3

The derivation of the equations for fluids and structure

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In this chapter we derive the equations governing the mechanics of the fluid (the blood) and the structure (the vessel wall). The derivation will be made in a rigorous way, yet trying to provide the reader a physical understanding. Reason of space obliged us to focus on the most important models for haemodynamic computations and to omit several details and the proofs of the propositions. For instance, all issues related to energy conservation principles have been ignored and we have eventually considered only incompressible fluids.

The reader interested in a more in-depth analysis may refer to several books on continuum mechanics available in the literature. We give here some, non exhaustive, indications. Introductory general texts are [211,310] as well as [221], which is more focused on solid mechanics. More mathematical oriented texts are [92, 313], while a general introduction on non-linear mechanics is found in [512]. For what concerns shell theory, of which we are here giving just an outline, a rigorous mathematical introduction is found in [93,94], while a text more oriented on the numerical aspects is [75]. As for fluid mechanics, we mention [90].

The first part of the chapter is dedicated to the kinematics of continuum media. Kinematics is the part of mechanics that describe the motion. It forms the background enabling to derive the differential equations which "translate" into mathematical terms some fundamental principles of physics. Namely, mass and momentum balance. Up to this point there is no need to distinguish between solids and fluids (and gases, for what matters).

It is only when we characterise how the medium reacts internally to an exterior action that the behaviour of the two types of media diverge and we are able to finalise the derivation of the mathematical models.

In haemodynamic applications often fluid and structure interact, for instance when blood flows in a compliant vessel. This fact prompts the introduction of a particular point of view: the so-called Arbitrary Lagrangian Eulerian formulation, which is particularly convenient for the numerical computation of this type of problems. Reduced models for the structure are often used

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in fluid/structure interaction computations to cut down on computational complexity. They will be illustrated together with the general formulation of the fluid-structure problem.

We will write the equations in the so called *tensorial form*, which is independent of the particular coordinate system used. Whenever is needed we will report the corresponding expression in terms of Cartesian components. When dealing with shell models we will need to introduce a curvilinear system of coordinates. We have tried to keep the notation as standard as possible, avoiding too technical details, yet the reader may find in appendix 12.4.4 an explanation of the main mathematical symbols used throughout the chapter.

3.1 The kinematics of continuum media

Let $\widehat{\Omega} \subset \mathbb{R}^3$ be a domain, that is a bounded, open and simply connected subset of $\mathbb{R}^3$, with smooth boundary, filled by a *continuum medium*. We shall refer to $\widehat{\Omega}$ as the *reference configuration* of the medium under consideration.

A *deformation* of $\widehat{\Omega}$ is a smooth one-to-one mapping

$$\widehat{\boldsymbol{\phi}} : \widehat{\Omega} \longrightarrow \Omega, \quad \widehat{\boldsymbol{x}} \longrightarrow \boldsymbol{x} = \widehat{\boldsymbol{\phi}}(\widehat{\boldsymbol{x}}),$$

associating each point $\widehat{\boldsymbol{x}}$ of $\widehat{\Omega}$ to new position $\boldsymbol{x} = \widehat{\boldsymbol{\phi}}(\widehat{\boldsymbol{x}})$ in the current configuration $\Omega \subset \mathbb{R}^3$. The vector quantity

$$\widehat{\boldsymbol{\eta}}(\widehat{\boldsymbol{x}}) = \widehat{\boldsymbol{\phi}}(\widehat{\boldsymbol{x}}) - \widehat{\boldsymbol{x}} \tag{3.1}$$

is called *displacement* of the *material point* $\widehat{\boldsymbol{x}}$. The local deformation is linked to the *deformation gradient*, defined as

$$\widehat{\boldsymbol{F}}(\widehat{\boldsymbol{x}}) = \boldsymbol{\nabla}_{\widehat{\boldsymbol{x}}} \widehat{\boldsymbol{\phi}}. \tag{3.2}$$

Here, the symbol $\boldsymbol{\nabla}_{\widehat{\boldsymbol{x}}}$ indicates the gradient with respect to the $\widehat{\boldsymbol{x}} = (\widehat{x}_1, \widehat{x}_2, \widehat{x}_3)$ coordinates. Sometimes we will omit the suffix when it is clear from the context which coordinate system we are adopting. The deformation gradient is a second order tensor field, therefore $\widehat{\boldsymbol{F}} : \widehat{\Omega} \to \mathbb{R}^{3\times 3}$ being $\mathbb{R}^{3\times 3}$ the space of three dimensional matrices. In Cartesian coordinates its value is given by the 3×3 matrix of components[1]

$$\widehat{F}_{ij} = \frac{\partial x_i}{\partial \widehat{x}_j}, \quad i, j = 1, 2, 3.$$

We also assume that its determinant

$$\widehat{J} = \det \widehat{\boldsymbol{F}}, \tag{3.3}$$

[1] In fact, to be a tensor the components have to satisfy certain rules with respect to coordinate transformation, see any elementary book on tensor analysis like [47].

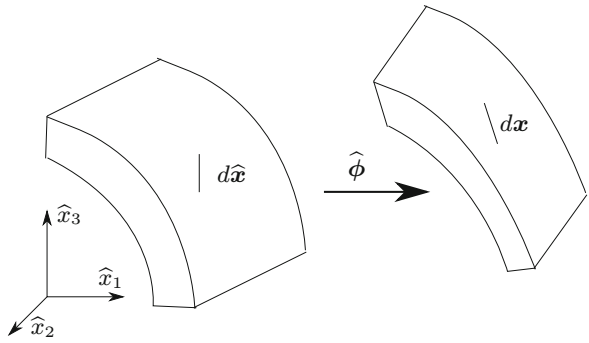

Fig. 3.1. Local deformation of a material neighbourhood. An infinitesimal vector $d\widehat{\boldsymbol{x}}$ is deformed into $d\boldsymbol{x}$

called the *Jacobian* of the deformation, is everywhere strictly positive. It means that the mapping is *orientation preserving*.

The link between $\widehat{\boldsymbol{F}}$ and the local deformation is made clear if we consider two arbitrary points $\widehat{\boldsymbol{a}}$ and $\widehat{\boldsymbol{b}} = \widehat{\boldsymbol{a}} + \widehat{\boldsymbol{\delta}}$ of $\widehat{\Omega}$, separated by the "small" vector $\widehat{\boldsymbol{\delta}}$ (refer to Fig. 3.1).

Let $\boldsymbol{a} = \widehat{\boldsymbol{\phi}}(\widehat{\boldsymbol{a}})$ and $\boldsymbol{b} = \widehat{\boldsymbol{\phi}}(\widehat{\boldsymbol{b}})$ be the corresponding points in Ω. The regularity of the map $\widehat{\boldsymbol{\phi}}$ allows us to write $\boldsymbol{b} = \boldsymbol{a} + \boldsymbol{\delta} = \widehat{\boldsymbol{a}} + \widehat{\boldsymbol{F}}(\widehat{\boldsymbol{a}})\widehat{\boldsymbol{\delta}} + o(\widehat{\boldsymbol{\delta}})$, where the symbol $o(\boldsymbol{h})$ stands for an infinitesimal of higher order than $\boldsymbol{h}$ for $||\boldsymbol{h}|| \to 0^+$, being $||\cdot||$ the Euclidean norm. The length of $\boldsymbol{\delta} = \boldsymbol{b} - \boldsymbol{a}$ is given by

$$||\boldsymbol{\delta}|| = \sqrt{\boldsymbol{\delta}^T \boldsymbol{\delta}} = \left[\widehat{\boldsymbol{\delta}}^T \widehat{\boldsymbol{F}}^T(\widehat{\boldsymbol{a}})\widehat{\boldsymbol{F}}(\widehat{\boldsymbol{a}})\widehat{\boldsymbol{\delta}}\right]^{1/2} + o(||\widehat{\boldsymbol{\delta}}||).$$

This relation is often expressed in the form

$$||d\boldsymbol{x}|| = \sqrt{d\widehat{\boldsymbol{x}}^T \widehat{\boldsymbol{F}}^T \widehat{\boldsymbol{F}} d\widehat{\boldsymbol{x}}}, \tag{3.4}$$

and it gives the change of the length of the "infinitesimal vector" $d\widehat{\boldsymbol{x}}$ due to the deformation. The tensor $\widehat{\boldsymbol{C}} = \widehat{\boldsymbol{F}}^T \widehat{\boldsymbol{F}}$ is called the *right Cauchy-Green* tensor.

In the following we will often indicate by $\widehat{V}$ a subdomain of $\widehat{\Omega}$ and by V its image $V = \widehat{\boldsymbol{\phi}}(\widehat{V}) = \{\boldsymbol{x} \in \Omega : \widehat{\boldsymbol{\phi}}^{-1}(\boldsymbol{x}) \in \widehat{V}\}$.

Then,

$$|V| = \int_V d\boldsymbol{x} = \int_{\widehat{V}} \widehat{J}(\widehat{\boldsymbol{x}}) d\widehat{\boldsymbol{x}} \tag{3.5}$$

is the *measure* of V (i.e. its volume). The Jacobian thus measures the variation of volume due to the deformation.

To derive the equations of continuum mechanics we need to relate differential operators acting on the two configurations. For instance, by applying the usual rules for the gradient of composite functions we have

Proposition 3.1. *Let* $\widehat{f} : \widehat{\Omega} \to \mathbb{R}$ *be a regular function and* $f : \Omega \to \mathbb{R}$ *defined as* $f(\boldsymbol{x}) = \widehat{f}(\widehat{\boldsymbol{\phi}}^{-1}(\boldsymbol{x}))$*; then*

$$\boldsymbol{\nabla}_{\widehat{\boldsymbol{x}}}\widehat{f} = \widehat{\boldsymbol{F}}\boldsymbol{\nabla} f.$$

An important role in this context is played by the *Piola transformation*. Let us refer to Fig. 3.2, where we display a generic volume $\widehat{V} \subset \widehat{\Omega}$ and its image V, together with the corresponding normals to the boundary, indicated by $\widehat{\boldsymbol{n}}$ and $\boldsymbol{n}$, respectively. Let also assume that we have a sufficiently regular second order tensor field $\boldsymbol{\sigma} : \Omega \longrightarrow \mathbb{R}^{3\times 3}$, defined on the deformed configuration.

The *Piola transformation* of $\boldsymbol{\sigma}$ associated to the given deformation $\widehat{\boldsymbol{\phi}}$ is the second order tensor field $\widehat{\boldsymbol{\Pi}} = \mathcal{P}_{\widehat{\boldsymbol{\phi}}}(\boldsymbol{\sigma}) : \widehat{\Omega} \to \mathbb{R}^{3\times 3}$ given by

$$\widehat{\boldsymbol{\Pi}}(\widehat{\boldsymbol{x}}) = \widehat{J}(\widehat{\boldsymbol{x}})\boldsymbol{\sigma}\big(\widehat{\boldsymbol{\phi}}(\widehat{\boldsymbol{x}})\big)\widehat{\boldsymbol{F}}^{-T}(\widehat{\boldsymbol{x}}), \tag{3.6}$$

for all $\widehat{\boldsymbol{x}} \in \widehat{\Omega}$. Using a short hand notation we may write $\widehat{\boldsymbol{\Pi}} = \widehat{J}\widehat{\boldsymbol{\sigma}}\widehat{\boldsymbol{F}}^{-T}$.

The *inverse Piola transformation* of $\widehat{\boldsymbol{\Pi}}$ returns the tensor $\boldsymbol{\sigma}(\boldsymbol{x})$ according to

$$\boldsymbol{\sigma}(\boldsymbol{x}) = \widehat{J}^{-1}\big(\widehat{\boldsymbol{\phi}}^{-1}(\boldsymbol{x})\big)\widehat{\boldsymbol{\Pi}}\big(\widehat{\boldsymbol{\phi}}^{-1}(\boldsymbol{x})\big)\widehat{\boldsymbol{F}}^{T}\big(\widehat{\boldsymbol{\phi}}^{-1}(\boldsymbol{x})\big), \tag{3.7}$$

or, more simply, $\boldsymbol{\sigma} = J^{-1}\widehat{\boldsymbol{\Pi}}\boldsymbol{F}^{T}$.

The main property of the Piola transformation is given by the following important formula (see [92] for a proof).

Proposition 3.2. *Let* $\boldsymbol{\sigma}$ *be a regular tensor field in* Ω *and* $\widehat{\boldsymbol{\Pi}}$ *its Piola transformation, we have*

$$\mathbf{div}_{\widehat{\boldsymbol{x}}}\widehat{\boldsymbol{\Pi}} = J\,\mathbf{div}\boldsymbol{\sigma}, \tag{3.8}$$

where $\mathbf{div}_{\widehat{\boldsymbol{x}}}$ *is the divergence with respect to the* $\widehat{\boldsymbol{x}}$ *coordinates and the equality has to be understood on corresponding points in* $\widehat{\Omega}$ *and* Ω*, respectively.*

As a result, by the application of the divergence theorem, we have

$$\int_{\partial\widehat{V}} \widehat{\boldsymbol{\Pi}}\widehat{\boldsymbol{n}}\,\mathrm{d}\hat{\gamma} = \int_{\partial V} \boldsymbol{\sigma}\boldsymbol{n}\,\mathrm{d}\gamma, \tag{3.9}$$

whenever $\widehat{\boldsymbol{\Pi}}$ and $\boldsymbol{\sigma}$ are related by (3.6). The use of this equality will be made clear in the next sections.

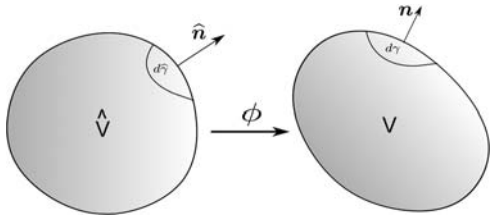

Fig. 3.2. The transformation of a material volume under deformation

It is also possible to derive the following relation for any scalar field f in Ω and $\widehat{f} = f \circ \widehat{\boldsymbol{\phi}}$

$$\int_{\partial V} f\boldsymbol{n}\,\mathrm{d}\boldsymbol{\gamma} = \int_{\partial \widehat{V}} \widehat{f}\widehat{J}\widehat{\boldsymbol{F}}^{-T}\widehat{\boldsymbol{n}}\,\mathrm{d}\hat{\boldsymbol{\gamma}}, \tag{3.10}$$

where

$$\mathrm{d}\boldsymbol{\gamma} = \widehat{J}||\widehat{\boldsymbol{F}}^{-T}\widehat{\boldsymbol{n}}||\,\mathrm{d}\hat{\boldsymbol{\gamma}} \quad \text{and } \boldsymbol{n} = \frac{\widehat{\boldsymbol{F}}^{-T}\widehat{\boldsymbol{n}}}{||\widehat{\boldsymbol{F}}^{-T}\widehat{\boldsymbol{n}}||}. \tag{3.11}$$

The first relation in (3.11) is often called *Nanson's formula* and relates the measure of a surface element in the reference configuration to that of the corresponding element in the current configuration. The second expression relates the corresponding normals.

3.1.1 The motion

What we have shown so far is a static picture, to have motion we need to bring the time into play. A *motion* is a smooth map

$$\widehat{\boldsymbol{\varphi}} : \widehat{\Omega} \times \mathbb{R}^+ \longrightarrow \mathbb{R}^3, \quad (\widehat{\boldsymbol{x}}, t) \longrightarrow \boldsymbol{x} = \widehat{\boldsymbol{\varphi}}(\widehat{\boldsymbol{x}}, t),$$

such that, at any $t \geq 0$, $\widehat{\boldsymbol{\varphi}}_t = \widehat{\boldsymbol{\varphi}}(\cdot, t)$ is a deformation. In other words, a motion is one-parameter family of deformations, the parameter t being the time. Without loss of generality we have assumed here that the motion starts at $t = 0$ (initial time). The reference configuration $\widehat{\Omega}$ is in principle arbitrary, yet often it coincides with the initial configuration, i.e $\widehat{\Omega} = \Omega(0)$. When not otherwise stated, we will implicitly make this assumption.

The point $\boldsymbol{x} = \widehat{\boldsymbol{\varphi}}(\widehat{\boldsymbol{x}}, t)$ is the position at time t of the *material point* (also called *material particle*) identified by $\widehat{\boldsymbol{x}}$, while $\Omega(t) = \widehat{\boldsymbol{\varphi}}(\widehat{\Omega}, t)$ denotes the *current configuration* at time t.

In this context, the displacement is now also function of time, $\widehat{\boldsymbol{\eta}}(\widehat{\boldsymbol{x}}, t) = \widehat{\boldsymbol{\varphi}}(\widehat{\boldsymbol{x}}, t) - \widehat{\boldsymbol{x}}$ being the displacement at time t.

All the kinematic quantities defined in the previous section can be extended to a motion. In particular, $\widehat{\boldsymbol{F}}$ and $\widehat{J}$ still indicate the deformation gradient and Jacobian, respectively, yet are now function also of time. For instance, $\widehat{\boldsymbol{F}}(\widehat{\boldsymbol{x}}, t) = \boldsymbol{\nabla}_{\widehat{\boldsymbol{x}}}\widehat{\boldsymbol{\varphi}}(\widehat{\boldsymbol{x}}, t)$.

Given a subdomain $\widehat{V} \subset \widehat{\Omega}$, the set $V(t) = \{\boldsymbol{x} \in \Omega(t) : \boldsymbol{x} = \widehat{\boldsymbol{\varphi}}(\widehat{\boldsymbol{x}}, t), \widehat{\boldsymbol{x}} \in \widehat{V}\}$ is formed by the same material particles as $\widehat{V}$ and is called a *material (sub)domain*, or also *material volume*. Thanks to (3.5), if $\widehat{J}$ is constant in time (i.e $\frac{\partial \widehat{J}}{\partial t} = 0$) the material subdomain does not change its measure during motion.

The *velocity* is indeed a major kinematic quantity and is the time derivative of the displacement:

$$\widehat{\boldsymbol{u}}(\widehat{\boldsymbol{x}}, t) = \frac{\partial}{\partial t}\widehat{\boldsymbol{\eta}}(\widehat{\boldsymbol{x}}, t) = \frac{\partial}{\partial t}\widehat{\boldsymbol{\varphi}}(\widehat{\boldsymbol{x}}, t), \tag{3.12}$$

the last equality is obtained by using the definition (3.1), now referred to time t.

3.1.2 Lagrangian, Eulerian and ALE formulations

We can define all physical quantities alternatively on the reference or on the current configuration, the choice being a matter of convenience. For instance the field $\widehat{\rho} : \widehat{\Omega} \times \mathbb{R}^+ \to \mathbb{R}^+$ indicates the density, i.e. $\widehat{\rho}(\widehat{\boldsymbol{x}}, t)$ is the density at time t in the material point $\widehat{\boldsymbol{x}}$. Yet, the invertibility of the mapping allows us to refer the same quantity to the current configuration: for all $t > 0$

$$\rho(\boldsymbol{x}, t) = \widehat{\rho}(\widehat{\boldsymbol{\varphi}}_t^{-1}(\boldsymbol{x}), t), \quad \boldsymbol{x} \in \Omega(t),$$

is the density at the point $\boldsymbol{x} \in \Omega(t)$ occupied by the material particle $\widehat{\boldsymbol{x}}$ at time t.

The interplay between these two "points of view" is crucial in continuum mechanics. When we adopt $(\widehat{\boldsymbol{x}}, t)$ as independent variables we use a *Lagrangian formulation*, while when we refer to the $(\boldsymbol{x}, t)$ pair we employ the *Eulerian formulation*. In the Lagrangian formulation we focus on the material particle $\widehat{\boldsymbol{x}}$ and its evolution; in the Eulerian formulation we observe what happens at a given point $\boldsymbol{x}$ in the physical space. When a field is expressed in the Eulerian coordinates it is referred to as an *Eulerian field*, while a *Lagrangian field*, also called *material field*, is a field expressed in Lagrangian coordinates.

