2

A mathematical justification of the eddy current model

The aim of this chapter is to analyze in which sense the eddy current model is a proper approximation of the full Maxwell system. As explained in the previous chapter, the eddy current problem is a simplified model derived from the full system of Maxwell equations by neglecting the displacement currents, namely, the term $i\omega\varepsilon\mathbf{E}$. Therefore it can be seen either as the low electric permittivity limit or as the low-frequency limit of the full Maxwell system. The analysis is mainly based on the $\mathbf{E}$-based formulation of Maxwell equations obtained by eliminating the magnetic field.

2.1 The E-based formulation of Maxwell equations

In this chapter we are not concerned with the problem of well-posedness of the eddy current model, an aspect that is dealt with in Chapter 3. We simply assume that a solution of the eddy current equations exists, and that a solution of the full Maxwell system exists as well, both solutions being smooth enough to justify all the computations we will perform. Moreover, we focus on the magnetic boundary value problem (1.22), leaving to the reader the modifications needed for treating the electric boundary value case (1.20).

The geometrical situation is that described in Section 1.3. Moreover, as already indicated, in agreement with well-known physical considerations we suppose that the matrix $\boldsymbol{\mu}$ is symmetric and uniformly positive definite in Ω, with entries in $L^\infty(\Omega)$, the matrix ε_I is symmetric and uniformly positive definite in Ω_I, with entries in $L^\infty(\Omega_I)$, and the matrix $\boldsymbol{\sigma}$ is symmetric and uniformly positive definite in Ω_C, with entries in $L^\infty(\Omega_C)$, whereas it is vanishing in Ω_I. Finally, the current density $\mathbf{J}_e \in (L^2(\Omega))^3$ satisfies the necessary conditions (1.23).

In the Maxwell system, and also in the eddy current model, it is possible to eliminate either the electric field (as it will be done in the first part of Chapter 3) or the magnetic field. For the full Maxwell system the two formulations are quite similar, but this is not the case for the eddy current model. In particular, in order to compare the two problems it is convenient to use the electric approach, eliminating the magnetic field.

A. Alonso Rodríguez, A. Valli: Eddy Current Approximation of Maxwell Equations.

From the Faraday law in (1.22) one has $\boldsymbol{\mu}^{-1} \operatorname{curl} \mathbf{E} = -i\omega\mathbf{H}$, then substituting in the Ampère law we obtain the $\mathbf{E}$-based formulation of the eddy current problem

$$\begin{cases} \operatorname{curl}(\boldsymbol{\mu}^{-1} \operatorname{curl} \mathbf{E}) + i\omega\boldsymbol{\sigma}\mathbf{E} = -i\omega\mathbf{J}_e & \text{in } \Omega \\ \operatorname{div}(\boldsymbol{\varepsilon}_I \mathbf{E}_I) = 0 & \text{in } \Omega_I \\ \boldsymbol{\mu}^{-1} \operatorname{curl} \mathbf{E} \times \mathbf{n} = \mathbf{0} & \text{on } \partial\Omega \\ \boldsymbol{\varepsilon}_I \mathbf{E}_I \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \\ \int_{\Gamma_j} \boldsymbol{\varepsilon}_I \mathbf{E}_I \cdot \mathbf{n}_I = 0 & \forall\, j = 1, \ldots, p_\Gamma \\ \int_{\Omega_I} \boldsymbol{\varepsilon}_I \mathbf{E}_I \cdot \boldsymbol{\pi}_{k,I} = 0 & \forall\, k = 1, \ldots, n_{\partial\Omega} \,. \end{cases} \tag{2.1}$$

To help the reader, we remark that if the boundary of the conductor Ω_C is connected then $p_\Gamma = 0$, and that if Ω is simply-connected then $n_{\partial\Omega} = 0$. An example of this simplified geometry is that of a connected conductor (possibly with “handles”) contained in a computational domain similar to a “box”.

Using integration by parts it is easily seen that a solution $\mathbf{E}$ to (2.1) satisfies

$$\begin{aligned} \int_\Omega \boldsymbol{\mu}^{-1} \operatorname{curl} \mathbf{E} \cdot \operatorname{curl} \overline{\mathbf{z}} \; &+ i\omega \int_{\Omega_C} \boldsymbol{\sigma}\mathbf{E}_C \cdot \overline{\mathbf{z}_C} \\ &+ c_0^* \int_{\Omega_I} \operatorname{div}(\boldsymbol{\varepsilon}_I \mathbf{E}_I)\, \operatorname{div}(\boldsymbol{\varepsilon}_I \overline{\mathbf{z}_I}) = -i\omega \int_\Omega \mathbf{J}_e \cdot \overline{\mathbf{z}} \end{aligned} \tag{2.2}$$

for all $\mathbf{z} \in H(\operatorname{curl}; \Omega)$ with $\operatorname{div}(\boldsymbol{\varepsilon}_I \mathbf{z}_I) \in L^2(\Omega_I)$, where $c_0^* > 0$ is an arbitrarily chosen dimensional constant.

Let us consider the space

$$\begin{aligned} W_{\boldsymbol{\varepsilon}_I}(\Omega_I; \Omega) := \{\mathbf{z} \in H(\operatorname{curl}; \Omega) \,|\, &\mathbf{z}_I \in H_{0,\partial\Omega}(\boldsymbol{\varepsilon}_I, \operatorname{div}; \Omega_I), \\ &\mathbf{z}_I \perp^{\boldsymbol{\varepsilon}_I} \mathcal{H}_{\boldsymbol{\varepsilon}_I}(\Gamma, \partial\Omega; \Omega_I)\} \end{aligned} \tag{2.3}$$

where the symbol $\perp^{\boldsymbol{\varepsilon}_I}$ denotes the orthogonality with respect to the scalar product

$$(\mathbf{w}_I, \mathbf{z}_I)_{\boldsymbol{\varepsilon}_I, \Omega_I} := \int_{\Omega_I} \boldsymbol{\varepsilon}_I \mathbf{w}_I \cdot \overline{\mathbf{z}_I}$$

(for the other notations see Sections 1.4 and A.1). In $W_{\boldsymbol{\varepsilon}_I}(\Omega_I; \Omega)$ we define the norm

$$\|\mathbf{z}\|_{W_{\boldsymbol{\varepsilon}_I}(\Omega_I;\Omega)} := \left(\| \operatorname{curl} \mathbf{z}\|_{0,\Omega}^2 + \|\mathbf{z}\|_{0,\Omega}^2 + \| \operatorname{div}(\boldsymbol{\varepsilon}_I \mathbf{z}_I)\|_{0,\Omega_I}^2\right)^{1/2} .$$

Recalling that, as shown in Section 1.5, the integral conditions in (2.1) are orthogonality conditions with respect to the space of harmonic fields $\mathcal{H}_{\boldsymbol{\varepsilon}_I}(\Gamma, \partial\Omega; \Omega_I)$, with respect to the scalar product $(\cdot, \cdot)_{\boldsymbol{\varepsilon}_I, \Omega_I}$, we have in particular that $\mathbf{E} \in W_{\boldsymbol{\varepsilon}_I}(\Omega_I; \Omega)$.

In the space $W_{\boldsymbol{\varepsilon}_I}(\Omega_I; \Omega)$ let us define the sesquilinear form

$$\begin{aligned} a_e^*(\mathbf{w}, \mathbf{z}) := \int_\Omega \boldsymbol{\mu}^{-1} \operatorname{curl} \mathbf{w} \cdot \operatorname{curl} \overline{\mathbf{z}} &+ i\omega \int_{\Omega_C} \boldsymbol{\sigma}\mathbf{w}_C \cdot \overline{\mathbf{z}_C} \\ &+ c_0^* \int_{\Omega_I} \operatorname{div}(\boldsymbol{\varepsilon}_I \mathbf{w}_I)\, \operatorname{div}(\boldsymbol{\varepsilon}_I \overline{\mathbf{z}_I}) \,. \end{aligned} \tag{2.4}$$

With the aim of analyzing the asymptotic behaviour of the solution $\mathbf{E}$ as the electric permittivity $\boldsymbol{\varepsilon}$ tends to 0 or the angular frequency ω tends to 0, an important point is to show that the sesquilinear form $a_e^*(\cdot, \cdot)$ is coercive in $W_{\boldsymbol{\varepsilon}_I}(\Omega_I; \Omega)$.

