7 Coupled FEM–BEM approaches

In this chapter we focus on some procedures for solving eddy current problems that are based on a strategy which couples the finite element method (FEM) and the boundary element method (BEM). This kind of coupling allows the numerical approximation of the solution in unbounded domains, a typical situation in electromagnetism. The boundary element method is used for the approximation in the complement of a bounded domain: either the conductor Ω_C or else an artificial computational domain Ω, containing Ω_C but in general not very large. Instead, in the bounded domain the solution is approximated using the finite element method. Compared with the formulations presented in the previous chapters, the coupled FEM–BEM approaches compute the FEM approximation of the solution in a smaller region (say, the conductor), not required to be so large that the use of homogeneous boundary conditions is justified. This can be done because the BEM method takes into account the behaviour of the solution in the external region.

The idea of coupling a variational approach in one region with a potential approach in another region of the computational domain has been first proposed by engineers for the Laplace operator (see, e.g., Zienkiewicz et al. [248], Jami and Lenoir [138]), and then widely analyzed from the mathematical point of view, starting from the pioneering works of Brezzi and Johnson [66] and Johnson and Nédélec [141]. An important improvement has been furnished by the papers of Costabel [85], [86], that, for elliptic boundary value problems, show how to obtain a symmetric (or else to a positive definite) problem. Extensions to the full Maxwell equations are due to Ammari and Nédélec [25], [26].

Coming to the eddy current problem, the first FEM–BEM couplings have been proposed by Bossavit and Vérité [62], [63] (for the magnetic field, and using the Steklov–Poincaré operator) and Mayergoyz et al. [174] (for the electric field, and using special basis functions near Γ). A more recent result in FEM–BEM coupling, for axisymmetric problems associated to the modeling of induction furnaces, is due to Bermúdez et al. [40].

The approach of Bossavit and Vérité [62], [63] has led them to devise a very popular numerical code, named TRIFOU, that has been often used in engineering com-

A. Alonso Rodríguez, A. Valli: Eddy Current Approximation of Maxwell Equations.

putations. A complete presentation of this approach can be found in Bossavit [59], Sect. 8.2; we describe its basic ideas in Section 7.6.1.

Symmetric formulations à la Costabel have been proposed for eddy current problems by Hiptmair [127] (unknowns: $\mathbf{E}_C$ in Ω_C, $\mathbf{H} \times \mathbf{n}$ on Γ) and Meddahi and Selgas [176] (unknowns: $\mathbf{H}_C$ in Ω_C, $\boldsymbol{\mu}\mathbf{H} \cdot \mathbf{n}$ on Γ), and are briefly presented in Sections 7.6.3 and 7.6.2, respectively.

The chapter begins with Sections 7.1–7.5, where we describe a FEM–BEM formulation proposed by Alonso Rodríguez and Valli [19], based on a vector magnetic potential and a scalar electric potential in the conductor, and on a scalar magnetic potential in the external part. An approach in terms of magnetic vector potentials has been also proposed for magnetostatics by Kuhn et al. [159] and Kuhn and Steinbach [160]; with respect to the choice of potentials, the presentation in Sections 7.1–7.5 is close to these last ones.

The reader mainly interested in numerical approximation and implementation can focus on problems (7.12), (7.30) and (7.31) ($(\mathbf{A}_C, V_C, q)$ formulation), on problem (7.36) (TRIFOU formulation), on problem (7.42) ($(\mathbf{H}_C, \lambda)$ formulation), and on problem (7.52) ($(\mathbf{E}_C, \mathbf{p}_\Gamma)$ formulation).

Let us focus now on a different aspect: not all the known methods devised for studying the Maxwell equations are robust enough to be used, without any modification, for both the time-harmonic case and the static case (namely, the case in which the electric and magnetic inductions are assumed to be time-independent; in other words, in the equations one has to set $\omega = 0$). In Sections 7.1–7.5 we show how one can treat without distinction the cases $\omega \neq 0$ and $\omega = 0$. Moreover, the numerical approximation there proposed is quite simple, since we use standard Lagrange nodal finite elements in the conductor, while a cheap formulation based on boundary elements is proposed in the external insulator.

Being simple, robust and cheap, this method can be therefore a suitable direct solver for some inverse problems in electromagnetism, for instance in electroencephalography (EEG) or magnetoencephalography (MEG) (see Section 9.2). In this respect, though in many papers devoted to these topics only the static case is considered (see, e.g., Sarvas [220], Hämäläinen et al. [117]), recently some researchers have focused on the time-harmonic case, which is a more precise model for describing the electric and magnetic activities in the brain (see Ammari et al. [22]). Clearly, the static case is much easier to solve, as, due to the irrotationality condition, one can reduce the problem to the sole determination of a scalar potential of the electric field in Ω_C (a suitable Neumann condition on Γ is the correct boundary condition to add). However, in no way that simple approach can be extended to the time-harmonic case, as irrotationality no longer holds.

In this chapter the geometrical assumptions on the conductor Ω_C are more restrictive than in the preceding chapters. In fact, we consider a bounded simply-connected open set $\Omega_C \subset \mathbb{R}^3$, with a Lipschitz boundary Γ (for EEG and MEG applications, Ω_C represents the human head). For simplicity, as in the preceding chapters we also assume that $\Omega_I := \mathbb{R}^3 \setminus \overline{\Omega_C}$ is connected, so that Γ is connected, too. The unit outward normal vector on Γ will be denoted by $\mathbf{n}_C = -\mathbf{n}_I$.

As usual, we assume that the electric conductivity $\boldsymbol{\sigma}$ and the magnetic permeability $\boldsymbol{\mu}_C$ are uniformly positive definite symmetric matrices in Ω_C, with entries belonging to $L^\infty(\Omega_C)$. The electric conductivity $\boldsymbol{\sigma}$ and the applied current density $\mathbf{J}_e \in (L^2(\mathbb{R}^3))^3$ are vanishing in Ω_I. Moreover, the magnetic permeability $\boldsymbol{\mu}_I$ and the electric permittivity $\boldsymbol{\varepsilon}_I$ are assumed to be a positive constant in Ω_I, say $\mu_0 > 0$ and $\varepsilon_0 > 0$.

7.1 The $(\mathbf{A}_C, V_C) - \psi_I$ formulation

In the present situation the eddy current problem in terms of the magnetic field $\mathbf{H}$ and the electric field $\mathbf{E}_C$ reads (see (3.25))

$$\begin{cases} \operatorname{curl} \mathbf{E}_C + i\omega\boldsymbol{\mu}_C\mathbf{H}_C = \mathbf{0} & \text{in } \Omega_C \\ \operatorname{curl} \mathbf{H}_C - \boldsymbol{\sigma}\mathbf{E}_C = \mathbf{J}_{e,C} & \text{in } \Omega_C \\ \operatorname{curl} \mathbf{H}_I = \mathbf{0} & \text{in } \Omega_I \\ \operatorname{div}(\mu_0\mathbf{H}_I) = 0 & \text{in } \Omega_I \\ \boldsymbol{\mu}_C\mathbf{H}_C \cdot \mathbf{n}_C + \mu_0\mathbf{H}_I \cdot \mathbf{n}_I = 0 & \text{on } \Gamma \\ \mathbf{H}_C \times \mathbf{n}_C + \mathbf{H}_I \times \mathbf{n}_I = \mathbf{0} & \text{on } \Gamma \\ \mathbf{H}_I(\mathbf{x}) = O(|\mathbf{x}|^{-1}) & \text{as } |\mathbf{x}| \to \infty\,. \end{cases} \tag{7.1}$$

If needed, but here we are not dealing with this aspect, the electric field $\mathbf{E}_I$ can be computed after having determined $\mathbf{H}_I$ and $\mathbf{E}_C$ in (7.1), by solving

$$\begin{cases} \operatorname{curl} \mathbf{E}_I = -i\omega\mu_0\mathbf{H}_I & \text{in } \Omega_I \\ \operatorname{div}(\varepsilon_0\mathbf{E}_I) = 0 & \text{in } \Omega_I \\ \mathbf{E}_I \times \mathbf{n}_I = -\mathbf{E}_C \times \mathbf{n}_C & \text{on } \Gamma \\ \int_\Gamma \varepsilon_0\mathbf{E}_I \cdot \mathbf{n}_I = 0 & \\ \mathbf{E}_I(\mathbf{x}) = O(|\mathbf{x}|^{-1}) & \text{as } |\mathbf{x}| \to \infty\,. \end{cases} \tag{7.2}$$

Since Ω_I is unbounded, note that we have to impose the no-flux condition on Γ, though it is a connected surface.

As proposed by Pillsbury [193], Rodger and Eastham [211], Emson and Simkin [100], we look for a vector magnetic potential $\mathbf{A}_C$, a scalar electric potential V_C and a scalar magnetic potential ψ_I such that

$$\boldsymbol{\mu}_C\mathbf{H}_C = \operatorname{curl} \mathbf{A}_C \ \ , \ \ \mathbf{E}_C = -i\omega\mathbf{A}_C - \operatorname{grad} V_C \ \ , \ \ \mathbf{H}_I = \operatorname{grad} \psi_I\,. \tag{7.3}$$

In this way one has $\operatorname{curl} \mathbf{E}_C = -i\omega \operatorname{curl} \mathbf{A}_C = -i\omega\boldsymbol{\mu}_C\mathbf{H}_C$, and therefore the Faraday equation in Ω_C is satisfied. Note that, in particular, when $\omega = 0$ one finds $\mathbf{E}_C = -\operatorname{grad} V_C$, therefore for the static case the usual formulation in terms of a scalar electric potential is recovered.

As usual, in order to have a unique vector potential $\mathbf{A}_C$, it is necessary to impose some gauge conditions: here we are considering the Coulomb gauge $\operatorname{div} \mathbf{A}_C = 0$ in Ω_C, with $\mathbf{A}_C \cdot \mathbf{n}_C = 0$ on Γ. Moreover, we also impose that

$$|\psi_I(\mathbf{x})| = O(|\mathbf{x}|^{-1}) \quad \text{as } |\mathbf{x}| \to \infty\,.$$

In conclusion, we are left with the problem

$$
\begin{cases}
\operatorname{curl}(\boldsymbol{\mu}_C^{-1} \operatorname{curl} \mathbf{A}_C) \\
\qquad\qquad + i\omega\boldsymbol{\sigma}\mathbf{A}_C + \boldsymbol{\sigma} \operatorname{grad} V_C = \mathbf{J}_{e,C} & \text{in } \Omega_C \\
\Delta\psi_I = 0 & \text{in } \Omega_I \\
\operatorname{div} \mathbf{A}_C = 0 & \text{in } \Omega_C \\
\mathbf{A}_C \cdot \mathbf{n}_C = 0 & \text{on } \Gamma \\
\operatorname{curl} \mathbf{A}_C \cdot \mathbf{n}_C + \mu_0 \operatorname{grad} \psi_I \cdot \mathbf{n}_I = 0 & \text{on } \Gamma \\
(\boldsymbol{\mu}_C^{-1} \operatorname{curl} \mathbf{A}_C) \times \mathbf{n}_C + \operatorname{grad} \psi_I \times \mathbf{n}_I = \mathbf{0} & \text{on } \Gamma \\
|\psi_I(\mathbf{x})| + |\operatorname{grad} \psi_I(\mathbf{x})| = O(|\mathbf{x}|^{-1}) & \text{as } |\mathbf{x}| \to \infty\,,
\end{cases}
\tag{7.4}
$$

where V_C is determined up to an additive constant.

In order to obtain a problem which is stable also in the case $\omega = 0$, and for which Lagrange nodal finite elements can be used for approximation, it is well-known (see, e.g., Coulomb [91], Morisue [180], Bìrò and Preis [49] and Chapter 6) that the Coulomb gauge condition $\operatorname{div} \mathbf{A}_C = 0$ in Ω_C can be incorporated as a penalization term in the Ampère equation. Introducing the constant $\mu_* > 0$, that for physical consistency can be chosen, for example, as a suitable average in Ω_C of the entries of the matrix $\boldsymbol{\mu}_C$, one considers

$$
\begin{cases}
\operatorname{curl}(\boldsymbol{\mu}_C^{-1} \operatorname{curl} \mathbf{A}_C) - \mu_*^{-1} \operatorname{grad} \operatorname{div} \mathbf{A}_C \\
\qquad\qquad + i\omega\boldsymbol{\sigma}\mathbf{A}_C + \boldsymbol{\sigma} \operatorname{grad} V_C = \mathbf{J}_{e,C} & \text{in } \Omega_C \\
\Delta\psi_I = 0 & \text{in } \Omega_I \\
\operatorname{div}(i\omega\boldsymbol{\sigma}\mathbf{A}_C + \boldsymbol{\sigma} \operatorname{grad} V_C) = \operatorname{div} \mathbf{J}_{e,C} & \text{in } \Omega_C \\
(i\omega\boldsymbol{\sigma}\mathbf{A}_C + \boldsymbol{\sigma} \operatorname{grad} V_C) \cdot \mathbf{n}_C = \mathbf{J}_{e,C} \cdot \mathbf{n}_C & \text{on } \Gamma \\
\mathbf{A}_C \cdot \mathbf{n}_C = 0 & \text{on } \Gamma \\
\operatorname{curl} \mathbf{A}_C \cdot \mathbf{n}_C + \mu_0 \operatorname{grad} \psi_I \cdot \mathbf{n}_I = 0 & \text{on } \Gamma \\
(\boldsymbol{\mu}_C^{-1} \operatorname{curl} \mathbf{A}_C) \times \mathbf{n}_C + \operatorname{grad} \psi_I \times \mathbf{n}_I = \mathbf{0} & \text{on } \Gamma \\
|\psi_I(\mathbf{x})| + |\operatorname{grad} \psi_I(\mathbf{x})| = O(|\mathbf{x}|^{-1}) & \text{as } |\mathbf{x}| \to \infty\,,
\end{cases}
\tag{7.5}
$$

the two additional equations appearing in (7.5) being necessary as the modification in the Ampère equation does not ensure now that $\mathbf{E}_C = -i\omega\mathbf{A}_C - \operatorname{grad} V_C$ satisfies the necessary conditions $\operatorname{div}(\boldsymbol{\sigma}\mathbf{E}_C) = -\operatorname{div} \mathbf{J}_{e,C}$ in Ω_C and $\boldsymbol{\sigma}\mathbf{E}_C \cdot \mathbf{n}_C = -\mathbf{J}_{e,C} \cdot \mathbf{n}_C$ on Γ.

Moreover, taking the divergence of $(7.5)_1$ and using $(7.5)_3$, we have $\Delta \operatorname{div} \mathbf{A}_C = 0$ in Ω_C, and, taking the scalar product of $(7.5)_1$ by $\mathbf{n}_C$, using $(7.5)_4$ and $(7.5)_7$, we find

$$
\begin{aligned}
\mu_*^{-1} \operatorname{grad} \operatorname{div} \mathbf{A}_C \cdot \mathbf{n}_C &= \operatorname{curl}(\boldsymbol{\mu}^{-1} \operatorname{curl} \mathbf{A}_C) \cdot \mathbf{n}_C \\
&= \operatorname{div}_\tau[(\boldsymbol{\mu}^{-1} \operatorname{curl} \mathbf{A}_C) \times \mathbf{n}_C] = -\operatorname{div}_\tau(\operatorname{grad} \psi_I \times \mathbf{n}_I) \\
&= -\operatorname{curl} \operatorname{grad} \psi_I \cdot \mathbf{n}_I = 0 \quad \text{on } \Gamma\,.
\end{aligned}
$$

Therefore $\operatorname{div} \mathbf{A}_C$ is constant in Ω_C, and this constant is 0 as a consequence of $(7.5)_5$. In conclusion, any solution to (7.5) satisfies $\operatorname{div} \mathbf{A}_C = 0$ in Ω_C, and thus (7.4) and (7.5) are equivalent.

7.2 The $(\mathbf{A}_C, V_C) - \psi_\Gamma$ weak formulation

In this chapter we have assumed that $\boldsymbol{\mu}_I$ is a positive constant μ_0 and we are looking for a scalar magnetic potential ψ_I. Therefore for determining this potential we have to solve the Laplace equation in Ω_I. This allows us to use potential theory, transforming the problem for ψ_I into a problem on the interface Γ and reducing in a significative way the number of unknowns in numerical computations.

Referring for notation to Section A.1, it is well-known from potential theory (see, e.g., McLean [175], Nédélec [187]) that we can introduce on Γ the single layer and double layer potentials

$$\mathcal{S} : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma) \ , \ \ \mathcal{S}(\xi)(\mathbf{x}) := \int_\Gamma \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}\, \xi(\mathbf{y}) dS_y \tag{7.6}$$

$$\mathcal{D} : H^{1/2}(\Gamma) \to H^{1/2}(\Gamma) \ , \ \ \mathcal{D}(\eta)(\mathbf{x}) := \int_\Gamma \frac{\mathbf{x}-\mathbf{y}}{4\pi|\mathbf{x}-\mathbf{y}|^3} \cdot \eta(\mathbf{y})\mathbf{n}_C(\mathbf{y}) dS_y , \tag{7.7}$$

and the hypersingular integral operator

$$\begin{aligned} &\mathcal{H} : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma) \ , \\ &\qquad \mathcal{H}(\eta)(\mathbf{x}) := -\operatorname{grad}\left(\int_\Gamma \tfrac{\mathbf{x}-\mathbf{y}}{4\pi|\mathbf{x}-\mathbf{y}|^3} \cdot \eta(\mathbf{y})\mathbf{n}_C(\mathbf{y}) dS_y\right) \cdot \mathbf{n}_C(\mathbf{x}) \ . \end{aligned} \tag{7.8}$$

We also recall that the adjoint operator $\mathcal{D}' : H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ reads

$$\mathcal{D}'(\xi)(\mathbf{x}) = \left(\int_\Gamma \frac{\mathbf{y}-\mathbf{x}}{4\pi|\mathbf{x}-\mathbf{y}|^3}\, \xi(\mathbf{y}) dS_y\right) \cdot \mathbf{n}_C(\mathbf{x}) \ . \tag{7.9}$$

Since we have that $\Delta\psi_I = 0$ in Ω_I and $\operatorname{grad}\psi_I \cdot \mathbf{n}_I = -\frac{1}{\mu_0}\operatorname{curl}\mathbf{A}_C \cdot \mathbf{n}_C$ on Γ, a first result is that the trace $\psi_\Gamma := \psi_{I|\Gamma}$ satisfies

$$\frac{1}{2}\psi_\Gamma - \mathcal{D}(\psi_\Gamma) + \frac{1}{\mu_0}\mathcal{S}(\operatorname{curl}\mathbf{A}_C \cdot \mathbf{n}_C) = 0 \ \text{ on } \Gamma \tag{7.10}$$

$$\frac{1}{2\mu_0}\operatorname{curl}\mathbf{A}_C \cdot \mathbf{n}_C + \frac{1}{\mu_0}\mathcal{D}'(\operatorname{curl}\mathbf{A}_C \cdot \mathbf{n}_C) + \mathcal{H}(\psi_\Gamma) = 0 \ \text{ on } \Gamma \tag{7.11}$$

(see, e.g., McLean [175], Nédélec [187]).

