1

Derivatives and arbitrage pricing

A financial derivative is a contract whose value depends on one or more securities or assets, called underlying assets. Typically the underlying asset is a stock, a bond, a currency exchange rate or the quotation of commodities such as gold, oil or wheat.

1.1 Options

An option is the simplest example of a derivative instrument. An option is a contract that gives the right (but not the obligation) to its holder to buy or sell some amount of the underlying asset at a future date, for a prespecified price. Therefore in an option contract we need to specify:

- an underlying asset;
- an exercise price K, the so-called *strike price*;
- a date T, the so-called *maturity*.

A *Call* option gives the right to buy, whilst a *Put* option gives the right to sell. An option is called *European* if the right to buy or sell can be exercised only at maturity, and it is called *American* if it can be exercised at any time before maturity.

Let us consider a European Call option with strike K, maturity T and let us denote the price of the underlying asset at maturity by S_T. At time T we have two possibilities (cf. Figure 1.1): if $S_T > K$, the payoff of the option is equal to $S_T - K$, corresponding to the profit obtained by exercising the option (i.e. by buying the underlying asset at price K and then selling it at the market price S_T). If $S_T < K$, exercising the option is not profitable and the payoff is zero. In conclusion the payoff of a European Call option is

$$(S_T - K)^+ = \max\{S_T - K, 0\}.$$

Figure 1.2 represents the graph of the payoff as a function of S_T: notice that the payoff increases with S_T and gives a potentially unlimited profit. Analo-

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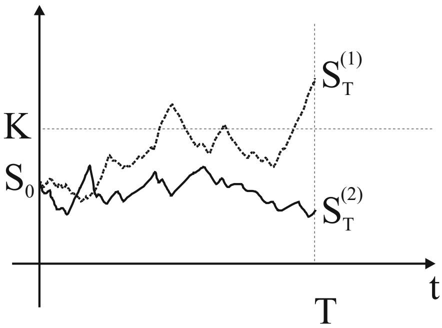

Fig. 1.1. Different scenarios for a European Call option

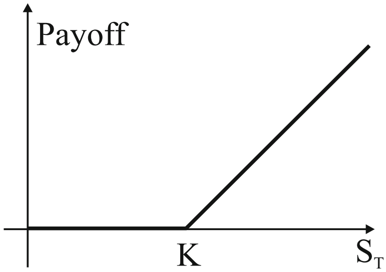

Fig. 1.2. Payoff of a European Call option

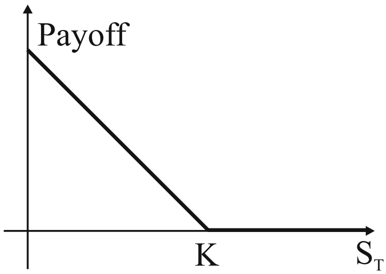

Fig. 1.3. Payoff of a European Put option

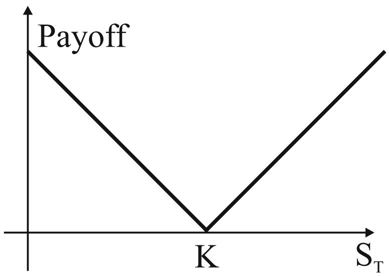

Fig. 1.4. Payoff of a Straddle

gously, we see that the payoff of a European Put option is

$$(K - S_T)^+ = \max\{K - S_T, 0\}.$$

Call and Put options are the basic derivative instruments and for this reason they are often called *plain vanilla* options. Combining such types of options it is possible to build new derivatives: for example, by buying a Call and a Put option with the same underlying asset, strike and maturity we obtain a derivative, the so-called Straddle, whose payoff increases the more S_T is far from the strike. This kind of derivative is interesting when one expects a wide movement of the price of the underlying asset without being able to foresee the direction. Evidently the pricing of this option can be reformulated in terms of the pricing of plain vanilla options. On the other hand, in the real-world markets there exists a great deal of derivatives (usually called *exotic*) having very complicated structures: the market of such derivatives is in continuous expansion and development. One can consult, for example, Zhang [344] for an encyclopedic exposition of exotic derivatives.

1.1.1 Main purposes

The use of derivatives serves mainly two purposes:

- hedging the risk;
- speculation.