We will adopt the same symbol for a given physical quantity. Yet, the superscript $\widehat{\ }$ will denote a Lagrangian field. To summarise, for a quantity q we have

$$\widehat{q}(\widehat{\boldsymbol{x}}, t) = q(\boldsymbol{x}, t), \quad \text{with } \boldsymbol{x} = \widehat{\boldsymbol{\varphi}}(\widehat{\boldsymbol{x}}, t),\ \widehat{\boldsymbol{x}} \in \widehat{\Omega},\ t > 0. \tag{3.13}$$

We will also make use of the composition operator: $\widehat{q}(\cdot, t) = q(\cdot, t) \circ \widehat{\boldsymbol{\varphi}}_t$. Conversely,

$$q(\boldsymbol{x}, t) = \widehat{q}(\widehat{\boldsymbol{x}}, t), \quad \text{with } \widehat{\boldsymbol{x}} = \widehat{\boldsymbol{\varphi}}_t^{-1}(\boldsymbol{x}),\ \boldsymbol{x} \in \Omega(t),\ t > 0, \tag{3.14}$$

or, more simply, $q(\cdot, t) = \widehat{q}(\cdot, t) \circ \widehat{\boldsymbol{\varphi}}_t^{-1}$ (see Fig. 3.3 and Fig. 3.4).

Therefore, the velocity $\boldsymbol{u}$ in the Eulerian frame is simply obtained by mapping $\widehat{\boldsymbol{u}}$ in the current configuration, i.e.

$$\boldsymbol{u}(\boldsymbol{x}, t) = \widehat{\boldsymbol{u}}(\widehat{\boldsymbol{\varphi}}_t^{-1}(\boldsymbol{x}), t), \quad (\boldsymbol{x}, t) \in \Omega(t) \times \mathbb{R}^+. \tag{3.15}$$

Sometimes it is useful to describe the displacement as an Eulerian field, obtaining

$$\boldsymbol{\eta}(\boldsymbol{x}, t) = \boldsymbol{x} - \widehat{\boldsymbol{\varphi}}_t^{-1}(\boldsymbol{x}). \tag{3.16}$$

One formulation may be more convenient than the other, depending on the context. Let us make this aspect more precise. As already illustrated in Chapter 2 when we want to solve the differential equations governing the motion of a fluid or a solid we need to identify the *computational domain* where we want

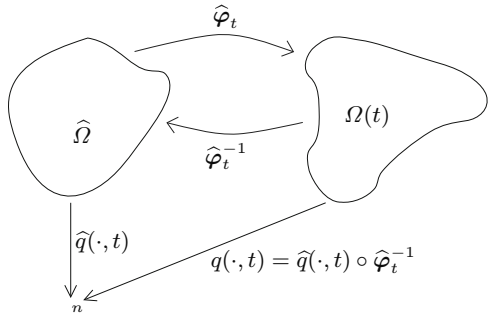

Fig. 3.3. Eulerian description of a Lagrangian field

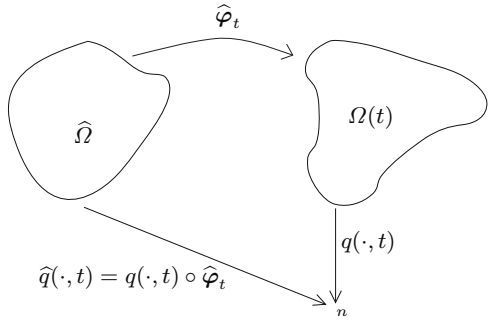

Fig. 3.4. Lagrangian description of an Eulerian field

to solve the equations, on the boundary of which we need to provide suitable boundary conditions. In a solid, where the displacements are often relatively small, the computational domain is often taken to be $\widehat{\Omega}$ and the Lagrangian formulation is thus preferred. This situation is sketched in Fig. 3.5.

In a fluid the situation is rather different. The displacements are extremely large and, moreover, usually irrelevant, since, as we will make clear in a later

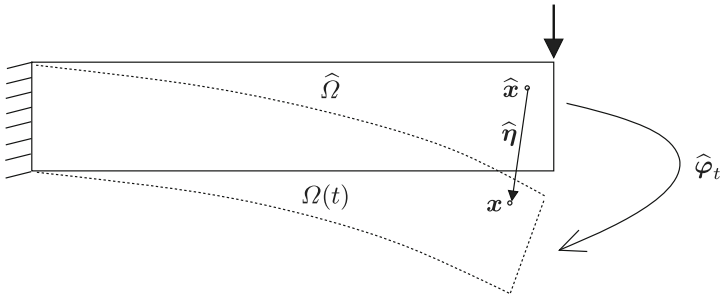

Fig. 3.5. Lagrangian description of the motion of a solid. The differential problem will be posed on $\widehat{\Omega}$, using a Lagrangian description

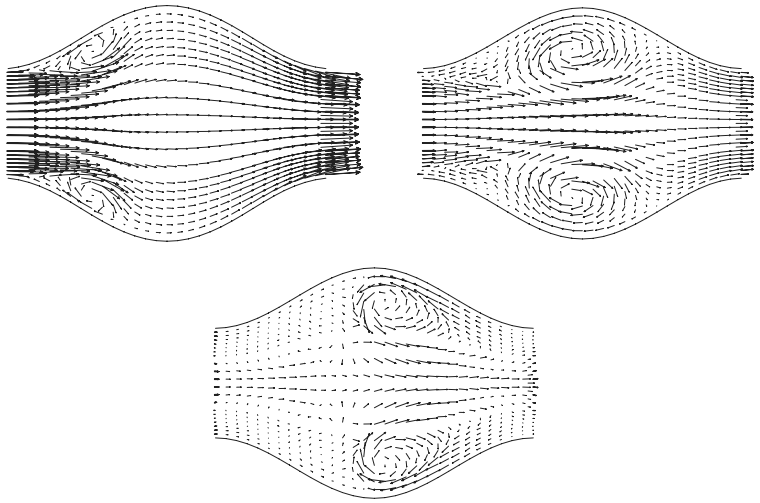

Fig. 3.6. Example of velocity field in the same region at three time instants

section, for a fluid we are normally interested in the velocity field, like the one in Fig. 3.6, or other related quantities, rather than the displacement itself. Therefore, the computational domain is normally chosen as a fixed, open bounded set $\Omega \subset \mathbb{R}^3$ located where we are interested to compute the solution. No special requirements is made on Ω apart that is should be "filled by the fluid", that is $\Omega \subset \Omega(t)$ for all times t we are observing the motion[2], see Fig. 3.7. The Eulerian framework is then here preferable. However, the Lagrangian frame is still useful as a tool to formally derive the equations from fundamental principles.

As already mentioned in Chapter 2, in many cases of practical interest in haemodynamics, such as blood flowing in a compliant artery, the computational domain for the fluid cannot be fixed in time, as it has to follow the displacements of the fluid-wall interface[3]. Yet, the Lagrangian frame is not of help here, since certainly we do not wish to follow the evolution of the blood particles as they circulate along the whole cardiovascular system! We usually wish to compute the flow field in a domain confined in the area of interest, yet following the movement of the wall interface, (see for instance Fig. 3.8).

The computational domain, which we will now indicate with $\omega(t)$, is neither fixed nor a material subdomain, since its evolution is not governed by the fluid motion, but has to comply by that of the boundary $\partial\omega(t)$, which

[2] For the sake of simplicity we have set as the time interval for our equations the whole positive real line, yet in practical computations the time interval of interest is obviously finite.

[3] Special computational techniques, like the immersed boundary method [385], may get around this fact, at the price of using more complex equations, see Chapter 9.

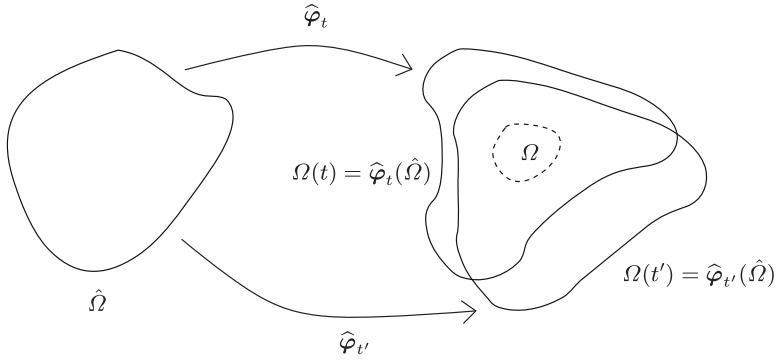

Fig. 3.7. The computational domain Ω in the Eulerian formulation. It is a fixed portion of space filled by the medium during its motion

is either given or the result of the coupling with a structural model. It is then necessary to introduce another, intermediate, frame of reference, called *Arbitrary Lagrangian Eulerian* (ALE).

We will show in Section 3.5 how it is possible to build from the evolution of $\partial\omega(t)$ an auxiliary motion

$$\widetilde{\mathcal{A}} : \widetilde{\omega} \times \mathbb{R}^+ \to \mathbb{R}^3 \quad (\widetilde{\boldsymbol{x}}, t) \to \boldsymbol{x} = \widetilde{\mathcal{A}}(\widetilde{\boldsymbol{x}}, t),$$

such that $\omega(t) = \widetilde{\mathcal{A}}(\widetilde{\omega}, t)$, for all $t > 0$, see Fig. 3.9. Here, $\widetilde{\omega} \subset \mathbb{R}^3$ is a reference (fixed) configuration, which in general (yet not necessarily) corresponds to the initial position at $t = 0$, i.e. $\omega(0)$. Fig. 3.9 gives a sketch of the situation.

In the ALE formulation we have then the interplay of (at least) two motions: the one of the medium under consideration and that of the computational domain. The former is governed by physical laws, the latter is rather arbitrary, provided that the given law for the domain boundary movement be respected.

Given an *ALE field*, that is a field defined in the ALE reference domain, $\widetilde{q} : \widetilde{\omega} \times \mathbb{R}^+ \longrightarrow \mathbb{R}$, its *Eulerian description* is given by

$$q(\boldsymbol{x}, t) = \widetilde{q}(\widetilde{\mathcal{A}}_t^{-1}(\boldsymbol{x}), t), \quad \forall \boldsymbol{x} \in \omega(t), \quad t > 0,$$

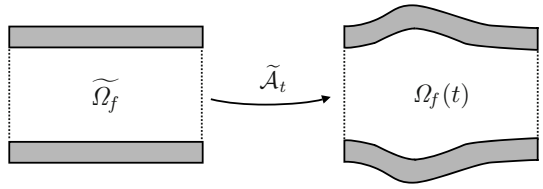

Fig. 3.8. The computational domain Ω_f for the fluid in a compliant artery. It deforms to follow the arterial wall movement, yet the axial position of its proximal and distal boundary is kept fixed. Its evolution is described by the ALE map

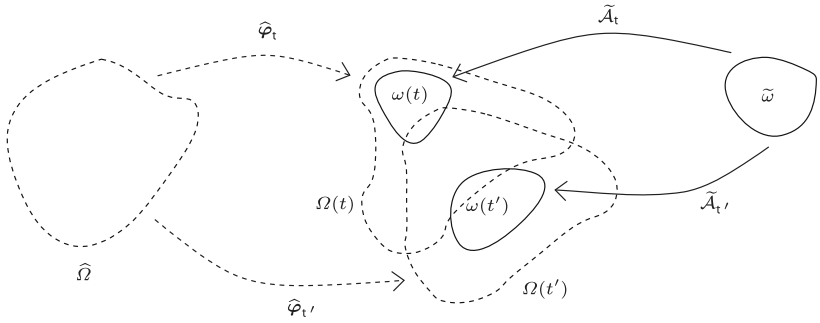

Fig. 3.9. The moving computational domain $\omega(t)$ and the ALE map. Here, for generality, we show an arbitrary reference computational domain $\widetilde{\omega}$. Most of the times, however, it is chosen to coincide with $\omega(0)$

also indicated as $q(\cdot,t) = \widetilde{q}(\cdot,t) \circ \widetilde{\mathcal{A}}_t^{-1}$. Conversely,

$$\widetilde{q}(\widetilde{\boldsymbol{x}},t) = q(\widetilde{\mathcal{A}}(\widetilde{\boldsymbol{x}},t),t), \quad \forall \widetilde{\boldsymbol{x}} \in \widetilde{\omega}, \quad t > 0, \tag{3.17}$$

or, equivalently $\widetilde{q}(\cdot,t) = q(\cdot,t) \circ \widetilde{\mathcal{A}}_t$. Here we have taken the case of a scalar field, yet the same rule applies to vector and tensor fields.

Analogously to what done before, we can define the *computational domain velocity*, also called *ALE velocity*, as

$$\widetilde{\boldsymbol{w}}(\widetilde{\boldsymbol{x}},t) = \frac{\partial \widetilde{\mathcal{A}}}{\partial t}(\widetilde{\boldsymbol{x}},t), \quad \forall \widetilde{\boldsymbol{x}} \in \widetilde{\omega}, \tag{3.18}$$

which can be mapped to the Eulerian frame by means of (3.17), in short hand notation $\boldsymbol{w}(.,t) = \widetilde{\boldsymbol{w}}(\cdot,t) \circ \widetilde{\mathcal{A}}_t^{-1}$.

Remark 3.1.1 *In general, $\boldsymbol{w}(\boldsymbol{x},t) \neq \boldsymbol{u}(\boldsymbol{x},t)$. However, we can note two particular cases:*

- $\boldsymbol{w} = \boldsymbol{0}$*: the computational domain is fixed as $\omega(t) = \omega(0)$ for all times; we recover the Eulerian formulation;*
- $\boldsymbol{w} = \boldsymbol{u}$*: the computational domain $\omega(t)$ is now a material domain; we recover the Lagrangian formulation.*

In analogy with what already done for the Lagrangian frame, we can define the Jacobian of the ALE movement $\widetilde{J}_{\widetilde{\mathcal{A}}} = \det \frac{\partial \widetilde{\mathcal{A}}}{\partial \widetilde{\boldsymbol{x}}}$ and with $J_{\widetilde{\mathcal{A}}}$ the corresponding quantity in the current configuration, obtained by composition with the inverse ALE map. Recasting the Euler expansion formula (3.26) to the ALE mapping we obtain

$$\frac{\partial J_{\widetilde{\mathcal{A}}}}{\partial t}_{|\widetilde{\mathcal{A}}} = J_{\widetilde{\mathcal{A}}} \operatorname{div} \boldsymbol{w}. \tag{3.19}$$

For the sake of notation, we will follow the convention that if $F = F(\boldsymbol{x},t)$ then F_t indicates the function of the space variable only, defined as $F_t(\boldsymbol{x}) = F(\boldsymbol{x},t)$, at any fixed time t.

Eulerian, Lagrangian and ALE time-derivatives

We have already seen how to transform some space differential operators from different configurations, once we know the deformation. To complete the picture, we need now to understand how to relate time derivatives in the different formulations.

For a given scalar Eulerian field q (the discussion applies also to vector or tensor fields), we define the *Eulerian time-derivative* as simply

$$\frac{\partial q}{\partial t}(\boldsymbol{x}, t), \quad \boldsymbol{x} \in \Omega(t). \tag{3.20}$$

In other words, we look at the rate of change of q at a fixed point $\boldsymbol{x}$ in the physical space, where the current configuration lives. It is nothing else than the classical partial derivative.

Let now $\widehat{q}$ be the Lagrangian description of q. We define the *material time-derivative* $\dfrac{Dq}{Dt}$ of q as

$$\frac{Dq}{Dt}(\cdot, t) = \frac{\partial \widehat{q}}{\partial t}(\cdot, t) \circ \widehat{\boldsymbol{\varphi}}_t^{-1}. \tag{3.21}$$

We can give a different interpretation of the material derivative, with a more immediate physical meaning. Let us note that by recalling relation (3.13) we may write that

$$\begin{aligned}
\frac{\partial}{\partial t}\widehat{q}(\widehat{\boldsymbol{x}}, t) &= \lim_{h \to 0} \frac{\widehat{q}(\widehat{\boldsymbol{x}}, t+h) - \widehat{q}(\widehat{\boldsymbol{x}}, t)}{h} \\
&= \lim_{h \to 0} \frac{q(\widehat{\boldsymbol{\varphi}}(\widehat{\boldsymbol{x}}, t+h), t+h) - q(\widehat{\boldsymbol{\varphi}}(\widehat{\boldsymbol{x}}, t), t)}{h} \\
&= \frac{\mathrm{d}}{\mathrm{d}\, t}(\widehat{\boldsymbol{\varphi}}(\widehat{\boldsymbol{x}}, t), t).
\end{aligned}$$

Therefore, using (3.21),

$$\frac{D}{Dt}q(\boldsymbol{x}, t) = \frac{\mathrm{d}}{\mathrm{d}\, t}q(\widehat{\boldsymbol{\varphi}}(\widehat{\boldsymbol{x}}, t), t), \quad \text{with } \boldsymbol{x} = \widehat{\boldsymbol{\varphi}}(\widehat{\boldsymbol{x}}, t). \tag{3.22}$$

The material derivative of q at $(\boldsymbol{x}, t)$ is thus the rate of variation in time of q "experienced" by an observer which moves with the particle $\widehat{\boldsymbol{x}}$ located at time t in the point $\boldsymbol{x}$.

Standard application of the chain rule for the composition of functions in (3.22) yields the following result.

Proposition 3.3. *For any given Eulerian field q, the following identity holds*

$$\frac{Dq}{Dt} = \boldsymbol{u} \cdot \boldsymbol{\nabla} q + \frac{\partial q}{\partial t}. \tag{3.23}$$

It follows that the Lagrangian derivative is made of two contributions. A *transport term* $\boldsymbol{u} \cdot \boldsymbol{\nabla} q$ accounting for variations of q due to changes in the position of the particle and the standard Eulerian time derivative.

The same type of considerations may be extended to the ALE formulation. In particular the *ALE time-derivative* $\dfrac{\partial q}{\partial t}_{|\widetilde{\mathcal{A}}}$ of a field q may be defined in a way analogous to the material derivative. In particular, for each $\boldsymbol{x} \in \omega(t)$ and $t > 0$ we have

$$\frac{\partial q}{\partial t}_{|\widetilde{\mathcal{A}}} = \frac{\mathrm{d}}{\mathrm{d}\, t} q(\widetilde{\mathcal{A}}(\widetilde{\boldsymbol{x}}, t), t), \quad \text{with } \boldsymbol{x} = \widetilde{\mathcal{A}}(\widetilde{\boldsymbol{x}}, t). \tag{3.24}$$

In other words, we look at the rate of change of q in a point that moves with the computational domain. The importance of this relation emerges in the context of the numerical discretisation. When computing numerically a solution in a moving domain we are usually interested in the variation of quantities collocated at the nodes of a computational mesh, and the latter necessarily follows the evolution of the computational domain. In Fig. 3.10 we show one node at two different times, namely $\boldsymbol{x}_i(t)$ and $\boldsymbol{x}_i(t+\delta t)$, being i the node index. If q_i indicates the quantity of interest at the given node, its value at the two different times, $q_i(t)$ and $q_i(t+\delta t)$, have to be understood as

$$q_i(t) = q(\boldsymbol{x}_i(t), t), \quad q_i(t+\delta t) = q(\boldsymbol{x}_i(t+\delta t), t+\delta t).$$

As a result, their difference is related to the ALE time derivative, since

$$q_i(t+\delta t) - q_i(t) = q(\boldsymbol{x}_i(t+\delta t), t+\delta t) - q(\boldsymbol{x}_i(t), t) = \int_t^{t+\delta t} \frac{\partial q}{\partial t}_{|\widetilde{\mathcal{A}}}(\boldsymbol{x}_i, t)\, \mathrm{d}t.$$

The use of the Eulerian time-derivative would be in this case troublesome, because a fixed point $\boldsymbol{x}$ which at time t is inside a moving computational domain may well have fallen outside at time $t + \delta t$! This remark points out the advantage of using the ALE framework.

We can use the same arguments used to derive (3.23) to obtain the following proposition.