As a second step, we can devise a weak formulation in terms of $(\mathbf{A}_C, V_C) - \psi_\Gamma$. A standard integration by parts yields

$$\int_\Gamma \mathbf{n}_I \times \operatorname{grad}\psi_I \cdot \overline{\mathbf{w}_C} = \int_\Gamma \psi_\Gamma \operatorname{curl}\overline{\mathbf{w}_C} \cdot \mathbf{n}_C \ .$$

Moreover, multiplying $(7.5)_1$, $(7.5)_3$ and (7.11) by suitable test functions $(\mathbf{w}_C, Q_C, \eta)$ with $\mathbf{w}_C \cdot \mathbf{n}_C = 0$ on Γ, integrating in Ω_C and Γ, and integrating by parts, from the other matching condition

$$\boldsymbol{\mu}_C^{-1}\operatorname{curl}\mathbf{A}_C \times \mathbf{n}_C + \operatorname{grad}\psi_I \times \mathbf{n}_I = \mathbf{0} \text{ on } \Gamma,$$

and the interface equation (7.10) we end up with the following weak problem

$$
\begin{aligned}
&\int_{\Omega_C} (\boldsymbol{\mu}_C^{-1} \operatorname{curl} \mathbf{A}_C \cdot \operatorname{curl} \overline{\mathbf{w}_C} + \mu_*^{-1} \operatorname{div} \mathbf{A}_C \operatorname{div} \overline{\mathbf{w}_C}) \\
&\qquad + \int_{\Omega_C} (i\omega \boldsymbol{\sigma} \mathbf{A}_C \cdot \overline{\mathbf{w}_C} + \boldsymbol{\sigma} \operatorname{grad} V_C \cdot \overline{\mathbf{w}_C}) \\
&\qquad + \int_\Gamma [-\tfrac{1}{2}\psi_\Gamma - \mathcal{D}(\psi_\Gamma) + \tfrac{1}{\mu_0} \mathcal{S}(\operatorname{curl} \mathbf{A}_C \cdot \mathbf{n}_C)] \operatorname{curl} \overline{\mathbf{w}_C} \cdot \mathbf{n}_C \\
&\quad = \int_{\Omega_C} \mathbf{J}_{e,C} \cdot \overline{\mathbf{w}_C} \\
&\int_{\Omega_C} (i\omega \boldsymbol{\sigma} \mathbf{A}_C \cdot \operatorname{grad} \overline{Q_C} + \boldsymbol{\sigma} \operatorname{grad} V_C \cdot \operatorname{grad} \overline{Q_C}) \\
&\quad = \int_{\Omega_C} \mathbf{J}_{e,C} \cdot \operatorname{grad} \overline{Q_C} \\
&\int_\Gamma [\tfrac{1}{2} \operatorname{curl} \mathbf{A}_C \cdot \mathbf{n}_C + \mathcal{D}'(\operatorname{curl} \mathbf{A}_C \cdot \mathbf{n}_C) + \mu_0 \mathcal{H}(\psi_\Gamma)] \overline{\eta} = 0 \, .
\end{aligned}
$$

We note that, for the ease of notation, as usual here above we have written the integration symbol on Γ instead of the pairing between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$; the same notation will be used in the sequel.

Since the hypersingular operator $\mathcal{H}(\cdot)$ is coercive in the constrained space $H^{1/2}(\Gamma)/\mathbb{C}$, it is convenient to rewrite the preceding problem for test functions $\eta \in H^{1/2}(\Gamma)/\mathbb{C}$, looking for $q \in H^{1/2}(\Gamma)/\mathbb{C}$, which differs from ψ_Γ by an additive constant.

We know that $\mathcal{H}(1) = 0$ and $\mathcal{D}(1) = -\frac{1}{2}$ (see, e.g., McLean [175], Nédélec [187]), and that $\int_\Gamma \mathcal{H}(\eta) = 0$ for each $\eta \in H^{1/2}(\Gamma)$ (see, e.g., Nédélec [187], Theor. 3.3.2). Hence $\mathcal{H}(\psi_\Gamma + c_0) = \mathcal{H}(\psi_\Gamma)$,

$$
-\frac{1}{2}(\psi_\Gamma + c_0) - \mathcal{D}(\psi_\Gamma + c_0) = -\frac{1}{2}\psi_\Gamma - \mathcal{D}(\psi_\Gamma),
$$

and

$$
\begin{aligned}
&\int_\Gamma [\tfrac{1}{2} \operatorname{curl} \mathbf{A}_C \cdot \mathbf{n}_C + \mathcal{D}'(\operatorname{curl} \mathbf{A}_C \cdot \mathbf{n}_C) + \mu_0 \mathcal{H}(\psi_\Gamma)] \\
&\quad = \int_\Gamma [\tfrac{1}{2} \operatorname{curl} \mathbf{A}_C \cdot \mathbf{n}_C + \operatorname{curl} \mathbf{A}_C \cdot \mathbf{n}_C \, \mathcal{D}(1) + \mu_0 \mathcal{H}(\psi_\Gamma)] = 0 \, .
\end{aligned}
$$

In conclusion, introducing the space

$$
W_C := H(\operatorname{curl}; \Omega_C) \cap H_0(\operatorname{div}; \Omega_C)
$$

we are looking for the solution of the following coupled problem

$$
\begin{aligned}
&\text{Find } (\mathbf{A}_C, V_C, q) \in W_C \times H^1(\Omega_C)/\mathbb{C} \times H^{1/2}(\Gamma)/\mathbb{C} \text{ such that} \\
&\quad \int_{\Omega_C} (\boldsymbol{\mu}_C^{-1} \operatorname{curl} \mathbf{A}_C \cdot \operatorname{curl} \overline{\mathbf{w}_C} + \mu_*^{-1} \operatorname{div} \mathbf{A}_C \operatorname{div} \overline{\mathbf{w}_C}) \\
&\qquad\quad + \int_{\Omega_C} (i\omega \boldsymbol{\sigma} \mathbf{A}_C \cdot \overline{\mathbf{w}_C} + \boldsymbol{\sigma} \operatorname{grad} V_C \cdot \overline{\mathbf{w}_C}) \\
&\qquad\quad + \int_\Gamma [-\tfrac{1}{2} q - \mathcal{D}(q) + \tfrac{1}{\mu_0} \mathcal{S}(\operatorname{curl} \mathbf{A}_C \cdot \mathbf{n}_C)] \operatorname{curl} \overline{\mathbf{w}_C} \cdot \mathbf{n}_C \\
&\qquad = \int_{\Omega_C} \mathbf{J}_{e,C} \cdot \overline{\mathbf{w}_C} \\
&\quad \int_{\Omega_C} (i\omega \boldsymbol{\sigma} \mathbf{A}_C \cdot \operatorname{grad} \overline{Q_C} + \boldsymbol{\sigma} \operatorname{grad} V_C \cdot \operatorname{grad} \overline{Q_C}) \\
&\qquad = \int_{\Omega_C} \mathbf{J}_{e,C} \cdot \operatorname{grad} \overline{Q_C} \\
&\quad \int_\Gamma [\tfrac{1}{2} \operatorname{curl} \mathbf{A}_C \cdot \mathbf{n}_C + \mathcal{D}'(\operatorname{curl} \mathbf{A}_C \cdot \mathbf{n}_C) + \mu_0 \mathcal{H}(q)] \overline{\eta} = 0 \\
&\text{for all } (\mathbf{w}_C, Q_C, \eta) \in W_C \times H^1(\Omega_C)/\mathbb{C} \times H^{1/2}(\Gamma)/\mathbb{C} \, .
\end{aligned}
\tag{7.12}
$$

Now we want to prove that from a solution to (7.12) we can construct a solution to the strong problem (7.4). Let us note that the condition $\mathbf{A}_C \cdot \mathbf{n}_C = 0$ on Γ in (7.4) is already contained in the definition of the space W_C.

Lemma 7.1. *Suppose that* $(\mathbf{A}_C, V_C, q)$ *is a solution to* (7.12). *Then* $\operatorname{div} \mathbf{A}_C = 0$ *in* Ω_C.

Proof. Since $\int_{\Omega_C} \operatorname{div} \mathbf{A}_C = \int_\Gamma \mathbf{A}_C \cdot \mathbf{n}_C = 0$, we can consider the solution $v_C \in H^1(\Omega_C)/\mathbb{C}$ to the Neumann problem

$$\begin{cases} \Delta v_C = \operatorname{div} \mathbf{A}_C & \text{in } \Omega_C \\ \operatorname{grad} v_C \cdot \mathbf{n}_C = 0 & \text{on } \Gamma \ . \end{cases}$$

Setting $\mathbf{w}_C = \operatorname{grad} v_C$, clearly we have $\mathbf{w}_C \in W_C$. Using in $(7.12)_1$ and $(7.12)_2$ the test function $(\mathbf{w}_C, v_C)$ we find $\int_{\Omega_C} |\operatorname{div} \mathbf{A}_C|^2 = 0$, therefore $\operatorname{div} \mathbf{A}_C = 0$ in Ω_C. □

Concerning the interface equations (7.10) and (7.11) we have:

Lemma 7.2. *Suppose that* $(\mathbf{A}_C, V_C, q)$ *is a solution to* (7.12). *Then*

$$\frac{1}{2} q - \mathcal{D}(q) + \frac{1}{\mu_0} \mathcal{S}(\operatorname{curl} \mathbf{A}_C \cdot \mathbf{n}_C) = \text{const} \ \ \textit{on } \Gamma \tag{7.13}$$

$$\frac{1}{2\mu_0} \operatorname{curl} \mathbf{A}_C \cdot \mathbf{n}_C + \frac{1}{\mu_0} \mathcal{D}'(\operatorname{curl} \mathbf{A}_C \cdot \mathbf{n}_C) + \mathcal{H}(q) = 0 \ \ \textit{on } \Gamma \ . \tag{7.14}$$

Proof. As already seen, we have $\int_\Gamma \mathcal{H}(\eta) = 0$ for each $\eta \in H^{1/2}(\Gamma)$ and $\mathcal{D}(1) = -\frac{1}{2}$, thus

$$\begin{aligned} &\int_\Gamma [\tfrac{1}{2} \operatorname{curl} \mathbf{A}_C \cdot \mathbf{n}_C + \mathcal{D}'(\operatorname{curl} \mathbf{A}_C \cdot \mathbf{n}_C) + \mu_0 \mathcal{H}(q)] \\ &\quad = \int_\Gamma [\tfrac{1}{2} \operatorname{curl} \mathbf{A}_C \cdot \mathbf{n}_C + \operatorname{curl} \mathbf{A}_C \cdot \mathbf{n}_C \, \mathcal{D}(1) + \mu_0 \mathcal{H}(q)] = 0 \ . \end{aligned}$$

Therefore equation $(7.12)_3$ is satisfied not only for all $\eta \in H^{1/2}(\Gamma)/\mathbb{C}$, but also for all $\eta \in H^{1/2}(\Gamma)$, and equation (7.14) follows at once.

Consequently, it is well-known from potential theory that we also obtain (7.13). □

We need now to introduce the single layer and double layer operators in the interior of Ω_I (namely, the exterior of Ω_C). For $\mathbf{x} \in \Omega_I$ we can define (see, e.g., McLean [175], Nédélec [187])

$$\mathcal{S}_I : H^{-1/2}(\Gamma) \to W^1(\Omega_I) \ , \ \ \mathcal{S}_I(\xi)(\mathbf{x}) := \int_\Gamma \frac{1}{4\pi |\mathbf{x} - \mathbf{y}|} \, \xi(\mathbf{y}) dS_y \tag{7.15}$$

$$\begin{aligned} &\mathcal{D}_I : H^{1/2}(\Gamma)/\mathbb{C} \to W^1(\Omega_I) \ , \\ &\qquad \mathcal{D}_I(\eta)(\mathbf{x}) := \int_\Gamma \tfrac{(\mathbf{x}-\mathbf{y})}{4\pi |\mathbf{x}-\mathbf{y}|^3} \cdot \eta(\mathbf{y}) \mathbf{n}_C(\mathbf{y}) dS_y \ , \end{aligned} \tag{7.16}$$

where

$$\begin{aligned} W^1(\Omega_I) := \{ &\chi_I \in (C_0^\infty(\Omega_I))' \mid \\ &(1 + |\mathbf{x}|^2)^{-1/2} \chi_I \in L^2(\Omega_I), \operatorname{grad} \chi_I \in (L^2(\Omega_I))^3 \} \ . \end{aligned} \tag{7.17}$$

We conclude our argument by showing that:

Lemma 7.3. *Suppose that* $(\mathbf{A}_C, V_C, q)$ *is a solution to* (7.12). *In the domain* Ω_I *define the function* $\psi_I := \mathcal{D}_I(q) - \frac{1}{\mu_0}\mathcal{S}_I(\operatorname{curl}\mathbf{A}_C \cdot \mathbf{n}_C)$. *Then*

$$\begin{aligned}\int_{\Omega_C}(\boldsymbol{\mu}_C^{-1}\operatorname{curl}\mathbf{A}_C\cdot\operatorname{curl}\overline{\mathbf{w}_C^*} + i\omega\boldsymbol{\sigma}\mathbf{A}_C\cdot\overline{\mathbf{w}_C^*} + \boldsymbol{\sigma}\operatorname{grad}V_C\cdot\overline{\mathbf{w}_C^*})\\ + \int_\Gamma \mathbf{n}_C\times\operatorname{grad}\psi_I\cdot\overline{\mathbf{w}_C^*} = \int_{\Omega_C}\mathbf{J}_{e,C}\cdot\overline{\mathbf{w}_C^*}\end{aligned} \tag{7.18}$$

for all $\mathbf{w}_C^* \in H(\operatorname{curl};\Omega_C)$. *Therefore,*

$$\begin{cases}\operatorname{curl}(\boldsymbol{\mu}_C^{-1}\operatorname{curl}\mathbf{A}_C) + i\omega\boldsymbol{\sigma}\mathbf{A}_C + \boldsymbol{\sigma}\operatorname{grad}V_C = \mathbf{J}_{e,C} & \text{in } \Omega_C\\ \Delta\psi_I = 0 & \text{in } \Omega_I\\ \operatorname{curl}\mathbf{A}_C\cdot\mathbf{n}_C + \mu_0\operatorname{grad}\psi_I\cdot\mathbf{n}_I = 0 & \text{on } \Gamma\\ (\boldsymbol{\mu}_C^{-1}\operatorname{curl}\mathbf{A}_C)\times\mathbf{n}_C + \operatorname{grad}\psi_I\times\mathbf{n}_I = \mathbf{0} & \text{on } \Gamma\\ |\psi_I(\mathbf{x})| + |\operatorname{grad}\psi_I(\mathbf{x})| = O(|\mathbf{x}|^{-1}) & \text{as } |\mathbf{x}|\to\infty\,.\end{cases} \tag{7.19}$$

Proof. Well-known results of potential theory imply that ψ_I is a harmonic function with $|\psi_I(\mathbf{x})|$ and $|\operatorname{grad}\psi_I(\mathbf{x})|$ decaying at infinity as $O(|\mathbf{x}|^{-1})$. Moreover, ψ_I satisfies the trace relations

$$\psi_{I|\Gamma} = \frac{1}{2}q + \mathcal{D}(q) - \frac{1}{\mu_0}\mathcal{S}(\operatorname{curl}\mathbf{A}_C\cdot\mathbf{n}_C), \tag{7.20}$$

and

$$\operatorname{grad}\psi_I\cdot\mathbf{n}_I = \mathcal{H}(q) + \mu_0^{-1}\Big[-\frac{1}{2}\operatorname{curl}\mathbf{A}_C\cdot\mathbf{n}_C + \mathcal{D}'(\operatorname{curl}\mathbf{A}_C\cdot\mathbf{n}_C)\Big] \tag{7.21}$$

(see, e.g., McLean [175], Nédélec [187]).