For example, let us consider an investor holding the stock S: buying a Put option on S, the investor gets the right to sell S in the future at the strike price and therefore he/she hedges the risk of a crash of the price of S. Analogously, a firm using oil in its business might purchase a Call option to have the right to buy oil in the future at the fixed strike price: in this way the firm hedges the risk of a rise of the price of oil.

In recent years the use of derivatives has become widespread: not long ago a home loan was available only with fixed or variable rate, while now the offer is definitely wider. For example, it is not hard to find “protected” loans with capped variable rate: this kind of structured products contains one or more derivative instruments and pricing such objects is not really straightforward.

Derivatives can be used to speculate as well: for instance, buying Put options is the simplest way to get a profit in case of a market crash. We also remark that options have a so-called *leverage effect*: relatively minor movements in stock price can result in a huge change of the option price. For example, let us denote by S_0 the current price of the underlying asset and let us suppose that \$1 is the price of a Call option with $K = S_0 = \$10$ and maturity one year. We suppose that, at maturity, $S_T = \$13$: if we buy one unit of the underlying asset, i.e. we invest \$10, we would have a \$3 profit (i.e. 30%); if we buy a Call option, i.e. we invest only \$1, we would have a \$2 profit (i.e. 200%). On the other hand, we must also bear in mind that, if $S_T = \$10$, by investing in the Call option we would lose all our money!

1.1.2 Main problems

An option is a contract whose final value is given, this depending on the price of the underlying asset at maturity which is not known at present. Therefore the non-trivial problem of *pricing* arises, i.e. the determination of the "rational" or fair price of the option: this price is the *premium* that the buyer of the option has to pay at the initial time to get the right guaranteed by the contract.

The second problem is that of *hedging*: we have already pointed out that a Call option has a potentially unlimited payoff and consequently the institution that *sells* a Call option exposes itself to the risk of a potentially unlimited loss. A bank selling a derivative faces therefore the problem of finding an investment strategy that, by using the premium (i.e. the money received when the derivative was sold), can replicate the payoff at maturity, whatever the final value of the underlying asset will be. As we are going to see shortly, the problems of pricing and hedging are deeply connected.

1.1.3 Rules of compounding

Before going any further, it is good to recall some notions on the *time value of money* in finance: receiving \$1 today is not like receiving it after a month. We point out also that it is common practice to consider as the unit of time one year and so, for example, $T = 0.5$ corresponds to six months.

The rules of compounding express the dynamics of an investment with fixed risk-free interest rate: to put it simply, this corresponds to deposit the money on a savings account. In the financial modeling, it is always assumed that a (locally[1]) risk-free asset, the so-called *bond*, exists. If B_t denotes the value of the bond at time $t \in [0, T]$, the following rule of simple compounding with annual interest rate r

$$B_T = B_0(1 + rT),$$

states that the final value B_T is equal to the initial value B_0 plus the interest $B_0 rT$, corresponding to the interest over the period $[0, T]$ accrued on the initial wealth. Therefore, by the rule of simple compounding, the interest is only paid on the initial wealth.

Alternatively we may consider the period $[0, T]$, divide it into N sub-intervals $[t_{n-1}, t_n]$ whose common length is $\frac{T}{N}$ and assume that the simple interest is paid at the end of every sub-interval: we get

$$B_T = B_{t_{N-1}} \left(1 + r\frac{T}{N}\right) = B_{t_{N-2}} \left(1 + r\frac{T}{N}\right)^2 = \cdots = B_0 \left(1 + r\frac{T}{N}\right)^N.$$

[1] This means that the official interest rate is fixed and risk-free over a brief period of time (e.g. some weeks) but in the long term it is random as well.

By taking the limit as $N \to \infty$, i.e. by assuming that the simple interest is paid more and more frequently, we obtain the formula of continuous compounding with annual interest rate r:

$$B_T = B_0 \, e^{rT}. \tag{1.1}$$

Formula (1.1) expresses the final wealth in terms of the initial investment. Conversely, since to obtain a final wealth (at time T) equal to B, it is necessary to invest the amount Be^{-rT} at the initial time, this amount is usually called *discounted value* of B.

While the rule of simple compounding is the one used in the market, the rule of continuous compounding is generally used in theoretical contexts and particularly in continuous-time models.