We need some preliminary results. It is known that there are several ways of writing a vector function belonging to $(L^2(\Omega_I))^3$ as the sum of a curl, a gradient and a harmonic field. In particular, let us recall (see Theorem A.6) that $\mathbf{z}_I \in (L^2(\Omega_I))^3$ can be represented as

$$\mathbf{z}_I = \boldsymbol{\varepsilon}_I^{-1} \operatorname{curl} \mathbf{q}_I + \operatorname{grad} \varphi_I + \mathbf{h}_I \ , \tag{2.5}$$

where $\mathbf{q}_I \in H_{0,\partial\Omega}(\mathrm{curl}; \Omega_I) \cap H^0_{0,\Gamma}(\mathrm{div}; \Omega_I) \cap \mathcal{H}(\partial\Omega, \Gamma; \Omega_I)^\perp$, $\varphi_I \in H^1_{0,\Gamma}(\Omega_I)$ and $\mathbf{h}_I \in \mathcal{H}_{\varepsilon_I}(\Gamma, \partial\Omega; \Omega_I)$. Moreover, we also know that if $\mathbf{z}_I \perp^{\varepsilon_I} \mathcal{H}_{\varepsilon_I}(\Gamma, \partial\Omega; \Omega_I)$ we have $\mathbf{h}_I = \mathbf{0}$.

A second useful result is the following one.

Lemma 2.1. *There exists a constant* $C > 0$ *such that*

$$\begin{aligned}\|\mathbf{z}_I\|_{0,\Omega_I} \le C\big(&\|\operatorname{curl} \mathbf{z}_I\|_{0,\Omega_I} + \|\operatorname{div}(\boldsymbol{\varepsilon}_I \mathbf{z}_I)\|_{0,\Omega_I} \\ &+\|\mathbf{z}_I \times \mathbf{n}_I\|_{H^{-1/2}(\mathrm{div}_\tau;\Gamma)} + \|\boldsymbol{\varepsilon}_I \mathbf{z}_I \cdot \mathbf{n}_I\|_{-1/2,\partial\Omega}\big)\end{aligned}$$

for all $\mathbf{z}_I \in H(\mathrm{curl}; \Omega_I) \cap H(\varepsilon_I, \mathrm{div}; \Omega_I)$ *with* $\mathbf{z}_I \perp^{\varepsilon_I} \mathcal{H}_{\varepsilon_I}(\Gamma, \partial\Omega; \Omega_I)$.

Proof. Since $\mathbf{z}_I \perp^{\varepsilon_I} \mathcal{H}_{\varepsilon_I}(\Gamma, \partial\Omega; \Omega_I)$, from (2.5) we can write

$$\mathbf{z}_I = \boldsymbol{\varepsilon}_I^{-1} \operatorname{curl} \mathbf{q}_I + \operatorname{grad} \varphi_I \ .$$

Then we estimate the norm of the two terms on the right hand side. Looking at problem (A.14), we start by considering $\int_{\Omega_I} \mathbf{z}_I \cdot \operatorname{curl} \overline{\mathbf{q}_I}$. Integrating by parts we have

$$\begin{aligned}\Big|\int_{\Omega_I} \mathbf{z}_I \cdot \operatorname{curl} \overline{\mathbf{q}_I}\Big| &= \Big|\int_{\Omega_I} \operatorname{curl} \mathbf{z}_I \cdot \overline{\mathbf{q}_I} + \int_\Gamma \mathbf{z}_I \times \mathbf{n}_I \cdot \overline{\mathbf{q}_I}\Big| \\ &\le C(\|\operatorname{curl} \mathbf{z}_I\|_{0,\Omega_I} + \|\mathbf{z}_I \times \mathbf{n}_I\|_{H^{-1/2}(\mathrm{div}_\tau;\Gamma)})(\|\mathbf{q}_I\|_{0,\Omega_I} + \|\operatorname{curl} \mathbf{q}_I\|_{0,\Omega_I}) \ ,\end{aligned}$$

where we have used the duality estimate

$$\Big|\int_\Gamma \mathbf{z}_I \times \mathbf{n}_I \cdot \overline{\mathbf{q}_I}\Big| \le C\|\mathbf{z}_I \times \mathbf{n}_I\|_{H^{-1/2}(\mathrm{div}_\tau;\Gamma)} \|\mathbf{n}_I \times \mathbf{q}_I \times \mathbf{n}_I\|_{H^{-1/2}(\mathrm{curl}_\tau;\Gamma)}$$

(see Section A.1) and the tangential trace inequality (A.11)

$$\|\mathbf{n}_I \times \mathbf{q}_I \times \mathbf{n}_I\|_{H^{-1/2}(\mathrm{curl}_\tau;\Gamma)} \le C(\|\mathbf{q}_I\|_{0,\Omega_I} + \|\operatorname{curl} \mathbf{q}_I\|_{0,\Omega_I}) \ .$$

On the other hand, from the Poincaré-like inequality (A.15) and taking also into account that $\operatorname{div} \mathbf{q}_I = \mathbf{0}$ in Ω_I we find

$$\int_{\Omega_I} |\mathbf{q}_I|^2 \le C \int_{\Omega_I} (|\operatorname{curl} \mathbf{q}_I|^2 + |\operatorname{div} \mathbf{q}_I|^2) = C \int_{\Omega_I} |\operatorname{curl} \mathbf{q}_I|^2 \, .$$

Summing up, choosing $\mathbf{p}_I = \mathbf{q}_I$ in (A.14) gives

$$\|\boldsymbol{\varepsilon}^{-1} \operatorname{curl} \mathbf{q}_I\|_{0,\Omega_I} \le C(\|\operatorname{curl} \mathbf{z}_I\|_{0,\Omega_I} + \|\mathbf{z}_I \times \mathbf{n}_I\|_{H^{-1/2}(\mathrm{div}_\tau;\Gamma)}) \ .$$

Another integration by parts in the right hand side of (A.17) furnishes

$$\begin{aligned}\Big|\int_{\Omega_I} \boldsymbol{\varepsilon}_I \mathbf{z}_I \cdot \operatorname{grad} \overline{\varphi_I}\Big| &= \Big|-\int_{\Omega_I} \operatorname{div}(\boldsymbol{\varepsilon}_I \mathbf{z}_I)\, \overline{\varphi_I} + \int_{\partial\Omega} \boldsymbol{\varepsilon}_I \mathbf{z}_I \cdot \mathbf{n}\, \overline{\varphi_I}\Big| \\ &\le C(\|\operatorname{div}(\boldsymbol{\varepsilon}_I \mathbf{z}_I)\|_{0,\Omega_I} + \|\boldsymbol{\varepsilon}_I \mathbf{z}_I \cdot \mathbf{n}_I\|_{-1/2,\partial\Omega})(\|\varphi_I\|_{0,\Omega_I} + \|\operatorname{grad} \varphi_I\|_{0,\Omega_I}) \ ,\end{aligned}$$

having used the duality estimate

$$\left| \int_{\partial\Omega} \boldsymbol{\varepsilon}_I \mathbf{z}_I \cdot \mathbf{n} \overline{\varphi_I} \right| \leq C \|\boldsymbol{\varepsilon}_I \mathbf{z}_I \cdot \mathbf{n}\|_{-1/2,\partial\Omega} \|\varphi_{I|\partial\Omega}\|_{1/2,\partial\Omega}$$

(see Section A.1) and the trace inequality (A.8)

$$\|\varphi_{I|\partial\Omega}\|_{1/2,\partial\Omega} \leq C(\|\varphi_I\|_{0,\Omega_I} + \|\operatorname{grad}\varphi_I\|_{0,\Omega_I}) \, .$$

Since the Poincaré inequality (A.18)

$$\int_{\Omega_I} |\varphi_I|^2 \leq C \int_{\Omega_I} |\operatorname{grad}\varphi_I|^2$$

holds in $H^1_{0,\Gamma}(\Omega_I)$, choosing $\eta_I = \varphi_I$ in (A.17) we have

$$\|\operatorname{grad}\varphi_I\|_{0,\Omega_I} \leq C(\|\operatorname{div}(\boldsymbol{\varepsilon}_I \mathbf{z}_I)\|_{0,\Omega_I} + \|\boldsymbol{\varepsilon}_I \mathbf{z}_I \cdot \mathbf{n}_I\|_{-1/2,\partial\Omega}) \, ,$$

and the thesis follows. □

As a consequence we have the following lemma, that is the main point for proving the coerciveness of the sesquilinear form $a_e^*(\cdot,\cdot)$.