From (7.14) and (7.21) we see that the interface condition $(7.19)_3$ is satisfied. Moreover, from Lemma 7.1, $(7.12)_1$ and (7.20) we find that

$$\begin{aligned}\int_{\Omega_C}(\boldsymbol{\mu}_C^{-1}\operatorname{curl}\mathbf{A}_C\cdot\operatorname{curl}\overline{\mathbf{w}_C} + i\omega\boldsymbol{\sigma}\mathbf{A}_C\cdot\overline{\mathbf{w}_C} + \boldsymbol{\sigma}\operatorname{grad}V_C\cdot\overline{\mathbf{w}_C})\\ - \int_\Gamma \psi_{I|\Gamma}\operatorname{curl}\overline{\mathbf{w}_C}\cdot\mathbf{n}_C = \int_{\Omega_C}\mathbf{J}_{e,C}\cdot\overline{\mathbf{w}_C}\,.\end{aligned}$$

Since we have $-\int_\Gamma \psi_{I|\Gamma}\operatorname{curl}\overline{\mathbf{w}_C}\cdot\mathbf{n}_C = \int_\Gamma \mathbf{n}_C\times\operatorname{grad}\psi_I\cdot\overline{\mathbf{w}_C}$, for each $\mathbf{w}_C\in W_C$ we have obtained

$$\begin{aligned}\int_{\Omega_C}(\boldsymbol{\mu}_C^{-1}\operatorname{curl}\mathbf{A}_C\cdot\operatorname{curl}\overline{\mathbf{w}_C} + i\omega\boldsymbol{\sigma}\mathbf{A}_C\cdot\overline{\mathbf{w}_C} + \boldsymbol{\sigma}\operatorname{grad}V_C\cdot\overline{\mathbf{w}_C})\\ + \int_\Gamma \mathbf{n}_C\times\operatorname{grad}\psi_I\cdot\overline{\mathbf{w}_C} = \int_{\Omega_C}\mathbf{J}_{e,C}\cdot\overline{\mathbf{w}_C}\,.\end{aligned} \tag{7.22}$$

If $\mathbf{w}_C^*\in H(\operatorname{curl};\Omega_C)$, consider the solution $v_C^*\in H^1(\Omega_C)/\mathbb{C}$ of the Neumann problem $\Delta v_C^* = \operatorname{div}\mathbf{w}_C^*$ in Ω_C with $\operatorname{grad}v_C^*\cdot\mathbf{n}_C = \mathbf{w}_C^*\cdot\mathbf{n}_C$ on Γ. Setting $\mathbf{w}_C = \mathbf{w}_C^* - \operatorname{grad}v_C^*$, we have $\mathbf{w}_C\in W_C$, and using it in (7.22) we obtain

$$\begin{aligned}&\int_{\Omega_C}(\boldsymbol{\mu}_C^{-1}\operatorname{curl}\mathbf{A}_C\cdot\operatorname{curl}\overline{\mathbf{w}_C^*} + i\omega\boldsymbol{\sigma}\mathbf{A}_C\cdot\overline{\mathbf{w}_C^*} + \boldsymbol{\sigma}\operatorname{grad}V_C\cdot\overline{\mathbf{w}_C^*})\\ &\qquad + \int_\Gamma \mathbf{n}_C\times\operatorname{grad}\psi_I\cdot\overline{\mathbf{w}_C^*}\\ &\quad = \int_{\Omega_C}(\boldsymbol{\mu}_C^{-1}\operatorname{curl}\mathbf{A}_C\cdot\operatorname{curl}\overline{\mathbf{w}_C} + i\omega\boldsymbol{\sigma}\mathbf{A}_C\cdot\overline{\mathbf{w}_C} + \boldsymbol{\sigma}\operatorname{grad}V_C\cdot\overline{\mathbf{w}_C})\\ &\qquad + \int_\Gamma \mathbf{n}_C\times\operatorname{grad}\psi_I\cdot\overline{\mathbf{w}_C}\\ &\qquad + \int_{\Omega_C}(i\omega\boldsymbol{\sigma}\mathbf{A}_C\cdot\operatorname{grad}\overline{v_C^*} + \boldsymbol{\sigma}\operatorname{grad}V_C\cdot\operatorname{grad}\overline{v_C^*})\\ &\qquad + \int_\Gamma \mathbf{n}_C\times\operatorname{grad}\psi_I\cdot\operatorname{grad}\overline{v_C^*}\\ &\quad = \int_{\Omega_C}\mathbf{J}_{e,C}\cdot\overline{\mathbf{w}_C} + \int_{\Omega_C}\mathbf{J}_{e,C}\cdot\operatorname{grad}\overline{v_C^*} = \int_{\Omega_C}\mathbf{J}_{e,C}\cdot\overline{\mathbf{w}_C^*}\,,\end{aligned}$$

having used $(7.12)_2$ and the fact that

$$\begin{aligned}&\int_\Gamma \mathbf{n}_C \times \operatorname{grad}\psi_I \cdot \operatorname{grad}\overline{v_C^*} = -\int_\Gamma \operatorname{div}_\tau(\mathbf{n}_C \times \operatorname{grad}\psi_I)\,\overline{v_C^*}\\ &\quad = -\int_\Gamma \operatorname{curl}\operatorname{grad}\psi_I \cdot \mathbf{n}_I\,\overline{v_C^*} = 0\,.\end{aligned}$$

Taking a test function $\mathbf{w}_C^* \in (C_0^\infty(\Omega_C))^3$ and integrating by parts we verify that $(7.19)_1$ is satisfied; repeating the same argument for $\mathbf{w}_C^* \in H(\mathrm{curl};\Omega_C)$, we see that the interface condition $(7.19)_4$ is satisfied as well. □

Remark 7.4. The function $q \in H^{1/2}(\Gamma)/\mathbb{C}$ determined in (7.12) is defined up to an additive constant. It is easily seen that, as functions in $H^{1/2}(\Gamma)/\mathbb{C}$, q and the trace on Γ of the harmonic scalar potential ψ_I, namely, what we have called ψ_Γ, are the same function. Indeed, from (7.13) and (7.20) we see that $\psi_\Gamma + \mathrm{const} = q$. □

7.3 Existence and uniqueness of the weak solution

In order to prove the existence and uniqueness of the solution to (7.12), let us introduce the following sesquilinear forms: for $\omega \neq 0$

$$\begin{aligned}&\mathcal{A}_{(\omega\neq 0)}[(\mathbf{A}_C, V_C, q), (\mathbf{w}_C, Q_C, \eta)]\\ &\quad = \int_{\Omega_C}(\boldsymbol{\mu}_C^{-1}\operatorname{curl}\mathbf{A}_C \cdot \operatorname{curl}\overline{\mathbf{w}_C} + \mu_*^{-1}\operatorname{div}\mathbf{A}_C\ \operatorname{div}\overline{\mathbf{w}_C})\\ &\qquad + i\omega^{-1}\int_{\Omega_C}\boldsymbol{\sigma}(i\omega\mathbf{A}_C + \operatorname{grad}V_C)\cdot(-i\omega\overline{\mathbf{w}_C} + \operatorname{grad}\overline{Q_C})\\ &\qquad + \int_\Gamma[-\tfrac{1}{2}q - \mathcal{D}(q)]\operatorname{curl}\overline{\mathbf{w}_C}\cdot\mathbf{n}_C\\ &\qquad + \int_\Gamma[\tfrac{1}{2}\operatorname{curl}\mathbf{A}_C\cdot\mathbf{n}_C + \mathcal{D}'(\operatorname{curl}\mathbf{A}_C\cdot\mathbf{n}_C)]\overline{\eta}\\ &\qquad + \int_\Gamma[\tfrac{1}{\mu_0}\mathcal{S}(\operatorname{curl}\mathbf{A}_C\cdot\mathbf{n}_C)\ \operatorname{curl}\overline{\mathbf{w}_C}\cdot\mathbf{n}_C + \mu_0\mathcal{H}(q)\,\overline{\eta}]\,,\end{aligned}\tag{7.23}$$

and for $\omega = 0$

$$\begin{aligned}&\mathcal{A}_{(\omega=0)}[(\mathbf{A}_C, V_C, q), (\mathbf{w}_C, Q_C, \eta)]\\ &\quad = \int_{\Omega_C}(\boldsymbol{\mu}_C^{-1}\operatorname{curl}\mathbf{A}_C \cdot \operatorname{curl}\overline{\mathbf{w}_C} + \mu_*^{-1}\operatorname{div}\mathbf{A}_C\ \operatorname{div}\overline{\mathbf{w}_C})\\ &\qquad + \int_{\Omega_C}(\boldsymbol{\sigma}\operatorname{grad}V_C\cdot\overline{\mathbf{w}_C} + \beta\boldsymbol{\sigma}\operatorname{grad}V_C\cdot\operatorname{grad}\overline{Q_C})\\ &\qquad + \int_\Gamma[-\tfrac{1}{2}q - \mathcal{D}(q)]\operatorname{curl}\overline{\mathbf{w}_C}\cdot\mathbf{n}_C\\ &\qquad + \int_\Gamma[\tfrac{1}{2}\operatorname{curl}\mathbf{A}_C\cdot\mathbf{n}_C + \mathcal{D}'(\operatorname{curl}\mathbf{A}_C\cdot\mathbf{n}_C)]\overline{\eta}\\ &\qquad + \int_\Gamma[\tfrac{1}{\mu_0}\mathcal{S}(\operatorname{curl}\mathbf{A}_C\cdot\mathbf{n}_C)\ \operatorname{curl}\overline{\mathbf{w}_C}\cdot\mathbf{n}_C + \mu_0\mathcal{H}(q)\,\overline{\eta}]\,.\end{aligned}\tag{7.24}$$

These forms are obtained by adding the left hand sides in (7.12): however, in the case $\omega \neq 0$ we have multiplied the second equation by $i\omega^{-1}$, obtaining $\mathcal{A}_{(\omega\neq 0)}[\cdot,\cdot]$, while in the case $\omega = 0$ we have multiplied the second equation by $\beta > 0$, to be chosen in the sequel, obtaining $\mathcal{A}_{(\omega=0)}[\cdot,\cdot]$.

The main result of this section is:

Theorem 7.5. *The sesquilinear form $\mathcal{A}_{(\omega\neq 0)}[\cdot,\cdot]$ is coercive in the space $W_C \times H^1(\Omega_C)/\mathbb{C} \times H^{1/2}(\Gamma)/\mathbb{C}$, uniformly as $\omega \to 0$; namely, there exists a constant $\kappa > 0$, independent of ω, such that for each $(\mathbf{w}_C, Q_C, \eta) \in W_C \times H^1(\Omega_C) \times H^{1/2}(\Gamma)$ with*

$\int_{\Omega_C} Q_C = 0$ *and* $\int_\Gamma \eta = 0$ *one has*

$$
\begin{aligned}
&|\mathcal{A}_{(\omega\neq 0)}[(\mathbf{w}_C, Q_C, \eta), (\mathbf{w}_C, Q_C, \eta)]| \\
&\quad \geq \kappa\Big(\int_{\Omega_C} (|\mathbf{w}_C|^2 + |\operatorname{curl} \mathbf{w}_C|^2 + |\operatorname{div} \mathbf{w}_C|^2) \\
&\qquad + ||\eta||^2_{1/2,\Gamma} + \chi(\omega) \int_{\Omega_C} (|Q_C|^2 + |\operatorname{grad} Q_C|^2)\Big) \ ,
\end{aligned}
\tag{7.25}
$$

where the constant $\chi(\omega) > 0$ *is equal to* $|\omega|^{-1}$ *in the case* $0 < |\omega| < 1$ *and is equal to* ω^{-2} *in the case* $|\omega| \geq 1$.

Moreover, the sesquilinear form $\mathcal{A}_{(\omega=0)}[\cdot,\cdot]$ *is coercive in the space* $W_C \times H^1(\Omega_C)/\mathbb{C} \times H^{1/2}(\Gamma)/\mathbb{C}$, *namely, there exists a constant* $\kappa_0 > 0$ *such that for each* $(\mathbf{w}_C, Q_C, \eta) \in W_C \times H^1(\Omega_C) \times H^{1/2}(\Gamma)$ *with* $\int_{\Omega_C} Q_C = 0$ *and* $\int_\Gamma \eta = 0$ *one has*

$$
\begin{aligned}
&|\mathcal{A}_{(\omega=0)}[(\mathbf{w}_C, Q_C, \eta), (\mathbf{w}_C, Q_C, \eta)]| \\
&\quad \geq \kappa_0\Big(\int_{\Omega_C} (|\mathbf{w}_C|^2 + |\operatorname{curl} \mathbf{w}_C|^2 + |\operatorname{div} \mathbf{w}_C|^2) \\
&\qquad + ||\eta||^2_{1/2,\Gamma} + \int_{\Omega_C} (|Q_C|^2 + |\operatorname{grad} Q_C|^2)\Big) \ .
\end{aligned}
\tag{7.26}
$$

As a consequence, for each $\mathbf{J}_{e,C} \in (L^2(\Omega_C))^3$, *existence and uniqueness of the solution to* (7.12) *follow from the Lax–Milgram lemma.*

Proof. First of all, let us recall that the operators $\mathcal{S}$ and $\mathcal{H}$ are continuous from $H^{-1/2}(\Gamma)$ into $H^{1/2}(\Gamma)$ and from $H^{1/2}(\Gamma)$ into $H^{-1/2}(\Gamma)$, respectively, and satisfy

$$
\int_\Gamma \mathcal{S}(\xi)\,\overline{\xi} \geq \kappa_1 ||\xi||^2_{-1/2,\Gamma} \ , \quad \int_\Gamma \mathcal{H}(\eta)\,\overline{\eta} \geq \kappa_2 ||\eta||^2_{1/2,\Gamma}
\tag{7.27}
$$

for each $\xi \in H^{-1/2}(\Gamma)$ and $\eta \in H^{1/2}(\Gamma)$ with $\int_\Gamma \eta = 0$, and moreover that the operator $\mathcal{D}$ is continuous from $H^{1/2}(\Gamma)$ into itself (see, e.g., McLean [175], Nédélec [187]).

The sesquilinear form $\mathcal{A}_{(\omega\neq 0)}[\cdot,\cdot]$ satisfies

$$
\begin{aligned}
&\mathcal{A}_{(\omega\neq 0)}[(\mathbf{w}_C, Q_C, \eta), (\mathbf{w}_C, Q_C, \eta)] \\
&\quad = \int_{\Omega_C} (\boldsymbol{\mu}_C^{-1} \operatorname{curl} \mathbf{w}_C \cdot \operatorname{curl} \overline{\mathbf{w}_C} + \mu_*^{-1} |\operatorname{div} \mathbf{w}_C|^2) \\
&\qquad + i\omega^{-1} \int_{\Omega_C} \boldsymbol{\sigma}(i\omega \mathbf{w}_C + \operatorname{grad} Q_C) \cdot (-i\omega \overline{\mathbf{w}_C} + \operatorname{grad} \overline{Q_C}) \\
&\qquad + \int_\Gamma [-\tfrac{1}{2}\eta - \mathcal{D}(\eta)] \operatorname{curl} \overline{\mathbf{w}_C} \cdot \mathbf{n}_C \\
&\qquad + \int_\Gamma [\tfrac{1}{2} \operatorname{curl} \mathbf{w}_C \cdot \mathbf{n}_C + \mathcal{D}'(\operatorname{curl} \mathbf{w}_C \cdot \mathbf{n}_C)] \overline{\eta} \\
&\qquad + \int_\Gamma [\tfrac{1}{\mu_0} \mathcal{S}(\operatorname{curl} \mathbf{w}_C \cdot \mathbf{n}_C) \operatorname{curl} \overline{\mathbf{w}_C} \cdot \mathbf{n}_C + \mu_0 \mathcal{H}(\eta)\,\overline{\eta}] \ .
\end{aligned}
$$

Since

$$
\int_\Gamma \mathcal{D}'(\operatorname{curl} \mathbf{w}_C \cdot \mathbf{n}_C)]\overline{\eta} = \int_\Gamma \mathcal{D}(\overline{\eta}) \operatorname{curl} \mathbf{w}_C \cdot \mathbf{n}_C,
$$

and

$$
\begin{aligned}
&[-\tfrac{1}{2}\eta - \mathcal{D}(\eta)] \operatorname{curl} \overline{\mathbf{w}_C} \cdot \mathbf{n}_C + \operatorname{curl} \mathbf{w}_C \cdot \mathbf{n}_C\, [\tfrac{1}{2}\overline{\eta} + \mathcal{D}(\overline{\eta})] \\
&\qquad = -2\, i \operatorname{Im}\Big([\tfrac{1}{2}\eta + \mathcal{D}(\eta)] \operatorname{curl} \overline{\mathbf{w}_C} \cdot \mathbf{n}_C\Big) \ ,
\end{aligned}
$$

we have

$$\begin{aligned}
&\mathrm{Re}\,\mathcal{A}_{(\omega\neq 0)}[(\mathbf{w}_C,Q_C,\eta),(\mathbf{w}_C,Q_C,\eta)]\\
&\quad=\int_{\Omega_C}(\boldsymbol{\mu}_C^{-1}\,\mathrm{curl}\,\mathbf{w}_C\cdot\mathrm{curl}\,\overline{\mathbf{w}_C}+\mu_*^{-1}|\,\mathrm{div}\,\mathbf{w}_C|^2)\\
&\qquad+\int_\Gamma[\tfrac{1}{\mu_0}\mathcal{S}(\mathrm{curl}\,\mathbf{w}_C\cdot\mathbf{n}_C)\ \mathrm{curl}\,\overline{\mathbf{w}_C}\cdot\mathbf{n}_C+\mu_0\mathcal{H}(\eta)\,\overline{\eta}],
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{Im}\,\mathcal{A}_{(\omega\neq 0)}[(\mathbf{w}_C,Q_C,\eta),(\mathbf{w}_C,Q_C,\eta)]\\
&\quad=\omega^{-1}\int_{\Omega_C}\boldsymbol{\sigma}(i\omega\mathbf{w}_C+\mathrm{grad}\,Q_C)\cdot(-i\omega\overline{\mathbf{w}_C}+\mathrm{grad}\,\overline{Q_C})\\
&\qquad-2\,\mathrm{Im}\,\int_\Gamma[\tfrac{1}{2}\eta+\mathcal{D}(\eta)]\,\mathrm{curl}\,\overline{\mathbf{w}_C}\cdot\mathbf{n}_C\,.
\end{aligned}$$

Hence, for a suitable constant $\kappa_3>0$, independent of ω, we find

$$\begin{aligned}
&\mathrm{Re}\,\mathcal{A}_{(\omega\neq 0)}[(\mathbf{w}_C,Q_C,\eta),(\mathbf{w}_C,Q_C,\eta)]\\
&\quad\geq\kappa_3\Big(\int_{\Omega_C}(|\,\mathrm{curl}\,\mathbf{w}_C|^2+|\,\mathrm{div}\,\mathbf{w}_C|^2)+||\,\mathrm{curl}\,\mathbf{w}_C\cdot\mathbf{n}_C||^2_{-1/2,\Gamma}+||\eta||^2_{1/2,\Gamma}\Big)\,,
\end{aligned}$$

and moreover, taking into account that the operator $\mathcal{D}$ is continuous from $H^{1/2}(\Gamma)$ into itself, for a suitable constant $C_1>0$, independent of ω, we obtain

$$\begin{aligned}
&\Big|2\,\mathrm{Im}\,\int_\Gamma[\tfrac{1}{2}\eta+\mathcal{D}(\eta)]\,\mathrm{curl}\,\overline{\mathbf{w}_C}\cdot\mathbf{n}_C\Big|\leq C_1\,||\eta||_{1/2,\Gamma}\,||\,\mathrm{curl}\,\mathbf{w}_C\cdot\mathbf{n}_C||_{-1/2,\Gamma}\\
&\qquad\leq\tfrac{C_1}{2}||\eta||^2_{1/2,\Gamma}+\tfrac{C_1}{2}||\,\mathrm{curl}\,\mathbf{w}_C\cdot\mathbf{n}_C||^2_{-1/2,\Gamma}\,.
\end{aligned}$$

Hence, proceeding as in the proof of the coerciveness of the sesquilinear form $\mathcal{A}[\cdot,\cdot]$ in Section 6.1.2, we find, for each $0<\tau\leq 1$,

$$\begin{aligned}
&|\mathrm{Im}\,\mathcal{A}_{(\omega\neq 0)}[(\mathbf{w}_C,Q_C,\eta),(\mathbf{w}_C,Q_C,\eta)]|\\
&\quad\geq\tau|\mathrm{Im}\,\mathcal{A}_{(\omega\neq 0)}[(\mathbf{w}_C,Q_C,\eta),(\mathbf{w}_C,Q_C,\eta)]|\\
&\quad\geq\tfrac{1}{2}\tau|\omega|^{-1}\sigma_{\min}\int_{\Omega_C}|\,\mathrm{grad}\,Q_C|^2-\tau|\omega|\sigma_{\min}\int_{\Omega_C}|\mathbf{w}_C|^2\\
&\qquad-\tau\tfrac{C_1}{2}||\eta||^2_{1/2,\Gamma}-\tau\tfrac{C_1}{2}||\,\mathrm{curl}\,\mathbf{w}_C\cdot\mathbf{n}_C||^2_{-1/2,\Gamma}\,,
\end{aligned}$$

where $\sigma_{\min}$ is a uniform lower bound in Ω_C for the minimum eigenvalues of $\boldsymbol{\sigma}(\mathbf{x})$.