1.1.4 Arbitrage opportunities and Put-Call parity formula

Broadly speaking an arbitrage opportunity is the possibility of carrying out a financial operation without any investment, but leading to profit without any risk of a loss. In real-world markets arbitrage opportunities do exist, even though their life span is very brief: as soon as they arise, the market will reach a new equilibrium because of the actions of those who succeed in exploiting such opportunities. From a theoretical point of view it is evident that a sensible market model must avoid this type of profit. As a matter of fact, the no-arbitrage principle has become one of the main criteria to price financial derivatives.

The idea on which arbitrage pricing is built is that, if two financial instruments will *certainly* have the same value[2] at future date, then also in this moment they must have the same value. If this were not the case, an obvious arbitrage opportunity would arise: by selling the instrument that is more expensive and by buying the less expensive one, we would have an immediate risk-free profit since the selling position *(short position)* on the more more expensive asset is going to cancel out the buying position *(long position)* on the cheaper asset. Concisely, we can express the no-arbitrage principle in the following way:

$$X_T \leq Y_T \qquad \Longrightarrow \qquad X_t \leq Y_t, \quad t \leq T, \tag{1.2}$$

where X_t and Y_t are the values of the two financial instruments respectively. From (1.2) in particular it follows that

$$X_T = Y_T \qquad \Longrightarrow \qquad X_t = Y_t, \quad t \leq T. \tag{1.3}$$

Now let us consider a financial-market model that is free from arbitrage opportunities and consists of a bond and a stock S, that is the underlying asset

[2] We note that we need not know the future values of the two financial instruments, but only that they will certainly be equal.

of a Call option c and of a Put option p, both of European type with maturity T and strike K:

$$c_T = (S_T - K)^+, \qquad p_T = (K - S_T)^+.$$

We denote by r the risk-free interest rate and we assume that the bond follows the dynamics given by (1.1). On the basis of arbitrage arguments, we get the classical *Put-Call parity formula*, which establishes a relation between the prices c and p, and some upper and lower estimates for such prices. It is remarkable that the following formulas are “universal”, i.e. independent of the market model and based only on the general no-arbitrage principle.

Corollary 1.1 (Put-Call parity) *Under the previous assumptions, we have*

$$c_t = p_t + S_t - Ke^{-r(T-t)}, \qquad t \in [0, T]. \tag{1.4}$$

Proof. It suffices to note that the investments

$$X_t = c_t + \frac{K}{B_T} B_t \quad \text{and} \quad Y_t = p_t + S_t,$$

have the same final value

$$X_T = Y_T = \max\{K, S_T\}.$$

The claim follows from (1.3). □

If the underlying asset pays a dividend D at a date between t and T, the Put-Call parity formula becomes

$$c_t = p_t + S_t - D - Ke^{-r(T-t)}.$$

Corollary 1.2 (Estimates from above and below for European options) *For every $t \in [0, T]$ we have*

$$\begin{aligned} \left(S_t - Ke^{-r(T-t)}\right)^+ &< c_t < S_t, \\ \left(Ke^{-r(T-t)} - S_t\right)^+ &< p_t < Ke^{-r(T-t)}. \end{aligned} \tag{1.5}$$

Proof. By (1.2)

$$c_t, p_t > 0. \tag{1.6}$$

Consequently by (1.4) we get

$$c_t > S_t - Ke^{-r(T-t)}.$$

Moreover, since $c_t > 0$, we get the first estimate from below. Finally $c_T < S_T$ and so by (1.2) we get the first estimate from above. The second estimate can be proved analogously and it is left as an exercise. □

1.2 Risk-neutral price and arbitrage pricing

In order to illustrate the fundamental ideas of derivative pricing by arbitrage arguments, it is useful to examine a simplified model in which we consider only two moments in time, the initial date 0 and the maturity T. As usual we assume that there exists a bond with risk-free rate r and initial value $B_0 = 1$. Further, we assume that there is a risky asset S whose final value depends on some random event: to consider the simplest possible model, we assume that the event can assume only two possible states E_1 and E_2 in which S_T takes the values S^+ and S^- respectively. To fix the ideas, let us consider the outcome of a throw of a die and let us put, for example,

$$E_1 = \{1, 2, 3, 4\}, \qquad E_2 = \{5, 6\}.$$

In this case S represents a bet on the outcome of a throw of a die: if we get a number between 1 and 4 the bet pays S^+, otherwise it pays S^-. The model can be summarized by the following table:

Time	0	T
Bond	1	e^{rT}
Risky asset	?	$S_T = \begin{cases} S^+ & \text{if } E_1, \\ S^- & \text{if } E_2. \end{cases}$

The problem is to determine the value S_0, i.e. the price of the bet.