Lemma 2.2. *There exists a constant* $C > 0$ *such that for each* $\mathbf{z} \in W_{\varepsilon_I}(\Omega_I;\Omega)$

$$\|\mathbf{z}_I\|_{0,\Omega_I} \leq C\big(\|\operatorname{curl}\mathbf{z}\|_{0,\Omega} + \|\operatorname{div}(\boldsymbol{\varepsilon}_I \mathbf{z}_I)\|_{0,\Omega_I} + \|\mathbf{z}_C\|_{0,\Omega_C}\big) \, .$$

Proof. First we recall that $\mathbf{z} \in H(\mathrm{curl};\Omega)$ if and only if $\mathbf{z}_C \in H(\mathrm{curl};\Omega_C)$, $\mathbf{z}_I \in H(\mathrm{curl};\Omega_I)$ and $\mathbf{z}_C \times \mathbf{n}_C = -\mathbf{z}_I \times \mathbf{n}_I$ on Γ.

From Lemma 2.1 we have

$$\begin{aligned} \|\mathbf{z}_I\|_{0,\Omega_I} &\leq C\left(\|\operatorname{curl}\mathbf{z}_I\|_{0,\Omega_I} + \|\operatorname{div}(\boldsymbol{\varepsilon}_I \mathbf{z}_I)\|_{0,\Omega_I} + \|\mathbf{z}_I \times \mathbf{n}_I\|_{H^{-1/2}(\mathrm{div}_\tau;\Gamma)}\right) \\ &= C\left(\|\operatorname{curl}\mathbf{z}_I\|_{0,\Omega_I} + \|\operatorname{div}(\boldsymbol{\varepsilon}_I \mathbf{z}_I)\|_{0,\Omega_I} + \|\mathbf{z}_C \times \mathbf{n}_C\|_{H^{-1/2}(\mathrm{div}_\tau;\Gamma)}\right) \, . \end{aligned}$$

Taking into account the tangential trace inequality (A.10), namely,

$$\|\mathbf{z}_C \times \mathbf{n}_C\|_{H^{-1/2}(\mathrm{div}_\tau;\Gamma)} \leq \kappa \, \|\mathbf{z}_C\|_{H(\mathrm{curl};\Omega_C)} \, ,$$

the proof is complete. □

Now we are in condition to prove the main result on this section.

Theorem 2.3. *The sesquilinear form* $a_e^*(\cdot,\cdot)$ *is coercive in* $W_{\varepsilon_I}(\Omega_I;\Omega)$*, i.e., there exists a constant* $C_0 > 0$ *such that*

$$|a_e^*(\mathbf{z},\mathbf{z})| \geq C_0 \|\mathbf{z}\|^2_{W_{\varepsilon_I}(\Omega_I;\Omega)} \qquad \textit{for all } \mathbf{z} \in W_{\varepsilon_I}(\Omega_I;\Omega) \, . \tag{2.6}$$

Proof. As a consequence of Lemma 2.2 we have, for all $\mathbf{z} \in W_{\varepsilon_I}(\Omega_I;\Omega)$

$$\|\mathbf{z}\|^2_{W_{\varepsilon_I}(\Omega_I;\Omega)} \leq C_1 \left(\|\operatorname{curl}\mathbf{z}\|^2_{0,\Omega} + \|\mathbf{z}_C\|^2_{0,\Omega_C} + \|\operatorname{div}(\boldsymbol{\varepsilon}_I \mathbf{z}_I)\|^2_{0,\Omega_I} \right) , \tag{2.7}$$

for some positive constant C_1. Since $\boldsymbol{\nu} := \boldsymbol{\mu}^{-1}$ and $\boldsymbol{\sigma}$ are symmetric and uniformly positive definite in Ω and Ω_C, respectively, we have

$$\begin{aligned}
|a_e^*(\mathbf{z},\mathbf{z})|^2 &= \left(\int_\Omega \boldsymbol{\mu}^{-1} \operatorname{curl}\mathbf{z} \cdot \operatorname{curl}\overline{\mathbf{z}} + c_0^* \int_{\Omega_I} \operatorname{div}(\boldsymbol{\varepsilon}_I \mathbf{z}_I)\, \operatorname{div}(\boldsymbol{\varepsilon}_I \overline{\mathbf{z}_I}) \right)^2 \\
&\qquad\qquad + \left(\omega \int_{\Omega_C} \boldsymbol{\sigma} \mathbf{w}_C \cdot \overline{\mathbf{z}_C} \right)^2 \\
&\geq \left(\nu_{\min} \|\operatorname{curl}\mathbf{z}\|^2_{0,\Omega} + c_0^* \|\operatorname{div}(\boldsymbol{\varepsilon}_I \mathbf{z}_I)\|^2_{0,\Omega_I} \right)^2 + \left(\omega \sigma_{\min} \|\mathbf{z}_C\|^2_{0,\Omega_C} \right)^2 \\
&\geq C_2 \left(\|\operatorname{curl}\mathbf{z}\|^2_{0,\Omega} + \|\operatorname{div}(\boldsymbol{\varepsilon}_I \mathbf{z}_I)\|^2_{0,\Omega_I} + \|\mathbf{z}_C\|^2_{0,\Omega_C} \right)^2 \\
&\geq C_2 C_1^{-2} \|\mathbf{z}\|^4_{W_{\varepsilon_I}(\Omega_I;\Omega)} ,
\end{aligned}$$

where $\nu_{\min}$ is a uniform lower bound in Ω for the minimum eigenvalues of $\boldsymbol{\nu}(\mathbf{x})$, $\sigma_{\min}$ is a uniform lower bound in Ω_C for the minimum eigenvalues of $\boldsymbol{\sigma}(\mathbf{x})$, and $C_2 = \frac{1}{2} \min(\nu^2_{\min}, c_0^{*2}, \omega^2 \sigma^2_{\min})$. □

Remark 2.4. The proof of the coerciveness of the sesquilinear form $a_e^*(\cdot,\cdot)$ is the crucial point in showing, via the Lax–Milgram lemma, that the weak problem

$$\begin{aligned}
&\text{Find } \mathbf{E} \in W_{\varepsilon_I}(\Omega_I;\Omega) \text{ such that} \\
&\quad \int_\Omega \boldsymbol{\mu}^{-1} \operatorname{curl}\mathbf{E} \cdot \operatorname{curl}\overline{\mathbf{z}} + i\omega \int_{\Omega_C} \boldsymbol{\sigma} \mathbf{E}_C \cdot \overline{\mathbf{z}_C} \\
&\qquad\qquad + c_0^* \int_{\Omega_I} \operatorname{div}(\boldsymbol{\varepsilon}_I \mathbf{E}_I)\, \operatorname{div}(\boldsymbol{\varepsilon}_I \overline{\mathbf{z}_I}) = -i\omega \int_\Omega \mathbf{J}_e \cdot \overline{\mathbf{z}} \\
&\text{for each } \mathbf{z} \in W_{\varepsilon_I}(\Omega_I;\Omega)
\end{aligned}$$

is well-posed.

Starting from this result, some additional work gives that also the solution to (2.1) exists and is unique. However, we do not want to consider this aspect here, and we refer to Chapter 3 for a systematic approach to the existence and uniqueness theory for eddy current problems, where the result is based on a simpler weak formulation in terms of the magnetic field $\mathbf{H}$ only, and to Section 6.1.5 for a complete analysis of the $\mathbf{E}$-based formulation (2.1). □

2.2 The eddy current model as the low electric permittivity limit

By eliminating the magnetic field in the time-harmonic Maxwell equations we obtain the following boundary value problem for the electric field $\mathbf{E}^M$ (with the magnetic boundary condition)

$$\begin{cases} \operatorname{curl}(\boldsymbol{\mu}^{-1} \operatorname{curl}\mathbf{E}^M) - \omega^2 \boldsymbol{\varepsilon} \mathbf{E}^M + i\omega\boldsymbol{\sigma}\mathbf{E}^M = -i\omega \mathbf{J}_e & \text{in } \Omega \\ \boldsymbol{\mu}^{-1} \operatorname{curl}\mathbf{E}^M \times \mathbf{n} = \mathbf{0} & \text{on } \partial\Omega\ . \end{cases} \tag{2.8}$$

Here $\boldsymbol{\varepsilon}$ is a symmetric tensor, uniformly positive definite in Ω, with coefficients in $L^\infty(\Omega)$ and, as usual, $\boldsymbol{\varepsilon}_{|\Omega_I} = \boldsymbol{\varepsilon}_I$. As we have already explained in Section 1.2, the eddy current model is obtained by neglecting the term $\omega^2\varepsilon\mathbf{E}^M$, that corresponds to the displacement currents.

In this section we consider an electric permittivity of the form $\varepsilon_\delta := \delta\varepsilon$, where $\delta > 0$ is a real number. Clearly, the physical problem described by the full Maxwell system corresponds to the case $\delta = 1$. We want to show that the eddy current model is the limit as δ tends to 0 of the problem with electric permittivity ε_δ. This is the notion of eddy current limit presented in Bossavit [58], Chap. 4.