Let us recall now the Poincaré inequalities (6.38) and (6.39): there exist constants $\kappa_5>0$ and $\kappa_6>0$ such that

$$\int_{\Omega_C}|\,\mathrm{grad}\,Q_C|^2\geq\kappa_5\int_{\Omega_C}(|\,\mathrm{grad}\,Q_C|^2+|Q_C|^2)$$

for all $Q_C\in H^1(\Omega_C)$ with $\int_{\Omega_C}Q_C=0$, and

$$\begin{aligned}
&\int_{\Omega_C}(|\,\mathrm{curl}\,\mathbf{w}_C|^2+|\,\mathrm{div}\,\mathbf{w}_C|^2)\\
&\qquad\qquad\geq\kappa_6\int_{\Omega_C}(|\,\mathrm{curl}\,\mathbf{w}_C|^2+|\,\mathrm{div}\,\mathbf{w}_C|^2+|\mathbf{w}_C|^2)
\end{aligned}$$

for all $\mathbf{w}_C\in W_C$. Coerciveness follows by choosing τ small enough to have $\tau|\omega|\sigma_{\min}<\kappa_3\kappa_6$ and $\tau\frac{C_1}{2}<\kappa_3$. In particular, we have $\tau=O(1)$ for $0<|\omega|<1$ and $\tau=O(|\omega|^{-1})$ for $|\omega|\geq 1$. Thus the constant κ in (7.25) can be clearly chosen independent of ω, and the constant $\chi(\omega)$ is $O(|\omega|^{-1})$ for $0<|\omega|<1$ and $O(\omega^{-2})$ for $|\omega|\geq 1$.

In the case $\omega = 0$, the sesquilinear form satisfies

$$
\begin{aligned}
&\mathcal{A}_{(\omega=0)}[(\mathbf{w}_C, Q_C, \eta), (\mathbf{w}_C, Q_C, \eta)] \\
&\quad = \int_{\Omega_C} (\boldsymbol{\mu}_C^{-1} \operatorname{curl} \mathbf{w}_C \cdot \operatorname{curl} \overline{\mathbf{w}_C} + \mu_*^{-1} |\operatorname{div} \mathbf{w}_C|^2 \\
&\qquad\quad + \boldsymbol{\sigma} \operatorname{grad} Q_C \cdot \overline{\mathbf{w}_C} + \beta \boldsymbol{\sigma} \operatorname{grad} Q_C \cdot \operatorname{grad} \overline{Q_C}) \\
&\qquad + \int_\Gamma [-\tfrac{1}{2}\eta - \mathcal{D}(\eta)] \operatorname{curl} \overline{\mathbf{w}_C} \cdot \mathbf{n}_C \\
&\qquad + \int_\Gamma [\tfrac{1}{2} \operatorname{curl} \mathbf{w}_C \cdot \mathbf{n}_C + \mathcal{D}'(\operatorname{curl} \mathbf{w}_C \cdot \mathbf{n}_C)] \overline{\eta} \\
&\qquad + \int_\Gamma [\tfrac{1}{\mu_0} \mathcal{S}(\operatorname{curl} \mathbf{w}_C \cdot \mathbf{n}_C)\ \operatorname{curl} \overline{\mathbf{w}_C} \cdot \mathbf{n}_C + \mu_0 \mathcal{H}(\eta)\, \overline{\eta}]\ .
\end{aligned}
$$

We split $\int_{\Omega_C} \boldsymbol{\sigma} \operatorname{grad} Q_C \cdot \overline{\mathbf{w}_C}$ into its real and imaginary part, and, for each $\delta > 0$ and suitable constants $\kappa_7 > 0$ and $C_2 > 0$, we end up with

$$
\begin{aligned}
&\left|\operatorname{Re} \mathcal{A}_{(\omega=0)}[(\mathbf{w}_C, Q_C, \eta), (\mathbf{w}_C, Q_C, \eta)]\right| \\
&\quad \geq \kappa_7 \Big(\int_{\Omega_C} (|\operatorname{curl} \mathbf{w}_C|^2 + |\operatorname{div} \mathbf{w}_C|^2 + \beta |\operatorname{grad} Q_C|^2) \\
&\qquad\qquad\qquad + ||\operatorname{curl} \mathbf{w}_C \cdot \mathbf{n}_C||^2_{-1/2,\Gamma} + ||\eta||^2_{1/2,\Gamma} \Big) \\
&\qquad - C_2 \delta^{-1} \int_{\Omega_C} |\operatorname{grad} Q_C|^2 - \delta \int_{\Omega_C} |\mathbf{w}_C|^2 ,
\end{aligned}
$$

thus the conclusion follows by choosing δ so small that $\kappa_7 \kappa_6 - \delta > 0$, and then β large enough to have $\kappa_7 \beta - C_2 \delta^{-1} > 0$. □

7.4 Stability as ω goes to 0

We are now interested in showing that the solution to problem (7.12) is stable with respect to ω, namely, if we denote by $(\mathbf{A}_C^\omega, V_C^\omega, q^\omega)$ the solution to (7.12) corresponding to the angular frequency ω, we have $(\mathbf{A}_C^\omega, V_C^\omega, q^\omega) \to (\mathbf{A}_C^0, V_C^0, q^0)$ as $\omega \to 0$.

Theorem 7.6. *There exists a constant $K > 0$, independent of ω, such that for each ω with $0 < |\omega| < 1$, the solutions to (7.12) satisfy*

$$
\int_{\Omega_C} (|\mathbf{A}_C^\omega - \mathbf{A}_C^0|^2 + |\operatorname{curl} \mathbf{A}_C^\omega - \operatorname{curl} \mathbf{A}_C^0|^2) \leq K\, \omega^2
$$

$$
\int_{\Omega_C} (|V_C^\omega - V_C^0|^2 + |\operatorname{grad} V_C^\omega - \operatorname{grad} V_C^0|^2) \leq K\, \omega^2
$$

$$
||q^\omega - q^0||^2_{1/2,\Gamma} \leq K\, \omega^2\ ,
$$

having chosen V_C^ω, V_C^0, q^ω and q^0 such that $\int_{\Omega_C} V_C^\omega = \int_{\Omega_C} V_C^0 = 0$ and $\int_\Gamma q^\omega = \int_\Gamma q^0 = 0$.

Proof. By linearity, the difference $(\mathbf{Z}_C, N_C, p) := (\mathbf{A}^\omega_C, V^\omega_C, q^\omega) - (\mathbf{A}^0_C, V^0_C, q^0)$ satisfies

$$\begin{aligned}
&\int_{\Omega_C}(\boldsymbol{\mu}_C^{-1}\operatorname{curl}\mathbf{Z}_C\cdot\operatorname{curl}\overline{\mathbf{w}_C}+\mu_*^{-1}\operatorname{div}\mathbf{Z}_C\operatorname{div}\overline{\mathbf{w}_C})\\
&\qquad+\int_{\Omega_C}(i\omega\boldsymbol{\sigma}\mathbf{Z}_C\cdot\overline{\mathbf{w}_C}+\boldsymbol{\sigma}\operatorname{grad}N_C\cdot\overline{\mathbf{w}_C})\\
&\qquad+\int_\Gamma[-\tfrac{1}{2}p-\mathcal{D}(p)+\tfrac{1}{\mu_0}\mathcal{S}(\operatorname{curl}\mathbf{Z}_C\cdot\mathbf{n}_C)]\operatorname{curl}\overline{\mathbf{w}_C}\cdot\mathbf{n}_C\\
&\quad=-\int_{\Omega_C}i\omega\boldsymbol{\sigma}\mathbf{A}^0_C\cdot\overline{\mathbf{w}_C}\\
&\int_{\Omega_C}(-\boldsymbol{\sigma}\mathbf{Z}_C\cdot\operatorname{grad}\overline{Q_C}+i\omega^{-1}\boldsymbol{\sigma}\operatorname{grad}N_C\cdot\operatorname{grad}\overline{Q_C})\\
&\quad=\int_{\Omega_C}\boldsymbol{\sigma}\mathbf{A}^0_C\cdot\operatorname{grad}\overline{Q_C}\\
&\int_\Gamma[\tfrac{1}{2}\operatorname{curl}\mathbf{Z}_C\cdot\mathbf{n}_C+\mathcal{D}'(\operatorname{curl}\mathbf{Z}_C\cdot\mathbf{n}_C)+\mu_0\mathcal{H}(p)]\overline{\eta}=0
\end{aligned}\tag{7.28}$$

(here, we have $\operatorname{div}\mathbf{Z}_C = 0$ in Ω_C by Lemma 7.1; however, we prefer to write everything in terms of the sesquilinear form $\mathcal{A}_{(\omega\neq 0)}[\cdot,\cdot]$).

Therefore, from the coerciveness of $\mathcal{A}_{(\omega\neq 0)}[\cdot,\cdot]$ and taking into account that $0 < |\omega| < 1$, from (7.25) we obtain at once that

$$\begin{aligned}
&\int_{\Omega_C}(|\mathbf{Z}_C|^2+|\operatorname{curl}\mathbf{Z}_C|^2+|\operatorname{div}\mathbf{Z}_C|^2)\\
&\qquad+||p||^2_{1/2,\Gamma}+\chi(\omega)\int_{\Omega_C}(|N_C|^2+|\operatorname{grad}N_C|^2)\\
&\quad\le\kappa^{-1}c_1\Big[|\omega|\big(\int_{\Omega_C}|\mathbf{A}^0_C|^2\big)^{1/2}\big(\int_{\Omega_C}|\mathbf{Z}_C|^2\big)^{1/2}\\
&\qquad\qquad+\big(\int_{\Omega_C}|\mathbf{A}^0_C|^2\big)^{1/2}\big(\int_{\Omega_C}|\operatorname{grad}N_C|^2\big)^{1/2}\Big]\\
&\quad\le\kappa^{-1}c_2|\omega|^2\alpha_1^{-1}\int_{\Omega_C}|\mathbf{A}^0_C|^2+\kappa^{-1}c_2\alpha_2^{-1}\int_{\Omega_C}|\mathbf{A}^0_C|^2\\
&\qquad+\alpha_1\int_{\Omega_C}|\mathbf{Z}_C|^2+\alpha_2\int_{\Omega_C}|\operatorname{grad}N_C|^2
\end{aligned}\tag{7.29}$$

for each $\alpha_1 > 0$ and $\alpha_2 > 0$. Choosing $\alpha_1 = 1/2$ and $\alpha_2 = \chi(\omega)/2 = O(|\omega|^{-1})$ (see Theorem 7.5), we have that the left hand side in (7.29) is $O(|\omega|)$. In particular,

$$\int_{\Omega_C}(|N_C|^2+|\operatorname{grad}N_C|^2)=[\chi(\omega)]^{-1}O(|\omega|)=O(\omega^2),$$

and

$$\int_{\Omega_C}(|\mathbf{Z}_C|^2+|\operatorname{curl}\mathbf{Z}_C|^2+|\operatorname{div}\mathbf{Z}_C|^2)+||p||^2_{1/2,\Gamma}=O(|\omega|)\,.$$

Rewriting the first equation in (7.28) as

$$\begin{aligned}
&\int_{\Omega_C}(\boldsymbol{\mu}_C^{-1}\operatorname{curl}\mathbf{Z}_C\cdot\operatorname{curl}\overline{\mathbf{w}_C}+\mu_*^{-1}\operatorname{div}\mathbf{Z}_C\operatorname{div}\overline{\mathbf{w}_C})\\
&\qquad+\int_\Gamma[-\tfrac{1}{2}p-\mathcal{D}(p)+\tfrac{1}{\mu_0}\mathcal{S}(\operatorname{curl}\mathbf{Z}_C\cdot\mathbf{n}_C)]\operatorname{curl}\overline{\mathbf{w}_C}\cdot\mathbf{n}_C\\
&\quad=-\int_{\Omega_C}(i\omega\boldsymbol{\sigma}\mathbf{Z}_C\cdot\overline{\mathbf{w}_C}+\boldsymbol{\sigma}\operatorname{grad}N_C\cdot\overline{\mathbf{w}_C})-\int_{\Omega_C}i\omega\boldsymbol{\sigma}\mathbf{A}^0_C\cdot\overline{\mathbf{w}_C}\,,
\end{aligned}$$

using $(7.28)_3$ and proceeding as in the proof of Theorem 7.5, we obtain that the sesquilinear form at the left hand side is coercive (with coerciveness constant $K_0 > 0$ independent of ω). Hence we have

$$\begin{aligned}
&\int_{\Omega_C}(|\mathbf{Z}_C|^2+|\operatorname{curl}\mathbf{Z}_C|^2+|\operatorname{div}\mathbf{Z}_C|^2)+||\operatorname{curl}\mathbf{Z}_C\cdot\mathbf{n}_C||^2_{-1/2,\Gamma}+||p||^2_{1/2,\Gamma}\\
&\quad\le K_0^{-1}c_3\Big[|\omega|\int_{\Omega_C}|\mathbf{Z}_C|^2+\big(\int_{\Omega_C}|\operatorname{grad}N_C|^2\big)^{1/2}\big(\int_{\Omega_C}|\mathbf{Z}_C|^2\big)^{1/2}\\
&\qquad\qquad+|\omega|\big(\int_{\Omega_C}|\mathbf{A}^0_C|^2\big)^{1/2}\big(\int_{\Omega_C}|\mathbf{Z}_C|^2\big)^{1/2}\Big]\\
&\quad=O(\omega^2)+O(|\omega|)\big(\int_{\Omega_C}|\mathbf{Z}_C|^2\big)^{1/2}\le O(\omega^2)+\tfrac{1}{2}\int_{\Omega_C}|\mathbf{Z}_C|^2\,.
\end{aligned}$$

The result thus follows. □

Remark 7.7. As we recalled in Section 2.3.1, an analysis of the asymptotic behaviour of the solution of the eddy current model with respect to $\omega \to 0$ has been presented in Ammari et al. [23]. In particular they prove, by a formal asymptotic expansion, that the electric and the magnetic fields solutions of the eddy current problem converge linearly to the corresponding solutions of the static problem in the L^2-norm. Expressing the electric and magnetic fields in terms of $\mathbf{A}_C$, V_C and ψ_I, it can be easily checked that Theorem 7.6 is in agreement with their result. □

7.5 Numerical approximation

In this section we deal with the numerical approximation of problem (7.12). In the sequel we assume that Ω_C is a Lipschitz polyhedral domain, and that $\mathcal{T}_{C,h}$ and $\mathcal{T}_{\Gamma,h}$ are two regular families of triangulations of Ω_C and Γ, respectively. For the sake of simplicity, we suppose that each element K of $\mathcal{T}_{C,h}$ is a tetrahedron and each element T of $\mathcal{T}_{\Gamma,h}$ is a triangle; however, the results below also hold for hexahedral and rectangular elements, respectively. Let us note that the mesh induced on Γ by $\mathcal{T}_{C,h}$ is not assumed to coincide with $\mathcal{T}_{\Gamma,h}$.

Let $\mathbb{P}_k$, $k \geq 1$, be the space of polynomials of degree less than or equal to k. For $r \geq 1$, $s \geq 1$ and $t \geq 1$ we employ the discrete spaces given by Lagrange nodal elements

$$W^r_{C,h} := \{\mathbf{w}_{C,h} \in (C^0(\Omega_C))^3 \\ \mid \mathbf{w}_{C,h|K} \in (\mathbb{P}_r)^3 \; \forall\, K \in \mathcal{T}_h \;,\; \mathbf{w}_{C,h} \cdot \mathbf{n}_C = 0 \text{ on } \Gamma\} \;,$$

$$L^s_{C,h} := \{Q_{C,h} \in C^0(\Omega_C) \mid Q_{C,h|K} \in \mathbb{P}_s \; \forall\, K \in \mathcal{T}_{C,h}\} \;,$$

and

$$L^t_{\Gamma,h} := \{\eta_h \in C^0(\Gamma) \mid \eta_{h|T} \in \mathbb{P}_t \; \forall\, T \in \mathcal{T}_{\Gamma,h}\} \;.$$

Clearly, we have $W^r_{C,h} \subset W_C$, $L^s_{C,h} \subset H^1(\Omega_C)$ and $L^t_{\Gamma,h} \subset H^{1/2}(\Gamma)$, therefore we are ready to consider a conforming finite element approximation.