1.2.1 Risk-neutral price

The first approach is to assign a probability to the events:

$$P(E_1) = p \quad \text{and} \quad P(E_2) = 1 - p, \tag{1.7}$$

where $p \in]0, 1[$. For example, if we roll a die it seems natural to set $p = \frac{4}{6}$. In this way we can have an estimate of the final average value of the bet

$$S_T = pS^+ + (1 - p)S^-.$$

By discounting that value at the present time, we get the so-called *risk-neutral price*:

$$\widetilde{S}_0 = e^{-rT}\left(pS^+ + (1 - p)S^-\right). \tag{1.8}$$

This price expresses the value that a risk-neutral investor assigns to the risky asset (i.e. the bet): indeed the current price is equal to the future discounted expected profit. On the basis of this pricing rule (that depends on the probability p of the event E_1), the investor is neither inclined nor adverse to buy the asset.

1.2.2 Risk-neutral probability

Let us suppose now that S_0 is the price given by the market and therefore it is a known quantity. The fact that S_0 is observable gives information on the random event that we are considering. Indeed by imposing that $S_0 = \widetilde{S}_0$, i.e. that the risk-neutral pricing formula holds with respect to some probability defined in terms of $q \in]0,1[$ as in (1.7), we have

$$S_0 = e^{-rT}\left(qS^+ + (1-q)S^-\right),$$

whence we get

$$q = \frac{e^{rT}S_0 - S^-}{S^+ - S^-}, \qquad 1-q = \frac{S^+ - e^{rT}S_0}{S^+ - S^-}. \tag{1.9}$$

Evidently $q \in]0,1[$ if and only if

$$S^- < e^{rT}S_0 < S^+,$$

and, on the other hand, if this were not the case, obvious arbitrage opportunities would arise. The probability defined in (1.9) is called *risk-neutral probability and it represents the unique probability to be assigned to the events* E_1, E_2 *so that* S_0 *is a risk-neutral price.*

Therefore, in this simple setting there exists a bijection between prices and risk-neutral probabilities: by calculating the probabilities of the events, we determine a "rational" price for the risky asset; conversely, given a market price, there exists a unique probability of events that is consistent with the observed price.

1.2.3 Arbitrage price

Let us suppose now that there are two risky assets S and C, both depending on the same random event:

Time	0	T
Bond	1	e^{rT}
Risky asset S	S_0	$S_T = \begin{cases} S^+ & \text{if } E_1, \\ S^- & \text{if } E_2, \end{cases}$
Risky asset C	?	$C_T = \begin{cases} C^+ & \text{if } E_1, \\ C^- & \text{if } E_2. \end{cases}$

To fix the ideas, we can think of C as an option with underlying the risky asset S. If the price S_0 is quoted by the market, we can infer the corresponding risk-neutral probability q defined as in (1.9) and then find the neutral-risk price of C under the probability q:

$$\widetilde{C}_0 = e^{-rT}\left(qC^+ + (1-q)C^-\right). \tag{1.10}$$

This pricing procedure seems reasonable and consistent with the market price of the underlying asset. We emphasize the fact that *the price* $\widetilde{C}_0$ *in* (1.10) *does not depend on a subjective estimation of the probabilities of the events* E_1, E_2*, but it is implicitly contained in the quoted market value of the underlying asset.* In particular this pricing method *does not require to estimate in advance the probability of random events.* We say that $\widetilde{C}_0$ is the *risk-neutral price* of the derivative C.