We will show that the norm in $H(\mathrm{curl};\Omega)$ of the difference between the electric field solution of the full Maxwell system and the electric field solution of the eddy current problem is of order δ. This result has been proved in Costabel et al. [90] who impose the electric boundary condition $\mathbf{E}\times\mathbf{n} = \mathbf{0}$ on $\partial\Omega$. Here we report a proof of this result in the case of the magnetic boundary condition $\mathbf{H}\times\mathbf{n} = \mathbf{0}$ on $\partial\Omega$, that, in terms of the electric field, is the boundary condition considered in (2.1) and (2.8).

Let us denote by $\mathbf{E}$ the solution of the eddy current problem (2.1) and by $\mathbf{E}_\delta^M$ the solution of the full Maxwell system

$$\begin{cases} \mathrm{curl}(\boldsymbol{\mu}^{-1}\,\mathrm{curl}\,\mathbf{E}_\delta^M) - \omega^2\delta\varepsilon\mathbf{E}_\delta^M + i\omega\boldsymbol{\sigma}\mathbf{E}_\delta^M = -i\omega\mathbf{J}_e & \text{in } \Omega \\ \boldsymbol{\mu}^{-1}\,\mathrm{curl}\,\mathbf{E}_\delta^M \times \mathbf{n} = \mathbf{0} & \text{on } \partial\Omega\,. \end{cases} \tag{2.9}$$

For the existence and uniqueness of solution of (2.9) see, for instance, Alonso and Valli [8], Alonso and Raffetto [15].

As shown in Section 1.5, the current density $\mathbf{J}_e$ has to satisfy conditions (1.23), that are necessary conditions for the solvability of the eddy current problem. Let us rewrite them here

$$\begin{array}{ll} \mathrm{div}\,\mathbf{J}_{e,I} = 0 & \text{in } \Omega_I \\ \mathbf{J}_{e,I}\cdot\mathbf{n} = 0 & \text{on } \partial\Omega \\ \int_{\Gamma_j}\mathbf{J}_{e,I}\cdot\mathbf{n}_I = 0 & \forall\, j = 1,\dots,p_\Gamma \\ \int_{\Omega_I}\mathbf{J}_{e,I}\cdot\boldsymbol{\pi}_{k,I} = 0 & \forall\, k = 1,\dots,n_{\partial\Omega}\,. \end{array} \tag{2.10}$$

We notice that from these conditions it follows that $\mathbf{J}_{e,I} \in \mathcal{H}_{\varepsilon_I}(\Gamma,\partial\Omega;\Omega_I)^\perp$ and that the solution $\mathbf{E}_\delta^M$ of (2.9) belongs to $W_{\varepsilon_I}(\Omega_I;\Omega)$. In fact, by a direct computation from (2.10) we have $\mathrm{div}(\varepsilon_I\mathbf{E}_{\delta,I}^M) = 0$, and moreover $\varepsilon_I\mathbf{E}_{\delta,I}^M\cdot\mathbf{n} = 0$ on $\partial\Omega$, as from the boundary condition $\boldsymbol{\mu}^{-1}\,\mathrm{curl}\,\mathbf{E}_\delta^M\times\mathbf{n} = \mathbf{0}$ on $\partial\Omega$ it follows that $\mathrm{curl}(\boldsymbol{\mu}^{-1}\,\mathrm{curl}\,\mathbf{E}_\delta^M)\cdot\mathbf{n} = 0$ on $\partial\Omega$. Finally, it is clear that $\mathbf{E}_{\delta,I}^M \perp^{\varepsilon_I} \mathcal{H}_{\varepsilon_I}(\Gamma,\partial\Omega;\Omega_I)$, since for all $\mathbf{h}_I \in \mathcal{H}_{\varepsilon_I}(\Gamma,\partial\Omega;\Omega_I)$ we have

$$\begin{aligned} \int_{\Omega_I}\varepsilon_I\mathbf{E}_{\delta,I}^M\cdot\mathbf{h}_I &= \frac{1}{\omega^2\delta}\int_{\Omega_I}\mathrm{curl}(\boldsymbol{\mu}_I^{-1}\,\mathrm{curl}\,\mathbf{E}_{\delta,I}^M)\cdot\mathbf{h}_I + \frac{i}{\omega\delta}\int_{\Omega_I}\mathbf{J}_{e,I}\cdot\mathbf{h}_I \\ &= \frac{1}{\omega^2\delta}\int_{\Omega_I}\boldsymbol{\mu}_I^{-1}\,\mathrm{curl}\,\mathbf{E}_{\delta,I}^M\cdot\mathrm{curl}\,\mathbf{h}_I \\ &\quad + \frac{1}{\omega^2\delta}\omega\int_{\Gamma\cup\partial\Omega}\mathbf{n}_I\times\boldsymbol{\mu}_I^{-1}\,\mathrm{curl}\,\mathbf{E}_{\delta,I}^M\cdot\mathbf{h}_I = 0\,. \end{aligned}$$

We are thus in a position to prove the main result of this section.

Theorem 2.5. *There exists $\delta^* > 0$ such that if $0 < \delta \le \delta^*$ one has*

$$\|\mathbf{E} - \mathbf{E}_\delta^M\|_{H(\mathrm{curl};\Omega)} \le C\delta\,,$$

for some constant $C > 0$ independent of δ.

Proof. Taking the difference between the first equations in (2.1) and (2.9), multiplying by a test function $\mathbf{z} \in H(\text{curl};\Omega)$ and integrating by parts one obtains

$$\int_\Omega \boldsymbol{\mu}^{-1} \operatorname{curl}(\mathbf{E} - \mathbf{E}^M_\delta) \cdot \operatorname{curl} \overline{\mathbf{z}} + i\omega \int_{\Omega_C} \boldsymbol{\sigma}(\mathbf{E}_C - \mathbf{E}^M_{\delta,C}) \cdot \overline{\mathbf{z}_C} = -\delta\omega^2 \int_\Omega \boldsymbol{\varepsilon}\mathbf{E}^M_\delta \cdot \overline{\mathbf{z}}\,.$$

Since $\operatorname{div}(\boldsymbol{\varepsilon}_I(\mathbf{E}_I - \mathbf{E}^M_{\delta,I})) = 0$ in Ω_I, from the coerciveness of the sesquilinear form $a^*_e(\cdot,\cdot)$ in $W_{\varepsilon_I}(\Omega_I;\Omega)$ it follows that there exists a constant $C_0 > 0$, independent of δ, such that

$$\begin{aligned} C_0 \|\mathbf{E} - \mathbf{E}^M_\delta\|^2_{H(\text{curl};\Omega)} &\le |a^*_e(\mathbf{E} - \mathbf{E}^M_\delta, \mathbf{E} - \mathbf{E}^M_\delta)| = \left|\delta\omega^2 \textstyle\int_\Omega \boldsymbol{\varepsilon}\mathbf{E}^M_\delta \cdot (\overline{\mathbf{E}} - \overline{\mathbf{E}^M_\delta})\right| \\ &\le \delta\omega^2 \varepsilon_{\max} \|\mathbf{E}^M_\delta\|_{0,\Omega} \|\mathbf{E} - \mathbf{E}^M_\delta\|_{0,\Omega}\,, \end{aligned}$$

where $\varepsilon_{\max}$ is a uniform upper bound in Ω for the maximum eigenvalues of $\boldsymbol{\varepsilon}(\mathbf{x})$. Therefore

$$\|\mathbf{E} - \mathbf{E}^M_\delta\|_{H(\text{curl};\Omega)} \le \delta \frac{\omega^2 \varepsilon_{\max}}{C_0} \|\mathbf{E}^M_\delta\|_{0,\Omega}\,. \tag{2.11}$$

Now we need to show that $\|\mathbf{E}^M_\delta\|_{0,\Omega}$ is bounded uniformly with respect to δ. To do that we proceed as follows: first of all $\mathbf{E}^M_\delta$ satisfies

$$\int_\Omega \boldsymbol{\mu}^{-1} \operatorname{curl} \mathbf{E}^M_\delta \cdot \operatorname{curl} \overline{\mathbf{z}} + i\omega \int_{\Omega_C} \boldsymbol{\sigma}\mathbf{E}^M_{\delta,C} \cdot \overline{\mathbf{z}_C} = \int_\Omega (-i\omega \mathbf{J}_e + \delta\omega^2 \boldsymbol{\varepsilon}\mathbf{E}^M_\delta) \cdot \overline{\mathbf{z}}$$

for all $\mathbf{z} \in H(\text{curl};\Omega)$. Then, since $\operatorname{div}(\boldsymbol{\varepsilon}_I \mathbf{E}^M_{\delta,I}) = 0$ in Ω_I, using again the coerciveness of $a^*_e(\cdot,\cdot)$ in $W_{\varepsilon_I}(\Omega_I;\Omega)$ we have

$$C_0 \|\mathbf{E}^M_\delta\|_{H(\text{curl};\Omega)} \le |\omega| \|\mathbf{J}_e\|_{0,\Omega} + \delta\omega^2 \varepsilon_{\max} \|\mathbf{E}^M_\delta\|_{0,\Omega}\,.$$