The discrete problem is given by

$$\begin{array}{l}
\text{Find } (\mathbf{A}_{C,h}, V_{C,h}, q_h) \in W^r_{C,h} \times L^s_{C,h}/\mathbb{C} \times L^t_{\Gamma,h}/\mathbb{C} \text{ such that} \\
\quad \int_{\Omega_C} (\boldsymbol{\mu}_C^{-1} \operatorname{curl} \mathbf{A}_{C,h} \cdot \operatorname{curl} \overline{\mathbf{w}_{C,h}} + \mu_*^{-1} \operatorname{div} \mathbf{A}_{C,h} \operatorname{div} \overline{\mathbf{w}_{C,h}} \\
\qquad\qquad + i\omega\boldsymbol{\sigma} \mathbf{A}_{C,h} \cdot \overline{\mathbf{w}_{C,h}} + \boldsymbol{\sigma} \operatorname{grad} V_{C,h} \cdot \overline{\mathbf{w}_{C,h}}) \\
\qquad + \int_\Gamma [-\frac{1}{2} q_h - \mathcal{D}(q_h) + \frac{1}{\mu_0} \mathcal{S}(\operatorname{curl} \mathbf{A}_{C,h} \cdot \mathbf{n}_C)] \operatorname{curl} \overline{\mathbf{w}_{C,h}} \cdot \mathbf{n}_C \\
\qquad = \int_{\Omega_C} \mathbf{J}_{e,C} \cdot \overline{\mathbf{w}_{C,h}} \\
\quad \int_{\Omega_C} (i\omega\boldsymbol{\sigma} \mathbf{A}_{C,h} \cdot \operatorname{grad} \overline{Q_{C,h}} + \boldsymbol{\sigma} \operatorname{grad} V_{C,h} \cdot \operatorname{grad} \overline{Q_{C,h}}) \\
\qquad = \int_{\Omega_C} \mathbf{J}_{e,C} \cdot \operatorname{grad} \overline{Q_{C,h}} \\
\quad \int_\Gamma [\frac{1}{2} \operatorname{curl} \mathbf{A}_{C,h} \cdot \mathbf{n}_C + \mathcal{D}'(\operatorname{curl} \mathbf{A}_{C,h} \cdot \mathbf{n}_C) + \mu_0 \mathcal{H}(q_h)] \overline{\eta_h} = 0 \\
\text{for all } (\mathbf{w}_{C,h}, Q_{C,h}, \eta_h) \in W^r_{C,h} \times L^s_{C,h}/\mathbb{C} \times L^t_{\Gamma,h}/\mathbb{C} \,.
\end{array} \tag{7.30}$$

Existence and uniqueness of the discrete solution follow by the Lax–Milgram lemma, applied in $W^r_{C,h} \times L^s_{C,h}/\mathbb{C} \times L^t_{\Gamma,h}/\mathbb{C}$.

We also have

Theorem 7.8. *Assume that Ω_C is a convex polyhedron, or else that the solution $(\mathbf{A}_C, V_C, q)$ is smooth enough. Then the discrete solution $(\mathbf{A}_{C,h}, V_{C,h}, q_h)$ converges in $W_C \times H^1(\Omega_C)/\mathbb{C} \times H^{1/2}(\Gamma)/\mathbb{C}$ to the exact solution $(\mathbf{A}_C, V_C, q)$.*

Proof. Let us start noting that, as proved in Lemma 7.2, $(\mathbf{A}_C, V_C, q)$ and $(\mathbf{A}_{C,h}, V_{C,h}, q_h)$ are solutions to problems (7.12) and (7.30), respectively, also for all test functions $\eta \in H^{1/2}(\Gamma)$ and $\eta_h \in L^t_{\Gamma,h}$. Similarly, it is obvious that (7.12) and (7.30) also hold for all test functions $Q_C \in H^1(\Omega_C)$ and $Q_{C,h} \in L^s_{C,h}$.

Therefore, finite element interpolants can be used as test functions, and, if the solution $(\mathbf{A}, V_C, q)$ is smooth enough, the convergence follows by applying Céa lemma and standard interpolation results.

If the domain Ω_C is convex, it is known (see Costabel et al. [89]) that smooth functions with vanishing normal component are dense in W_C, and the same arguments can be applied. □

Remark 7.9. As noted in Remark 6.6, if Ω_C is a non-convex polyhedral domain it can happen that the solution $\mathbf{A}_C$ is non-smooth, namely, not even an element of $(H^1(\Omega_C))^3$, and that $H^1_\tau(\Omega_C) := (H^1(\Omega_C))^3 \cap H_0(\text{div}; \Omega_C)$ is a closed proper subspace of W_C. Since the finite element space $W^r_{C,h}$ is contained in $H^1_\tau(\Omega_C)$, in that case a convergence result in W_C cannot hold. For non-convex domains, an alternative approch is presented in Section 7.5.1. □

The determination of a precise order of convergence requires the knowledge of the regularity of the solution: as usual, if $(\mathbf{A}_C, V_C, q) \in H^{k+1}(\Omega_C) \times H^{k+1}(\Omega_C) \times H^{k+1/2}(\Gamma)$, where the integer $k \geq 1$ is equal to $r = s = t$, the degree of polynomial approximation, we have

$$\Big(\int_{\Omega_C} (|\mathbf{A}_C - \mathbf{A}_{C,h}|^2 + |\operatorname{curl}(\mathbf{A}_C - \mathbf{A}_{C,h})|^2 + |\operatorname{div}(\mathbf{A}_C - \mathbf{A}_{C,h})|^2) + \int_{\Omega_C} (|V_C - V_{C,h}|^2 + |\operatorname{grad}(V_C - V_{C,h})|^2) + ||q - q_h||^2_{1/2,\Gamma} \Big)^{1/2} = O(h^k) ,$$

having chosen V_C, $V_{C,h}$, q and q_h such that $\int_{\Omega_C} V_C = \int_{\Omega_C} V_{C,h} = 0$ and $\int_\Gamma q = \int_\Gamma q_h = 0$.

On the other hand, in EEG and MEG applications a typical assumption for $\boldsymbol{\sigma}$, the human head conductivity, is that it is a piecewise-smooth (but not globally continuous) positive definite symmetric matrix. In this case, it is not clear if the solution is regular as required above. In general, one could expect that the solution belongs to $H^{1+\gamma}(\Omega_C) \times H^{1+\gamma}(\Omega_C) \times H^{1/2+\gamma}(\Gamma)$ for some γ with $0 < \gamma < 1/2$; however, we do not know a proof of this result.

It is worth noting that the same difficulty arises if one assumes $\omega = 0$, namely, one just considers the problem of electrostatics. In this case one has to approximate the solution V_C (determined up to an additive constant) of

$$\begin{cases} \operatorname{div}(\boldsymbol{\sigma} \operatorname{grad} V_C) = \operatorname{div} \mathbf{J}_{e,C} & \text{in } \Omega_C \\ \boldsymbol{\sigma} \operatorname{grad} V_C \cdot \mathbf{n}_C = \mathbf{J}_{e,C} \cdot \mathbf{n}_C & \text{on } \Gamma \ , \end{cases}$$

and the regularity of V_C is not easily determined for a piecewise-smooth positive definite symmetric matrix $\boldsymbol{\sigma}$. Therefore, also in this case the rate of convergence of a finite element approximation scheme is not easily determined.

Concerning the behaviour with respect to the angular frequency ω, in the discrete case we can repeat the proof of Theorem 7.6 and obtain (with obvious notation):

Theorem 7.10. *There exists a constant $K > 0$, independent of ω and h, such that for each ω with $0 < |\omega| < 1$, the solutions to (7.30) satisfy*

$$\begin{aligned}\int_{\Omega_C}(|\mathbf{A}^{\omega}_{C,h} - \mathbf{A}^{0}_{C,h}|^2 + |\operatorname{curl}\mathbf{A}^{\omega}_{C,h} - \operatorname{curl}\mathbf{A}^{0}_{C,h}|^2 \\ +|\operatorname{div}\mathbf{A}^{\omega}_{C,h} - \operatorname{div}\mathbf{A}^{0}_{C,h}|^2) \leq K\,\omega^2\end{aligned}$$

$$\int_{\Omega_C}(|V^{\omega}_{C,h} - V^{0}_{C,h}|^2 + |\operatorname{grad}V^{\omega}_{C,h} - \operatorname{grad}V^{0}_{C,h}|^2) \leq K\,\omega^2$$

$$||q^{\omega}_h - q^{0}_h||^2_{1/2,\Gamma} \leq K\,\omega^2 \ ,$$

having chosen $V^{\omega}_{C,h}$, $V^{0}_{C,h}$, q^{ω}_h and q^{0}_h such that $\int_{\Omega_C} V^{\omega}_{C,h} = \int_{\Omega_C} V^{0}_{C,h} = 0$ and $\int_{\Gamma} q^{\omega}_h = \int_{\Gamma} q^{0}_h = 0$.

An important point of the above result is that the behaviour in ω is uniform with respect to h; it is not evident that this is true for other finite element approximation schemes, as it is not always possible to show that the associated sesquilinear form is coercive uniformly with respect to ω (for our approach, this has been proved in Theorem 7.5).

Remark 7.11. A delicate point of the discretization is the efficient computation of the terms involving the single layer and double layer potentials and the hypersingular integral operator: an extensive literature is devoted to analyze this problem.

By integration by parts it is possible to restrict the problem to the computation of terms of the form

$$\int_{T\times T'} \frac{1}{|\mathbf{x}-\mathbf{y}|}\ p(\mathbf{y})q(\mathbf{x})\ dS_y dS_x$$

or

$$\int_{T\times T'} \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^3}\cdot \mathbf{n}(\mathbf{y})\ p(\mathbf{y})q(\mathbf{x})\ dS_y dS_x \ ,$$

where p, q are polynomials and T, T' are triangles of the mesh $\mathcal{T}_{\Gamma,h}$. If $T \cap T' = \emptyset$ the integrands are regular functions and standard cubature methods can be used. On the other hand, if $T = T'$ or $T \cap T'$ is an edge or a vertex the integrands have a singular behavior. As indicated in Börm and Hackbusch [56], different techniques can be applied to evaluate these terms. One possibility is to use quadrature rules adapted to the singularity of the kernel (see Schwab and Wendland [224]). Another possibility is to apply a suitable regularizing coordinate transformation that renders regular the integrand, and then to use standard cubature formulas (see Duffy [98], Erichsen and Sauter [101], Sauter and Lage [221]). Finally, semi-analytical approaches apply an exact integration at least for the inner integral (see Sauter and Schwab [222], Gray et al. [113]). □

7.5.1 The non-convex case

As we noted in Remark 7.9, if the conductor Ω_C is a polyhedral non-convex set it can happen that the convergence of the finite element approximation does not hold. Therefore, it is suitable to follow an alternative approach.

We start by recalling that, when the conductor has a complex geometry, it is usual to enclose it into a "simpler" set, and in this new region to look for a vector potential of the magnetic induction. This procedure, that is generally called the $(\mathbf{A}_C, V_C)-\mathbf{A}_I-\psi_I$ formulation, has been described in Section 6.3.

In our case, we assume that the conductor $\overline{\Omega_C}$ is included into a *polyhedral convex* bounded open set Ω_A, as small as possible. Setting now $\Omega_I := \mathbb{R}^3 \setminus \overline{\Omega_A}$, $\Gamma_A := \partial\Omega_A$,

$$W_A := H(\text{curl}; \Omega_A) \cap H_0(\text{div}; \Omega_A) \, ,$$

and denoting by $\mathbf{n}_A$ the unit outward normal vector on Γ_A, the weak formulation reads

$$\begin{aligned}
&\text{Find } (\mathbf{A}, V_C, q) \in W_A \times H^1(\Omega_C)/\mathbb{C} \times H^{1/2}(\Gamma_A)/\mathbb{C} \text{ such that} \\
&\quad \textstyle\int_{\Omega_A} (\boldsymbol{\mu}^{-1} \,\text{curl}\, \mathbf{A} \cdot \text{curl}\, \overline{\mathbf{w}} + \mu_*^{-1} \,\text{div}\, \mathbf{A} \,\text{div}\, \overline{\mathbf{w}}) \\
&\qquad + \textstyle\int_{\Omega_C} (i\omega\boldsymbol{\sigma} \mathbf{A}_C \cdot \overline{\mathbf{w}_C} + \boldsymbol{\sigma} \,\text{grad}\, V_C \cdot \overline{\mathbf{w}_C}) \\
&\qquad + \textstyle\int_{\Gamma_A} [-\frac{1}{2} q - \mathcal{D}(q) + \frac{1}{\mu_0} \mathcal{S}(\text{curl}\, \mathbf{A} \cdot \mathbf{n}_A)] \,\text{curl}\, \overline{\mathbf{w}} \cdot \mathbf{n}_A \\
&\qquad = \textstyle\int_{\Omega_C} \mathbf{J}_{e,C} \cdot \overline{\mathbf{w}_C} \\
&\quad \textstyle\int_{\Omega_C} (i\omega\boldsymbol{\sigma} \mathbf{A}_C \cdot \text{grad}\, \overline{Q_C} + \boldsymbol{\sigma} \,\text{grad}\, V_C \cdot \text{grad}\, \overline{Q_C}) \\
&\qquad = \textstyle\int_{\Omega_C} \mathbf{J}_{e,C} \cdot \text{grad}\, \overline{Q_C} \\
&\quad \textstyle\int_{\Gamma_A} [\frac{1}{2} \,\text{curl}\, \mathbf{A} \cdot \mathbf{n}_A + \mathcal{D}'(\text{curl}\, \mathbf{A} \cdot \mathbf{n}_A) + \mu_0 \mathcal{H}(q)] \overline{\eta} = 0 \\
&\text{for all } (\mathbf{w}, Q_C, \eta) \in W_A \times H^1(\Omega_C)/\mathbb{C} \times H^{1/2}(\Gamma_A)/\mathbb{C} \, .
\end{aligned} \tag{7.31}$$

The results presented in Section 7.3, as well as those in Sections 7.4 and 7.5, can be easily obtained also for this formulation, with essentially the same proofs. In particular, the finite element approximation scheme converges, as stated in Theorem 7.8, since the domain Ω_A is convex. All the details concerning this approach have been given in Alonso Rodríguez and Valli [19].

7.6 Other FEM–BEM approaches

Among the FEM–BEM formulations that we mentioned at the beginning of this chapter, in this section we briefly present those due to Bossavit and Vérité [62], [63], Meddahi and Selgas [176] and Hiptmair [127].

7.6.1 The code TRIFOU

The first authors who proposed a coupled FEM–BEM formulation of the eddy current problem are Bossavit and Vérité [62], [63]. Based on this coupled approach, they have also developed a popular numerical code, named TRIFOU, widely used at Electricité de France since 1980.

We recall that, for the sake of simplicity, we are assuming that Ω_C is simply-connected and that $\Omega_I = \mathbb{R}^3 \setminus \overline{\Omega_C}$ is connected, so that the boundary $\Gamma = \partial\Omega_C$ is connected. As a consequence, as in Chapter 5 we can write $\mathbf{H}_I = \operatorname{grad}\psi_I$.

As in (3.9), for each test function $\mathbf{v} \in H(\operatorname{curl};\mathbb{R}^3)$ with $\operatorname{curl}\mathbf{v}_I = \mathbf{0}$ in Ω_I we have

$$\int_{\Omega_C} \boldsymbol{\sigma}^{-1} \operatorname{curl}\mathbf{H}_C \cdot \operatorname{curl}\overline{\mathbf{v}_C} + \int_{\mathbb{R}^3} i\omega\boldsymbol{\mu}\mathbf{H}\cdot\overline{\mathbf{v}} = \int_{\Omega_C} \boldsymbol{\sigma}^{-1}\mathbf{J}_{e,C}\cdot\operatorname{curl}\overline{\mathbf{v}_C} \,. \quad (7.32)$$

On the other hand, writing $\mathbf{v}_I = \operatorname{grad}\chi_I$ and remembering that ψ_I is a harmonic function vanishing at infinity, we find by integration by parts

$$\begin{aligned} \textstyle\int_{\Omega_I} i\omega\mu_0\mathbf{H}_I\cdot\overline{\mathbf{v}_I} &= \textstyle\int_{\Omega_I} i\omega\mu_0 \operatorname{grad}\psi_I\cdot\operatorname{grad}\overline{\chi_I} \\ &= \textstyle\int_\Gamma i\omega\mu_0 \operatorname{grad}\psi_I\cdot\mathbf{n}_I\,\overline{\chi_I} \,. \end{aligned} \quad (7.33)$$

We introduce the linear and continuous Steklov–Poincaré operator $\mathcal{R}$ as

$$\mathcal{R} : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma) \ , \ \ \mathcal{R}(\chi_\Gamma) := \operatorname{grad}\chi_I\cdot\mathbf{n}_I \quad \text{on } \Gamma \,, \quad (7.34)$$

where χ_I belongs to the space $W^1(\Omega_I)$ introduced in (7.17) and satisfies $\Delta\chi_I = 0$ in Ω_I and $\chi_{I|\Gamma} = \chi_\Gamma$. We also set

$$\begin{aligned} \widetilde{W} := \{(\mathbf{v}_C,\chi_\Gamma) \in H(\operatorname{curl};\Omega_C)\times H^{1/2}(\Gamma) \\ \mid \mathbf{v}_C\times\mathbf{n}_C + \operatorname{grad}_\tau\chi_\Gamma\times\mathbf{n}_I = \mathbf{0} \text{ on } \Gamma\} \,. \end{aligned} \quad (7.35)$$

We can thus rewrite the eddy current problem as

$$\begin{aligned} &\text{Find } (\mathbf{H}_C,\psi_\Gamma) \in \widetilde{W} \text{ such that} \\ &\quad \textstyle\int_{\Omega_C} \boldsymbol{\sigma}^{-1}\operatorname{curl}\mathbf{H}_C\cdot\operatorname{curl}\overline{\mathbf{v}_C} + \int_{\Omega_C} i\omega\boldsymbol{\mu}_C\mathbf{H}_C\cdot\overline{\mathbf{v}_C} \\ &\qquad\qquad + i\omega\mu_0 \textstyle\int_\Gamma \mathcal{R}(\psi_\Gamma)\,\overline{\chi_\Gamma} \\ &\qquad = \textstyle\int_{\Omega_C} \boldsymbol{\sigma}^{-1}\mathbf{J}_{e,C}\cdot\operatorname{curl}\overline{\mathbf{v}_C} \\ &\text{for each } (\mathbf{v}_C,\chi_\Gamma) \in \widetilde{W} \,. \end{aligned} \quad (7.36)$$

By the trace inequality (A.8) and the Poincaré inequality in $W^1(\Omega_I)$ (see, e.g., Nédélec [187], Theor. 2.5.13) we have

$$\int_\Gamma \mathcal{R}(\chi_\Gamma)\,\overline{\chi_\Gamma} = \int_{\Omega_I} \operatorname{grad}\chi_I\cdot\operatorname{grad}\overline{\chi_I} \geq \kappa_0\|\chi_\Gamma\|_{1/2,\Gamma} \,,$$

hence the sesquilinear form $a_T(\cdot,\cdot)$ at the left hand side of (7.36) is clearly coercive in $\widetilde{W}$, and the problem is well-posed.