An alternative approach is based upon the assumption of absence of arbitrage opportunities. We recall that the two main problems of the theory and practice of derivatives are pricing and hedging. Let us suppose to be able to determine an investment strategy on the riskless asset and on the risky asset S replicating the payoff of C. If we denote the value of this strategy by V, the replication condition is

$$V_T = C_T. \tag{1.11}$$

From the no-arbitrage condition (1.3) it follows that

$$C_0 = V_0$$

is the only price guaranteeing the absence of arbitrage opportunities. In other terms, in order to price correctly (without giving rise to arbitrage opportunities) a financial instrument, it suffices to *determine an investment strategy with the same final value* (payoff): by definition, the *arbitrage price* of the financial instrument is the current value of the replicating strategy. This price can be interpreted also as the premium that the bank receives by selling the derivative and this amount coincides with the wealth to be invested in the replicating portfolio.

Now let us show how to construct a replicating strategy for our simple model. We consider a portfolio which consists in holding a number α of shares of the risky asset and a number β of bonds. The value of such a portfolio is given by

$$V = \alpha S + \beta B.$$

By imposing the replicating condition (1.11) we have

$$\begin{cases} \alpha S^+ + \beta e^{rT} = C^+ & \text{if } E_1, \\ \alpha S^- + \beta e^{rT} = C^- & \text{if } E_2, \end{cases}$$

which is a linear system, with a unique solution under the assumption $S^+ \neq S^-$. The solution of the system is

$$\alpha = \frac{C^+ - C^-}{S^+ - S^-}, \qquad \beta = e^{-rT}\frac{S^+C^- - C^+S^-}{S^+ - S^-};$$

therefore the arbitrage price is equal to

$$C_0 = \alpha S_0 + \beta = S_0 \frac{C^+ - C^-}{S^+ - S^-} + e^{-rT} \frac{S^+ C^- - C^+ S^-}{S^+ - S^-}$$
$$= e^{-rT} \left(C^+ \frac{e^{rT} S_0 - S^-}{S^+ - S^-} + C^- \frac{S^+ - e^{rT} S_0}{S^+ - S^-} \right) =$$

(recalling the expression (1.9) of the risk-neutral probability)

$$= e^{-rT} \left(C^+ q + C^- (1 - q) \right) = \widetilde{C}_0,$$

where $\widetilde{C}_0$ is the risk-neutral price in (1.10). The results obtained so far can be expressed in this way: *in an arbitrage-free and complete market (i.e. in which every financial instrument is replicable) the arbitrage price and the risk-neutral price coincide: they are determined by the quoted price S_0, observable on the market.*

In particular *the arbitrage price does not depend on the subjective estimation of the probability p of the event E_1*. Intuitively, the choice of p is bound to the subjective vision on the future behaviour of the risky asset: the fact of choosing p equal to 50% or 99% is due to different estimations on the events E_1, E_2. As we have seen, different choices of p determine different prices for S and C on the basis of formula (1.8) of risk-neutral valuation. Nevertheless, the only choice of p that is consistent with the market price S_0 is that corresponding to $p = q$ in (1.9). Such a choice is also the only one that avoids the introduction of arbitrage opportunities.

1.2.4 A generalization of the Put-Call parity

Let us consider again a market with two risky assets S and C, but S_0 and C_0 are not quoted:

Time	0	T
Riskless asset	1	e^{rT}
Risky asset S	?	$S_T = \begin{cases} S^+ & \text{if } E_1, \\ S^- & \text{if } E_2, \end{cases}$
Risky asset C	?	$C_T = \begin{cases} C^+ & \text{if } E_1, \\ C^- & \text{if } E_2. \end{cases}$

We consider an investment on the two risky assets

$$V = \alpha S + \beta C$$

and we impose that it replicates at maturity the riskless asset, $V_T = e^{rT}$:

$$\begin{cases} \alpha S^+ + \beta C^+ = e^{rT} & \text{if } E_1, \\ \alpha S^- + \beta C^- = e^{rT} & \text{if } E_2. \end{cases}$$

As we have seen earlier, we obtain a linear system that has a unique solution (provided that C and S do not coincide):

$$\bar{\alpha} = e^{rT} \frac{C^+ - C^-}{C^+ S^- - C^- S^+}, \qquad \bar{\beta} = -e^{rT} \frac{S^+ - S^-}{C^+ S^- - C^- S^+}.$$

By the no-arbitrage condition (1.3), we must have $V_0 = 1$ i.e.

$$\bar{\alpha} S_0 + \bar{\beta} C_0 = 1. \tag{1.12}$$

Condition (1.12) gives a relation between the prices of the two risky assets that must hold in order not to introduce arbitrage opportunities. For fixed S_0, the price C_0 is uniquely determined by (1.12), in line with the results of the previous section. This fact must not come as a surprise: since the two assets “depend” on the same random phenomenon, the relative prices must move consistently.