Now, taking for instance $\delta_* = \frac{C_0}{2\omega^2 \varepsilon_{\max}}$, for all $\delta \le \delta_*$ we find

$$\|\mathbf{E}^M_\delta\|_{H(\text{curl};\Omega)} \le \frac{2|\omega|}{C_0} \|\mathbf{J}_e\|_{0,\Omega}\,,$$

and by substituting in (2.11) we obtain the desired result. □

2.3 The eddy current model as the low-frequency limit

The eddy current model can also be considered as the low-frequency limit of the full Maxwell system. This statement must be properly understood, since the limit problem obtained by formally taking the frequency equal to 0 is in fact the magneto-electrostatic problem, where induced eddy currents are not present. The interpretation of the limit procedure we are interested in is that the difference between the solution of the full Maxwell system and the solution of the eddy current model is vanishing as the frequency is going to 0. A different asymptotic analysis is performed when focusing on the difference between the eddy current solution and the magneto-electrostatic solution: this problem is considered in Section 7.4.

In this section we assume that all the material parameters are fixed and we consider the asymptotic behaviour as the frequency goes to 0 of the difference between the solution of the full Maxwell system (2.8), denoted by $\mathbf{E}^M$, and the solution of the eddy current problem (2.1), denoted by $\mathbf{E}$. We focus on the magnetic boundary condition $\mathbf{H} \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$, but we recall that this result also holds true when considering the electric boundary condition $\mathbf{E} \times \mathbf{n} = \mathbf{E}^M \times \mathbf{n} = \mathbf{0}$ in $\partial\Omega$, as proved in Alonso [5].

We show that the norm in $L^2(\Omega)$ of the difference $\mathbf{E} - \mathbf{E}^M$ is of order $|\omega|$. We also give an estimate in terms of ω of the $L^2(\Omega)$-norm of the difference between the magnetic fields.

As a first step we obtain an estimate of the energy norm of $\mathbf{E} - \mathbf{E}^M$ in terms of a power of $|\omega|$ times the $L^2(\Omega)$-norm of $\mathbf{E}^M$. Since the solution $\mathbf{E}^M$ depends on ω, a second step is the proof that the $L^2(\Omega)$-norm of $\mathbf{E}^M$ is uniformly bounded in $|\omega|$.

Lemma 2.6. *There exists a constant $C > 0$, independent of ω, such that*

$$\| \operatorname{curl}(\mathbf{E} - \mathbf{E}^M)\|^2_{0,\Omega} + |\omega| \|\mathbf{E}_C - \mathbf{E}^M_C\|^2_{0,\Omega_C} \leq C|\omega|^3(|\omega| + 1)\|\mathbf{E}^M\|^2_{0,\Omega} \,.$$

Proof. As we have seen in the previous section, from (2.10) we know that $(\mathbf{E}-\mathbf{E}^M) \in W_\varepsilon(\Omega_I; \Omega)$ and that

$$\int_\Omega \boldsymbol{\mu}^{-1} \operatorname{curl}(\mathbf{E} - \mathbf{E}^M) \cdot \operatorname{curl} \overline{\mathbf{z}} + i\omega \int_{\Omega_C} \boldsymbol{\sigma}(\mathbf{E}_C - \mathbf{E}^M_C) \cdot \overline{\mathbf{z}_C} = -\omega^2 \int_\Omega \boldsymbol{\varepsilon}\mathbf{E}^M \cdot \overline{\mathbf{z}}$$

for all $\mathbf{z} \in H(\operatorname{curl}; \Omega)$.

Taking $\mathbf{z} = \mathbf{E} - \mathbf{E}^M$ we have

$$\begin{aligned} &\textstyle\int_\Omega \boldsymbol{\mu}^{-1} \operatorname{curl}(\mathbf{E} - \mathbf{E}^M) \cdot \operatorname{curl}(\overline{\mathbf{E}} - \overline{\mathbf{E}^M}) + i\omega \int_{\Omega_C} \boldsymbol{\sigma}(\mathbf{E}_C - \mathbf{E}^M_C) \cdot (\overline{\mathbf{E}_C} - \overline{\mathbf{E}^M_C}) \\ &\qquad = -\omega^2 \textstyle\int_\Omega \boldsymbol{\varepsilon}\mathbf{E}^M \cdot (\overline{\mathbf{E}} - \overline{\mathbf{E}^M}) \,, \end{aligned}$$

hence

$$\begin{aligned} &\nu^2_{\min} \| \operatorname{curl}(\mathbf{E} - \mathbf{E}^M)\|^4_{0,\Omega} + \omega^2\sigma^2_{\min} \|\mathbf{E}_C - \mathbf{E}^M_C\|^4_{0,\Omega_C} \\ &\qquad \leq \omega^4\varepsilon^2_{\max} \|\mathbf{E}^M\|^2_{0,\Omega} \|\mathbf{E} - \mathbf{E}^M\|^2_{0,\Omega} \,. \end{aligned}$$

Since $\operatorname{div}(\boldsymbol{\varepsilon}_I(\mathbf{E}_I - \mathbf{E}^M_I)) = 0$ in Ω_I, from (2.7) we know that there exists a constant $C_1 > 0$, independent of ω, such that

$$\|\mathbf{E} - \mathbf{E}^M\|^2_{0,\Omega} \leq C_1 \left(\| \operatorname{curl}(\mathbf{E} - \mathbf{E}^M)\|^2_{0.\Omega} + \|\mathbf{E}_C - \mathbf{E}^M_C\|^2_{0,\Omega_C}\right) \,, \tag{2.12}$$

therefore we find, for a constant $C_3 > 0$ independent of ω,

$$\begin{aligned} &\| \operatorname{curl}(\mathbf{E} - \mathbf{E}^M)\|^4_{0,\Omega} + \omega^2 \|\mathbf{E}_C - \mathbf{E}^M_C\|^4_{0,\Omega_C} \\ &\qquad \leq C_3\omega^4 \|\mathbf{E}^M\|^2_{0,\Omega} (\| \operatorname{curl}(\mathbf{E} - \mathbf{E}^M)\|^2_{0,\Omega} + \|\mathbf{E}_C - \mathbf{E}^M_C\|^2_{0,\Omega_C}) \,. \end{aligned}$$

Using that for each $\delta > 0$ it holds $AB \leq \frac{1}{2\delta}A^2 + \frac{\delta}{2}B^2$, we have

$$\begin{aligned} \| \operatorname{curl}(\mathbf{E} - \mathbf{E}^M)\|^4_{0,\Omega} + \omega^2 \|\mathbf{E}_C - \mathbf{E}^M_C\|^4_{0,\Omega_C} &\leq \left(\tfrac{1}{2\delta_1} + \tfrac{1}{2\delta_2}\right) C_3^2\omega^8 \|\mathbf{E}^M\|^4_{0,\Omega} \\ &\quad + \tfrac{\delta_1}{2} \| \operatorname{curl}(\mathbf{E} - \mathbf{E}^M)\|^4_{0,\Omega} + \tfrac{\delta_2}{2} \|\mathbf{E}_C - \mathbf{E}^M_C\|^4_{0,\Omega_C} \,, \end{aligned}$$

for $\delta_1 > 0$ and $\delta_2 > 0$. Taking in particular $\delta_1 = 1$ and $\delta_2 = \omega^2$ one finds

$$\|\operatorname{curl}(\mathbf{E} - \mathbf{E}^M)\|^4_{0,\Omega} + \omega^2 \|\mathbf{E}_C - \mathbf{E}^M_C\|^4_{0,\Omega_C} \leq C_3^2(\omega^2+1)\omega^6 \|\mathbf{E}^M\|^4_{0,\Omega} ,$$

hence the desired result. □

The following result provides a bound for $\|\mathbf{E}^M\|_{0,\Omega}$ that is uniform with respect to $|\omega|$.

Lemma 2.7. *There exists a constant* ω^*, $0 < \omega^* \leq 1$, *such that for* $|\omega| \leq \omega^*$ *one has*

$$\|\mathbf{E}^M\|_{0,\Omega} \leq C ,$$

for some constant $C > 0$ *independent of* ω.