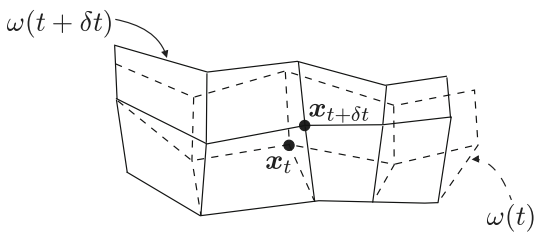

Fig. 3.10. Example of a moving mesh

Proposition 3.4. *The following identity holds*

$$\frac{\partial}{\partial t}_{|\widetilde{\mathcal{A}}} q = \boldsymbol{w} \cdot \boldsymbol{\nabla} q + \frac{\partial q}{\partial t}. \tag{3.25}$$

The transport term $\boldsymbol{w} \cdot \boldsymbol{\nabla} q$ accounts for the variations of q caused by the motion of the computational domain. It is clearly zero if the domain is fixed, while it coincides with the transport term in the material derivative (3.23) if $\boldsymbol{w} = \boldsymbol{u}$.

The Reynolds transport formula

An interesting property of the Jacobian is that its time derivative is linked to the divergence of the velocity field.

Proposition 3.5. *Let J denote the Jacobian (3.3) in the Eulerian frame. We have the relation*

$$\frac{D}{Dt} J = J \operatorname{div} \boldsymbol{u}, \tag{3.26}$$

sometimes called the *Euler expansion formula*. It allows to obtain the following fundamental result.

Proposition 3.6 (Reynolds transport formula). *Let $V(t)$ be a material domain, i.e. $V(t) = \{\boldsymbol{x} : \boldsymbol{x} = \widehat{\boldsymbol{\varphi}}(\widehat{\boldsymbol{x}}, t),\, \widehat{\boldsymbol{x}} \in \widehat{V}\}$, and f a continuously differentiable field. Then,*

$$\frac{\mathrm{d}}{\mathrm{d}\, t} \int_{V(t)} f \,\mathrm{d}\boldsymbol{x} = \int_{V(t)} \left(\frac{Df}{Dt} + f \operatorname{div} \boldsymbol{u} \right) \mathrm{d}\boldsymbol{x} = \int_{V(t)} \left(\frac{\partial f}{\partial t} + \operatorname{div} (f \boldsymbol{u}) \right) \mathrm{d}\boldsymbol{x}. \tag{3.27}$$

When working with the ALE formulation it might be useful to consider the Reynolds formula acting on the moving computational domain.

Proposition 3.7 (ALE transport formula). *Let $\widetilde{\omega}_0 \subset \widetilde{\omega}$ be a subdomain in the ALE reference configuration and $\omega_0(t) = \{\boldsymbol{x} : \boldsymbol{x} = \widetilde{\mathcal{A}}(\widetilde{\boldsymbol{x}}, t),\, \widetilde{\boldsymbol{x}} \in \widetilde{\omega}\}$ its image by the ALE map. Clearly $\omega_0(t)$ is always contained in the computational domain $\omega(t)$. We have that*

$$\frac{\mathrm{d}}{\mathrm{d}\, t} \int_{\omega_0(t)} f \,\mathrm{d}\boldsymbol{x} = \int_{\omega_0(t)} \left(\frac{\partial}{\partial t}_{|\widetilde{\mathcal{A}}} f + f \operatorname{div} \boldsymbol{w} \right) \mathrm{d}\boldsymbol{x} = \int_{\omega_0(t)} \left(\frac{\partial f}{\partial t} + \operatorname{div} (f \boldsymbol{w}) \right) \mathrm{d}\boldsymbol{x}\,, \tag{3.28}$$

for any continuously differentiable field f. Here $\boldsymbol{w}$ indicates the domain velocity defined in (3.18).

3.2 The equations of continuum mechanics

The basic equations of continuum mechanics provide some well known conservation principles in the form of differential problems.

3.2.1 Mass conservation

The *mass* of an arbitrary material domain $V(t)$ at time t is given by

$$\int_{V(t)} \rho \, \mathrm{d}\boldsymbol{x}, \tag{3.29}$$

being ρ the density (or volume mass) of the continuum medium. The units of measurement of density are $[\rho] = \mathrm{kg/m^3}$.

In classical mechanics the mass of a body does not change during the motion, a principle known as the *mass conservation*. Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}\,t} \int_{V(t)} \rho \, \mathrm{d}\boldsymbol{x} = 0, \tag{3.30}$$

holds true for any $V(t)$ at any time. This is an integral statement, we want to express it "point-wise". To is aim, we use the Reynolds transport formula (3.6) to obtain

$$\frac{\mathrm{d}}{\mathrm{d}\,t} \int_{V(t)} \rho \, \mathrm{d}\boldsymbol{x} = \int_{V(t)} \left(\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \boldsymbol{u}) \right) \mathrm{d}\boldsymbol{x},$$

by which, due to the arbitrariness of $V(t)$, we get the following

Proposition 3.8 (Continuity equation). *If ρ indicates the density of a continuum medium, mass conservation implies that*

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \boldsymbol{u}) = 0, \quad \textit{in} \quad \Omega(t), \tag{3.31}$$

for all $t > 0$, that is

$$\frac{\partial \rho}{\partial t} + \sum_{i=1}^{3} \frac{\partial}{\partial x_i}(\rho u_i) = 0.$$

If the fluid has *constant density* then (3.31) implies the well known *incompressibility equation*

$$\mathrm{div}\, \boldsymbol{u} = 0 \quad \text{in} \quad \Omega(t), \quad t > 0. \tag{3.32}$$

On the other hand $\mathrm{div}\, \boldsymbol{u} = 0$ implies $\frac{DJ}{Dt} = 0$, thanks to the Euler expansion formula and the definition of material derivative. In turn, this is equivalent to $\dfrac{\mathrm{d}}{\mathrm{d}\,t}|V(t)| = 0$ for any material domain $V(t)$. That is, in a constant density fluid the volume of a material domain does not change during motion. In haemodynamics applications, blood is usually considered a constant density fluid. The continuity equation can be rewritten in the Lagrangian frame, we will give more details when we deal with the dynamics of a solid.

3.2.2 Conservation of momentum

The *conservation of (linear) momentum* is in fact the well known Newton's law. The rate of change of the momentum of a material domain $V(t)$, given by $\int_{V(t)} \rho \boldsymbol{u}\, \mathrm{d}\boldsymbol{x}$ equals the resultant of the external forces acting on it, that is

$$\frac{\mathrm{d}}{\mathrm{d}\,t} \int_{V(t)} \rho \boldsymbol{u}\, \mathrm{d}\boldsymbol{x} = \boldsymbol{F} = \boldsymbol{F}_v + \boldsymbol{F}_s.$$

Referring to Fig. 3.11, the force $\boldsymbol{F}$ is the composition of two terms: a *volume force* $\boldsymbol{F}_v$, and a *surface force* $\boldsymbol{F}_s$. The former acts on each particle of $V(t)$ (like the force of gravity) and is expressed as the integral of the density times a *specific force* (i.e. force per unit of weight) $\boldsymbol{f}$ which has the dimension of an acceleration, $[\boldsymbol{f}] = \mathrm{m/s^2}$.

The latter is instead responsible for the mutual interaction between the material contained in $V(t)$ and the exterior, through the boundary $\partial V(t)$. More precisely, $\boldsymbol{F}_s$ is equal to the surface integral of the so called *Cauchy stress* $\boldsymbol{t}$, which has the dimension of force per unit area, $[\boldsymbol{t}] = \mathrm{N/m^2}$, that is $\boldsymbol{F}_s = \int_{\partial V(t)} \boldsymbol{t} d\boldsymbol{\gamma}$.

It was indeed Cauchy who also postulated that $\boldsymbol{t}$ can be computed by applying to the normal $\boldsymbol{n}$ of $\partial V(t)$ a symmetric second-order tensor[4]

$\boldsymbol{\sigma} : \Omega(t) \to \mathbb{R}^{3\times 3}$, called the *Cauchy stress tensor*, i.e.

$$\boldsymbol{t} = \boldsymbol{\sigma}\boldsymbol{n} \quad \text{on } \partial V(t), \quad \text{componentwise } t_i = \sum_{j=1}^{3} \sigma_{ij} n_j. \tag{3.33}$$

The *momentum conservation law* can then be expressed by the following equation,

$$\frac{\mathrm{d}}{\mathrm{d}\,t} \int_{V(t)} \rho \boldsymbol{u}\, \mathrm{d}\boldsymbol{x} = \int_{V(t)} \rho \boldsymbol{f}\, \mathrm{d}\boldsymbol{x} + \int_{\partial V(t)} \boldsymbol{\sigma}\boldsymbol{n}\, \mathrm{d}\boldsymbol{\gamma} = \int_{V(t)} \rho \boldsymbol{f}\, \mathrm{d}\boldsymbol{x} + \int_{V(t)} \mathbf{div}\boldsymbol{\sigma}\, d\boldsymbol{x}, \tag{3.34}$$

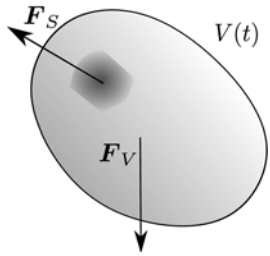

Fig. 3.11. Forces acting on a material volume $V(t)$

[4] The symmetry is in fact an implication of the conservation of angular momentum. We will not pursue this issue here, the interested reader may refer to the given bibliography. Note that the Cauchy postulate implies that the dependence of $\boldsymbol{t}$ on the geometry of $\partial V(t)$ is only through its normal. This holds true in most situations.

valid for all material domains $V(t)$. To obtain the last equality we have used the divergence theorem. Finally, by exploiting the Reynolds transport formula (3.6) we obtain

Proposition 3.9 (Momentum conservation). *Assume that* (3.30) *holds. Then* (3.34) *is equivalent to*

$$\rho\frac{\partial \boldsymbol{u}}{\partial t} + \rho(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} - \mathbf{div}\boldsymbol{\sigma} = \rho\boldsymbol{f}, \quad in \quad \Omega(t), \quad t > 0. \tag{3.35}$$

Componentwise, we have

$$\rho\frac{\partial u_i}{\partial t} + \rho\sum_{j=1}^{3} u_j\frac{\partial u_i}{\partial x_j} - \rho\sum_{j=1}^{3}\frac{\partial \sigma_{ij}}{\partial x_j} = f_i, \quad i = 1,2,3.$$

The equations may be written in conservation form as

$$\frac{\partial(\rho\boldsymbol{u})}{\partial t} + \mathbf{div}\big(\rho\boldsymbol{u}\otimes\boldsymbol{u} - \boldsymbol{\sigma}\big) = \rho\boldsymbol{f}, \quad in \quad \Omega(t), \quad t > 0, \tag{3.36}$$

which componentwise reads

$$\frac{\partial(\rho u_i)}{\partial t} + \sum_{j=1}^{3}\frac{\partial}{\partial x_j}\big(\rho u_i u_j - \sigma_{ij}\big) = \rho f_i, \quad i = 1,2,3.$$

In contrast, (3.35) is generally said to be in *quasi-linear* form or sometimes in the *gradient form*.

Remark 3.2.1 *The transport term* $(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}$ *(or* $\mathbf{div}\boldsymbol{u}\otimes\boldsymbol{u}$ *in the conservation form), is a* non-linear *term. This aspect makes the analysis, as well as the numerical solution more complex. Only in flow at very low Reynolds numbers* (≤ 10) *the non-linear term may be neglected.*

At each point of the boundary of a material domain $V(t)$ the Cauchy stress $\boldsymbol{t}$ can be decomposed into its components normal and tangential to the surface, given respectively by

$$t_n = \boldsymbol{t}\cdot\boldsymbol{n} = (\boldsymbol{\sigma}\boldsymbol{n})\cdot\boldsymbol{n}, \quad \text{and } \boldsymbol{t}_t = \boldsymbol{t} - t_n\boldsymbol{n}. \tag{3.37}$$

The latter is indeed a vector laying on the tangential plane and is called the *shear stress* vector. Componentwise, it may be computed as

$$[t_t]_i = \sum_{j=1}^{3}\sigma_{ij}n_j - \sum_{k,j=1}^{3}\sigma_{kj}n_j n_k n_i, \quad i = 1,\ldots 3.$$

It is an important parameter in haemodynamics since the endothelium cells are very sensitive to the shear stress at the vessel walls. Let us note that $(\boldsymbol{\sigma}\boldsymbol{n})\cdot\boldsymbol{n}$ is a scalar quantity which may also be written as $\boldsymbol{n}^T\boldsymbol{\sigma}\boldsymbol{n}$. In a Cartesian

coordinate system it is readily computed as $\sum_{i,j=1}^{3} \sigma_{ij} n_i n_j$. The equations have been here written in Eulerian formulation.

In the case of a fixed computational domain Ω they can be used directly, just replacing $\Omega(t)$ with Ω (we recall that Ω is a subset of $\Omega(t)$ for all t). In the case of a moving domain it is preferable to use the ALE formulation. To this aim it is sufficient to employ identity (3.25). If instead one wants to use a full Lagrangian formulation it is necessary to transform also the space differential operators, in order to write the equations on $\widehat{\Omega}$ instead of $\Omega(t)$. The Piola formula (3.8) can then become handy. We postpone these issues to Section 3.3.2.

3.3 Fluids and solids

We need now to make precise how the Cauchy stress tensor is linked to the kinematics. It is indeed at this point where the behaviour of solids and fluids diverges.

As solids react to deformations, the Cauchy stress must depend on $\widehat{\boldsymbol{F}}$ (or on quantities which are directly related to $\widehat{\boldsymbol{F}}$). The reference configuration plays here an important role.

Fluids instead can adapt to a deformation, as a fluid can fill freely any arbitrary shape. Yet it takes time to fill it. And oil takes more time than water. It means that fluids react mechanically not to the deformation itself but to its rate. More precisely, the relevant quantity is here the *strain rate tensor* $\boldsymbol{D}$ defined in (2.2) of Chapter 2, and whose dimensions are $[\boldsymbol{D}] = \mathrm{s}^{-1}$. Componentwise, the strain rate reads

$$D_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad i, j = 1, \ldots 3.$$

In a fluid then $\boldsymbol{\sigma}$ is a function of $\boldsymbol{D}$, while it is independent of $\widehat{\boldsymbol{F}}$. A consequence is that the reference configuration is a concept useful for the derivation of the equations, yet it does not play any particular role for a fluid. Intermediate behaviours, like that of visco-elastic fluids, for instance, are possible; they will be addressed in detail in Chapter 6.

The relation between the Cauchy stress tensor $\boldsymbol{\sigma}$ and the kinematic quantities is called *constitutive relation*, or constitutive law, and is a characteristic of the type of material under consideration. To be physically correct, a constitutive relation must obey certain rules, like the principle of material frame indifference [512] which states that the relation should be invariant under a change of frame of reference. More details may be found also in the literature cited at the beginning of the chapter.

3.3.1 The Navier Stokes equations for a constant density fluid

We consider here the case of a constant density incompressible Newtonian fluid. As anticipated in Chapter 2, this is a limitation usually accepted for blood flow in large arteries, and we have discussed its consequences already in Section 3.2.1.

In a *Newtonian incompressible fluid*, the Cauchy stress tensor depends linearly on the strain rate. More precisely, we have

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{u}, P) = -P\boldsymbol{I} + 2\mu\boldsymbol{D}(\boldsymbol{u}) = -P\boldsymbol{I} + \mu(\boldsymbol{\nabla u} + \boldsymbol{\nabla u}^T), \tag{3.38}$$

where P is the *pressure*, $\boldsymbol{I}$ is the identity matrix, μ is the *dynamic viscosity* of the fluid and is a positive quantity.

The term $2\mu\boldsymbol{D}(\boldsymbol{u})$ is the *viscous stress* component of the stress tensor. We have that $[P] = \mathrm{N/m^2}$ and $[\mu] = \mathrm{kg/ms}$.

The viscosity may vary, for example it may depend on the fluid temperature. The assumption of Newtonian fluid, however, implies that μ is *independent of kinematic quantities*. Simple models for non-Newtonian fluids, often used for blood flow simulations, express the viscosity as function of the strain rate, that is $\mu = \mu(\boldsymbol{D}(\boldsymbol{u}))$. The treatment of such cases is considered in Chapter 6 and will not be covered here.

We now recall that, if P is a scalar and $\boldsymbol{\Sigma}$ a tensor field, then the following identity holds,

$$\mathbf{div}(P\boldsymbol{\Sigma}) = \boldsymbol{\nabla} P\boldsymbol{\Sigma} + P\mathbf{div}\boldsymbol{\Sigma},$$

and, therefore,

$$\mathbf{div}(P\boldsymbol{I}) = \boldsymbol{\nabla} P\boldsymbol{I} + P\mathbf{div}\boldsymbol{I} = \boldsymbol{\nabla} P.$$

The momentum equation(3.36) may then be written as

$$\rho\frac{\partial \boldsymbol{u}}{\partial t} + \rho\mathbf{div}(\rho\boldsymbol{u}\otimes\boldsymbol{u}) + \boldsymbol{\nabla} P - \mathbf{div}(\mu\boldsymbol{D}(\boldsymbol{u})) = \rho\boldsymbol{f}. \tag{3.39}$$

Since ρ is constant, it is sometimes convenient to introduce the kinematic viscosity $\nu = \mu/\rho$, with $[\nu] = \mathrm{m^2/s}$, and write

$$\frac{\partial \boldsymbol{u}}{\partial t} + \mathbf{div}(\boldsymbol{u}\otimes\boldsymbol{u}) + \boldsymbol{\nabla} P - \mathbf{div}[\nu(\boldsymbol{\nabla u} + \boldsymbol{\nabla u}^T)] = \boldsymbol{f}, \tag{3.40}$$

where $P = P/\rho$ is a scaled pressure (with $[P] = \mathrm{m^2/s^2}$).

Under the additional hypothesis that ν is constant, the momentum equation may be further elaborated by considering that $\mathbf{div}\boldsymbol{\nabla u} = \Delta\boldsymbol{u}$ and $\mathbf{div}\boldsymbol{\nabla u}^T = \boldsymbol{\nabla}(\mathbf{div}\boldsymbol{u}) = \mathbf{0}$, producing the alternative formulation

$$\frac{\partial \boldsymbol{u}}{\partial t} + \mathbf{div}(\boldsymbol{u}\otimes\boldsymbol{u}) + \boldsymbol{\nabla} P - \nu\Delta\boldsymbol{u} = \boldsymbol{f}. \tag{3.41}$$

However, for reasons that will appear clear in Chapter9, and have to do with the different natural boundary conditions associated with the two formulations, for fluid-structure interaction problems it is more convenient to use the momentum equation in the form (3.39), even when the viscosity is constant.

The Navier-Stokes equations in the Eulerian frame

The set of differential equations formed by the continuity equation and the momentum equation in the form derived in the previous section provides the *Navier-Stokes equations* for a constant density fluid.

They have been written in the Eulerian frame and if the computational domain Ω is time-independent they can be recast as the following system of equations for the unknowns velocity $\boldsymbol{u}$ and pressure P,

$$\begin{aligned} &\rho\frac{\partial \boldsymbol{u}}{\partial t} + \rho(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} + \boldsymbol{\nabla}P - 2\mathbf{div}(\mu\boldsymbol{D}(\boldsymbol{u})) = \rho\boldsymbol{f}, \\ &\operatorname{div}\boldsymbol{u} = 0, \end{aligned} \tag{3.42}$$

for any $t > 0$ and in Ω. Alternatively, one may use the conservative form (3.39).

Furthermore, we need to prescribe the initial status of the fluid velocity, for instance

$$\boldsymbol{u}(t = t_0, \boldsymbol{x}) = \boldsymbol{u}_0(\boldsymbol{x}) \qquad \boldsymbol{x} \in \Omega. \tag{3.43}$$

There is *no initial condition* for the pressure.