Formula (1.12) also suggests that *the pricing of a derivative does not necessarily require that the underlying asset is quoted, since we can price a derivative using the quoted price of another derivative on the same underlying asset.* A particular case of (1.12) is the Put-Call parity formula expressing the link between the price of a Call and a Put option on the same underlying asset.

1.2.5 Incomplete markets

Let us go back to the example of die rolling and suppose that the risky assets have final values according to the following table:

Time	0	T
Riskless asset	1	e^{rT}
Risky asset S	S_0	$S_T = \begin{cases} S^+ & \text{if } \{1,2,3,4\}, \\ S^- & \text{if } \{5,6\}, \end{cases}$
Riskless asset C	?	$C_T = \begin{cases} C^+ & \text{if } \{1,2\}, \\ C^- & \text{if } \{3,4,5,6\}. \end{cases}$

Now we set

$$E_1 = \{1,2\}, \qquad E_2 = \{3,4\}, \qquad E_3 = \{5,6\}.$$

If we suppose to be able to assign the probabilities to the events

$$P(E_1) = p_1, \qquad P(E_2) = p_2, \qquad P(E_3) = 1 - p_1 - p_2,$$

where $p_1, p_2 > 0$ and $p_1 + p_2 < 1$, then the risk-neutral prices are defined just as in Section 1.2.1:

$$\begin{aligned}
\widetilde{S}_0 =& e^{-rT}\left(p_1 S^+ + p_2 S^+ + (1 - p_1 - p_2) S^-\right) \\
=& e^{-rT}\left((p_1 + p_2) S^+ + (1 - p_1 - p_2) S^-\right) \\
\widetilde{C}_0 =& e^{-rT}\left(p_1 C^+ + p_2 C^- + (1 - p_1 - p_2) C^-\right) \\
=& e^{-rT}\left(p_1 C^+ + (1 - p_1) C^-\right).
\end{aligned}$$

Conversely, if S_0 is quoted on the market, by imposing $S_0 = \widetilde{S}_0$, we obtain

$$S_0 = e^{-rT}\left(q_1 S^+ + q_2 S^+ + (1 - q_1 - q_2) S^-\right)$$

and so *there exist infinitely many*[3] *risk-neutral probabilities.*

Analogously, by proceeding as in Section 1.2.3 to determine a replicating strategy for C, we obtain

$$\begin{cases} \alpha S^+ + \beta e^{rT} = C^+ & \text{if } E_1, \\ \alpha S^+ + \beta e^{rT} = C^- & \text{if } E_2, \\ \alpha S^- + \beta e^{rT} = C^- & \text{if } E_3. \end{cases} \tag{1.13}$$

In general this system is not solvable and therefore the asset C is not replicable: we say that the market model is *incomplete*. In this case it is not possible to price C on the basis of replication arguments: since we can only solve two out of three equations, we cannot build a strategy replicating C in all the possible cases and *we are able to hedge the risk only partially.*

We note that, if (α, β) solves the first and the third equation of the system (1.13), then the terminal value V_T of the corresponding strategy is equal to

$$V_T = \begin{cases} C^+ & \text{if } E_1, \\ C^+ & \text{if } E_2, \\ C^- & \text{if } E_3. \end{cases}$$

With this choice (and assuming that $C^+ > C^-$) we obtain a strategy that *super-replicates* C.

Summing up:

- *in a market model that is free from arbitrage opportunities and complete, on one hand there exists a unique the risk-neutral probability measure;*

[3] Actually, it is possible to determine a unique risk-neutral probability if we assume that both S_0 and C_0 are observable.

on the other hand, for every derivative there exists a replicating strategy. Consequently there exists a unique risk-neutral price which coincides with the arbitrage price;

- *in a market model that is free from arbitrage opportunities and incomplete, on one hand there exist infinitely many risk-neutral probabilities; on the other hand not every derivative is replicable. Consequently there exist infinitely many risk-neutral prices but it is not possible, in general, to define the arbitrage price.*