Proof. Multiplying the first equation of (2.8) by $\mathbf{E}^M$ and integrating by parts we obtain

$$\begin{aligned} &\int_\Omega \boldsymbol{\mu}^{-1} \operatorname{curl} \mathbf{E}^M \cdot \operatorname{curl} \overline{\mathbf{E}^M} + i\omega \int_{\Omega_C} \boldsymbol{\sigma} \mathbf{E}^M_C \cdot \overline{\mathbf{E}^M_C} \\ &\qquad = -i\omega \int_\Omega \mathbf{J}_e \cdot \overline{\mathbf{E}^M} + \omega^2 \int_\Omega \boldsymbol{\varepsilon} \mathbf{E}^M \cdot \overline{\mathbf{E}^M} , \end{aligned}$$

hence

$$\begin{aligned} &\nu^2_{\min} \|\operatorname{curl} \mathbf{E}^M\|^4_{0.\Omega} + \omega^2 \sigma^2_{\min} \|\mathbf{E}^M_C\|^4_{0,\Omega_C} \\ &\qquad \leq 2\omega^2 \|\mathbf{J}_e\|^2_{0,\Omega} \|\mathbf{E}^M\|^2_{0,\Omega} + 2\omega^4 \varepsilon^2_{\max} \|\mathbf{E}^M\|^4_{0,\Omega} , \end{aligned}$$

or simply, for a suitable constant $\hat{C} > 0$, independent of ω,

$$\begin{aligned} &\|\operatorname{curl} \mathbf{E}^M\|^2_{0.\Omega} + |\omega| \|\mathbf{E}^M_C\|^2_{0,\Omega_C} \\ &\qquad \leq \hat{C}(|\omega| \|\mathbf{J}_e\|_{0,\Omega} \|\mathbf{E}^M\|_{0,\Omega} + \omega^2 \|\mathbf{E}^M\|^2_{0,\Omega}) . \end{aligned}$$

Since $\operatorname{div}(\boldsymbol{\varepsilon}_I \mathbf{E}^M_I) = 0$ in Ω_I, as in (2.12) we have

$$\|\mathbf{E}^M\|^2_{0,\Omega} \leq C_1 \left(\|\operatorname{curl} \mathbf{E}^M\|^2_{0.\Omega} + \|\mathbf{E}^M_C\|^2_{0,\Omega_C} \right) .$$

Then, for $|\omega| \leq 1$,

$$\begin{aligned} \|\mathbf{E}^M\|^2_{0,\Omega} &\leq C_1 \tfrac{1}{|\omega|} (\|\operatorname{curl} \mathbf{E}^M\|^2_{0,\Omega} + |\omega| \|\mathbf{E}^M_C\|^2_{0,\Omega_C}) \\ &\leq \hat{C} C_1 \tfrac{1}{|\omega|} (|\omega| \|\mathbf{J}_e\|_{0,\Omega} \|\mathbf{E}^M\|_{0,\Omega} + \omega^2 \|\mathbf{E}^M\|^2_{0,\Omega}) \\ &\leq \tfrac{1}{2} \hat{C}^2 C_1^2 \|\mathbf{J}_e\|^2_{0,\Omega} + \tfrac{1}{2} \|\mathbf{E}^M\|^2_{0,\Omega} + \hat{C} C_1 |\omega| \|\mathbf{E}^M\|^2_{0,\Omega} . \end{aligned} \tag{2.13}$$

To finish the proof we have only to choose $|\omega| \leq \min\{1, \frac{1}{4\hat{C}C_1}\}$. □

In conclusion, we have obtained the following result.

Theorem 2.8. *There exists a constant* ω^*, $0 < \omega^* \leq 1$, *such that for* $|\omega| \leq \omega^*$ *one has*

$$\begin{aligned} \|\mathbf{E} - \mathbf{E}^M\|_{0,\Omega} &\leq C|\omega| \\ \|\mathbf{H} - \mathbf{H}^M\|_{0,\Omega} &\leq C|\omega|^{1/2} , \end{aligned}$$

for some constant $C > 0$ *independent of* ω.

Proof. From Lemma 2.6 and Lemma 2.7 for $|\omega| \leq \omega^*$ we have

$$\|\operatorname{curl}(\mathbf{E}-\mathbf{E}^M)\|^2_{0,\Omega} + |\omega|\|\mathbf{E}_C - \mathbf{E}^M_C\|^2_{0,\Omega_C} \leq C_*|\omega|^3(|\omega|+1) \tag{2.14}$$

for some constant $C_* > 0$ independent of ω. Hence proceeding as in (2.13), for $|\omega| \leq \omega^*$ we find

$$\|\mathbf{E}-\mathbf{E}^M\|^2_{0,\Omega} \leq C_1 C_* \omega^2(|\omega|+1) \leq 2C_1 C_* \omega^2 \, .$$

From (2.14) it also follows that

$$\|\operatorname{curl}(\mathbf{E}-\mathbf{E}^M)\|_{0,\Omega} \leq C|\omega|^{3/2} \, .$$

Finally, from the Faraday law $\operatorname{curl}(\mathbf{E}-\mathbf{E}^M) = -i\omega\boldsymbol{\mu}(\mathbf{H}-\mathbf{H}^M)$ we have also obtained

$$\|\mathbf{H}-\mathbf{H}^M\|_{0,\Omega} \leq C|\omega|^{1/2} \, ,$$

which ends the proof. □

2.3.1 Higher order convergence

Under suitable additional assumptions the order of convergence can be improved. The following result can be found in Schmidt et al. [223], where the eddy current modeling error has been investigated under different points of view.

Lemma 2.9. *Suppose that* $\operatorname{div}\mathbf{J}_e = 0$ *in* Ω *and that* Ω_C *is simply-connected. There exists a constant* ω^*, $0 < \omega^* \leq 1$, *such that for* $|\omega| \leq \omega^*$ *one has*

$$\|\mathbf{E}^M\|_{0,\Omega} \leq C|\omega| \, , \tag{2.15}$$

for some constant $C > 0$, *independent of* ω.

Proof. For a while, let us proceed without making use of the assumptions that $\operatorname{div}\mathbf{J}_e = 0$ in Ω and Ω_C is simply-connected.

Since from Theorem 2.8 we have $\|\mathbf{E}-\mathbf{E}^M\|_{0,\Omega} \leq C|\omega|$, it is enough to show that $\|\mathbf{E}\|_{0,\Omega} \leq C|\omega|$. From the Ampère equation we have $\operatorname{div}(\boldsymbol{\sigma}\mathbf{E}+\mathbf{J}_e) = 0$ in Ω, hence

$$\operatorname{div}(\boldsymbol{\sigma}\mathbf{E}_C + \mathbf{J}_{e,C}) = 0 \qquad \text{in } \Omega_C$$

and

$$(\boldsymbol{\sigma}\mathbf{E}_C + \mathbf{J}_{e,C}) \cdot \mathbf{n}_C = -\mathbf{J}_{e,I} \cdot \mathbf{n}_I \qquad \text{on } \Gamma \, .$$

Proceeding as in Lemma 2.1 we obtain

$$\begin{aligned}
\|\mathbf{E}_C\|_{0,\Omega_C} &\leq C\Big(\|\operatorname{curl}\mathbf{E}_C\|_{0,\Omega_C} + \|\operatorname{div}(\boldsymbol{\sigma}\mathbf{E}_C)\|_{0,\Omega_C} \\
&\qquad + \|\boldsymbol{\sigma}\mathbf{E}_C\cdot\mathbf{n}_C\|_{-1/2,\Gamma} + \textstyle\sum_{\beta=1}^{n_{\Omega_C}} \Big|\int_{\Omega_C} \boldsymbol{\sigma}\mathbf{E}_C \cdot \boldsymbol{\rho}^*_{\beta,C}\Big|\Big) \\
&\leq C\Big(\|\operatorname{curl}\mathbf{E}_C\|_{0,\Omega_C} + \|\operatorname{div}\mathbf{J}_{e,C}\|_{0,\Omega_C} \\
&\qquad + \|\mathbf{J}_{e,C}\cdot\mathbf{n}_C + \mathbf{J}_{e,I}\cdot\mathbf{n}_I\|_{-1/2,\Gamma} + \textstyle\sum_{\beta=1}^{n_{\Omega_C}} \Big|\int_{\Omega_C} \boldsymbol{\sigma}\mathbf{E}_C \cdot \boldsymbol{\rho}^*_{\beta,C}\Big|\Big),
\end{aligned}$$

where $\boldsymbol{\rho}^*_{\beta,C}$, $\beta = 1, \ldots, n_{\Omega_C}$, are the basis functions of the space of harmonic fields $\mathcal{H}_\sigma(m; \Omega_C)$ defined as follows

$$\mathcal{H}_\sigma(m; \Omega_C) := \{\mathbf{z}_C \in (L^2(\Omega_C))^3 \mid \operatorname{curl} \mathbf{z}_C = \mathbf{0}, \operatorname{div}(\boldsymbol{\sigma}\mathbf{z}_C) = 0, \\ \boldsymbol{\sigma}\mathbf{z}_C \cdot \mathbf{n}_C = 0 \text{ on } \Gamma\} \, .$$