If one is interested in finding also the magnetic field in Ω_I, one has to set

$$\psi_I = \mathcal{D}_I(\psi_\Gamma) - \frac{1}{\mu_0}\mathcal{S}_I(\boldsymbol{\mu}_C\mathbf{H}_C\cdot\mathbf{n}_C) \,,$$

where the operators $\mathcal{S}_I$ and $\mathcal{D}_I$ have been introduced in (7.15) and (7.16).

When considering numerical approximation we assume that Ω_C is a Lipschitz polyhedral domain, and we denote by $\mathcal{T}_{C,h}$ a regular family of triangulations of Ω_C and by $\mathcal{T}_{\partial,h}$ the mesh induced on Γ by $\mathcal{T}_{C,h}$. We also suppose that each element K of $\mathcal{T}_{C,h}$ is a tetrahedron. We consider

$$\begin{aligned}\widetilde{W}_h := \{(\mathbf{v}_{C,h}, \chi_{\Gamma,h}) \in N^1_{C,h} \times C^0(\Gamma) \,|\, \chi_{\Gamma,h|T} \in \mathbb{P}_1 \; \forall\, T \in \mathcal{T}_{\partial,h}, \\ \mathbf{v}_{C,h} \times \mathbf{n}_C + \operatorname{grad}_\tau \chi_{\Gamma,h} \times \mathbf{n}_I = \mathbf{0} \text{ on } \Gamma\} \ ,\end{aligned}$$

where $N^1_{C,h}$ is the space of Nédélec curl-conforming edge elements of the lowest order in Ω_C (see Section A.2).

Due to the constraint on Γ, any function $(\mathbf{v}_{C,h}, \chi_{\Gamma,h})$ in $\widetilde{W}_h$ can be clearly written as

$$\mathbf{v}_{C,h} = \sum_{e \in \mathcal{E}^0_{C,h}} \alpha_e \mathbf{q}_e + \sum_{v \in \mathcal{V}_{\Gamma,h}} \alpha_v \operatorname{grad} \varphi_v \ \ , \ \ \chi_{\Gamma,h} = \sum_{v \in \mathcal{V}_{\Gamma,h}} \alpha_v \varphi_v \ ,$$

where $\mathcal{E}^0_{C,h}$ is the set of edges $e \in \mathcal{T}_{C,h}$ that are internal to Ω_C, $\mathcal{V}_{\Gamma,h}$ is the set of vertices $v \in \mathcal{T}_{\partial,h}$, and we have denoted by $\mathbf{q}_e$ the edge basis function defined in Ω_C and associated to the edge e, and by φ_v the nodal basis function defined in Ω_C and associated to the vertex v.

For a suitable implementation it is necessary to find a sound and computationally cheap approximation of the Steklov–Poincaré operator $\mathcal{R}$. Recalling the definition of the operators $\mathcal{S}$ in (7.6) and $\mathcal{S}_I$ in (7.15), we can write $\psi_I = \mathcal{S}_I(\lambda_\Gamma)$ and $\psi_\Gamma = \mathcal{S}(\lambda_\Gamma)$, where, as a consequence of well-known results in potential theory, $\lambda_\Gamma \in H^{-1/2}(\Gamma)$ satisfies

$$\operatorname{grad} \psi_I \cdot \mathbf{n}_I = \frac{1}{2}\lambda_\Gamma - \mathcal{D}'(\lambda_\Gamma) \qquad \text{on } \Gamma$$

(see, e.g., McLean [175], Nédélec [187]). Passing to a variational formulation, we are looking for $\psi_\Gamma \in H^{1/2}(\Gamma)$ and $\lambda_\Gamma \in H^{-1/2}(\Gamma)$ such that

$$\begin{aligned}&\textstyle\int_\Gamma \mathcal{S}(\lambda_\Gamma)\, \overline{\xi_\Gamma} = \int_\Gamma \psi_\Gamma \, \overline{\xi_\Gamma} \\ &\textstyle\int_\Gamma \mathcal{R}(\psi_\Gamma)\, \overline{\chi_\Gamma} = \int_\Gamma [\frac{1}{2}\lambda_\Gamma - \mathcal{D}'(\lambda_\Gamma)]\, \overline{\chi_\Gamma}\end{aligned}$$

for all $\chi_\Gamma \in H^{1/2}(\Gamma)$ and $\xi_\Gamma \in H^{-1/2}(\Gamma)$.

In matrix form we can write, with obvious notation,

$$\begin{aligned}&S\boldsymbol{\lambda}_\Gamma = B^T_\Gamma \boldsymbol{\psi}_\Gamma \\ &R\boldsymbol{\psi}_\Gamma = \tfrac{1}{2} B_\Gamma \boldsymbol{\lambda}_\Gamma - D'\boldsymbol{\lambda}_\Gamma \ ,\end{aligned}$$

where the vector unknowns are complex-valued, while the matrices are real-valued (as we can choose real-valued finite element basis functions). We also see at once that the matrix S is symmetric and positive definite, hence we can rewrite

$$R = \Big(\frac{1}{2}B_\Gamma - D'\Big) S^{-1} B^T_\Gamma \ .$$

Unfortunately, this matrix is not symmetric, though the Steklov–Poincaré operator $\mathcal{R}$ is hermitian. Therefore, in the TRIFOU code the following symmetric matrix

$$R_\star := \frac{1}{2}(R + R^T)$$

has been proposed as an approximation of the operator $\mathcal{R}$.

Though a complete analysis of the convergence of the method is not available, the TRIFOU code has been used in many engineering applications with satisfactory results (for a deeper insight and additional comments, see Bossavit and Vérité [62], [63], Bossavit [57]; in particular, note that the matrix $R_\star$ may even happen to be singular: see Bossavit [59], p. 214).

7.6.2 An approach based on the magnetic field $\mathbf{H}_C$

In Meddahi and Selgas [176], following an approach that is close to that presented in the preceding section, the authors choose as unknowns $\mathbf{H}_C$ in Ω_C and $\boldsymbol{\mu}_C \mathbf{H}_C \cdot \mathbf{n}_C$ on Γ, and derive a symmetric formulation. Again we assume, for the sake of simplicity, that Ω_C is simply-connected and that the boundary $\Gamma = \partial\Omega_C$ is connected (for the general not simply-connected case, see Meddahi and Selgas [176]). Consequently, we can write $\mathbf{H}_I = \operatorname{grad} \psi_I$.

As before, we obtain (7.32) and (7.33), and, using the interface condition $\boldsymbol{\mu}_C \mathbf{H}_C \cdot \mathbf{n}_C + \mu_0 \operatorname{grad} \psi_I \cdot \mathbf{n}_I = 0$ on Γ, we also find

$$\begin{aligned} \textstyle\int_{\Omega_I} i\omega\mu_0 \mathbf{H}_I \cdot \overline{\mathbf{v}_I} &= \textstyle\int_\Gamma i\omega\mu_0 \operatorname{grad} \psi_I \cdot \mathbf{n}_I \overline{\chi_I} \\ &= -\textstyle\int_\Gamma i\omega \boldsymbol{\mu}_C \mathbf{H}_C \cdot \mathbf{n}_C \overline{\chi_I} \,. \end{aligned}$$

Furthermore, we can rewrite (7.10) and (7.11) as

$$\frac{1}{2}\psi_\Gamma - \mathcal{D}(\psi_\Gamma) + \frac{1}{\mu_0}\mathcal{S}(\boldsymbol{\mu}_C \mathbf{H}_C \cdot \mathbf{n}_C) = 0 \text{ on } \Gamma \tag{7.37}$$

$$\frac{1}{2\mu_0}\boldsymbol{\mu}_C \mathbf{H}_C \cdot \mathbf{n}_C + \frac{1}{\mu_0}\mathcal{D}'(\boldsymbol{\mu}_C \mathbf{H}_C \cdot \mathbf{n}_C) + \mathcal{H}(\psi_\Gamma) = 0 \text{ on } \Gamma \,. \tag{7.38}$$

Thus, setting $\chi_\Gamma := \chi_{I|\Gamma}$, we easily find

$$\begin{aligned} &\textstyle\int_{\Omega_C} \boldsymbol{\sigma}^{-1} \operatorname{curl} \mathbf{H}_C \cdot \operatorname{curl} \overline{\mathbf{v}_C} + \int_{\Omega_C} i\omega \boldsymbol{\mu}_C \mathbf{H}_C \cdot \overline{\mathbf{v}_C} \\ &\qquad + i\omega \textstyle\int_\Gamma [-\frac{1}{2}\boldsymbol{\mu}_C \mathbf{H}_C \cdot \mathbf{n}_C + \mathcal{D}'(\boldsymbol{\mu}_C \mathbf{H}_C \cdot \mathbf{n}_C) + \mu_0 \mathcal{H}(\psi_\Gamma)] \overline{\chi_\Gamma} \\ &\quad = \textstyle\int_{\Omega_C} \boldsymbol{\sigma}^{-1} \mathbf{J}_{e,C} \cdot \operatorname{curl} \overline{\mathbf{v}_C} \,, \end{aligned} \tag{7.39}$$

and for any test function ξ on Γ we also have

$$\int_\Gamma [\frac{1}{2}\psi_\Gamma - \mathcal{D}(\psi_\Gamma) + \frac{1}{\mu_0}\mathcal{S}(\boldsymbol{\mu}_C \mathbf{H}_C \cdot \mathbf{n}_C)] \overline{\xi} = 0 \,. \tag{7.40}$$

Let us set $\lambda := \mu_0^{-1} \boldsymbol{\mu}_C \mathbf{H}_C \cdot \mathbf{n}_C$. From the Stokes theorem for closed surfaces we have $\lambda \in H_\sharp^{-1/2}(\Gamma)$, where

$$H_\sharp^{-1/2}(\Gamma) := \left\{ \xi \in H^{-1/2}(\Gamma) \mid \int_\Gamma \xi = 0 \right\} .$$

Moreover, as in Section 4.4, define

$$\widetilde{X}_C := \{\mathbf{v}_C \in H(\mathrm{curl}; \Omega_C) \,|\, \mathrm{div}_\tau(\mathbf{v}_C \times \mathbf{n}_C) = 0 \text{ on } \Gamma\} \, ,$$

and set

$$\widetilde{X}_\Gamma := \{(\mathbf{v}_C \times \mathbf{n}_C)_{|\Gamma} \,|\, \mathbf{v}_C \in \widetilde{X}_C\} \, . \tag{7.41}$$

Introducing the operator

$$\mathrm{Curl}_\tau \chi_\Gamma := \mathrm{grad}\, \chi_I \times \mathbf{n}_I$$

(see also Section A.1), it is straightforward to verify that $\mathrm{Curl}_\tau \chi_\Gamma \in \widetilde{X}_\Gamma$.

We have seen in Section 7.2 that $\mathcal{D}(1) = -\frac{1}{2}$, so that

$$\begin{aligned} \textstyle\int_\Gamma [-\frac{1}{2}\boldsymbol{\mu}_C \mathbf{H}_C \cdot \mathbf{n}_C + \mathcal{D}'(\boldsymbol{\mu}_C \mathbf{H}_C \cdot \mathbf{n}_C)] &= \mu_0 \textstyle\int_\Gamma [-\frac{1}{2}\lambda + \mathcal{D}'(\lambda)] \\ &= \mu_0 \textstyle\int_\Gamma [-\frac{1}{2}\lambda + \lambda \mathcal{D}(1)] = -\mu_0 \int_\Gamma \lambda = 0 \, . \end{aligned}$$

Moreover, it holds $\int_\Gamma \mathcal{H}(\eta) = 0$ for each $\eta \in H^{1/2}(\Gamma)$ and $\mathcal{H}(1) = 0$, hence equation (7.39) does not change if we add a constant to ψ_Γ and χ_Γ. Instead, adding a constant to ψ_Γ we have

$$\frac{1}{2}(\psi_\Gamma + c_0) - \mathcal{D}(\psi_\Gamma + c_0) = \frac{1}{2}\psi_\Gamma - \mathcal{D}(\psi_\Gamma) + c_0 \, ,$$

therefore equation (7.40) does not change if we choose the test function $\xi \in H_\sharp^{-1/2}(\Gamma)$.

Meddahi and Selgas [176] proved that Curl_τ is an isomorphism from $H^{1/2}(\Gamma)/\mathbb{C}$ onto $\widetilde{X}_\Gamma$. Since $\mathbf{H}_C \times \mathbf{n}_C = -\,\mathrm{grad}\,\psi_I \times \mathbf{n}_I = -\mathrm{Curl}_\tau \psi_\Gamma$ on Γ, in (7.39) we can replace ψ_Γ and the test function χ_Γ with $\mathrm{Curl}_\tau^{-1}(\mathbf{H}_C \times \mathbf{n}_C)$ and $\mathrm{Curl}_\tau^{-1}(\mathbf{v}_C \times \mathbf{n}_C)$, respectively, and we finally obtain that the eddy current problem can be rewritten as

$$\begin{cases} \text{Find } (\mathbf{H}_C, \lambda) \in \widetilde{X}_C \times H_\sharp^{-1/2}(\Gamma) \text{ such that} \\ \int_{\Omega_C} \boldsymbol{\sigma}^{-1}\, \mathrm{curl}\, \mathbf{H}_C \cdot \mathrm{curl}\, \overline{\mathbf{v}_C} + \int_{\Omega_C} i\omega \boldsymbol{\mu}_C \mathbf{H}_C \cdot \overline{\mathbf{v}_C} \\ \qquad + i\omega\mu_0 \int_\Gamma [\frac{1}{2}\lambda - \mathcal{D}'(\lambda)]\, \mathrm{Curl}_\tau^{-1}(\overline{\mathbf{v}_C} \times \mathbf{n}_C) \\ \qquad + i\omega\mu_0 \int_\Gamma \mathcal{H}(\mathrm{Curl}_\tau^{-1}(\mathbf{H}_C \times \mathbf{n}_C))\, \mathrm{Curl}_\tau^{-1}(\overline{\mathbf{v}_C} \times \mathbf{n}_C) \\ \quad = \int_{\Omega_C} \boldsymbol{\sigma}^{-1} \mathbf{J}_{e,C} \cdot \mathrm{curl}\, \overline{\mathbf{v}_C} \\ i\omega\mu_0 \int_\Gamma [-\frac{1}{2}\mathrm{Curl}_\tau^{-1}(\mathbf{H}_C \times \mathbf{n}_C) + \mathcal{D}(\mathrm{Curl}_\tau^{-1}(\mathbf{H}_C \times \mathbf{n}_C))]\, \overline{\xi} \\ \qquad + i\omega\mu_0 \int_\Gamma \mathcal{S}(\lambda)\, \overline{\xi} = 0 \\ \text{for each } (\mathbf{v}_C, \xi) \in \widetilde{X}_C \times H_\sharp^{-1/2}(\Gamma) \, . \end{cases} \tag{7.42}$$

Note that

$$\begin{aligned} & i\omega\mu_0 \textstyle\int_\Gamma [\frac{1}{2}\xi - \mathcal{D}'(\xi)]\, \mathrm{Curl}_\tau^{-1}(\overline{\mathbf{v}_C} \times \mathbf{n}_C) \\ & \qquad + i\omega\mu_0 \textstyle\int_\Gamma [-\frac{1}{2}\mathrm{Curl}_\tau^{-1}(\mathbf{v}_C \times \mathbf{n}_C) + \mathcal{D}(\mathrm{Curl}_\tau^{-1}(\mathbf{v}_C \times \mathbf{n}_C))]\, \overline{\xi} \end{aligned}$$

is a real number, namely, it is equal to

$$-2\omega\mu_0 \,\mathrm{Im}\left(\int_\Gamma [\frac{1}{2}\xi - \mathcal{D}'(\xi)]\,\mathrm{Curl}_\tau^{-1}(\overline{\mathbf{v}_C} \times \mathbf{n}_C) \right) .$$

Moreover, taking into account that the operator $\mathcal{D}$ is continuous from $H^{1/2}(\Gamma)$ into itself and that the operator Curl_τ^{-1} is continuous from $\widetilde{X}_\Gamma$ into $H^{1/2}(\Gamma)/\mathbb{C}$, it follows that for each $0 < \delta < 1$ one has

$$\begin{aligned} 2|\omega|\mu_0 \left| \textstyle\int_\Gamma [\frac{1}{2}\xi - \mathcal{D}'(\xi)]\,\mathrm{Curl}_\tau^{-1}(\overline{\mathbf{v}_C} \times \mathbf{n}_C) \right| & \\ \leq c_* \|\xi\|_{-1/2,\Gamma} (\|\mathbf{v}_C\|_{0,\Omega_C} + \|\,\mathrm{curl}\,\mathbf{v}_C\|_{0,\Omega_C}) & \\ \leq \delta \|\,\mathrm{curl}\,\mathbf{v}_C\|^2_{0,\Omega_C} + C_* \delta^{-1} \|\xi\|^2_{-1/2,\Gamma} + C_* \|\mathbf{v}_C\|^2_{0,\Omega_C} & . \end{aligned}$$

Then, recalling (7.27) and adapting the proof of Theorem 7.5, by choosing δ small enough it is not difficult to show that the sesquilinear form $a_C^\Gamma(\cdot,\cdot)$ at the left hand side of (7.42) is coercive in $\widetilde{X}_C \times H_\sharp^{-1/2}(\Gamma)$. Problem (7.42) is therefore well-posed.