We have already introduced the issue of boundary conditions in Chapter 2. We here recall the more classical boundary conditions which are mathematically compatible with the Navier Stokes equations, namely

1. *Applied stresses* (or *Neumann* boundary condition)

$$\boldsymbol{\sigma}\cdot\boldsymbol{n} = -P\boldsymbol{n} + 2\mu\boldsymbol{D}(\boldsymbol{u})\cdot\boldsymbol{n} = \boldsymbol{h} \quad \text{on } \Gamma_N \subset \partial\Omega, \tag{3.44}$$

   where Γ_N is a measurable subset (possibly empty) of the whole boundary $\partial\Omega$ and $\boldsymbol{h} = \boldsymbol{h}(\boldsymbol{x}, t)$ a given vector defined on Γ_N and (possibly) function of time. This is a typical condition at distal boundaries, where often $\boldsymbol{h} = P_{ext}\boldsymbol{n}$, being $= P_{ext}$ a prescribed "external" pressure.
2. *Prescribed velocity* (or *Dirichlet* boundary condition).

$$\boldsymbol{u} = \boldsymbol{g} \qquad \text{on } \Gamma_D,$$

   where $\boldsymbol{g} : \Gamma_D \times \mathbb{R}^+ \to \mathbb{R}^3$ is a given function. Since $\operatorname{div}\boldsymbol{u} = 0$ in Ω, it must be noted that if $\Gamma_D = \partial\Omega$ then $\boldsymbol{g}$ must satisfy

$$\int_{\partial\Omega} \boldsymbol{g}\cdot\boldsymbol{n} = 0, \tag{3.45}$$

   at any time. This condition is applied at the vessel wall interface and usually also at the proximal boundaries.

Clearly for a proper boundary condition specification we must have $\Gamma_N \cup \Gamma_D = \partial\Omega$. Other boundary conditions are possible, they will be discussed whenever appropriate.

The conditions to apply are normally inspired by physical considerations. For instance, for a viscous fluid ($\mu > 0$) like the one we are considering here, it is appropriate to impose the Dirichlet condition $\boldsymbol{u} = \frac{\partial}{\partial t}\boldsymbol{\eta}$ at a solid boundary, being η the displacement of the boundary. In the case of a fixed wall the condition is homogeneous and is also called *no-slip* condition. When dealing with an artificial boundary the choice of appropriate conditions in the context of haemodynamic problems is more delicate and should in any case guarantee the well-posedness of the resulting differential problem. In particular, the imposition of a Neumann condition in a distal section may cause instabilities in the presence of flow reversal, since this condition is most appropriate in outflow sections.

As mentioned, at distal sections (like Γ^{out} in Fig. 2.1) one often imposes a constant (possibly homogeneous) Neumann condition. Yet, this would indeed simulate a discharge into a reservoir at constant pressure. A rather unphysical situation for the case of a human vessel as it neglects the presence of the remaining part of the circulatory system completely. In Chapter 11 we will address this problem in more depth presenting some possible solutions.

After having computed the solution using the numerical scheme of choice, we may wish to estimate the wall shear stresses using the second relation in (3.37). In a Newtonian fluid the Cauchy stress at each point of the surface of interest Γ may be computed from the flow field solution as

$$\boldsymbol{t} = -P\boldsymbol{n} + \mu\frac{\partial \boldsymbol{u}}{\partial n} + \mu\boldsymbol{\nabla u}^T\boldsymbol{n}. \tag{3.46}$$

Here, $\frac{\partial \boldsymbol{u}}{\partial n} = \boldsymbol{\nabla u}\,\boldsymbol{n}$ is the normal derivative of $\boldsymbol{u}$. If the surface is flat then $\boldsymbol{n}$ is constant on Γ and thus $\boldsymbol{\nabla u}^T\boldsymbol{n} = \boldsymbol{\nabla} u_n$, being $u_n = \boldsymbol{u}\cdot\boldsymbol{n}$ the component of the velocity normal to the surface. If in addition Γ is a fixed wall surface, then $\boldsymbol{u} = \boldsymbol{0}$ on Γ and thus $\boldsymbol{\nabla u}^T\boldsymbol{n} = \boldsymbol{0}$. We will see in Chapter 9 how it is possible to recover this quantity when adopting finite elements for the numerical solution. The shear stress acting on the wall is readily obtained from $\boldsymbol{t}$ by using relation (3.37). Note that the shear stress does not depend on the pressure term, since it cancels out.

The Navier-Stokes equations in the ALE frame

When dealing with a moving computational domain $\omega(t)$ it is preferable to use the Navier-Stokes equations in the ALE framework introduced in Section 3.1.2. By using (3.25) on the momentum equation we derive that

$$\begin{aligned} &\rho\frac{\partial \boldsymbol{u}}{\partial t}_{|\widetilde{\mathcal{A}}} + \rho[(\boldsymbol{u}-\boldsymbol{w})\cdot\boldsymbol{\nabla}]\boldsymbol{u} + \boldsymbol{\nabla}P - 2\mathbf{div}(\mu\boldsymbol{D}(\boldsymbol{u})) = \rho\boldsymbol{f}, \\ &\operatorname{div}\boldsymbol{u} = 0, \end{aligned} \tag{3.47}$$

in $\omega(t)$. We may note that the introduction of the ALE time-derivative induces a correction in the transport term by subtracting to the "transport velocity" $\boldsymbol{u}$ the domain velocity $\boldsymbol{w}$ given by (3.18).

A conservation form may be devised as well. Recalling (3.19) we have

$$J_{\tilde{\mathcal{A}}}\frac{\partial \boldsymbol{u}}{\partial t}_{|\tilde{\mathcal{A}}} = \frac{\partial}{\partial t}_{|\tilde{\mathcal{A}}}(J_{\tilde{\mathcal{A}}}\boldsymbol{u}) - J_{\tilde{\mathcal{A}}}\boldsymbol{u}\,\mathrm{div}\,\boldsymbol{w},$$

by which, with simple manipulations, we get the following conservation form for the Navier-Stokes equations in the ALE frame,

$$\begin{aligned} &J_{\tilde{\mathcal{A}}}^{-1}\rho\frac{\partial}{\partial t}_{|\tilde{\mathcal{A}}}(J_{\tilde{\mathcal{A}}}\boldsymbol{u}) + \mathbf{div}\big(\rho\boldsymbol{u}\otimes(\boldsymbol{u}-\boldsymbol{w})\big) + \boldsymbol{\nabla}P - 2\mathbf{div}(\mu\boldsymbol{D}(\boldsymbol{u})) = \rho\boldsymbol{f}, \\ &\mathrm{div}\,\boldsymbol{u} = 0. \end{aligned} \tag{3.48}$$

3.3.2 The equations for a solid

We describe now the motion of the structure in terms of its displacement field $\widehat{\boldsymbol{\eta}}$ with respect to a given material reference configuration $\widehat{\Omega}$. Mapping back the continuity equation in integral form (3.30) to the reference domain we obtain

$$0 = \frac{\mathrm{d}}{\mathrm{d}\,t}\int_{\widehat{V}}\widehat{J}\widehat{\rho}d\widehat{\boldsymbol{x}} = \int_{\widehat{V}}\frac{\partial(\widehat{J}\widehat{\rho})}{\partial t}d\widehat{\boldsymbol{x}}.$$

From the arbitrariness of $\widehat{V}$ we derive the continuity equation in the Lagrangian frame, namely

$$\frac{\partial}{\partial t}\widehat{\rho}_0 = 0, \quad \text{in } \widehat{\Omega}, \quad t > 0, \tag{3.49}$$

where we have set

$$\widehat{\rho}_0 = \widehat{J}\widehat{\rho}. \tag{3.50}$$

Note that for a constant density material (3.49) together with the definition of reference domain, implies that $\widehat{J} = 1$ for all $t \geq 0$.

The momentum equation in integral form (3.34) can also be rewritten in the Lagrangian frame by mapping all integrals back on the reference domain and using (3.12) to obtain

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\int_{\widehat{V}}\widehat{J}\widehat{\rho}\frac{\partial\widehat{\boldsymbol{\eta}}}{\partial t}\,\mathrm{d}\hat{\boldsymbol{x}} - \int_{\widehat{V}}\widehat{J}\widehat{\mathbf{div}\boldsymbol{\sigma}}\,\mathrm{d}\hat{\boldsymbol{x}} = \int_{\widehat{V}}\widehat{J}\widehat{\rho}\widehat{\boldsymbol{f}}\,\mathrm{d}\hat{\boldsymbol{x}}.$$

Yet, (3.50) and (3.49) give

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\int_{\widehat{V}}\widehat{J}\widehat{\rho}\frac{\partial\widehat{\boldsymbol{\eta}}}{\partial t}\,\mathrm{d}\hat{\boldsymbol{x}} = \int_{\widehat{V}}\widehat{\rho}_0\frac{\partial^2\widehat{\boldsymbol{\eta}}}{\partial t^2}\,\mathrm{d}\hat{\boldsymbol{x}}.$$

Thus, by considering the arbitrariness of $\widehat{V}$ we obtain the following differential equation

$$\widehat{\rho}_0\frac{\partial^2\widehat{\boldsymbol{\eta}}}{\partial t^2} - \widehat{J}\widehat{\mathbf{div}\boldsymbol{\sigma}} = \widehat{\rho}_0\widehat{\boldsymbol{f}}, \quad \text{in } \widehat{\Omega},\ t > 0.$$

This form is still not satisfactory as $\widehat{\mathbf{div}\boldsymbol{\sigma}}$ is in a "mixed form" because the divergence is still taken with respect to $\boldsymbol{x}$. We now use the Piola transform and Proposition 3.2 to get

$$\widehat{\rho}_0 \frac{\partial^2 \widehat{\boldsymbol{\eta}}}{\partial t^2} - \mathbf{div}_{\hat{\boldsymbol{x}}} \widehat{\boldsymbol{\Pi}} = \widehat{\rho}_0 \widehat{\boldsymbol{f}}, \quad \text{in } \widehat{\Omega}, \quad t > 0. \tag{3.51}$$

The tensor $\widehat{\boldsymbol{\Pi}} = \mathcal{P}_{\widehat{\boldsymbol{\varphi}}}(\boldsymbol{\sigma}) = \widehat{J}\widehat{\boldsymbol{\sigma}}\widehat{\boldsymbol{F}}^{-T}$ is called the *first Piola-Kirchhoff tensor* and (3.51) is the *momentum equation* written in the Lagrangian frame. It is also known as the equation of *elastodynamics*.

Unlike the Cauchy stress tensor $\boldsymbol{\sigma}$, the first Piola-Kirchhoff tensor $\widehat{\boldsymbol{\Pi}}$ is non-symmetric. Since constitutive laws are often better expressed in terms of symmetric stress tensors, it is natural to introduce the *second Piola-Kirchhoff tensor* $\widehat{\boldsymbol{\Sigma}}$

$$\widehat{\boldsymbol{\Sigma}} = \widehat{\boldsymbol{F}}^{-1}\widehat{\boldsymbol{\Pi}} = \widehat{J}\widehat{\boldsymbol{F}}^{-1}\widehat{\boldsymbol{\sigma}}\widehat{\boldsymbol{F}}^{-T} \tag{3.52}$$

which is symmetric.

For an *elastic material* the stress is a function of the deformation (and possibly of thermodynamic variables such the temperature) but is independent on the deformation history (and thus on time). The material characteristics may still vary in space. In an *homogeneous* material the mechanical properties do not vary with $\boldsymbol{x}$. As a consequence the strain energy function depends only on the deformation. A material is mechanically *isotropic* if its response to deformation is the same in all directions.

The constitutive equation is usually written in terms of the *Green-Lagrange strain tensor*, defined as

$$\widehat{\boldsymbol{E}} = \frac{1}{2}\left(\widehat{\boldsymbol{F}}^T\widehat{\boldsymbol{F}} - \boldsymbol{I}\right), \tag{3.53}$$

where $\boldsymbol{I}$ is the identity tensor. Componentwise,

$$\widehat{E}_{ij} = \frac{1}{2}\left(\sum_{l=1}^{3} \widehat{F}_{li}\widehat{F}_{lj} - \delta_{ij}\right),$$

being δ_{ij} the Kronecker's symbol. Applying (3.2) and (3.1) we have also

$$\widehat{\boldsymbol{E}} = \frac{1}{2}\left(\boldsymbol{\nabla}_{\hat{\boldsymbol{x}}}\widehat{\boldsymbol{\eta}} + \boldsymbol{\nabla}_{\hat{\boldsymbol{x}}}^T\widehat{\boldsymbol{\eta}}\right) + \frac{1}{2}\boldsymbol{\nabla}_{\hat{\boldsymbol{x}}}^T\widehat{\boldsymbol{\eta}}\boldsymbol{\nabla}_{\hat{\boldsymbol{x}}}\widehat{\boldsymbol{\eta}}, \tag{3.54}$$

which componentwise reads $\widehat{E}_{ij} = \frac{1}{2}\left(\dfrac{\partial\widehat{\eta}_i}{\partial\widehat{x}_j} + \dfrac{\partial\widehat{\eta}_j}{\partial\widehat{x}_i}\right) + \sum_{l=1}^{3}\dfrac{\partial\widehat{\eta}_l}{\partial\widehat{x}_i}\dfrac{\partial\widehat{\eta}_l}{\partial\widehat{x}_j}$.

The tensor $\widehat{\boldsymbol{E}}$ is not affected by a superimposed rigid body motion, and in particular by rigid rotations. Indeed, from a geometric point of view $\widehat{\boldsymbol{E}}$ is directly related to the difference of the squared length of a elemental vector $d\widehat{\boldsymbol{x}}$ and its image: by recalling (3.4) we have that

$$\frac{1}{2}(||d\boldsymbol{x}||^2 - ||d\widehat{\boldsymbol{x}}||^2) = d\widehat{\boldsymbol{x}}^T\widehat{\boldsymbol{E}}d\widehat{\boldsymbol{x}}.$$

Many constitutive laws can be devised for a solid. For a *hyperelastic material* we first define a *density of elastic energy* $W : \mathbb{R}^{3\times 3} \longrightarrow \mathbb{R}^+$, and then set

$$\widehat{\boldsymbol{\Sigma}}(\widehat{\boldsymbol{E}}) = \frac{\partial W}{\partial \widehat{\boldsymbol{E}}}(\widehat{\boldsymbol{E}}). \tag{3.55}$$

Componentwise, $\widehat{\Sigma}_{ij} = \frac{\partial W}{\partial \widehat{E}_{ij}}, \quad i,j = 1,2,3.$

A simple example of energy density for a *homogeneous isotropic material* whose reference configuration is the natural state[5] is the Saint-Venant Kirchhoff model, where

$$W(\widehat{\boldsymbol{E}}) = \frac{L_1}{2}(\operatorname{tr}\widehat{\boldsymbol{E}})^2 + L_2 \operatorname{tr}\widehat{\boldsymbol{E}}^2, \tag{3.56}$$

which componentwise reads (by exploiting the symmetry of $\widehat{\boldsymbol{E}}$)

$$W = \frac{L_1}{2}\left(\sum_{i=1}^{3}\widehat{E}_{ii}\right)^2 + L_2\sum_{i=1}^{3}\sum_{j=1}^{3}\widehat{E}_{ij}^2.$$

Here, L_1 and L_2 denote the first and second *Lamé coefficients*[6]. Correspondingly, we have

$$\widehat{\boldsymbol{\Sigma}}(\widehat{\boldsymbol{E}}) = L_1(\operatorname{tr}\widehat{\boldsymbol{E}})\boldsymbol{I} + 2L_2\widehat{\boldsymbol{E}}. \tag{3.57}$$

This relation is often written componentwise in terms of a fourth order symmetric tensor, called the elasticity tensor, $\boldsymbol{H} = (H_{ijkl})$, defined by

$$H_{ijkl} = L_1\delta_{ij}\delta_{kl} + L_2\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right), \tag{3.58}$$

so that

$$\widehat{\Sigma}_{ij} = \sum_{1\le i,j\le 3} H_{ijkl}E_{kl}, \tag{3.59}$$

which is commonly known as (generalised) *Hook's law*. In tensorial form it reads $\widehat{\boldsymbol{\Sigma}} = \mathbf{H} : \widehat{\boldsymbol{E}}$.

Note the in fact the only components of $\boldsymbol{H}$ which are different from zero are $H_{1111} = H_{2222} = H_{3333} = L_1 + 2L_2$ and $H_{stst} = H_{stts} = L_2$, for $s \neq t$.

More complex constitutive relations for hyperelastic materials may be found in [221], and in particular models specially tailored for biological tissues and blood vessels are reported in [178] and [222].

Often an elastic material is characterised by its *Young modulus* E and *Poisson coefficient* ξ. Indeed, these quantities are inferred from experiments more directly than the Lamé coefficients. We have the following relations

$$E = L_2\frac{3L_1 + 2L_2}{L_1 + L_2}, \qquad \xi = \frac{1}{2}\frac{L_1}{L_1 + L_2}, \tag{3.60}$$

[5] The natural state is a configuration where the Cauchy stress tensor is zero everywhere.

[6] In the literature L_1 and L_2 are usually denoted by λ and μ, respectively. We have used different symbols to avoid repetitions.

and

$$L_1 = \frac{E\xi}{(1-2\xi)(1+\xi)}, \qquad L_2 = \frac{E}{2(1+\xi)}. \tag{3.61}$$

Specific models for the arterial wall

A simple Saint-Venant Kirchhoff elasticity model can be adopted only when one is not really interested in the details of the stress inside the arterial wall but only on its effect of the Haemodynamics. The development of reduced models detailed in Section 3.4 does indeed make use of this type of modelling.

However, in reality the internal structure of the wall of an arterial vessel is rather complex, an account has been given in Chapter 1. The presence of different layers and of collagen fibres which activate when the strain has reached a critical level makes a homogeneous isotropic models clearly inadequate for a detailed analysis of stresses inside the arterial wall.

Furthermore, it is often assumed that biological tissues, and thus the wall of a blood vessel, are incompressible. The case of an incompressible material is rather special. Indeed the Saint-Venant Kirchhoff model in its original formulation fails, since $\xi = 1/2$ in an incompressible material. The constitutive relation has to take into account the incompressibility constraints, a general account is given in [243, 558]. We here only mention that it is convenient to factor the deformation gradient $\widehat{\boldsymbol{F}}$ into his spherical and distortional part,

$$\widehat{\boldsymbol{F}} = \left(\widehat{J}^{1/3}\boldsymbol{I}\right)\widehat{\boldsymbol{F}}^*,$$

and use this decomposition to define a modified strain measure $\widehat{\boldsymbol{E}}^*$ as

$$\widehat{\boldsymbol{E}}^* = \frac{1}{2}(\widehat{\boldsymbol{F}}^{*T}\widehat{\boldsymbol{F}}^* - \boldsymbol{I}) = \frac{1}{2}(\widehat{J}^{-2/3}\widehat{\boldsymbol{F}}^T\widehat{\boldsymbol{F}} - \boldsymbol{I}).$$

When the motion is incompressible $\widehat{J} = 1$ during the whole motion and indeed in this case $\widehat{\boldsymbol{F}} = \widehat{\boldsymbol{F}}^T$ and $\widehat{\boldsymbol{E}} = \widehat{\boldsymbol{E}}^*$. In general,

$$\widehat{\boldsymbol{E}}^* = \widehat{J}^{-2/3}\widehat{\boldsymbol{E}} + \frac{1}{2}(\widehat{J}^{-2/3} - 1)\boldsymbol{I},$$

and thus

$$\frac{\partial\widehat{\boldsymbol{E}}^*}{\partial\widehat{\boldsymbol{E}}} = \widehat{J}^{-2/3}\left[\mathcal{I} - \frac{1}{3}\widehat{\boldsymbol{F}}^T\widehat{\boldsymbol{F}} \otimes \left(\widehat{\boldsymbol{F}}^T\widehat{\boldsymbol{F}}\right)^{-1}\right].$$

where $\mathcal{I}$ is the fourth order identity tensor with components $I_{ijkl} = (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})/2$.