Moreover, from Lemma 2.2 we know that

$$\|\mathbf{E}_I\|_{0,\Omega_I} \leq C(\| \operatorname{curl} \mathbf{E}\|_{0,\Omega} + \|\mathbf{E}_C\|_{0,\Omega_C}) \, ,$$

so that we end up with

$$\|\mathbf{E}\|_{0,\Omega} \leq C\Big(\| \operatorname{curl} \mathbf{E}\|_{0,\Omega} + \| \operatorname{div} \mathbf{J}_{e,C}\|_{0,\Omega_C} \\ + \|\mathbf{J}_{e,C} \cdot \mathbf{n}_C + \mathbf{J}_{e,I} \cdot \mathbf{n}_I\|_{-1/2,\Gamma} \\ + \textstyle\sum_{\beta=1}^{n_{\Omega_C}} \left| \int_{\Omega_C} \boldsymbol{\sigma}\mathbf{E}_C \cdot \boldsymbol{\rho}^*_{\beta,C} \right| \Big) \, . \tag{2.16}$$

From the Ampère equation we obtain by integration by parts

$$\int_\Omega \boldsymbol{\mu}^{-1} \operatorname{curl} \mathbf{E} \cdot \operatorname{curl} \overline{\mathbf{E}} + i\omega \int_{\Omega_C} \boldsymbol{\sigma}\mathbf{E}_C \cdot \overline{\mathbf{E}_C} = -i\omega \int_\Omega \mathbf{J}_e \cdot \overline{\mathbf{E}} \, ,$$

hence

$$\| \operatorname{curl} \mathbf{E}\|^2_{0,\Omega} \leq C|\omega| \|\mathbf{J}_e\|_{0,\Omega} \, \|\mathbf{E}\|_{0,\Omega} \, .$$

In conclusion, assuming that $\operatorname{div} \mathbf{J}_{e,C} = 0$ in Ω_C and $\mathbf{J}_{e,C} \cdot \mathbf{n}_C + \mathbf{J}_{e,I} \cdot \mathbf{n}_I = 0$ on Γ (which is equivalent to require that $\operatorname{div} \mathbf{J}_e = 0$ in Ω, as we have already assumed $\operatorname{div} \mathbf{J}_{e,I} = 0$ in Ω_I) and that Ω_C is simply-connected (so that the space $\mathcal{H}_\sigma(m; \Omega_C)$ is trivial), from (2.16) it follows

$$\|\mathbf{E}\|_{0,\Omega} \leq C\| \operatorname{curl} \mathbf{E}\|_{0,\Omega} \leq C|\omega|^{1/2} \|\mathbf{J}_e\|^{1/2}_{0,\Omega} \, \|\mathbf{E}\|^{1/2}_{0,\Omega} \, ,$$

hence

$$\|\mathbf{E}\|_{0,\Omega} \leq C|\omega| \|\mathbf{J}_e\|_{0,\Omega} \, ,$$

which ends the proof. □

Corollary 2.10. *Under the assumptions of Lemma 2.9, for $|\omega| \leq \omega^*$ one has*

$$\|\mathbf{E} - \mathbf{E}^M\|_{0,\Omega} \leq C|\omega|^2 \\ \|\mathbf{H} - \mathbf{H}^M\|_{0,\Omega} \leq C|\omega|^{3/2} \, . \tag{2.17}$$

Proof. From (2.15) and Lemma 2.6 we find

$$\| \operatorname{curl}(\mathbf{E} - \mathbf{E}^M)\|^2_{0,\Omega} + |\omega| \|\mathbf{E}_C - \mathbf{E}^M_C\|^2_{0,\Omega_C} \leq C|\omega|^5 \, ,$$

and consequently, proceeding as in Theorem 2.8, the thesis follows. □

Remark 2.11. If we were able to prove that $\boldsymbol{\sigma}\mathbf{E}_C$ is $L^2(\Omega_C)$-orthogonal to $\mathcal{H}_\sigma(m;\Omega_C)$, in Lemma 2.9 we could avoid to require that Ω_C is simply-connected.

In Ammari et al. [23] an attempt is made to devise the necessary and sufficient conditions on $\mathbf{J}_e$ ensuring that $\boldsymbol{\sigma}\mathbf{E}_C$ is orthogonal to $\mathcal{H}_\sigma(m;\Omega_C)$; however, their argument is not conclusive, and to our knowledge a characterization of this orthogonality property in terms of $\mathbf{J}_e$ is not known. □

The estimate for the difference between the magnetic fields can be improved even if we do not impose additional conditions on $\mathbf{J}_e$ and Ω_C.

Theorem 2.12. *Suppose that the domain Ω is simply-connected. There exists a constant ω^*, $0 < \omega^* \leq 1$, such that for $|\omega| \leq \omega^*$ one has*

$$\|\mathbf{H} - \mathbf{H}^M\|_{0,\Omega} \leq C|\omega| \, . \tag{2.18}$$

Moreover, if estimate (2.15) *is satisfied one has*

$$\|\mathbf{H} - \mathbf{H}^M\|_{0,\Omega} \leq C\omega^2 \, . \tag{2.19}$$

In both cases the constant $C > 0$ is independent of ω.

Proof. To prove this result we use the formulation of the eddy current model in terms of a magnetic vector potential $\mathbf{A}$ and a electric scalar potential V_C (see Chapter 6). This means that we consider $\mathbf{A}$ and V_C such that

$$\operatorname{curl} \mathbf{A} = \boldsymbol{\mu}\mathbf{H} \quad \text{and} \quad \mathbf{E}_C = -i\omega\mathbf{A}_C - \operatorname{grad} V_C \, .$$

Since Ω is simply-connected, we can also do the same for the Maxwell equations, and introduce $\mathbf{A}^M$ and V_C^M such that

$$\operatorname{curl} \mathbf{A}^M = \boldsymbol{\mu}\mathbf{H}^M \quad \text{and} \quad \mathbf{E}_C^M = -i\omega\mathbf{A}_C^M - \operatorname{grad} V_C^M \, .$$

Setting now $(\mathbf{Z}, N_C) := (\mathbf{A} - \mathbf{A}^M, V_C - V_C^M)$, it is easily seen that it satisfies the problem

$$\begin{cases} \operatorname{curl}(\boldsymbol{\mu}^{-1} \operatorname{curl} \mathbf{Z}) - \mu_*^{-1} \operatorname{grad} \operatorname{div} \mathbf{Z} & \\ \qquad\qquad + i\omega\boldsymbol{\sigma}\mathbf{Z} + \boldsymbol{\sigma} \operatorname{grad} N_C = -i\omega\boldsymbol{\varepsilon}\mathbf{E}^M & \text{in } \Omega \\ \operatorname{div}(i\omega\boldsymbol{\sigma}\mathbf{Z}_C + \boldsymbol{\sigma} \operatorname{grad} N_C) = -i\omega \operatorname{div}(\boldsymbol{\varepsilon}\mathbf{E}^M) & \text{in } \Omega_C \\ (i\omega\boldsymbol{\sigma}\mathbf{Z}_C + \boldsymbol{\sigma} \operatorname{grad} N_C) \cdot \mathbf{n}_C & \\ \qquad\qquad = -i\omega(\boldsymbol{\varepsilon}_C\mathbf{E}_C^M \cdot \mathbf{n}_C + \boldsymbol{\varepsilon}_I\mathbf{E}_I^M \cdot \mathbf{n}_I) & \text{on } \Gamma \\ \mathbf{Z} \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \\ (\boldsymbol{\mu}^{-1} \operatorname{curl} \mathbf{Z}) \times \mathbf{n} = \mathbf{0} & \text{on } \partial\Omega \, , \end{cases} \tag{2.20}$$

where N_C is determined up to an additive constant in each connected component of Ω_C. The corresponding weak formulation is the same than that presented in (6.12), with $\mathbf{J}_e$ replaced by $-i\omega\boldsymbol{\varepsilon}\mathbf{E}^M$.