Having solved (7.42), one can determine ψ_I in Ω_I by setting

$$\psi_I = -\mathcal{D}_I(\mathrm{Curl}_\tau^{-1}(\mathbf{H}_C \times \mathbf{n}_C)) - \mathcal{S}_I(\lambda) .$$

The numerical approximation needs some remarks, as a conforming discretization requires that the finite element functions $\mathbf{v}_{C,h}$ satisfy the constraint $\mathrm{div}_\tau(\mathbf{v}_{C,h} \times \mathbf{n}_C) = \mathbf{0}$ on Γ. Instead of introducing a Lagrange multiplier, as done in Section 4.5, here we present an alternative approach, based on the explicit construction of a basis for the space

$$\widetilde{X}_{C,h} := \{\mathbf{v}_{C,h} \in N^1_{C,h} \mid \mathrm{div}_\tau(\mathbf{v}_{C,h} \times \mathbf{n}_C) = 0 \text{ on } \Gamma\} ,$$

where $N^1_{C,h}$ is the space of Nédélec curl-conforming edge elements of the lowest order (see Section A.2). Note that this construction could be used also for the approach presented in Section 4.5.

As in the preceding section, we assume that Ω_C is a Lipschitz polyhedral domain, and we denote by $\mathcal{T}_{C,h}$ and $\mathcal{T}_{\Gamma,h}$ two regular families of triangulations of Ω_C and Γ, respectively. We suppose that each element K of $\mathcal{T}_{C,h}$ is a tetrahedron and that each element T of $\mathcal{T}_{\Gamma,h}$ is a triangle. Let us also denote by $\mathcal{T}_{\partial,h}$ the mesh induced on Γ by $\mathcal{T}_{C,h}$; it is not assumed to coincide with $\mathcal{T}_{\Gamma,h}$. Finally, $\mathcal{E}^0_{C,h}$ denotes the set of edges $e \in \mathcal{T}_{C,h}$ that are internal to Ω_C, $\mathcal{V}_{\Gamma,h}$ the set of vertices $v \in \mathcal{T}_{\partial,h}$, $\mathbf{q}_e$ the lowest-order edge basis function defined in Ω_C and associated to the edge e, and φ_v the piecewise-linear nodal basis function defined in Ω_C and associated to the vertex v.

Proposition 7.12. *Let $v_0 \in \Gamma$ be a fixed vertex of $\mathcal{V}_{\Gamma,h}$. The set*

$$\widetilde{\mathcal{B}}_h := \{\mathbf{q}_e \mid e \in \mathcal{E}^0_{C,h}\} \cup \{\mathrm{grad}\,\varphi_v \mid v \in \mathcal{V}_{\Gamma,h}, v \neq v_0\}$$

is a basis of $\widetilde{X}_{C,h}$.

Proof. Let us start by showing that the elements of $\widetilde{\mathcal{B}}_h$ are linearly independent. Suppose that

$$\sum_{e\in\mathcal{E}^0_{C,h}} \alpha_e \mathbf{q}_e + \sum_{\substack{v\in\mathcal{V}_{\Gamma,h}\\ v\neq v_0}} \alpha_v \,\mathrm{grad}\,\varphi_v = \mathbf{0} \ .$$

Then on Γ we have

$$\mathbf{0} = \Big(\sum_{e\in\mathcal{E}^0_{C,h}} \alpha_e \mathbf{q}_e + \sum_{\substack{v\in\mathcal{V}_{\Gamma,h}\\ v\neq v_0}} \alpha_v \,\mathrm{grad}\,\varphi_v \Big) \times \mathbf{n}_C = \mathrm{grad}\Big(\sum_{\substack{v\in\mathcal{V}_{\Gamma,h}\\ v\neq v_0}} \alpha_v \varphi_v \Big) \times \mathbf{n}_C \ ,$$

so that

$$\sum_{\substack{v\in\mathcal{V}_{\Gamma,h}\\ v\neq v_0}} \alpha_v \varphi_v = k_0 \text{ on } \Gamma \ ,$$

where k_0 is a constant. Since $\varphi_v(v_0) = 0$ for each $v \neq v_0$, we have $k_0 = 0$ and therefore $\alpha_v = 0$ for each $v \in \mathcal{V}_{\Gamma,h}, v \neq v_0$. Then we are left with $\sum_{e\in\mathcal{E}^0_{C,h}} \alpha_e \mathbf{q}_e = \mathbf{0}$, which gives $\alpha_e = 0$ for each $e \in \mathcal{E}^0_{C,h}$.

On the other hand, the inclusion $\widetilde{\mathcal{B}}_h \subset \widetilde{X}_{C,h}$ is clearly true. Moreover, take $\mathbf{v}_{C,h} \in \widetilde{X}_{C,h}$: since $\mathrm{div}_\tau(\mathbf{v}_{C,h} \times \mathbf{n}_C) = \mathbf{0}$ on Γ, recalling that Γ is simply-connected it is possible to find a piecewise-linear function $\varphi_{\Gamma,h}$, defined on Γ, such that $\mathrm{grad}_\tau\, \varphi_{\Gamma,h} \times \mathbf{n}_C = \mathbf{v}_{C,h} \times \mathbf{n}_C$ on Γ. The function $\varphi_{\Gamma,h}$ is uniquely determined by requiring $\varphi_{\Gamma,h}(v_0) = 0$. The extension of $\varphi_{\Gamma,h}$ in Ω_C, obtained by setting all its internal nodal values equal to 0, will be denoted by φ_h. Clearly, $\mathrm{grad}\,\varphi_h$ belongs to the space spanned by the set of functions $\{\mathrm{grad}\,\varphi_v \,|\, v \in \mathcal{V}_{\Gamma,h}, v \neq v_0\}$. Since $\mathbf{v}_{C,h} \times \mathbf{n}_C = \mathrm{grad}\,\varphi_h \times \mathbf{n}_C$ on Γ, it follows that $(\mathbf{v}_{C,h} - \mathrm{grad}\,\varphi_h)$ is an edge element belonging to the space spanned by the set of functions $\{\mathbf{q}_e \,|\, e \in \mathcal{E}^0_{C,h}\}$, and the thesis follows. □

Thus we have a viable description of the finite element space $\widetilde{X}_{C,h}$, and, since we know that $\widetilde{X}_{C,h} \subset \widetilde{X}_C$, a conforming approximation scheme is readily devised. The finite element space used for approximating functions in $H^{-1/2}_\sharp(\Gamma)$ is typically

$$M_{\Gamma,h} := \{\xi_h \in L^2(\Gamma) \,|\, \xi_{h|T} \in \mathbb{P}_0 \ \forall\, T \in \mathcal{T}_{\Gamma,h} \ , \ \textstyle\int_\Gamma \xi_h = 0\} \ ,$$

and the convergence of the scheme is a straightforward consequence of Céa lemma.

It should also be noted that, in the implementation of the finite element scheme, the inverse of the tangential operator Curl_τ does not appear. In fact, let $\mathbf{b}_C \in \widetilde{\mathcal{B}}_h$ be a basis function. If $\mathbf{b}_C = \mathbf{q}_e$, one has $\mathbf{b}_C \times \mathbf{n}_C = \mathbf{0}$ on Γ; if $\mathbf{b}_C = \mathrm{grad}\,\varphi_v$, it holds $\mathrm{Curl}_\tau \varphi_v = -\,\mathrm{grad}\,\varphi_v \times \mathbf{n}_C = -\mathbf{b}_C \times \mathbf{n}_C$ on Γ, thus in the finite element approximation of (7.42) we can replace $\mathrm{Curl}_\tau^{-1}(\mathbf{b}_C \times \mathbf{n}_C)$ with $-\varphi_{v|\Gamma}$.

To illustrate the performance of this method, let us exhibit some numerical results presented in Selgas Buznego [226] for a couple of academic problems. In the first one the computational domain is the cube $\Omega_C = (-1,1)^3$, all the physical parameters are set equal to 1, and the current density $\mathbf{J}_e$ is computed starting from the exact solution

Table 7.1. Absolute errors for $\mathbf{H}_C$ and λ in the first example (courtesy of V. Selgas)

h	$\|\mathbf{H}_C - \mathbf{H}_{C,h}\|_{H(\mathrm{curl};\Omega_C)}$	$\|\lambda - \lambda_h\|_{0,\Gamma}$	α
0.7733	34.8247	0.3136	-
0.5330	25.0762	0.1634	0.8825
0.2989	14.0539	0.0257	1.0010
0.2337	11.4203	0.0131	0.8433

$\mathbf{E} = \mathrm{curl}(f, f, f)$, where

$$f(\mathbf{x}) := \begin{cases} (1-x_1^2)^4(1-x_2^2)^4(1-x_3^2)^4 & \text{in } \overline{\Omega_C} \\ 0 & \text{in } \Omega_I = \mathbb{R}^3 \setminus \overline{\Omega_C}. \end{cases}$$

In Table 7.1 and Figure 7.1 we report the absolute error and the convergence rate for $\mathbf{H}_C$ and λ for different value of the mesh size h. We have defined

$$\alpha := \frac{\log(\|\mathbf{H}_C - \mathbf{H}_{C,h_i}\|_{H(\mathrm{curl};\Omega_C)} / \|\mathbf{H}_C - \mathbf{H}_{C,h_{i+1}}\|_{H(\mathrm{curl};\Omega_C)})}{\log(h_i/h_{i+1})},$$

h_i and h_{i+1} being the mesh sizes of two consecutive computations.

In the second example the conductor Ω_C is the torus given by

$$\Omega_C := [(-1,1) \times (-1,1) \times (-1/2,1/2)] \setminus (-1/2,1/2)^3,$$

contained in the computational domain

$$\Omega := (-3/2,3/2) \times (-3/2,3/2) \times (-1,1)$$

(see Figure 7.2).

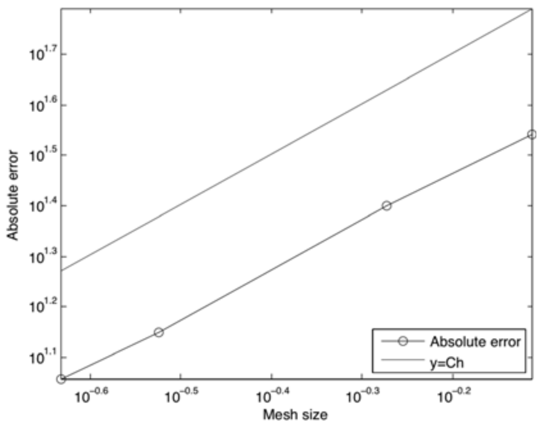

Fig. 7.1. Convergence rate for $\mathbf{H}_C$ in the first example (courtesy of V. Selgas)

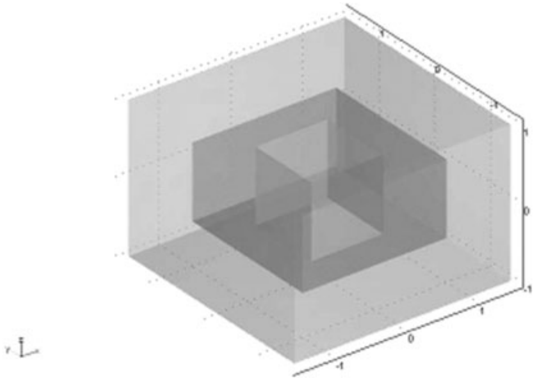

Fig. 7.2. The computational domain Ω for the second example (courtesy of V. Selgas)

In order to avoid the technical difficulties arising from the fact that Ω_C is not simply-connected, the variational formulation has been modified: to be precise, in $\mathbb{R}^3 \setminus \overline{\Omega}$ the usual approach based on the potential theory is used for reducing the contribution of the magnetic field $\mathbf{H}_I = \text{grad}\, \psi_I$ to suitable integrals on the boundary $\partial\Omega$, while the eddy current problem is solved in Ω by adopting the $\mathbf{H}$-based formulation described in Section 4.3.

Again, all the physical parameters are equal to 1, and the current density $\mathbf{J}_e$ is computed starting from the exact solution $\mathbf{E} = \text{curl}(g, g, g)$, where

$$g(\mathbf{x}) := \begin{cases} (1 - |\mathbf{x}|^2)^4 & \text{in } \overline{B(0,1)} \\ 0 & \text{in } \Omega_I = \mathbb{R}^3 \setminus \overline{B(0,1)} \end{cases} ,$$

where $B(0,1)$ is the ball of center 0 and radius 1. Note that in $\Omega \setminus \overline{\Omega_C}$ the curl of the magnetic field is not vanishing and consequently $\mathbf{H}_I$ is not the gradient of a scalar potential, while this is true outside Ω.

In Table 7.2 and Figure 7.3 the absolute error and the convergence rate for $\mathbf{H}_C$ and λ are presented for different value of the mesh size h.

Table 7.2. Absolute errors for $\mathbf{H}_C$ and λ in the second example (courtesy of V. Selgas)

h	$\|\mathbf{H}_C - \mathbf{H}_{C,h}\|_{H(\text{curl};\Omega_C)}$	$\|\lambda - \lambda_h\|_{0,\Gamma}$	α
0.9299	56.1051	0.7107	-
0.6601	39.3998	0.1413	1.0315
0.4572	28.6757	0.0620	0.8651
0.3653	22.0725	0.0173	1.1663

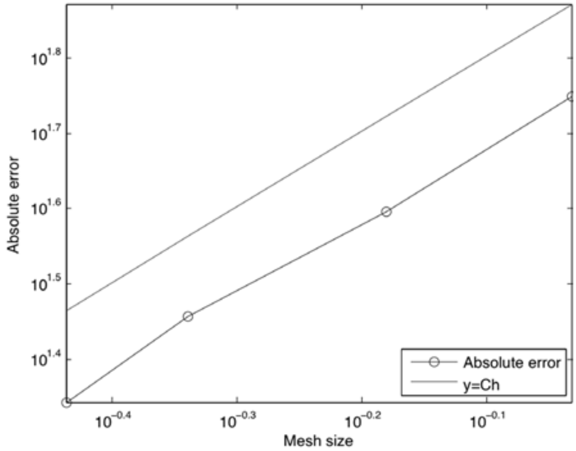

Fig. 7.3. Convergence rate for $\mathbf{H}_C$ in the second example (courtesy of V. Selgas)

7.6.3 An approach based on the electric field $\mathbf{E}_C$

Another FEM–BEM approach can be devised if one keeps the electric field $\mathbf{E}_C$ as principal unknown. Again, for the sake of simplicity, we assume that Ω_C is simply-connected and that the boundary $\Gamma = \partial\Omega_C$ is connected. Starting from the Ampère equation and inserting in it the Faraday equation, for a test function $\mathbf{z}$ that decays sufficiently fast at infinity we find

$$\begin{aligned}-i\omega \textstyle\int_{\Omega_C} \boldsymbol{\sigma}\mathbf{E}_C \cdot \overline{\mathbf{z}_C} - i\omega \int_{\Omega_C} \mathbf{J}_{e,C} \cdot \overline{\mathbf{z}_C} &= -i\omega \textstyle\int_{\mathbb{R}^3} \operatorname{curl} \mathbf{H} \cdot \overline{\mathbf{z}} \\ &= -i\omega \textstyle\int_{\mathbb{R}^3} \mathbf{H} \cdot \operatorname{curl} \overline{\mathbf{z}} = \int_{\mathbb{R}^3} \boldsymbol{\mu}^{-1} \operatorname{curl} \mathbf{E} \cdot \operatorname{curl} \overline{\mathbf{z}} \,.\end{aligned}$$

Since we know that $\mu_0^{-1} \operatorname{curl}\operatorname{curl} \mathbf{E}_I = -i\omega \operatorname{curl} \mathbf{H}_I = \mathbf{0}$ in Ω_I, we also have

$$\int_{\Omega_I} \mu_0^{-1} \operatorname{curl} \mathbf{E}_I \cdot \operatorname{curl} \overline{\mathbf{z}_I} = \int_\Gamma \mu_0^{-1} \operatorname{curl} \mathbf{E}_I \times \mathbf{n}_I \cdot \overline{\mathbf{z}_I} \,,$$

therefore we are left with

$$\begin{aligned}&\textstyle\int_{\Omega_C} \boldsymbol{\mu}_C^{-1} \operatorname{curl} \mathbf{E}_C \cdot \operatorname{curl} \overline{\mathbf{z}_C} + i\omega \int_{\Omega_C} \boldsymbol{\sigma}\mathbf{E}_C \cdot \overline{\mathbf{z}_C} \\ &\quad + \mu_0^{-1} \textstyle\int_\Gamma \operatorname{curl} \mathbf{E}_I \times \mathbf{n}_I \cdot \overline{\mathbf{z}_I} \\ &\quad = -i\omega \textstyle\int_{\Omega_C} \mathbf{J}_{e,C} \cdot \overline{\mathbf{z}_C} \,.\end{aligned} \tag{7.43}$$

Let us go on without giving all the details about the functional framework, but just presenting the main idea. Denote by R the vectorial Steklov–Poincaré operator given by

$$\mathsf{R}(\mathbf{q}) := \operatorname{curl} \mathbf{e}_I \times \mathbf{n}_I \qquad \text{on } \Gamma \,,$$

where $\mathbf{q} \cdot \mathbf{n}_I = 0$ on Γ and $\mathbf{e}_I$ is the solution to

$$\begin{cases} \operatorname{curl}\operatorname{curl}\mathbf{e}_I = \mathbf{0} & \text{in } \Omega_I \\ \operatorname{div}(\varepsilon_0 \mathbf{e}_I) = 0 & \text{in } \Omega_I \\ \mathbf{n}_I \times \mathbf{e}_I \times \mathbf{n}_I = \mathbf{q} & \text{on } \Gamma \\ \int_\Gamma \varepsilon_0 \mathbf{e}_I \cdot \mathbf{n}_I = 0 & \\ \mathbf{e}_I(\mathbf{x}) = O(|\mathbf{x}|^{-1}) & \text{as } |\mathbf{x}| \to \infty \, . \end{cases} \tag{7.44}$$

We thus have $\operatorname{curl}\mathbf{E}_I \times \mathbf{n}_I = \mathsf{R}(\mathbf{n}_C \times \mathbf{E}_C \times \mathbf{n}_C)$ on Γ, and we can rewrite (7.43) as

$$\begin{aligned} \int_{\Omega_C} \boldsymbol{\mu}_C^{-1} \operatorname{curl}\mathbf{E}_C \cdot \operatorname{curl}\overline{\mathbf{z}_C} + i\omega \int_{\Omega_C} \boldsymbol{\sigma}\mathbf{E}_C \cdot \overline{\mathbf{z}_C} \\ + \mu_0^{-1} \int_\Gamma \mathsf{R}(\mathbf{n}_C \times \mathbf{E}_C \times \mathbf{n}_C) \cdot \overline{\mathbf{z}_I} \\ = -i\omega \int_{\Omega_C} \mathbf{J}_{e,C} \cdot \overline{\mathbf{z}_C} \, . \end{aligned} \tag{7.45}$$

Bossavit [57] has extended the TRIFOU approach to this formulation, which is based on the electric field. We do not dwell on this here, referring the interested reader to the paper just quoted (see also Ren et al. [208]).

