy with Definition 4.31, we give the following:

Definition 4.45 *Let Z and V be continuous processes with bounded variation and $X, Y \in \mathscr{M}_{\rm c,loc}$. We call*

$$\langle X + Z, Y + V\rangle := \langle X, Y\rangle \tag{4.47}$$

the co-variation process of $X + Z$ and $Y + V$. In (4.47), $\langle X, Y\rangle$ is the unique (up to indistinguishability) continuous process with bounded variation, such that $\langle X, Y\rangle_0 = 0$ a.s. and $XY - \langle X, Y\rangle \in \mathscr{M}_{\rm c,loc}$.

Proposition 4.27 can be extended as follows:

Proposition 4.46 *Let $M \in \mathscr{M}_{\rm c,loc}$. For almost any ω such that $\langle M\rangle_T(\omega) > 0$, the function $t \mapsto M_t(\omega)$ does not have bounded variation over $[0, T]$. Moreover, for almost any ω such that $\langle M\rangle_T(\omega) = 0$ the function $t \mapsto M_t(\omega)$ is null.*

We conclude the paragraph stating[9] a classical result that claims that, for every $M \in \mathscr{M}_{\rm c,loc}$, the expected values

$$E\left[\langle M\rangle_T^p\right] \qquad \text{and} \qquad E\left[\sup_{t\in[0,T]} |M_t|^{2p}\right]$$

are comparable, for $p > 0$. More precisely, we have

Theorem 4.47 (Burkholder-Davis-Gundy's inequalities) *For any $p > 0$ there exist two positive constants λ_p, Λ_p such that*

$$\lambda_p E\left[\langle M\rangle_\tau^p\right] \le E\left[\sup_{t\in[0,\tau]} |M_t|^{2p}\right] \le \Lambda_p E\left[\langle M\rangle_\tau^p\right],$$

for every $M \in \mathscr{M}_{\rm c,loc}$ and stopping time τ.

As a consequence of Theorem 4.47 we prove a useful criterion to establish whether a stochastic integral of a process in $\mathbb{L}^2_{\rm loc}$ is a martingale.

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[9] For the proof we refer, for example, to Theorem 3.3.28 in [201].

Corollary 4.48 *If* $u \in \mathbb{L}^2_{\mathrm{loc}}$ *and*

$$E\left[\left(\int_0^T u_t^2 dt\right)^{\frac{1}{2}}\right] < \infty, \tag{4.48}$$

then the process

$$\int_0^t u_s dW_s, \qquad t \in [0,T],$$

is a martingale.

Proof. First of all we observe that, by Hölder's inequality, we have

$$E\left[\left(\int_0^T u_t^2 dt\right)^{\frac{1}{2}}\right] \le E\left[\int_0^T u_t^2 dt\right]^{\frac{1}{2}},$$

and so condition (4.48) is weaker than the integrability condition in the space $\mathbb{L}^2$.

By the second Burkholder-Davis-Gundy's inequality with $p = \frac{1}{2}$ and $\tau = T$, we have

$$E\left[\sup_{t\in[0,T]}\left|\int_0^t u_s dW_s\right|\right] \le \Lambda_{\frac{1}{2}} E\left[\left(\int_0^T u_t^2 dt\right)^{\frac{1}{2}}\right] < \infty$$

and so the claim follows from Proposition 4.39. □