Proceeding as in Section 6.1.2 (see in particular (6.36), (6.37), (6.38) and (6.39)), it can be proved that

$$\begin{aligned} \| \operatorname{curl} \mathbf{Z}\|_{0,\Omega}^2 + \| \operatorname{div} \mathbf{Z}\|_{0,\Omega}^2 + \|\mathbf{Z}\|_{0,\Omega}^2 + |\omega|^{-1}\tau\|N_C\|_{1,\Omega_C}^2 - C_4|\omega|\tau\|\mathbf{Z}\|_{0,\Omega}^2 \\ \leq C_4|\omega|\|\boldsymbol{\varepsilon}\mathbf{E}^M\|_{0,\Omega}\|\mathbf{Z}\|_{0,\Omega} + C_4\|\boldsymbol{\varepsilon}\mathbf{E}^M\|_{0,\Omega}\|N_C\|_{1,\Omega_C} \end{aligned}$$

for each $0 < \tau \le 1/2$ and a suitable positive constant C_4, independent of ω. Then for each $\delta_1 > 0$ and $\delta_2 > 0$

$$\begin{aligned}(1 - C_4|\omega|\tau)\|\mathbf{Z}\|^2_{0,\Omega} + \|\operatorname{curl}\mathbf{Z}\|^2_{0,\Omega} + \|\operatorname{div}\mathbf{Z}\|^2_{0,\Omega} + |\omega|^{-1}\tau\|N_C\|^2_{1,\Omega_C}\\ \le \tfrac{1}{2\delta_1}C_4^2\omega^2\|\boldsymbol{\varepsilon}\mathbf{E}^M\|^2_{0,\Omega} + \tfrac{\delta_1}{2}\|\mathbf{Z}\|^2_{0,\Omega} + \tfrac{1}{2\delta_2}C_4^2\|\boldsymbol{\varepsilon}\mathbf{E}^M\|^2_{0,\Omega} + \tfrac{\delta_2}{2}\|N_C\|^2_{1,\Omega_C} \, .\end{aligned}$$

Taking τ such that $1 - C_4|\omega|\tau > 0$ and choosing $\delta_1 = 1 - C_4|\omega|\tau$ and $\delta_2 = |\omega|^{-1}\tau$, we obtain that

$$|\omega|^{-1}\tau\|N_C\|^2_{1,\Omega_C} \le C_4^2\omega^2\tfrac{1}{1-C_4|\omega|\tau}\|\boldsymbol{\varepsilon}\mathbf{E}^M\|^2_{0,\Omega} + C_4^2|\omega|\tfrac{1}{\tau}\|\boldsymbol{\varepsilon}\mathbf{E}^M\|^2_{0,\Omega} \, .$$

If we choose $\tau = \min\{\frac{1}{2}, \frac{1}{2C_4|\omega|}\}$, for $|\omega| \le 1$ it is straightforward to verify that

$$\|N_C\|_{1,\Omega_C} \le C_5|\omega|\|\boldsymbol{\varepsilon}\mathbf{E}^M\|_{0,\Omega} \, ,$$

for some positive constant C_5, independent of ω.

Coming back to the weak formulation, we see that in particular we have

$$\begin{aligned}\int_\Omega \left(\boldsymbol{\mu}^{-1}\operatorname{curl}\mathbf{Z}\cdot\operatorname{curl}\overline{\mathbf{Z}} + \mu_*^{-1}|\operatorname{div}\mathbf{Z}|^2\right) + i\omega\int_{\Omega_C}\boldsymbol{\sigma}\mathbf{Z}_C\cdot\overline{\mathbf{Z}_C}\\ = -i\omega\int_\Omega \boldsymbol{\varepsilon}\mathbf{E}^M\cdot\overline{\mathbf{Z}} - \int_{\Omega_C}\boldsymbol{\sigma}\operatorname{grad}N_C\cdot\overline{\mathbf{Z}_C} \, .\end{aligned} \tag{2.21}$$

Hence, taking again into account (6.39), from (2.21) it is easy to see that

$$\begin{aligned}\|\boldsymbol{\mu}(\mathbf{H} - \mathbf{H}^M)\|_{0,\Omega} &= \|\operatorname{curl}\mathbf{Z}\|_{0,\Omega}\\ &\le C(|\omega|\|\mathbf{E}^M\|_{0,\Omega} + \|N_C\|_{1,\Omega_C}) \le C_6|\omega|\|\mathbf{E}^M\|_{0,\Omega}\end{aligned} \tag{2.22}$$

for some positive constant C_6, independent of ω. Thus, from Lemma 2.7,

$$\|\mathbf{H} - \mathbf{H}^M\|_{0,\Omega} = O(|\omega|) \, .$$

If we assume moreover that $\|\mathbf{E}^M\|_{0,\Omega} \le C|\omega|$ is satisfied (see for instance Lemma 2.9), from (2.22) one readily obtains (2.19). □

Remark 2.13. The geometrical assumption in Theorem 2.12 can be relaxed.

First, the solution of the Maxwell equations can be written in terms of $\mathbf{A}^M$ and V_C^M in more general geometric situations; for instance, it is surely true if the domain Ω_C is contained in a simply-connected domain $\widehat{\Omega}$ which is contained in Ω (hence, for example, if Ω is simply-connected).

Moreover, the results in Section 6.1.2 hold under the quite general geometrical assumptions that are described there by requiring that (6.2) is satisfied and that $n_{\partial\Omega} = n_\Omega$ (in particular, these assumptions hold true if Ω is simply-connected). □

Remark 2.14. In Ammari et al. [23], the full Maxwell problem and eddy current problem are considered in $\mathbb{R}^3$ with the following asymptotic conditions at infinity: for the full Maxwell system $\left(\mathbf{H}^M \times \frac{\mathbf{x}}{|\mathbf{x}|} - \mathbf{E}^M\right)$ tends to $\mathbf{0}$ uniformly as $|\mathbf{x}|$ goes to infinity, and for the eddy current problem $\mathbf{H}(\mathbf{x}) = O(1/|\mathbf{x}|)$ and $\mathbf{E}(\mathbf{x}) = O(1/|\mathbf{x}|)$ uniformly as $|\mathbf{x}|$ tends to infinity (see Chapter 7 for a more detailed presentation of the eddy current problem in the whole space $\mathbb{R}^3$).

Formally expanding the solutions of both problems in power series with respect to ω, they show that the eddy current model is a first order approximation of the full Maxwell system

$$\begin{aligned}\|\mathbf{E} - \mathbf{E}^M\|_{0,\Omega_R} &\leq C|\omega| \\ \|\mathbf{H} - \mathbf{H}^M\|_{0,\Omega_R} &\leq C|\omega| \ ,\end{aligned}$$

where $\Omega_R := (\mathbb{R}^3 \setminus \overline{\Omega_C}) \cap B_R$ and B_R is the open ball of radius R and center $\mathbf{0}$. In that paper the electric permittivity and the magnetic permeability are assumed to be constant outside B_R.

If additional conditions on the current source $\mathbf{J}_e$ and on Ω_C are fulfilled, they show that the eddy current model is in fact a second order approximation of the full Maxwell system

$$\begin{aligned}\|\mathbf{E} - \mathbf{E}^M\|_{0,\Omega_R} &\leq C\omega^2 \\ \|\mathbf{H} - \mathbf{H}^M\|_{0,\Omega_R} &\leq C\omega^2 \ .\end{aligned}$$

More precisely, having expanded $\mathbf{J}_e$ in the formal series

$$\mathbf{J}_e = \sum_{l\geq 0}^{\infty} \omega^l \mathbf{J}_e^l \ ,$$

where as usual for all $l \geq 0$ one has required that $\operatorname{div} \mathbf{J}_{e,I}^l = 0$ and $\int_{\Gamma_j} \mathbf{J}_{e,I}^l \cdot \mathbf{n}_I = 0$ for all $j = 1, \ldots, p_\Gamma + 1$, the additional assumption on the leading term $\mathbf{J}_e^0$ is the one we have already devised before for the complete field $\mathbf{J}_e$, namely, $\operatorname{div} \mathbf{J}_e^0 = 0$ in $\mathbb{R}^3$. Moreover, to complete the proof of the second order approximation, the conductor Ω_C is assumed to be simply-connected.

Let us also note that in this case $\Omega = \mathbb{R}^3$ is simply-connected, therefore the asymptotic behaviours obtained by Ammari et al. [23] are in perfect agreement with those established by resorting to the vector potentials $\mathbf{A}$ and $\mathbf{A}^M$: namely, first order approximation under general geometrical assumptions, in particular when Ω is simply-connected, and second order approximation under the additional assumptions that $\operatorname{div} \mathbf{J}_e = 0$ in Ω and the conductor Ω_C is simply-connected. □