The elastic energy density may then be expressed as the sum of two terms,

$$W(\widehat{\boldsymbol{E}}) = U(\widehat{J}) + W^*(\widehat{\boldsymbol{E}}^*),$$

where the dependence of $\widehat{\boldsymbol{E}}^*$ and $\widehat{J}$ on $\widehat{\boldsymbol{E}}$ is understood. The first term represent the volumetric elastic energy, which is constant in an incompressible

material, the second term is associated to the volume preserving deformations. Correspondingly, the second Piola Kirchhoff stress tensor becomes

$$\widehat{\boldsymbol{\Sigma}} = \frac{\partial U}{\partial \widehat{\boldsymbol{E}}} + \frac{\partial W^*}{\partial \widehat{\boldsymbol{E}}} = P\widehat{J}(\widehat{\boldsymbol{F}}^T\widehat{\boldsymbol{F}})^{-1} + \widehat{\boldsymbol{\Sigma}}^*, \tag{3.62}$$

and the Cauchy stress tensor

$$\boldsymbol{\sigma} = P\boldsymbol{I} + \boldsymbol{\sigma}^*,$$

P being the *pressure*, while $\boldsymbol{\sigma}^*$ and $\widehat{\boldsymbol{\Sigma}}^*$ are related by the inverse Piola transformation.

Pressure in an incompressible material plays the role of the Lagrange multiplier that enforces the incompressibility constraint $\widehat{J} = 1$. It is a role identical to that played by the fluid pressure in an incompressible flow. As for the distortional energy density W^*, several expressions specifically targeted to biological tissues are found in [23] and a critical review is available in [223].

The actual model to be adopted in practise may depend on the type of investigation to be carried out. For instance, in physiological situations the collagen fibres are not activated and the arterial wall behaves largely like an isotropic hyperelastic material. A possible constitutive law in this case can be derived from the model presented in [114], that is

$$W^* = \frac{a}{b}\left[e^{\frac{b}{2}(I_1-3)} - 1\right]. \tag{3.63}$$

Here, a and b are two parameters to be fitted by experiments and I_1 indicates the first invariant of the right Cauchy-Green tensor, which in terms of $\widehat{\boldsymbol{E}}^*$ is defined as $I_1 = 2\,\mathrm{tr}(\widehat{\boldsymbol{E}}^*) + 3$.

However, if we are interested to study situations where the strains go beyond the physiological range, like during balloon angioplasty, the hypothesis of isotropic behaviour is not realistic. The collagen fibres in this case enter into action, after a critical strain level. Since they are aligned principally along two directions, they introduce a markedly anisotropy. In [223] a model was proposed to account for this fact. The fibre directions are indicated by $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$, respectively, and are in general a function of $\widehat{\boldsymbol{x}}$. Correspondingly we may define the tensors $\boldsymbol{A}_1 = \boldsymbol{a}_1 \otimes \boldsymbol{a}_1$ and $\boldsymbol{A}_2 = \boldsymbol{a}_2 \otimes \boldsymbol{a}_2$. The energy density is expressed as function of the modified right Cauchy green tensor $\boldsymbol{C}^*$, which is related to $\widehat{\boldsymbol{E}}^*$ by $\boldsymbol{C}^* = 2\widehat{\boldsymbol{E}}^* + \boldsymbol{I}$. More precisely, the important variables are the first invariant $I_1 = \mathrm{tr}\,\boldsymbol{C}^*$, which we have already introduced, and $I_{a_1} = \boldsymbol{C}^* : \boldsymbol{A}_1$ and $I_{a_2} = \boldsymbol{C}^* : \boldsymbol{A}_2$. The distortional energy density is then expressed as $W^* = W^*_{iso} + W^*_{aniso}$, where

$$W^*_{iso} = W^*_{iso}(I_1) = \frac{c}{2}(I_1 - 3),$$

corresponds to a so-called neo-Hookean elestic model, and $c > 0$ is a suitable parameter. While

$$W^*_{aniso} = W^*_{aniso}(I_{a_1}, I_{a_2}) = \frac{k_1}{2k_2}\left[e^{k_2(I_{a_1}-1)^2} + e^{k_2(I_{a_2}-1)^2}\right].$$

Here, $k_1 > 0$ is a stress-like material parameter and $k_2 > 0$ a dimensionless parameter, both to be fitted by experiments. The dependance of W^*_{aniso} More details may be found in the cited reference.

We anticipate that another important characteristics of arterial walls is that they are normally in a pre-stressed state. We postpone the discussion on this issue to Section 3.3.2.

Finally, viscoelastic behaviour may be relevant in muscular arteries (i.e. arterioles) [180, 224]. Moreover, when undergoing major strains plasticity effects should be taken into account as well [183]. These types of modelling go beyond the scope of this book and will not be investigated any further.

Boundary conditions

Also in this case we have a second order system of partial differential equations and we need to provide proper boundary conditions on $\partial\widehat{\Omega}$. The two main conditions are again:

- *Dirichlet conditions.* The displacements are imposed on part of the boundary
$$\widehat{\boldsymbol{\eta}} = \widehat{\boldsymbol{g}}, \quad \text{on } \widehat{\Gamma}_D,$$
being $\widehat{\boldsymbol{g}}$ a given function;
- *Neumann conditions.* Surface stresses are applied on a portion of the boundary. Notice that often the given data are on the current configuration, so they have to be recast to $\partial\widehat{\Omega}$. For instance, we might want to enforce
$$\boldsymbol{\sigma n} = \boldsymbol{h}, \quad \text{on} \quad \Gamma_N(t) \subset \partial\Omega(t).$$
Using relations (3.9) and (3.11) we have
$$\widehat{\boldsymbol{\Pi}}_{\boldsymbol{\sigma}}\widehat{\boldsymbol{n}} = \widehat{J}||\widehat{\boldsymbol{F}}^{-T}\widehat{\boldsymbol{n}}||\widehat{\boldsymbol{h}} \quad \text{on } \widehat{\Gamma}_N, \tag{3.64}$$
which is the relation which is needed in practise to enforce the Neumann condition in the Lagrange frame.

Other conditions may be of interest for cardiovascular applications, for instance non-reflecting boundary conditions. They are meant to minimise the spurious reflections appearing when improper boundary conditions are imposed on a computational domain which is in fact representing a small portion of a larger body. For instance, an artery separated from the rest of the circulatory system. This issue is rather complex and its treatment goes beyond the scope of this chapter. Some considerations on special boundary conditions for cardiovascular problems may be found in Chapter 11 in the context of multiscale modelling.

The linear elasticity from a pre-stressed configuration

The equations written so far are rather general. Yet, even if we employ a linear relation between $\widehat{\boldsymbol{\Sigma}}$ and $\widehat{\boldsymbol{E}}$, like for instance (3.57), they give rise to a non-linear problem in the displacement $\widehat{\boldsymbol{\eta}}$, because of the presence of the deformation gradient in the relation between $\widehat{\boldsymbol{\Sigma}}$ and $\widehat{\boldsymbol{\Pi}}$, as well as the quadratic term in (3.54).

However, when both strains and displacements are small we may derive a simpler, linear form of the equation. In haemodynamics, the hypothesis of small displacements can be accepted in small arteries. Yet, it is sometimes used also in larger vessels when deriving reduced models of structure dynamics, since it is assumed that this approximation is of the same importance as the others introduced by the model reduction process.

Yet, things are complicated by the fact that usually displacements may be assumed to be small with respect to a configuration which is not a natural one. Indeed, it has been observed that a vessel when extracted from its natural site tends to shrink, and it opens up when cut longitudinally [177, 178]. This is an indication that the Cauchy stresses in an *in vivo* artery are not zero even when the artery is "at rest". The presence of a circumferential pre-stress may help in better equilibrating the stress state inside the arterial wall when the vessel is subject to the intramural pressure. Therefore, to be correct the linearisation procedure has to be carried out with respect to a pre-stressed reference state.

Since hyperelastic constitutive equations are written assuming instead a natural (i.e. zero stress) reference state, the problem is not straightforward. We proceed then by assuming the existence of a natural configuration $\widehat{\Omega}_0$ from which the actual reference configuration $\widehat{\Omega}$ is displaced by $\widehat{\boldsymbol{\eta}}_0 = \widehat{\boldsymbol{\eta}}_0(\widehat{\boldsymbol{x}}_0)$, being $\widehat{\boldsymbol{x}}_0 \in \widehat{\Omega}_0$. The current configuration $\Omega(t)$ is then obtained as usual from $\widehat{\Omega}$ by applying the displacement $\widehat{\boldsymbol{\eta}}$, which is assumed small and to have small gradients $\partial\widehat{\boldsymbol{\eta}}/\partial\widehat{\boldsymbol{x}}$. The total displacement from the natural configuration is $\widehat{\boldsymbol{\eta}}_t = \widehat{\boldsymbol{\eta}}_0 + \widehat{\boldsymbol{\eta}}$, and in general it is not small (see Fig. 3.12). For the sake of notation, here and in the following we will use the hat ($\widehat{\ }$) sign to indicate quantities referred either to the natural or the pre-stressed configuration, the ambiguity being resolved by the context.

The motion of $\Omega(t)$ is the superposition of a time-independent deformation from $\widehat{\Omega}_0$ to $\widehat{\Omega}$ and the motion from $\widehat{\Omega}$ to $\Omega(t)$. That is, a point $\boldsymbol{x}$ in the current configuration is associated to a point $\widehat{\boldsymbol{x}}$ in the natural configuration by $\boldsymbol{x} = \widehat{\boldsymbol{x}} + \widehat{\boldsymbol{\eta}}_0 + \widehat{\boldsymbol{\eta}} = \widehat{\boldsymbol{x}} + \widehat{\boldsymbol{\eta}}_t$. Finally, the Cauchy stress $\boldsymbol{\sigma}^0$ in the reference domain is self-equilibrating, i.e.

$$\mathbf{div}_{\widehat{\boldsymbol{x}}}\boldsymbol{\sigma}^0 = \mathbf{0}, \quad \text{in } \widehat{\Omega}. \tag{3.65}$$

To get the linearised equations we will write the elastodynamics equations with respect to $\widehat{\Omega}_0$ and then apply a linearisation procedure around the reference configuration $\widehat{\Omega}$. We define the deformation gradient with respect to the

natural state as $\widehat{\boldsymbol{F}}_0(\boldsymbol{\eta}) = \boldsymbol{I} + \boldsymbol{\nabla}_{\widehat{\boldsymbol{x}}_0}\boldsymbol{\eta}$, being $\boldsymbol{\nabla}_{\widehat{\boldsymbol{x}}_0}\boldsymbol{\eta} = \dfrac{\partial \boldsymbol{\eta}}{\partial \widehat{\boldsymbol{x}}_0}$, and $\widehat{\boldsymbol{F}}^0 = \widehat{\boldsymbol{F}}_0(\widehat{\boldsymbol{\eta}}_0)$ its value at the reference configuration. Accordingly, we have

$$\widehat{\boldsymbol{E}}_0(\boldsymbol{\eta}) = \frac{1}{2}(\widehat{\boldsymbol{F}}_0^T(\boldsymbol{\eta})\widehat{\boldsymbol{F}}_0(\boldsymbol{\eta}) - \boldsymbol{I}) = \frac{1}{2}(\boldsymbol{\nabla}_{\widehat{\boldsymbol{x}}_0}\boldsymbol{\eta} + \boldsymbol{\nabla}_{\widehat{\boldsymbol{x}}_0}^T\boldsymbol{\eta}) + \frac{1}{2}\boldsymbol{\nabla}_{\widehat{\boldsymbol{x}}_0}^T\boldsymbol{\eta}\boldsymbol{\nabla}_{\widehat{\boldsymbol{x}}_0}\boldsymbol{\eta}$$

and $\widehat{\boldsymbol{E}}^0 = \widehat{\boldsymbol{E}}_0(\widehat{\boldsymbol{\eta}}_0)$. We will consider a hyperelastic constitutive law given by

$$\widehat{\boldsymbol{\Sigma}}_0(\boldsymbol{\eta}_t) = \frac{\partial W}{\partial \widehat{\boldsymbol{E}}_0}(\widehat{\boldsymbol{E}}_0(\boldsymbol{\eta}_t)), \tag{3.66}$$

being W a suitable energy function. The equations of elastodynamics written with respect to the natural configuration then read

$$\widehat{\rho}^0\frac{\partial^2\widehat{\boldsymbol{\eta}}_t}{\partial t^2} - \mathbf{div}_{\widehat{\boldsymbol{x}}_0}\widehat{\boldsymbol{F}}_0(\widehat{\boldsymbol{\eta}}_t)\widehat{\boldsymbol{\Sigma}}_0(\widehat{\boldsymbol{\eta}}_t) = \widehat{\rho}^0\widehat{\boldsymbol{f}}, \quad \text{in } \widehat{\Omega}_0, \quad t > 0, \tag{3.67}$$

where $\widehat{\rho}^0$ is the density referred to the natural state configuration.

To linearize the differential operators in (3.67) around the reference configuration $\widehat{\Omega}$ we introduce the symbol $D_{\widehat{\boldsymbol{\eta}}_0}f(\boldsymbol{\eta})$ to indicate the Gateaux derivative in $\widehat{\boldsymbol{\eta}}_0$ and applied to $\boldsymbol{\eta}$, being $\boldsymbol{\eta}$ a displacement field on Ω_0 [92], i.e.

$$D_{\widehat{\boldsymbol{\eta}}_0}f(\widehat{\boldsymbol{\eta}}) = \lim_{\epsilon\to 0}\frac{f(\widehat{\boldsymbol{\eta}}_0 + \epsilon\widehat{\boldsymbol{\eta}}) - f(\widehat{\boldsymbol{\eta}}_0)}{\epsilon}.$$

We assume that the displacement are sufficiently regular to guarantee its existence and continuity everywhere.

The most troublesome term in (3.67) is the one containing the divergence operator, since we need to rewrite it with respect to the $\widehat{\boldsymbol{x}}$ coordinates. To this aim, we use the Piola transform (3.6) and property 3.2 to write

$$\mathbf{div}_{\widehat{\boldsymbol{x}}_0}\widehat{\boldsymbol{F}}_0\widehat{\boldsymbol{\Sigma}}_0 = \widehat{J}_0\mathbf{div}_{\widehat{\boldsymbol{x}}}\left[\widehat{J}_0^{-1}\left(\widehat{\boldsymbol{F}}_0\widehat{\boldsymbol{\Sigma}}_0\right)\widehat{\boldsymbol{F}}^{0T}\right],$$

where $\widehat{\boldsymbol{F}}^0$ and $\widehat{J}_0 = |\widehat{\boldsymbol{F}}^0|$ are independent from $\widehat{\boldsymbol{\eta}}$. Therefore,

$$D_{\widehat{\boldsymbol{\eta}}_0}\left(\mathbf{div}_{\widehat{\boldsymbol{x}}_0}\widehat{\boldsymbol{F}}_0\widehat{\boldsymbol{\Sigma}}_0\right)(\widehat{\boldsymbol{\eta}}) = \widehat{J}_0\mathbf{div}_{\widehat{\boldsymbol{x}}}\left[\widehat{J}_0^{-1}D_{\widehat{\boldsymbol{\eta}}_0}(\widehat{\boldsymbol{F}}_0\widehat{\boldsymbol{\Sigma}}_0)(\widehat{\boldsymbol{\eta}})\widehat{\boldsymbol{F}}^{0T}\right].$$

By standard derivation rule and setting $\widehat{\boldsymbol{\Sigma}}^0 = \widehat{\boldsymbol{\Sigma}}_0(\widehat{\boldsymbol{\eta}}_0)$ we can write

$$D_{\widehat{\boldsymbol{\eta}}_0}(\widehat{\boldsymbol{F}}_0\widehat{\boldsymbol{\Sigma}}_0)(\widehat{\boldsymbol{\eta}}) = D_{\widehat{\boldsymbol{\eta}}_0}\widehat{\boldsymbol{F}}_0(\widehat{\boldsymbol{\eta}})\widehat{\boldsymbol{\Sigma}}^0 + \widehat{\boldsymbol{F}}^0D_{\widehat{\boldsymbol{\eta}}_0}\widehat{\boldsymbol{\Sigma}}_0(\widehat{\boldsymbol{\eta}}). \tag{3.68}$$

Since

$$D_{\widehat{\boldsymbol{\eta}}_0}\widehat{\boldsymbol{F}}_0(\widehat{\boldsymbol{\eta}}) = \boldsymbol{\nabla}_{\widehat{\boldsymbol{x}}_0}\widehat{\boldsymbol{\eta}} = \boldsymbol{\nabla}_{\widehat{\boldsymbol{x}}}\widehat{\boldsymbol{\eta}}\widehat{\boldsymbol{F}}^0,$$

thanks to the inverse Piola transform and the relation between the Cauchy and the second Piola-Kirchhoff stress tensor, we have

$$\widehat{J}_0^{-1}D_{\widehat{\boldsymbol{\eta}}_0}\widehat{\boldsymbol{F}}_0(\widehat{\boldsymbol{\eta}})\widehat{\boldsymbol{\Sigma}}^0\widehat{\boldsymbol{F}}^{0T} = (\boldsymbol{\nabla}_{\widehat{\boldsymbol{x}}}\widehat{\boldsymbol{\eta}})(\widehat{J}_0^{-1}\widehat{\boldsymbol{F}}^0\widehat{\boldsymbol{\Sigma}}^0\widehat{\boldsymbol{F}}^{0T}) = \boldsymbol{\nabla}_{\widehat{\boldsymbol{x}}}\widehat{\boldsymbol{\eta}}\boldsymbol{\sigma}^0.$$

Indeed, $\boldsymbol{\sigma}^0 = \widehat{J}_0^{-1}\widehat{\boldsymbol{F}}_0\widehat{\boldsymbol{\Sigma}}^0\widehat{\boldsymbol{F}}^{0T}$.

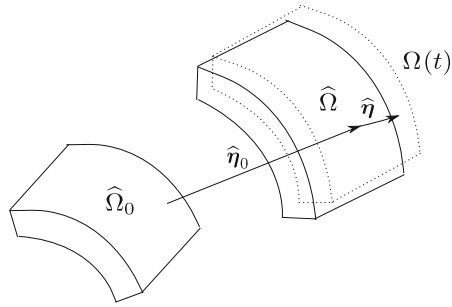

Fig. 3.12. The reference and natural state configurations together with the current configureation in dashed lines

The second term in (3.68) is elaborated further as

$$D_{\widehat{\boldsymbol{\eta}}_0}\widehat{\boldsymbol{\Sigma}}_0(\widehat{\boldsymbol{\eta}}) = \frac{\partial^2 W}{\partial \widehat{\boldsymbol{E}}^2}(\widehat{\boldsymbol{E}}^0) D_{\widehat{\boldsymbol{\eta}}_0}\widehat{\boldsymbol{E}}_0(\widehat{\boldsymbol{\eta}}) = \widehat{\boldsymbol{H}} : \left(\widehat{\boldsymbol{F}}^{0T}\boldsymbol{\varepsilon}(\widehat{\boldsymbol{\eta}})\widehat{\boldsymbol{F}}^0\right),$$

where

$$\widehat{\boldsymbol{H}} = \frac{\partial^2 W}{\partial \widehat{\boldsymbol{E}}^2}(\widehat{\boldsymbol{E}}_0(\widehat{\boldsymbol{\eta}}_0)), \quad \text{componentwise } \widehat{H}_{ijkl} = \frac{\partial^2 W}{\partial \widehat{E}_{ij}\widehat{E}_{kl}}(\widehat{\boldsymbol{E}}^0), \tag{3.69}$$

while

$$\boldsymbol{\varepsilon}(\widehat{\boldsymbol{\eta}}) = \frac{1}{2}\left(\boldsymbol{\nabla}_{\widehat{\boldsymbol{x}}}\widehat{\boldsymbol{\eta}} + \boldsymbol{\nabla}^T_{\widehat{\boldsymbol{x}}}\widehat{\boldsymbol{\eta}}\right) \tag{3.70}$$

is the well known *linearised* strain tensor.

Note that if we adopt the Saint-Venant Kirchhoff model (3.56) the components of the elasticity tensor defined in (3.69) are exactly those given in (3.58).