Instead, we present an alternative approach, proposed and analyzed by Hiptmair [127], which leads to a symmetric formulation (we also note that in that paper no restrictive assumption on the geometrical shape of the conducting domain Ω_C is imposed). First of all, as in (A.1), (A.3) and (A.4) introduce the trace spaces

$$H_T^{1/2}(\Gamma) := \{(\mathbf{n} \times \mathbf{v} \times \mathbf{n})_{|\Gamma} \,|\, \mathbf{v} \in (H^1(\Omega))^3\}$$

$$H^{-1/2}(\operatorname{div}_\tau; \Gamma) = \{(\mathbf{v}_C \times \mathbf{n}_C)_{|\Gamma} \,|\, \mathbf{v}_C \in H(\operatorname{curl}; \Omega_C)\},$$

and

$$H^{-1/2}(\operatorname{curl}_\tau; \Gamma) = \{(\mathbf{n}_C \times \mathbf{v}_C \times \mathbf{n}_C)_{|\Gamma} \,|\, \mathbf{v}_C \in H(\operatorname{curl}; \Omega_C)\} \, ;$$

note that the last two spaces are one the dual space of the other.

Let us define now on Γ the vectorial single layer and double layer potentials

$$\begin{aligned} &\mathsf{S} : (H_T^{1/2}(\Gamma))' \to H_T^{1/2}(\Gamma) \\ &\mathsf{S}(\mathbf{p})(\mathbf{x}) := \int_\Gamma \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|} \, \mathbf{p}(\mathbf{y}) dS_y \end{aligned} \tag{7.46}$$

$$\begin{aligned} &\mathsf{D} : H^{-1/2}(\operatorname{curl}_\tau; \Gamma) \to H^{-1/2}(\operatorname{curl}_\tau; \Gamma) \ , \\ &\mathsf{D}(\mathbf{q})(\mathbf{x}) := \int_\Gamma \frac{\mathbf{x}-\mathbf{y}}{4\pi|\mathbf{x}-\mathbf{y}|^3} \times [\mathbf{q}(\mathbf{y}) \times \mathbf{n}_C(\mathbf{y})] dS_y \ , \end{aligned} \tag{7.47}$$

the hypersingular integral operator

$$\begin{aligned} &\mathsf{H} : H^{-1/2}(\operatorname{curl}_\tau; \Gamma) \to H^{-1/2}(\operatorname{div}_\tau; \Gamma) \ , \\ &\mathsf{H}(\mathbf{q})(\mathbf{x}) := -\operatorname{curl}\left(\int_\Gamma \frac{\mathbf{x}-\mathbf{y}}{4\pi|\mathbf{x}-\mathbf{y}|^3} \times [\mathbf{q}(\mathbf{y}) \times \mathbf{n}_C(\mathbf{y})] dS_y \right) \times \mathbf{n}_C(\mathbf{x}), \end{aligned} \tag{7.48}$$

and the adjoint operator

$$\begin{aligned} &\mathsf{D}' : \widetilde{X}_\Gamma \to \widetilde{X}_\Gamma \ , \\ &\mathsf{D}'(\mathbf{p})(\mathbf{x}) := \left(\int_\Gamma \frac{\mathbf{y}-\mathbf{x}}{4\pi|\mathbf{x}-\mathbf{y}|^3} \times \mathbf{p}(\mathbf{y}) dS_y \right) \times \mathbf{n}_C(\mathbf{x}) \ , \end{aligned} \tag{7.49}$$

where the space $\widetilde{X}_\Gamma$ has been introduced in (7.41), and is given by the vector functions $\mathbf{p}$ belonging to $H^{-1/2}(\mathrm{div}_\tau;\Gamma)$ and such that $\mathrm{div}_\tau\,\mathbf{p}=0$ on Γ.

In Hiptmair [127] (see also Reissel [205], Hiptmair and Ostrowski [129]) it has been shown that these operators are continuous, and moreover that the solution $\mathbf{E}_I$ satisfies $\mathbf{n}_C\times\mathbf{E}_I\times\mathbf{n}_C=\mathbf{n}_C\times\mathbf{E}_C\times\mathbf{n}_C\in H^{-1/2}(\mathrm{curl}_\tau;\Gamma)$, $\mathrm{curl}\,\mathbf{E}_I\times\mathbf{n}_C\in\widetilde{X}_\Gamma$ and also

$$\begin{aligned}\tfrac{1}{2}\textstyle\int_\Gamma \mathbf{n}_C\times\mathbf{E}_C\times\mathbf{n}_C\cdot\overline{\mathbf{p}'}-\int_\Gamma \mathsf{D}(\mathbf{n}_C\times\mathbf{E}_C\times\mathbf{n}_C)\cdot\overline{\mathbf{p}'}\\ +\textstyle\int_\Gamma \mathsf{S}(\mathrm{curl}\,\mathbf{E}_I\times\mathbf{n}_C)\cdot\overline{\mathbf{p}'}=0\qquad\forall\,\mathbf{p}'\in\widetilde{X}_\Gamma,\end{aligned}\tag{7.50}$$

and

$$\begin{aligned}\tfrac{1}{2}\textstyle\int_\Gamma \mathrm{curl}\,\mathbf{E}_I\times\mathbf{n}_C\cdot\overline{\mathbf{q}'}+\int_\Gamma \mathsf{D}'(\mathrm{curl}\,\mathbf{E}_I\times\mathbf{n}_C)\cdot\overline{\mathbf{q}'}\\ +\textstyle\int_\Gamma \mathsf{H}(\mathbf{n}_C\times\mathbf{E}_C\times\mathbf{n}_C)\cdot\overline{\mathbf{q}'}=0\qquad\forall\,\mathbf{q}'\in H^{-1/2}(\mathrm{curl}_\tau;\Gamma)\,.\end{aligned}\tag{7.51}$$

Setting $\mathbf{p}_\Gamma:=\mathrm{curl}\,\mathbf{E}_I\times\mathbf{n}_C$, the term on Γ in equation (7.43) can be written as

$$\mu_0^{-1}\int_\Gamma \mathrm{curl}\,\mathbf{E}_I\times\mathbf{n}_I\cdot\overline{\mathbf{z}_I}=-\mu_0^{-1}\int_\Gamma \mathbf{p}_\Gamma\cdot\overline{\mathbf{z}_I}=-\mu_0^{-1}\int_\Gamma \mathbf{p}_\Gamma\cdot\mathbf{n}_C\times\overline{\mathbf{z}_C}\times\mathbf{n}_C\,.$$

Therefore, inserting (7.51) in (7.43), we see that the eddy current problem can be formulated as follows

$$\begin{aligned}&\text{Find } (\mathbf{E}_C,\mathbf{p}_\Gamma)\in H(\mathrm{curl};\Omega_C)\times\widetilde{X}_\Gamma \text{ such that}\\ &\textstyle\int_{\Omega_C}\boldsymbol{\mu}_C^{-1}\mathrm{curl}\,\mathbf{E}_C\cdot\mathrm{curl}\,\overline{\mathbf{z}_C}+i\omega\int_{\Omega_C}\boldsymbol{\sigma}\mathbf{E}_C\cdot\overline{\mathbf{z}_C}\\ &\qquad-\tfrac{1}{2}\mu_0^{-1}\textstyle\int_\Gamma \mathbf{p}_\Gamma\cdot\mathbf{n}_C\times\overline{\mathbf{z}_C}\times\mathbf{n}_C\\ &\qquad+\mu_0^{-1}\textstyle\int_\Gamma \mathsf{D}'(\mathbf{p}_\Gamma)\cdot\mathbf{n}_C\times\overline{\mathbf{z}_C}\times\mathbf{n}_C\\ &\qquad+\mu_0^{-1}\textstyle\int_\Gamma \mathsf{H}(\mathbf{n}_C\times\mathbf{E}_C\times\mathbf{n}_C)\cdot\mathbf{n}_C\times\overline{\mathbf{z}_C}\times\mathbf{n}_C\\ &\quad=-i\omega\textstyle\int_{\Omega_C}\mathbf{J}_{e,C}\cdot\overline{\mathbf{z}_C}\\ &\tfrac{1}{2}\mu_0^{-1}\textstyle\int_\Gamma \mathbf{n}_C\times\mathbf{E}_C\times\mathbf{n}_C\cdot\overline{\mathbf{p}'}-\mu_0^{-1}\int_\Gamma \mathsf{D}(\mathbf{n}_C\times\mathbf{E}_C\times\mathbf{n}_C)\cdot\overline{\mathbf{p}'}\\ &\qquad+\mu_0^{-1}\textstyle\int_\Gamma \mathsf{S}(\mathbf{p}_\Gamma)\cdot\overline{\mathbf{p}'}=0\\ &\text{for each } (\mathbf{z}_C,\mathbf{p}')\in H(\mathrm{curl};\Omega_C)\times\widetilde{X}_\Gamma\,.\end{aligned}\tag{7.52}$$

Note that

$$\begin{aligned}-\tfrac{1}{2}\textstyle\int_\Gamma \mathbf{p}'\cdot\mathbf{n}_C\times\overline{\mathbf{z}_C}\times\mathbf{n}_C+\int_\Gamma \mathsf{D}'(\mathbf{p}')\cdot\mathbf{n}_C\times\overline{\mathbf{z}_C}\times\mathbf{n}_C\\ +\tfrac{1}{2}\textstyle\int_\Gamma \mathbf{n}_C\times\mathbf{z}_C\times\mathbf{n}_C\cdot\overline{\mathbf{p}'}-\int_\Gamma \mathsf{D}(\mathbf{n}_C\times\mathbf{z}_C\times\mathbf{n}_C)\cdot\overline{\mathbf{p}'}\end{aligned}$$

is a purely imaginary number, and it is equal to

$$2\,i\,\mathrm{Im}\Big(\frac{1}{2}\int_\Gamma \mathbf{n}_C\times\mathbf{z}_C\times\mathbf{n}_C\cdot\overline{\mathbf{p}'}-\int_\Gamma \mathsf{D}(\mathbf{n}_C\times\mathbf{z}_C\times\mathbf{n}_C)\cdot\overline{\mathbf{p}'}\Big)\,.$$

Moreover, the boundedness of the operator D and the trace inequality for $H(\text{curl}; \Omega_C)$ (see (A.11)) give that, for each $0 < \delta < 1$,

$$\begin{aligned}
&\left|\tfrac{1}{2}\textstyle\int_\Gamma \mathbf{n}_C \times \mathbf{z}_C \times \mathbf{n}_C \cdot \overline{\mathbf{p}}' - \int_\Gamma \mathsf{D}(\mathbf{n}_C \times \mathbf{z}_C \times \mathbf{n}_C)\cdot \overline{\mathbf{p}}'\right| \\
&\quad \leq c_* \|\mathbf{p}'\|_{H^{-1/2}(\text{div}_\tau;\Sigma)}(\|\mathbf{z}_C\|_{0,\Omega_C} + \|\operatorname{curl}\mathbf{z}_C\|_{0,\Omega_C}) \\
&\quad \leq \delta \|\mathbf{z}_C\|^2_{0,\Omega_C} + C_*\delta^{-1}\|\mathbf{p}'\|^2_{H^{-1/2}(\text{div}_\tau;\Sigma)} + C_*\|\operatorname{curl}\mathbf{z}_C\|^2_{0,\Omega_C} \ .
\end{aligned}$$

Finally, in Hiptmair [127] it is shown that the operators S and H satisfies

$$\int_\Gamma \mathsf{H}(\mathbf{n}_C \times \mathbf{z}_C \times \mathbf{n}_C)\cdot \mathbf{n}_C \times \overline{\mathbf{z}_C} \times \mathbf{n}_C \geq 0 \ , \ \int_\Gamma \mathsf{S}(\mathbf{p}')\cdot \overline{\mathbf{p}}' \geq \kappa_0 \|\mathbf{p}'\|^2_{H^{-1/2}(\text{div}_\tau;\Sigma)} \ .$$

Thus, adapting the proof of Theorem 7.5, by choosing δ small enough it is not difficult to prove that the sesquilinear form $a^\Gamma_{e,C}(\cdot,\cdot)$ at the left hand side of (7.52) is coercive in $H(\text{curl}; \Omega_C) \times \widetilde{X}_\Gamma$, and we conclude that problem (7.52) is well-posed.

Having determined $\mathbf{E}_C$ and $\mathbf{p}_\Gamma = \operatorname{curl}\mathbf{E}_I \times \mathbf{n}_C = i\omega\mu_0 \mathbf{H}_I \times \mathbf{n}_I$, one can also find the magnetic field in Ω_I. In fact, setting

$$\mathsf{S}_I(\mathbf{p})(\mathbf{x}) := \int_\Gamma \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}\,\mathbf{p}(\mathbf{y})dS_y \ , \ \mathbf{x} \in \Omega_I \ ,$$

$$\mathsf{D}_I(\mathbf{q})(\mathbf{x}) := \int_\Gamma \frac{\mathbf{x}-\mathbf{y}}{4\pi|\mathbf{x}-\mathbf{y}|^3} \times [\mathbf{q}(\mathbf{y}) \times \mathbf{n}_C(\mathbf{y})]dS_y \ , \ \mathbf{x} \in \Omega_I \ ,$$

from well-known results of potential theory we easily obtain by integration by parts the representation formula

$$\mathbf{E}_I(\mathbf{x}) = \mathsf{D}_I(\mathbf{E}_C) - \mathsf{S}_I(\mathbf{p}_\Gamma) - \operatorname{grad}\mathcal{S}_I(\mathbf{E}_I \cdot \mathbf{n}_C) \ , \tag{7.53}$$

where the operator $\mathcal{S}_I$ has been introduced in (7.15). Then the magnetic field $\mathbf{H}_I = -(i\omega\mu_0)^{-1}\operatorname{curl}\mathbf{E}_I$ can be written as

$$\mathbf{H}_I = -(i\omega\mu_0)^{-1}\left(\operatorname{curl}\mathsf{D}_I(\mathbf{E}_C) - \operatorname{curl}\mathsf{S}_I(\mathbf{p}_\Gamma)\right) \ .$$

Instead, since we do not know the value of the normal component $\mathbf{E}_I \cdot \mathbf{n}_C$ on Γ, the electric field $\mathbf{E}_I$ cannot be computed through (7.53), and one has to solve (7.2).

The numerical approximation is quite similar to that presented in Section 7.6.2. In fact, Nédélec curl-conforming edge element of the lowest order can be used in Ω_C; instead, the conforming approximation of $\widetilde{X}_\Gamma$ is given by the space spanned by

$$\{\text{Curl}_\tau \varphi_v \,|\, v \in \mathcal{V}_{\Gamma,h}, v \neq v_0\} \ ,$$

where $\mathcal{V}_{\Gamma,h}$ is the set of vertices $v \in \mathcal{T}_{\partial,h}$, the mesh induced on Γ by $\mathcal{T}_{C,h}$, φ_v is the piecewise-linear nodal basis function defined on Γ and associated to the vertex v, and $v_0 \in \Gamma$ is a fixed vertex of $\mathcal{V}_{\Gamma,h}$.

In Hiptmair [127] the convergence of the approximation scheme, based on Céa lemma and suitable interpolation estimates, is completely proved. Moreover, the discrete problem is analyzed also when Ω_C is not simply-connected: this geometric situation has the drawback that the boundary element space for approximating $\widetilde{X}_\Gamma$ is more complicated. Finally, some remarks on implementation are also added: in particular, it is shown that the operators S, D and H can be expressed in terms of the analogous operators constructed for the Laplace operator. For example, one has

$$\int_\Gamma \mathsf{S}(\mathrm{Curl}_\tau \psi_{\Gamma,h}) \cdot \mathrm{Curl}_\tau \overline{\chi_{\Gamma,h}} = \int_\Gamma \mathcal{H}(\psi_{\Gamma,h})\, \overline{\chi_{\Gamma,h}} \,,$$

$$\begin{aligned} \textstyle\int_\Gamma \mathsf{D}'(\mathbf{p}_{\Gamma,h}) \cdot \mathbf{n}_C \times \overline{\mathbf{z}_{C,h}} \times \mathbf{n}_C &= \textstyle\int_\Gamma \mathbf{p}_{\Gamma,h} \cdot \mathcal{D}(\mathbf{n}_C \times \overline{\mathbf{z}_{C,h}} \times \mathbf{n}_C) \\ &+ \textstyle\int_\Gamma \int_\Gamma \frac{\mathbf{x}-\mathbf{y}}{4\pi|\mathbf{x}-\mathbf{y}|^3} \cdot [\mathbf{n}_C(\mathbf{x}) \times \overline{\mathbf{z}_{C,h}}(\mathbf{x}) \times \mathbf{n}_C(\mathbf{x})](\mathbf{p}_{\Gamma,h}(\mathbf{y}) \cdot \mathbf{n}_C(\mathbf{x})) dS_x dS_y \,, \end{aligned}$$

$$\int_\Gamma \mathsf{H}(\mathbf{n}_C \times \mathbf{E}_{C,h} \times \mathbf{n}_C) \cdot \mathbf{n}_C \times \overline{\mathbf{z}_{C,h}} \times \mathbf{n}_C = \int_\Gamma \mathcal{S}(\mathrm{div}_\tau(\mathbf{E}_{C,h} \times \mathbf{n}_C))\, \mathrm{div}_\tau(\overline{\mathbf{z}_{C,h}} \times \mathbf{n}_C) \,,$$

where $\mathcal{S}$, $\mathcal{D}$ and $\mathcal{H}$ are the operators introduced in (7.6), (7.7) and (7.8), respectively. Therefore, the techniques developed for Galerkin boundary element methods for the Laplace operator can be used in this framework (in this respect, see also Remark 7.11).