We now exploit the tensor identity $\widehat{\mathbf{H}} : (\widehat{\boldsymbol{F}}^{0T}\boldsymbol{\varepsilon}\widehat{\boldsymbol{F}}^0) = \widehat{\mathbf{H}} : (\widehat{\boldsymbol{F}}^{0T}\widehat{\boldsymbol{F}}^{0T}) : \boldsymbol{\varepsilon}$, which can be easily verified when written componentwise since

$$\widehat{H}_{ijkl}(\widehat{F}^0_{sk}\varepsilon_{st}\widehat{F}^0_{tl}) = (\widehat{H}_{ijkl}\widehat{F}^0_{sk}\widehat{F}^0_{tl})\varepsilon_{st}$$

to finally write,

$$\widehat{J}_0^{-1}\widehat{\boldsymbol{F}}^0 D_{\widehat{\boldsymbol{\eta}}_0}\widehat{\boldsymbol{\Sigma}}_0(\widehat{\boldsymbol{\eta}})\widehat{\boldsymbol{F}}^{0T} = \widehat{J}_0^{-1}\widehat{\boldsymbol{F}}^0\left[\widehat{\mathbf{H}} : \left(\widehat{\boldsymbol{F}}^{0T}\widehat{\boldsymbol{F}}^{0T}\right) : \boldsymbol{\varepsilon}(\widehat{\boldsymbol{\eta}})\right]\widehat{\boldsymbol{F}}^{0T} = \widehat{\mathbf{H}}^{\mathrm{p}} : \boldsymbol{\varepsilon}(\widehat{\boldsymbol{\eta}})$$

where

$$\widehat{\mathbf{H}}^{\mathrm{p}} = \widehat{J}_0^{-1}\left(\widehat{\boldsymbol{F}}^0\widehat{\boldsymbol{F}}^0\right) : \widehat{\mathbf{H}} : \left(\widehat{\boldsymbol{F}}^{0T}\widehat{\boldsymbol{F}}^{0T}\right) \tag{3.71}$$

is the linear elasticity fourth order tensor in the reference pre-stressed configuration $\widehat{\Omega}$. Componentwise, $\widehat{H}^{\mathrm{p}}_{ijkl} = \widehat{J}_0^{-1}\widehat{F}^0_{is}\widehat{F}^0_{jt}\widehat{H}_{stpq}\widehat{F}^0_{kp}\widehat{F}^0_{lq}$, where all repeated indexes are implicitly summed from 1 to 3.

We now recognise that $\widehat{\rho}^0/\widehat{J}^0$ is nothing else than $\widehat{\rho}_0$, the density in the reference configuration $\widehat{\Omega}$. Using the expressions found so far and exploiting (3.65), we are now able to write the linearised equations in the pre-stressed state as

$$\widehat{\rho}_0 \frac{\partial^2 \widehat{\boldsymbol{\eta}}}{\partial t^2} - \mathbf{div}_{\widehat{\boldsymbol{x}}} \left[\boldsymbol{\nabla}_{\widehat{\boldsymbol{x}}} \widehat{\boldsymbol{\eta}}\, \boldsymbol{\sigma}^0 + \widehat{\mathbf{H}}^{\mathrm{p}} : \boldsymbol{\varepsilon}(\widehat{\boldsymbol{\eta}}) \right] = \widehat{\rho}_0 \widehat{\boldsymbol{f}}, \quad \text{in } \widehat{\Omega}, \quad t > 0. \tag{3.72}$$

We can make the following remarks:

- Whenever $\widehat{\Omega}$ is a natural state $\boldsymbol{\sigma}^0 = \mathbf{0}$ and $\widehat{\mathbf{H}}^{\mathrm{p}}$ reduces to the standard linear elasticity tensor $\mathbf{H}$. System (3.72) becomes then the usual system of equations of linear elastodynamics,

$$\widehat{\rho}_0 \frac{\partial^2 \widehat{\boldsymbol{\eta}}}{\partial t^2} - \mathbf{div}_{\widehat{\boldsymbol{x}}} \big(\mathbf{H} : \boldsymbol{\varepsilon}(\widehat{\boldsymbol{\eta}}) \big) = \widehat{\rho}_0 \widehat{\boldsymbol{f}}, \tag{3.73}$$

  where if we adopt (3.58) we have

$$\mathbf{H} : \boldsymbol{\varepsilon}(\widehat{\boldsymbol{\eta}}) = L_1 (\operatorname{tr} \boldsymbol{\varepsilon}(\widehat{\boldsymbol{\eta}})) \boldsymbol{I} + 2 L_2 \boldsymbol{\varepsilon}(\widehat{\boldsymbol{\eta}}). \tag{3.74}$$

- Even if the material is homogeneous and isotropic with respect to the natural configuration, the same material in the pre-stressed configuration $\widehat{\Omega}$ is in general neither isotropic nor homogeneous. Indeed, these two properties depend not only on the material internal structure but also on the chosen reference state [92]. Homogeneity is retained whenever $\widehat{\boldsymbol{F}}^0$ (and thus $\boldsymbol{\sigma}^0$) is constant, while maintaining isotropy requires that $\widehat{\boldsymbol{F}}^0 = a\mathbf{I}$, for a non negative scalar field a, and (consequently) that $\boldsymbol{\sigma}^0$ be proportional to the identity tensor $\mathbf{I}$.
- An interesting case is when the deformation gradient $\widehat{\boldsymbol{F}}^0$ is *diagonal* and the material complies with the Saint-Venant Kirchhoff model (3.58) (w.r.t. the natural state). Then also $\boldsymbol{\sigma}^0$ is diagonal and it may be verified that $\widehat{\mathbf{H}}^{\mathrm{p}}$ conforms to that of an orthotropic material with axis of symmetry coinciding with the chosen coordinate axis.
  Now, it is well known that the mechanical behaviour of the arterial wall can be approximated as orthotropic [178]. The analysis carried out here suggests that this behaviour can be caused not only by the particular structure of the material composing a blood vessel, but also by its pre-stressed state.
- Thanks to the symmetry of $\boldsymbol{\sigma}^0$ we have $\boldsymbol{\nabla}_{\widehat{\boldsymbol{x}}} \widehat{\boldsymbol{\eta}} \boldsymbol{\sigma}^0 = \boldsymbol{\sigma}^0 \boldsymbol{\varepsilon}(\widehat{\boldsymbol{\eta}}) - \boldsymbol{\sigma}^0 \boldsymbol{\omega}(\widehat{\boldsymbol{\eta}})$, where $\boldsymbol{\omega}(\widehat{\boldsymbol{\eta}}) = \frac{1}{2}(\boldsymbol{\nabla}_{\widehat{\boldsymbol{x}}} \widehat{\boldsymbol{\eta}} - \boldsymbol{\nabla}_{\widehat{\boldsymbol{x}}}^T \widehat{\boldsymbol{\eta}})$ is the *rotation tensor*. It means that in a pre-stressed state the material may react also to pure rotations.

The results reported in this section are consistent with the findings in [23], where the authors considered an incompressible material and followed a different linearisation procedure by writing the elasticity tensor on the current configuration.

It is worth to point out that the expressions derived in this section are not relevant if one carries out a full non-linear analysis. However, even in this case it is necessary to account of pre-stress. What is done in common practise is to derive the pre-stress state from measurements of the opening angle and of the shrinking (when such invasive experiments are not feasible one may refer to reference data contained in the cited literature). How to introduce the pre-stress in the actual computation may depend on the numerical technique adopted.

Weak formulation of the equations of elastodynamics

We have introduced in Chapter 2 the weak formulation of a differential problem. We will here sketch that associated to (3.73). For the sake of simplicity we consider a homogeneous Dirichlet condition, i.e we set $\widehat{\boldsymbol{g}} = \mathbf{0}$ on $\widehat{\Gamma}_D$. Indeed the non-homogeneous case may be reduced to an homogeneous problem by a lifting technique [407] and more complex situations will be treated in Chapter 9. If we consider the space of vector functions

$$\widehat{\boldsymbol{V}} = [H_0^1(\widehat{\Omega})]^3 = \{\widehat{\boldsymbol{v}} \in [H^1(\widehat{\Omega})]^3 : \quad \widehat{\boldsymbol{v}} = \mathbf{0} \quad \text{on } \widehat{\Gamma}_D\},$$

following a route similar to that indicated in Chapter 2 we obtain the following formulation:

For any $t > 0$ *find* $\widehat{\boldsymbol{\eta}} = \widehat{\boldsymbol{\eta}}(t) \in \widehat{\boldsymbol{V}}$ *such that*

$$\int_{\widehat{\Omega}} \widehat{\rho}_0 \frac{\partial^2 \widehat{\boldsymbol{\eta}}}{\partial t^2} \cdot \widehat{\boldsymbol{v}} + a(\widehat{\boldsymbol{\eta}}, \widehat{\boldsymbol{v}}) = F(\widehat{\boldsymbol{v}}), \quad \forall \widehat{\boldsymbol{v}} \in \widehat{\boldsymbol{V}}, \tag{3.75}$$

where

$$a(\widehat{\boldsymbol{\eta}}, \widehat{\boldsymbol{v}}) = \int_{\widehat{\Omega}} \widehat{\boldsymbol{\Pi}} : \widehat{\boldsymbol{v}} d\Omega, \quad F(\widehat{\boldsymbol{v}}) = \int_{\widehat{\Omega}} \widehat{\boldsymbol{f}} \cdot \widehat{\boldsymbol{v}} d\Omega + \int_{\widehat{\Gamma}_N} \widehat{\boldsymbol{h}} \cdot \widehat{\boldsymbol{v}}\, d\widehat{\gamma}.$$

Here, given two tensors $\boldsymbol{A}$ ad $\boldsymbol{B}$, the symbol $\boldsymbol{A} : \boldsymbol{B}$ indicates the scalar $\sum_{i,j=1}^{3} A_{ij} B_{ij}$.

Relation (3.75) is indeed a scalar equation. We can however recover three equations for each component of $\widehat{\boldsymbol{\eta}}$ by selecting $\widehat{\boldsymbol{v}} = (\widehat{v}, 0, 0)^T$, $\widehat{\boldsymbol{v}} = (0, \widehat{v}, 0)^T$ and $\widehat{\boldsymbol{v}} = (0, 0, \widehat{v})^T$, respectively, with $\widehat{v} \in H_0^1(\widehat{\Omega})$.

3.4 Reduced structural models

As mentioned in Chapter 2, sometimes we can use reduced models for the structure. This choice may reduce computational costs when we are interested in the effects of the structure mechanics on the fluid, rather than on an accurate evaluation of the stresses inside the vessel tissue.

A possibility often used for haemodynamic computations is to adopt shell models, even if the thickness of the wall of blood vessels would not justify them in full, since they assume that the structure is very thin. The basic notions of linear shell theory can be found in [75] and the treatment of the non-linear case may be found in [286].

For the sake of completeness, we give a brief account of the derivation of the equations of linear shells in a rather general setting, yet at the same time we will specialise them for the case of a cylindrical surface, an important paradigm for blood vessels and discuss the effects of pre-stress, following the derivation found in [352], in the context of fluid-structure interaction problems.

3.4.1 The geometrical description of a shell

In this section we will use the following summation convection: quantities with repeated indexes appearing at the same side of an equation are automatically summed up. Furthermore, Greek indexes will run in the set $\{1, 2\}$ whereas Latin indexes in $\{1, 2, 3\}$.

A shell is a solid medium whose reference configuration, $\widehat{\Omega}$, can be defined by a mid-surface, $\mathcal{S}$, and a thickness $h_s > 0$. More precisely, we shall assume that the mid-surface is the image of a two-dimensional domain $\omega \subset \mathbb{R}^2$ by an injective mapping (or chart) $\boldsymbol{\psi} : \omega \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^3$, i.e.,

$$\mathcal{S} = \boldsymbol{\psi}(\omega),$$

see Fig. 3.13. We also assume that the chart $\boldsymbol{\psi}$ is such that the tangent vectors

$$\boldsymbol{a}_\alpha(\xi_1, \xi_2) = \frac{\partial \boldsymbol{\psi}}{\partial \xi_\alpha}(\xi_1, \xi_2), \quad (\xi_1, \xi_2) \in \omega,$$

are linearly independent. Thus we may define the unit normal vector to the mid-surface

$$\boldsymbol{a}_3 = \frac{\boldsymbol{a}_1 \times \boldsymbol{a}_2}{\|\boldsymbol{a}_1 \times \boldsymbol{a}_2\|}.$$

Finally, we parametrise the reference domain $\widehat{\Omega}$ by the mapping $\boldsymbol{\Psi}$ given by

$$\boldsymbol{\Psi}(\xi_1, \xi_2, \xi_3) = \boldsymbol{\psi}(\xi_1, \xi_2) + \xi_3 \boldsymbol{a}_3(\xi_1, \xi_2), \tag{3.76}$$

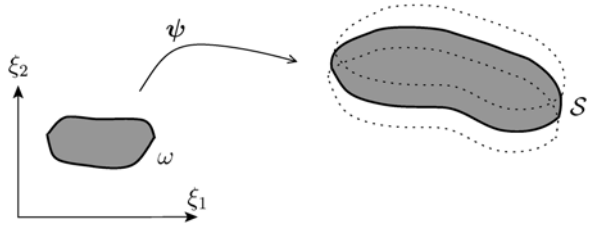

Fig. 3.13. Parametrisation of the shell mid-surface

for all $(\xi_1, \xi_2, \xi_3) \in \Theta$, where

$$\Theta = \left\{ (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 \ : \ (\xi_1, \xi_2) \in \omega, \, \xi_3 \in \left(-\frac{h_s}{2}, \frac{h_s}{2} \right) \right\}.$$

As a result, we have $\widehat{\Omega} = \boldsymbol{\Psi}(\Theta)$, see Fig. 3.14. Note that h_s could vary along the mid-surface, however, for the sake of clarity we assume that h_s is constant.

The parametrisation of the domain given by the map $\boldsymbol{\Psi}$ induces a three-dimensional curvilinear coordinate system. We can introduce the so-called covariant basis $\{\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3\}$ defined by

$$\boldsymbol{g}_i(\xi_1, \xi_2, \xi_3) = \frac{\partial \boldsymbol{\Psi}}{\partial \xi_i}(\xi_1, \xi_2, \xi_3), \quad \forall (\xi_1, \xi_2, \xi_3) \in \Theta.$$

and the associated local contravariant basis $\{\boldsymbol{g}^1, \boldsymbol{g}^2, \boldsymbol{g}^3\}$ defined by the relation

$$\boldsymbol{g}_i \cdot \boldsymbol{g}^j = \delta_i^j,$$

being δ_i^j the Kronecker's symbol. Tensors of arbitrary order can be expressed in terms of either basis. For instance, $\mathbf{f} = f_i \boldsymbol{g}^i = f^i \boldsymbol{g}_i$, f_i and f^i being called the *covariant* and *contravariant* components of the vector $\mathbf{f}$, respectively. The *metric tensor* $g_{ij} = \boldsymbol{g}_i \cdot \boldsymbol{g}_j$ and its inverse $g^{ij} = \boldsymbol{g}^i \cdot \boldsymbol{g}^j$ allow to perform the change of basis, whenever needed. For instance, we have $v_i = g^{ij} v^j$. If the coordinate system is orthonormal, then $g_{ij} = g^{ij} = \delta_{ij}$, i.e. the transformation reduces to the identity and there is no distinction between covariant and contravariant components.

We now introduce some symmetric surface tensors which are fundamental in shell analysis. The first, second and third fundamental forms of the mid-surface are tensors defined on the surface here indicated by $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$, whose covariant components are given by

$$a_{\alpha\beta} = \boldsymbol{a}_\alpha \cdot \boldsymbol{a}_\beta, \quad b_{\alpha\beta} = \boldsymbol{a}_3 \cdot \frac{\partial \boldsymbol{a}_\alpha}{\partial \xi_\beta} \quad \text{and } c_{\alpha\beta} = b_{\alpha\gamma} a^{\gamma\lambda} b_{\lambda\beta}, \tag{3.77}$$

respectively. The first fundamental form is also called *surface metric tensor*.

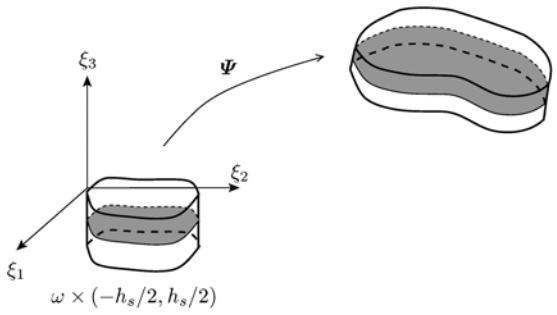

Fig. 3.14. Geometric description of a shell, which might represent a portion of the wall of a blood vessel

A function indexed by Greek letter is assumed to lay on the mid-surface $\xi_3 = 0$. Let us also note that at $\xi_3 = 0$ we have that $\boldsymbol{g}_\alpha = \boldsymbol{a}_\alpha$ and $g_{\alpha\beta} = a_{\alpha\beta}$.

There is the need to give a meaning to the derivation of a surface tensor field. Given a vector field $\boldsymbol{f} = \boldsymbol{f}(\xi_1, \xi_2)$ on the mid-surface we denote by $f_{\alpha|\beta}$ the *surface covariant derivative* of its covariant component f_α defined as

$$f_{\alpha|\beta} = \frac{\partial f_\alpha}{\partial \xi_\beta} - \Gamma^\lambda_{\alpha\beta} f_\lambda. \tag{3.78}$$

Here, $\Gamma^\lambda_{\alpha\beta}$ stands for the Christoffel's symbol defined as

$$\Gamma^\lambda_{\alpha\beta} = \frac{\partial \boldsymbol{a}_\alpha}{\partial \xi_\beta} \cdot \boldsymbol{a}^\lambda. \tag{3.79}$$

Relation (3.78) is readily extended to a tensor field of any order.

For the sake of illustration, let us consider a cylindrical shell of constant radius R_0 and length L aligned along the canonical base vector $\boldsymbol{e}_3$. It can be parametrised by the chart

$$\boldsymbol{\Psi}(\xi_1, \xi_2, \xi_3) = \begin{bmatrix} (R_0 + \xi_3)\cos\xi_1 \\ (R_0 + \xi_3)\sin\xi_1 \\ \xi_2 \end{bmatrix}, \tag{3.80}$$

where

$$(\xi_1, \xi_2) \in \omega = [0, 2\pi] \times [0, L], \quad \text{and} \qquad \xi_3 \in [-h_s/2, h_s/2]\,.$$

One may immediately associate the selected independent variables with the standard cylindrical coordinates (r, θ, z): $\theta = \xi_1$, $r = R_0 + \xi_3$ and $z = \xi_2$.

The covariant base vector are thus given by

$$\boldsymbol{g}_1 = \begin{bmatrix} -(R_0 + \xi_3)\sin\xi_1 \\ (R_0 + \xi_3)\cos\xi_1 \\ 0 \end{bmatrix}, \quad \boldsymbol{g}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \boldsymbol{g}_3 = \begin{bmatrix} \cos\xi_1 \\ \sin\xi_1 \\ 0 \end{bmatrix},$$

and the surface vectors by

$$\boldsymbol{a}_1 = \begin{bmatrix} -R_0\sin\xi_1 \\ R_0\cos\xi_1 \\ 0 \end{bmatrix}, \quad \boldsymbol{a}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \boldsymbol{a}_3 = \begin{bmatrix} \cos\xi_1 \\ \sin\xi_1 \\ 0 \end{bmatrix}.$$

It then follows that

$$\frac{\partial \boldsymbol{a}_1}{\partial \xi_1} = \begin{bmatrix} -R_0\cos\xi_1 \\ -R_0\sin\xi_1 \\ 0 \end{bmatrix}, \quad \frac{\partial \boldsymbol{a}_1}{\partial \xi_2} = \frac{\partial \boldsymbol{a}_2}{\partial \xi_1} = \frac{\partial \boldsymbol{a}_2}{\partial \xi_2} = \boldsymbol{0}.$$

On the other hand, as covariant surface vectors we have

$$\boldsymbol{a}^1 = \begin{bmatrix} -\frac{1}{R_0}\sin\xi_1 \\ \frac{1}{R_0}\cos\xi_1 \\ 0 \end{bmatrix}, \quad \boldsymbol{a}^2 = \boldsymbol{a}_2, \quad \boldsymbol{a}^3 = \boldsymbol{a}_3.$$

Therefore,

$$\begin{aligned} &a_{11} = R_0^2,\ a_{12} = a_{21} = 0,\ a_{22} = 1, \quad && a^{11} = R_0^{-2},\ a^{12} = a^{21} = 0,\ a^{22} = 1, \\ &b_{11} = -R_0,\ b_{12} = b_{21} = b_{22} = 0, && c_{11} = 1,\ c_{12} = c_{21} = c_{22} = 0. \end{aligned} \tag{3.81}$$

In particular, all the Christoffel's symbols are zero

$$\Gamma_{\alpha\beta}^{\lambda} = \frac{\partial \boldsymbol{a}_\alpha}{\partial \xi_\beta} \cdot \boldsymbol{a}^\lambda = 0,$$

so that the covariant and standard derivation coincide in this particular case.

3.4.2 Shell kinematics

In general, shell models are based on some kinematic assumptions relating the displacement of points located in a material line orthogonal to the mid-surface, i.e. when ξ_1, ξ_2 are kept fixed, while $\xi_3 \in (-h_s/2, h_s/2)$. For instance, under the *Reissner-Mindlin kinematics assumption* [335, 425], it is assumed that such material line remains unstretched during the motion. As a consequence the displacement $\widehat{\boldsymbol{\eta}}$ of a point belonging to the shell may be expressed by the following equation:

$$\widehat{\boldsymbol{\eta}}(\xi_1, \xi_3, \xi_3) = \boldsymbol{d}(\xi_1, \xi_2) + \xi_3 \theta_\lambda(\xi_1, \xi_2) \boldsymbol{a}^\lambda(\xi_1, \xi_2), \tag{3.82}$$

where $\boldsymbol{d}(\xi_1, \xi_2)$ is the displacement of the mid-surface and θ_λ the rotation of a line normal to the mid-surface.

Using this assumption, the covariant components of the linearised strain tensor $\boldsymbol{\varepsilon}(\widehat{\boldsymbol{\eta}})$, whose general definition in terms of the shell metric is

$$\varepsilon_{ij}(\widehat{\boldsymbol{\eta}}) = \varepsilon(\widehat{\boldsymbol{\eta}}) : \boldsymbol{g}_i \otimes \boldsymbol{g}_j = \frac{1}{2}\left(\frac{\partial \widehat{\boldsymbol{\eta}}}{\partial \xi_j} \cdot \boldsymbol{g}_i + \frac{\partial \widehat{\boldsymbol{\eta}}}{\partial \xi_i} \cdot \boldsymbol{g}_j \right),$$

becomes

$$\begin{aligned} &\varepsilon_{\alpha\beta}(\widehat{\boldsymbol{\eta}}) = \gamma_{\alpha\beta}(\boldsymbol{d}) + \xi_3 \chi_{\alpha\beta}(\boldsymbol{d}, \boldsymbol{\theta}) - \xi_3^2 \kappa_{\alpha\beta}(\boldsymbol{\theta}), \\ &\varepsilon_{\alpha 3}(\widehat{\boldsymbol{\eta}}) = \zeta_\alpha(\boldsymbol{d}, \boldsymbol{\theta}), \quad \text{and} \quad \varepsilon_{33}(\widehat{\boldsymbol{\eta}}) = 0, \end{aligned}$$

with $\boldsymbol{\gamma}$, $\boldsymbol{\chi}$ and $\boldsymbol{\kappa}$ being the so-called *membrane*, *bending* and *shear* parts of the strain tensor, whose covariant components are

$$\begin{aligned} \gamma_{\alpha\beta}(\boldsymbol{d}) &= \frac{1}{2}\left(d_{\alpha|\beta} + d_{\beta|\alpha} \right) - b_{\alpha\beta} d_3, \\ \chi_{\alpha\beta}(\boldsymbol{d}, \boldsymbol{\theta}) &= \frac{1}{2}\left(\theta_{\alpha|\beta} + \theta_{\beta|\alpha} - b_\beta^\lambda d_{\lambda|\alpha} - b_\alpha^\lambda d_{\lambda|\beta} \right) + c_{\alpha\beta} d_3, \\ \kappa_{\alpha\beta}(\boldsymbol{\theta}) &= \frac{1}{2}\left(b_\beta^\lambda \theta_{\lambda|\alpha} + b_\alpha^\lambda \theta_{\lambda|\beta} \right), \end{aligned}$$

respectively, while

$$\zeta_\alpha(\boldsymbol{d},\boldsymbol{\theta}) = \frac{1}{2}\left(\theta_\alpha + \frac{\partial d_3}{\partial \xi_\alpha} + b_\alpha^\lambda d_\lambda\right). \tag{3.83}$$

The tensor $\boldsymbol{\gamma}$ is also called *change of metric* tensor, since it is related to the change in the surface metric tensor caused by the deformation. Often an additional kinematic condition is adopted which takes the name of *Kirchhoff-Love* kinematics condition. It states that during the motion any material line orthogonal to the (reference) mid-surface remains straight, unstretched and orthogonal to the mid-surface. The Kirchhoff-Love assumption introduces the following relation between the translation and rotation displacements, see [75],

$$\theta_\lambda = -\frac{\partial d_3}{\partial \xi_\lambda} - b_\lambda^\mu d_\mu, \tag{3.84}$$

by which

$$\zeta_\alpha = 0, \quad \text{and} \quad \varepsilon_{\alpha 3} = 0.$$

Thanks to the Kirchhoff-Love conditions we have eliminated the rotations θ_α.

In addition, it is often assumed that the term of order ξ_3^2 may be neglected. This is the term responsible to shear strain, which can indeed be considered small for the target application. The resulting shell model is called *membrane-bending model*.

3.4.3 Shell dynamics

In curvilinear coordinates, Hook's law is still written in the form (3.59) where now to account for the fact that we are not using curvilinear coordinates we have

$$H^{ijkl} = L_1 g^{ij} g^{kl} + L_2\left(g^{ik}g^{jl} + g^{il}g^{jk}\right). \tag{3.85}$$

In shell models we make the further assumption of *plane stresses*. That is, because of the small thickness we assume zero stresses along the normal direction, namely $T^{33} = 0$. We thus obtain the modified constitutive equation

$$T^{\alpha\beta} = C^{\alpha\beta\lambda\mu}\varepsilon_{\lambda\mu}, \quad \sigma^{\alpha 3} = \frac{1}{2}D^{\alpha\lambda}\varepsilon_{\lambda 3},$$

with

$$C^{\alpha\beta\lambda\mu} = \frac{E}{(1+\xi)}\left(g^{\alpha\lambda}g^{\beta\mu} + \frac{\xi}{1-\xi}g^{\alpha\beta}g^{\lambda\mu}\right) \quad \text{and} \quad D^{\alpha\lambda} = \frac{2E}{1+\xi}g^{\alpha\lambda}.$$

We indicate by $\hat{\rho}_{s,0}$ the density of the material composing the shell, measured in kg/m^3. We assume that external stresses $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ act on the boundaries of $\widehat{\Omega}$ given by the image of $\omega \times \{-h_s/2\}$ and $\omega \times \{h_s/2\}$, respectively. Clearly we have $\boldsymbol{s}_1 = \boldsymbol{s}_1(\xi_1,\xi_2)$ and $\boldsymbol{s}_2 = \boldsymbol{s}_2(\xi_1,\xi_2)$, and with the symbol $\boldsymbol{f}$ we indicate the

resultant $\boldsymbol{f}(\xi_1, \xi_2) = \boldsymbol{s}_2(\xi_1, \xi_2) - \boldsymbol{s}_1(\xi_1, \xi_2)$, f being a force per unit area $[\boldsymbol{f}] = \mathrm{N/m^2}$ (in our applications $\boldsymbol{f}$ is directly related to the transmural pressure across the blood vessel),

The equations governing the dynamics of a shell is better written directly using the weak formulation. We here give just the sketch of the procedure. We assume a membrane-bending model and, for the sake of simplicity we take homogeneous Dirichlet conditions on the lateral boundaries, image of $\partial\omega \times (-h_s/2, h_s/2)$. On the remaining boundaries we have the action of the stresses $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$, which induce a Neumann-type boundary condition. Starting from (3.75)[7], since here $\widehat{\Omega} = \boldsymbol{\Psi}(\Theta)$ we can rewrite the integrals over Θ. Then, as $\Theta = \omega \times (-h_s/2, h_s/2)$ we can reduce all the integrals on ω by integrating along the ξ_3 direction. The final result is the variational formulation of the membrane-bending model, which reads: *For all* $t > 0$, *find* $\boldsymbol{d}$ *such that*

$$\int_\omega h_s \hat{\rho}_{s,0} \frac{\partial^2 \boldsymbol{d}}{\partial t^2} \cdot \boldsymbol{q} d\omega + \int_\omega \widetilde{C}^{\alpha\beta\lambda\mu} \left[h_s \gamma_{\alpha\beta}(\boldsymbol{d}) \gamma_{\lambda\mu}(\boldsymbol{q}) + \frac{h_s^3}{12} \rho_{\alpha\beta}(\boldsymbol{d}) \rho_{\lambda\mu}(\boldsymbol{q}) \right] d\omega = \int_\omega \boldsymbol{f} \cdot \boldsymbol{q} d\omega, \quad (3.86)$$

for all test function $\boldsymbol{q}$ regular enough and with zero trace on the Dirichlet portion of the boundary. Here,

$$\widetilde{C}^{\alpha\beta\lambda\mu} = \frac{E}{2(1+\xi)} \left(a^{\alpha\lambda} a^{\beta\mu} + a^{\alpha\mu} a^{\beta\lambda} + \frac{2\xi}{1-\xi} a^{\alpha\beta} a^{\lambda\mu} \right)$$

and

$$\rho_{\alpha\beta}(\boldsymbol{d}) = d_{3|\alpha\beta} + b^\mu_{\alpha|\beta} d_\mu + b^\mu_\alpha d_{\mu|\beta} + b^\mu_\beta d_{\mu|\alpha} - c_{\alpha\beta} d_3.$$

An advantage of the shell model is that now we are effectively operating on a two dimensional domain. The discretisation by finite elements of (3.86) then leads, in principle at least, to cheaper solution schemes than with a 3D formulation on $\widehat{\Omega}$. Examples of suitable finite element spaces for shell models can be found in the cited literature.

We consider again the example of the cylinder illustrated in the previous section and we make the additional hypotheses of axi-symmetric displacements, that is that $d_1 = 0$ as well as all derivatives w.r.t. ξ_1. Furthermore, for the sake of clarity, we will indicate with the suffixes θ and z the first and second component of the displacement vector: $\boldsymbol{d} = (d_\theta, d_r)$. Under these hypothesis we have

$$\gamma_{11}(\boldsymbol{d}) = R_0 d_r, \quad \gamma_{22}(\boldsymbol{d}) = \frac{\partial d_z}{\partial z}, \quad \gamma_{12}(\boldsymbol{d}) = \gamma_{21}(\boldsymbol{d}) = 0,$$

$$\rho_{11}(\boldsymbol{d}) = -d_r, \quad \rho_{22}(\boldsymbol{d}) = \frac{\partial^2 d_r}{\partial z^2}, \quad \rho_{12}(\boldsymbol{d}) = \rho_{21}(\boldsymbol{d}) = 0.$$

[7] In (3.75) $\boldsymbol{f}$ was a volume force, which is here taken equal to zero. The symbol $\boldsymbol{f}$ is here used instead to indicate the surface stress resultant acting on the shell.

Therefore, we have

$$\widetilde{C}^{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}(\boldsymbol{d})\gamma_{\lambda\mu}(\boldsymbol{q}) = \frac{E}{2(1+\xi)}\left(\frac{1}{R_0^2}d_r q_r + \frac{\partial d_z}{\partial z}\frac{\partial q_z}{\partial z}\right) + \frac{E\xi}{1-\xi^2}\left(\frac{d_r}{R_0^2}+\frac{\partial d_z}{\partial z}\right)\left(\frac{q_r}{R_0^2}+\frac{\partial q_z}{\partial z}\right),$$

while

$$\widetilde{C}^{\alpha\beta\lambda\mu}\rho_{\alpha\beta}(\boldsymbol{d})\rho_{\lambda\mu}(\boldsymbol{q}) = \frac{E}{2(1+\xi)}\left(\frac{1}{R_0^4}d_r q_r + \frac{\partial^2 d_r}{\partial z^2}\frac{\partial^2 q_r}{\partial z^2}\right) + \frac{E\xi}{1-\xi^2}\left(-\frac{d_r}{R_0^2}+\frac{\partial^2 d_r}{\partial z^2}\right)\left(-\frac{q_r}{R_0^2}+\frac{\partial^2 q_r}{\partial z^2}\right).$$

Replacing these expressions into (3.86) and counter-integrating by parts (for simplicity let us assume homogeneous Dirichlet boundary conditions) we may recognise that the corresponding strong differential form is nothing else than the well known system of equation for a cylindrical Koiter-type shell, that is

$$h_s\widehat{\rho}_{s,0}\frac{\partial^2 d_z}{\partial t^2} - \frac{h_s E}{1-\xi^2}\left(\frac{\partial^2 d_z}{\partial z^2} + \xi\frac{1}{R}\frac{\partial d_r}{\partial z}\right) = f_z,$$

$$h_s\rho_0\frac{\partial^2 d_r}{\partial t^2} + \frac{h_s E}{R(1-\xi^2)}\left(\xi\frac{\partial d_z}{\partial z}+\frac{d_r}{R}\right) + \frac{h_s^3 E}{12(1-\xi^2)}\left(\frac{\partial^4 d_r}{\partial z^4} - \frac{2\xi}{R^2}\frac{\partial^2 d_r}{\partial z^2} + \frac{d_r}{R^4}\right) = f_r. \qquad (3.87)$$

3.4.4 One-dimensional reduced structural models

One dimensional models are a very simple way to describe the dynamics of a single vessel. We assume that the artery is of cylindrical shape and the only space dimension considered is the axial one. There are different ways to derive them. For instance, directly from physical principles, as done in [401] or [406], or from shell models like in [522]. By further simplification assumptions that lead to simple algebraic relationship between the vessel section area and the average pressure, they are often used to develop 1D models for blood flow in compliant arteries like those illustrated in Chapter 10.

Here we will sketch the derivation starting from a shell model and accounting for pre-stress.

To this aim, we will follow [352], alternative derivations may be found, for instance, in [401], [406], or in [523], where some viscoelastic effects are accounted for.

We will consider the situation of Fig. 3.15 where the reference configuration is a cylindrical surface. In the case of a standard cylinder, the map

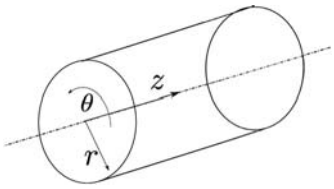

Fig. 3.15. Cylindrical map. $\xi_1 = \theta$, $\xi_2 = r$ and $\xi_3 = z$

will coincide with that indicated in (3.80). We may then identify ξ_1, ξ_2 and ξ_3 with the circumferential, the axial and the radial physical coordinates, and for this reason we will alternatively use the indexes θ, z, and r, respectively. Furthermore the surface is the image of $\theta \in (0, 2\pi)$, $z \in (0, L)$ and $r \in (R_0 - h_s/2, R_0 + h_s/2)$.
The main assumptions are:

- The ratio h_s/R_0 of the vessel wall is small so that we can neglect bending terms (which indeed scale with higher order than membrane terms with respect to this ratio). In other words we will set to zero the last term in the left hand side of (3.86) and consider only membrane effects.
- The wall displaces in the normal direction, i.e. $\boldsymbol{d} = (0, 0, \eta)$. Correspondingly we have

$$\gamma_{\alpha\beta}(\boldsymbol{d}) = -b_{\alpha\beta}\eta. \tag{3.88}$$

  The assumption that longitudinal and circumferential displacements are negligible compared to the radial ones is usually accepted in the biomedical literature.
- The vessel is subject to a time varying transmural pressure $\Delta P = P - P_{ext}$ across the surface. Consequently we take

$$\boldsymbol{f} = \Delta P \boldsymbol{n}, \tag{3.89}$$

  being $\boldsymbol{n}$ the outward oriented normal. For the sake of simplicity, we assume homogeneous Dirichlet boundary condition (i.e. a clamped vessel). Yet accounting for conditions of other type is rather straightforward.
- The reference configuration is pre-stressed, yet the deformation gradient $\widehat{\boldsymbol{F}}^0$ is assumed to be diagonal (in the shell local reference system) and constant. More precisely we set

$$\widehat{\boldsymbol{F}}^0 = \mathrm{diag}(F^0_{\theta\theta}, F^0_{zz}, 1),$$

  being $F^0_{\theta\theta}$ and F^0_{zz} constants. In correspondence we assume a diagonal (and constant) pre-stress tensor $\boldsymbol{\sigma}^0 = \mathrm{diag}(\sigma^0_{\theta\theta}, \sigma^0_{zz}, 0)$. The value of the parameters may be inferred from measurements of the opening angle and shrinking of an extracted vessel, or by considerations on how the pre-stressed configuration has been generated [23, 115, 391, 523].

We first have to find out the modifications induced by the pre-stress state to the standard shell model. We have seen that pre-stressing induces two modifications to the standard linearised elasticity equations: a change in the stress-strain tensor $\boldsymbol{H}$ and the appearance of an extra term in the stress, of the form $\boldsymbol{\nabla}_{\widehat{\boldsymbol{x}}}\widehat{\boldsymbol{\eta}}\boldsymbol{\sigma}^0$. Being $\widehat{\boldsymbol{F}}^0$ diagonal the former modification is readily accounted for by replacing g^{ij} with $F^0_{ii}F^0_{jj}g^{ij}$ (no implied sum) in (3.85). We are able to carry out the standard derivation that led to (3.86) also in this case, however now the tensor $\widetilde{\boldsymbol{C}}$ becomes

$$\widetilde{C}^{\alpha\beta\lambda\mu} = \frac{E}{2(1+\xi)}F^0_{\alpha\alpha}F^0_{\beta\beta}F^0_{\lambda\lambda}F^0_{\mu\mu}\left(a^{\alpha\lambda}a^{\beta\mu} + a^{\alpha\mu}a^{\beta\lambda} + \frac{2\xi}{1-\xi}a^{\alpha\beta}a^{\lambda\mu}\right).$$

Thanks to (3.88) we may derive that in (3.86)

$$h_s\widetilde{C}^{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}(\boldsymbol{d})\gamma_{\lambda\mu}(\boldsymbol{q}) = a_0\eta q_r, \tag{3.90}$$

where

$$a_0 = \frac{h_sE}{1-\xi^2}\left[(1-\xi)\widetilde{a}^{\alpha\lambda}\widetilde{a}^{\beta\delta}b_{\alpha\beta}b_{\lambda\delta} + \xi\widetilde{a}^{\alpha\beta}\widetilde{a}^{\lambda\delta}b_{\alpha\beta}b_{\lambda\delta}\right],$$

while

$$\widetilde{a}^{\alpha\beta} = F^0_{\alpha\alpha}F^0_{\beta\beta}a^{\alpha\beta} \quad \text{(no sum implied)}.$$

Finally, we account also for the term $\boldsymbol{\nabla}_{\widehat{\boldsymbol{x}}}\widehat{\boldsymbol{\eta}}\,\boldsymbol{\sigma}^0$ and we make the additional assumption of axial symmetry, i.e. we set to zero all derivatives w.r.t. ξ_1. Under this hypothesis the radial component of $\mathbf{div}(\boldsymbol{\nabla}\eta\widehat{\boldsymbol{\sigma}}^0)$ reduces to $T^0_{zz}\frac{\partial^2\eta}{\partial z^2}$.

We have then

$$h_s\hat{\rho}_{s,0}\frac{\partial^2\eta}{\partial t^2} + a\eta - b\frac{\partial^2\eta}{\partial z^2} = P - P_{ext}, \quad z\in(0,L), \quad t>0, \tag{3.91}$$

with $\eta = 0$ at $z = 0$ and $z = L$ (or other suitable boundary conditions) while $\eta = \eta_0$ and $\partial\eta/\partial t = \eta_1$ at $t = 0$, η_0 and η_1 being suitable initial data.

Here,

$$a = a_0 + \sigma^0_{\theta\theta}b_{\gamma1}b_{\gamma1} + \sigma^0_{zz}b_{\gamma2}b_{\gamma2}. \tag{3.92}$$

is the elastic coefficient modified to account for the pre-stress and $b = \sigma^0_{zz}$.

A notable case if that of the regular cylinder, where we may apply (3.81). We may for instance assume that the circumferential pre-stress has been caused by an external pressure P^+ originally applied to the unstressed cylinder (this value is sometimes assumed to be equal to the mean arterial pressure). By using the simple Poisson's law for the stress in a pressurised cylinder we have then $\sigma^0_{\theta\theta} = P^+R_0/h_s$. As for the term $F^0_{\theta\theta}$, it might be estimated from the measurements of the opening angle $\widehat{\Theta}$ of artery specimens opened up longitudinally so that they can recover a zero-stress configuration, i.e. $F^0_{\theta\theta} = 1+\frac{\pi}{\widehat{\Theta}}$.

Gathering all this information the expression for a simplifies into

$$a = (F^0_{\theta\theta})^4\frac{h_sE}{R_0(1-\xi^2)} + \frac{P^+}{R_0}.$$

In several works however, the effect of the pre-stress in the coefficient a is neglected. Either because the value of the a is estimated from measurements taken *in vivo*, thus already accounting for the pre-stress, or because it is felt that pre-stress introduces a correction of the same order of the incertitude on the value of E and R_0. In the latter case the simpler formula $a = \frac{h_s E}{R_0(1-\xi^2)}$ is used instead, and this is the expression we will adopt in the sequel of the book whenever model (3.91) is adopted.

Model (3.91) is called *string model*, since it is akin to the one governing the motion of a string under tension. Some authors link the presence of the second order space derivative in this model not to the pre-stress but to the Timoshenko shear factor, an empirical term introduced to better account for shear deformation and rotatory inertia effects in the theory of thin structures [510]. In fact, probably both effects are present at the same time.

A more complete reduced one-dimensional model for the vessel structure takes the general form (see [406], [525])

$$h_s\hat{\rho}_{s,0}\frac{\partial^2\eta}{\partial t^2} + a\eta - b\frac{\partial^2\eta}{\partial z^2} + c_1\frac{\partial\eta}{\partial t} - c_2\frac{\partial^3\eta}{\partial t\partial z^2} = P - P_{ext}, \quad z \in (0, L), \quad t > 0. \tag{3.93}$$

Here, c_1 and c_2 are two non-negative parameters accounting for the viscoelastic property of the vessel wall, see also [523], [33]. We will call (3.93) a *generalised string model* and it is often used as a simplified (yet rather complete) model for the study of fluid structure interaction problems in a single artery (see Chapters 8 and 9).

A whole class of models can be derived by setting to zero some of the parameters. In particular, the simplest model is obtained by neglecting all derivative terms (including the inertial term), obtaining the simple algebraic expression

$$a\eta = P - P_{ext} \tag{3.94}$$

used for the derivation of the basic one-dimensional model for blood flow in arteries, as illustrated in Chapter 10.

3.5 Modelling fluid-structure interaction problems

In this section we describe the general non-linear fluid-structure system in large displacements arising in blood flows in large arteries. We consider as computational domain a model of a portion of an artery, see Fig. 3.16. It consists of a deformable structure $\Omega_s(t)$ (vessel wall) surrounding a moving domain $\Omega_f(t)$ filled by a fluid under motion (blood). The fluid structure interface, *i.e.* the common boundary between $\Omega_s(t)$ and $\Omega_f(t)$, is denoted by $\Gamma(t) = \partial\Omega_f(t) \cap \partial\Omega_s(t)$. In the sequel, variables with a sub-script s or f shall refer to quantities within the fluid or the solid domains, respectively.

We will ignore body forces, i.e. we take $\boldsymbol{f} = \mathbf{0}$ both for the fluid and the structure. For haemodynamic applications this corresponds in practise to ignore the effects of gravity.

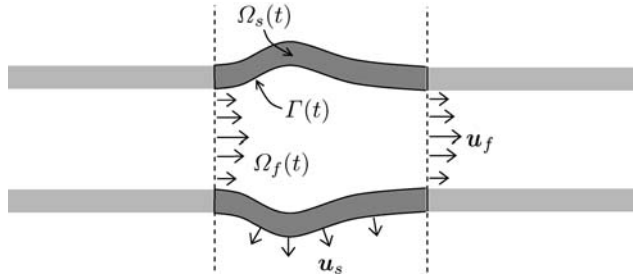

Fig. 3.16. Geometric configuration (2D section)

We assume the motion of the control volume $\Omega_f(t)$ to be parametrised by an ALE map $\widetilde{\mathcal{A}} : \widetilde{\Omega_f} \times \mathbb{R}^+ \longrightarrow \mathbb{R}^3$ (see Section 3.1.2), *i.e.* $\Omega_f(t) = \widetilde{\mathcal{A}}(\widetilde{\Omega_f}, t)$. The reference domain $\widetilde{\Omega_f}$ being the position of the control volume at the initial time. We assume that the inlet $\Gamma_{f,D}$ and outlet $\Gamma_{f,N}$ boundaries are at a fixed axial position along the artery model of Fig. 3.17.

As showed in Section 3.1.2, when dealing with moving domains it is natural to work with ALE time-derivatives. More precisely, we will use formulation (3.48).

The differential equations have to be completed with proper boundary conditions on $\partial\Omega_f(t)\backslash\Gamma(t)$. For instance, we can enforce

$$\begin{aligned} \boldsymbol{u}_f &= \boldsymbol{u}_D, \quad \text{on} \quad \Gamma_{f,D}, \\ \boldsymbol{\sigma}_f(\boldsymbol{u}_f, P)\boldsymbol{n}_f &= \boldsymbol{g}_N, \quad \text{on} \quad \Gamma_{f,N}, \end{aligned} \tag{3.95}$$

with $\boldsymbol{u}_D$ a given velocity and $\boldsymbol{g}_{f,N}$ a given density of surface load.

To summarise, we have

$$\begin{aligned} \rho_f \frac{\partial \boldsymbol{u}_f}{\partial t}_{|\widetilde{\mathcal{A}}} + \rho_f(\boldsymbol{u}_f - \boldsymbol{w}) \cdot \boldsymbol{\nabla}\boldsymbol{u}_f - \mathbf{div}\boldsymbol{\sigma}_f(\boldsymbol{u}_f, P) &= \mathbf{0}, \quad \text{in} \quad \Omega_f(t), \\ \operatorname{div} \boldsymbol{u}_f &= 0, \quad \text{in} \quad \Omega_f(t). \\ \boldsymbol{u}_f &= \boldsymbol{u}_{f,D}, \quad \text{on} \quad \Gamma_{f,D}, \\ \boldsymbol{\sigma}_f(\boldsymbol{u}_f, P)\boldsymbol{n}_f &= \boldsymbol{g}_{f,N}, \quad \text{on} \quad \Gamma_{f,N}. \end{aligned} \tag{3.96}$$

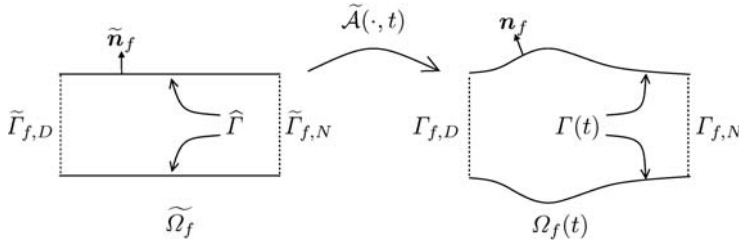

Fig. 3.17. Description of the motion of the computational domain for the fluid via the ALE map $\widetilde{\mathcal{A}}$

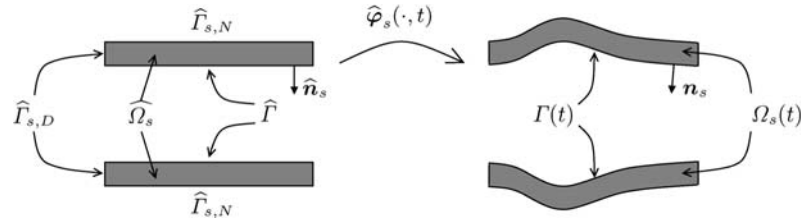

Fig. 3.18. Description of the motion of the solid (2D section)

Remark 3.5.1 *So far, we have assumed the ALE map $\widetilde{\mathcal{A}}$, and in particular the ALE velocity field $\boldsymbol{w}$, to be known. As wee shall see in the next section that we can define the ALE map as an extension of the boundary position with a technique that can be extended readily to the numerical computation.*

As anticipated in Section 3.3.2, we consider a Lagrangian description of the motion of the structure, where $\widehat{\Omega_s}$ is a given material reference configuration (see Fig. 3.18). We describe the motion of the structure in terms of its displacement field $\widehat{\boldsymbol{\eta}}_s : \widehat{\Omega_s} \times \mathbb{R}^+ \longrightarrow \mathbb{R}^3$. For the sake of simplicity, we here assume the structure to be clamped on the boundaries $\widehat{\Gamma}_{s,D}$.

The differential problem for the structure part then reads

$$\begin{aligned} \hat{\rho}_{s,0}\frac{\partial^2\widehat{\boldsymbol{\eta}}_s}{\partial t^2} - \mathbf{div}_{\hat{\boldsymbol{x}}}\big(\widehat{\boldsymbol{F}}_s\widehat{\boldsymbol{\Sigma}}\big) = \mathbf{0}, \quad &\text{in} \quad \widehat{\Omega_s}, \\ \widehat{\boldsymbol{\eta}}_s = \mathbf{0}, \quad &\text{on} \quad \widehat{\Gamma}_{s,D}, \\ \widehat{\boldsymbol{F}}_s\widehat{\boldsymbol{\Sigma}}\widehat{\boldsymbol{n}}_s = \widehat{J}_s\|\widehat{\boldsymbol{F}}^{-T}\widehat{\boldsymbol{n}}_s\|\widehat{\boldsymbol{g}}_{s,N}, \quad &\text{on} \quad \widehat{\Gamma}_{s,N}, \end{aligned} \tag{3.97}$$

with $\widehat{\boldsymbol{\Sigma}}$ related to $\widehat{\boldsymbol{\eta}}_s$ through a constitutive law of the form (3.55).

The fluid and solid problems (3.96) and (3.97) must be coupled ensuring a global energy balance. This is achieved by imposing three interface coupling conditions: geometry, velocity and stress.

Geometry coupling: construction of the ALE map

We first enforce that the moving control volume follows the interface motion, *i.e.*

$$\widetilde{\mathcal{A}} = \widehat{\boldsymbol{\varphi}}_s, \quad \text{on} \quad \widehat{\Gamma}, \tag{3.98}$$

this is a geometry coupling condition. Since we describe the motion of the solid in terms of its displacement $\widehat{\boldsymbol{\eta}}_s$, it is also useful to describe the ALE map in terms of the displacement of the control volume, $\widetilde{\boldsymbol{\eta}}_f : \widetilde{\Omega_f} \times \mathbb{R}^+ \longrightarrow \mathbb{R}^3$, defined by

$$\widetilde{\boldsymbol{\eta}}_f(\widetilde{\boldsymbol{x}}, t) = \widetilde{\mathcal{A}}(\widetilde{\boldsymbol{x}}, t) - \widetilde{\boldsymbol{x}},$$

for all $\widetilde{\boldsymbol{x}} \in \widetilde{\Omega_f}$. Thus, (3.98) reduces to

$$\widetilde{\boldsymbol{\eta}}_f = \widehat{\boldsymbol{\eta}}_s, \quad \text{on} \quad \widehat{\Gamma}. \tag{3.99}$$

By differentiating this equality with respect to t, it follows that

$$\widetilde{\boldsymbol{w}} = \widehat{\boldsymbol{u}_s}, \quad \text{on} \quad \widehat{\Gamma}. \tag{3.100}$$

On the other hand, since the inlet and outlet boundaries remain fixed along the motion, we also have

$$\widetilde{\boldsymbol{\eta}}_f = \mathbf{0}, \quad \text{on} \quad \widetilde{\Gamma}_{f,D} \cup \widetilde{\Gamma}_{f,N}. \tag{3.101}$$

Notice that (3.99) and (3.101) provide the value of $\widetilde{\boldsymbol{\eta}}_f$ on the boundary of $\widetilde{\Omega_f}$. However, inside $\widetilde{\Omega_f}$, f $\widetilde{\boldsymbol{\eta}}_f$ (and hence $\widetilde{\mathcal{A}}$) is arbitrary: it can be any reasonable extension of $\widehat{\boldsymbol{\eta}}_{s|\widehat{\Gamma}}$ over $\widetilde{\Omega_f}$ (subjected to (3.101)). In the sequel we will denote this operation by

$$\widetilde{\boldsymbol{\eta}}_f = \mathrm{Ext}(\widehat{\boldsymbol{\eta}}_{s|\widehat{\Gamma}}). \tag{3.102}$$

For instance, the operator Ext can be given in terms of an harmonic extension, by solving:

$$\begin{aligned} -\Delta\widetilde{\boldsymbol{\eta}}_f &= \mathbf{0}, \quad && \text{in} \quad \widetilde{\Omega_f}, \\ \widetilde{\boldsymbol{\eta}}_f &= \mathbf{0}, \quad && \text{on} \quad \widetilde{\Gamma}_{f,D} \cup \widetilde{\Gamma}_{f,N}, \\ \widetilde{\boldsymbol{\eta}}_f &= \widehat{\boldsymbol{\eta}}_s, \quad && \text{on} \quad \widehat{\Gamma}. \end{aligned} \tag{3.103}$$

Continuity of velocity and stress

Since the fluid is assumed to be viscous, it perfectly sticks to the interface (or solid) boundary. This means that the whole velocity field must be continuous at the interface. Thus, we set

$$\boldsymbol{u}_f = \boldsymbol{w}, \quad \text{on} \quad \Gamma(t). \tag{3.104}$$

Finally, in order to ensure the balance of stresses on the interface, we enforce the continuity of stress at the interface. Thus, using the properties of the Piola transform (Proposition 3.2) we get the coupling condition

$$\widehat{\boldsymbol{F}}_s \widehat{\boldsymbol{\Sigma}} \widehat{\boldsymbol{n}}_s + \widetilde{J}_{\widetilde{\mathcal{A}}} \widetilde{\boldsymbol{\sigma}}_f(\boldsymbol{u}_f, P) \widetilde{\boldsymbol{F}}_{\widetilde{\mathcal{A}}}^{-T} \widetilde{\boldsymbol{n}}_f = 0, \quad \text{on} \quad \widehat{\Gamma}. \tag{3.105}$$

The coupled fluid-structure problem

Using the coupling conditions (3.102), (3.104) and (3.105) the coupled fluid-structure interaction problem reads: find $\widetilde{\boldsymbol{\eta}}_f : \widetilde{\Omega_f} \times \mathbb{R}^+ \longrightarrow \mathbb{R}^3$, $\widetilde{\boldsymbol{u}_f} : \widetilde{\Omega_f} \times \mathbb{R}^+ \longrightarrow \mathbb{R}^3$, $\widetilde{p} : \widetilde{\Omega_f} \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ and $\widehat{\boldsymbol{\eta}}_s : \widehat{\Omega_s} \times \mathbb{R}^+ \longrightarrow \mathbb{R}^3$, such that

- Fluid sub-problem:

$$\begin{cases} \rho_f \dfrac{\partial \boldsymbol{u}_f}{\partial t}_{|\widetilde{\mathcal{A}}} + \rho_f(\boldsymbol{u}_f - \boldsymbol{w}) \cdot \boldsymbol{\nabla}\boldsymbol{u}_f - \mathbf{div}\boldsymbol{\sigma}_f(\boldsymbol{u}_f, P) = \mathbf{0}, & \text{in} \quad \Omega_f(t), \\ \qquad\qquad \mathrm{div}\, \boldsymbol{u}_f = 0, & \text{in} \quad \Omega_f(t), \\ \qquad\qquad \boldsymbol{\sigma}_f(\boldsymbol{u}_f, P)\boldsymbol{n}_f = \boldsymbol{g}_{f,N}, & \text{on} \quad \Gamma_{f,N}. \end{cases} \tag{3.106}$$

- Solid sub-problem:

$$\begin{cases} \hat{\rho}_{s,0}\dfrac{\partial^2 \widehat{\boldsymbol{\eta}}_s}{\partial t^2} - \mathbf{div}_{\hat{\boldsymbol{x}}}\left(\widehat{\boldsymbol{F}}_s \widehat{\boldsymbol{\Sigma}}\right) = \mathbf{0}, & \text{in} \quad \widehat{\Omega_s}, \\ \widehat{\boldsymbol{\eta}}_s = \mathbf{0}, & \text{on} \quad \Gamma_{s,D}, \\ \widehat{\boldsymbol{F}}_s \widehat{\boldsymbol{\Sigma}} \widehat{\boldsymbol{n}}_s = \mathbf{0}, & \text{on} \quad \widehat{\Gamma}_{s,N}. \end{cases} \tag{3.107}$$

- Coupling conditions:

$$\begin{cases} \widetilde{\boldsymbol{\eta}}_f = \mathrm{Ext}(\widehat{\boldsymbol{\eta}}_{s|\widehat{\Gamma}}), \quad \Omega_f(t) = \widetilde{\mathcal{A}}(\widetilde{\Omega_f}, t), \quad \widetilde{\boldsymbol{w}} = \dfrac{\partial \widetilde{\boldsymbol{\eta}}_f}{\partial t}, & \text{in} \quad \widetilde{\Omega_f}, \\ \boldsymbol{u}_f = \boldsymbol{w}, & \text{on} \quad \Gamma(t), \\ \widehat{\boldsymbol{F}}_s \widehat{\boldsymbol{\Sigma}} \widehat{\boldsymbol{n}}_s + \widetilde{J}_{\widetilde{\mathcal{A}}} \tilde{\boldsymbol{\sigma}}_f(\boldsymbol{u}_f, P) \widetilde{\boldsymbol{F}}_{\widetilde{\mathcal{A}}}^{-T} \widetilde{\boldsymbol{n}}_f = 0, & \text{on} \quad \widehat{\Gamma}. \end{cases} \tag{3.108}$$

This is a complex non-linear system of equations whose numerical treatment is addressed in Chapter 9. Some elements for its mathematical analysis, under some simplifying hypotheses, are reported in Chapter 8.

3.6 Conclusions

We have derived the equations that governs the dynamics of fluid and structure, in the hypothesis of a continuum media. If this is a reasonable assumption for the structure and for blood flow in the main vessels, the continuum hypothesis may become questionable for the flow in small capillaries, where the dimension of the vessel becomes comparable to that of blood cells.

This issue is treated partly in Chapter 6 and we will not investigate it further in this book. We will see in other chapters, in particular Chapter 10, how models of the global circulation may account for the haemodynamics in the capillary bed by using suitable lumped parameter models. Indeed, the details of the flow in the capillaries is needed only in specific microcirculation studies.

Even if we have assumed a Newtonian behaviour for the fluid, we have derived the flow equations in generality, and they may be easily adapted to the more complex rheological models presented in Chapter 6.

As for the models for the vessel wall, the actual structure of a blood vessel is rather complex, as explained in Chapter 1. We have preferred giving here the most basic models, giving reference to the interested reader of the specialised literature where more sophisticated modelling may be found. In fact, if one is interested mainly on the effect of the structure movements on the haemodynamics there is no need of using complex models for the latter. They are instead mandatory if one is interested in having the details of the stress fields inside the vessel wall.

We wish to warn the reader that usually the more complex a model is the higher number of parameters it requires. Experiments to determine those parameters are often complex, in particular if one wishes to fit them to a specific person. In this case only indirect measurements are at disposal, *ex vivo* experiments being obviously out of question.